

ELEMENTS BOOK 1

*Fundamentals of Plane Geometry Involving
Straight-Lines*

Ὅροι.

- α'. Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.
- β'. Γραμμὴ δὲ μῆκος ἀπλατές.
- γ'. Γραμμῆς δὲ πέρατα σημεῖα.
- δ'. Εὐθεῖα γραμμὴ ἐστίν, ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς σημείοις κεῖται.
- ε'. Ἐπιφάνεια δὲ ἐστίν, ὃ μῆκος καὶ πλάτος μόνον ἔχει.
- ς'. Ἐπιφανείας δὲ πέρατα γραμμαί.
- ζ'. Ἐπίπεδος ἐπιφάνειά ἐστίν, ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς εὐθείαις κεῖται.
- η'. Ἐπίπεδος δὲ γωνία ἐστίν ἡ ἐν ἐπιπέδῳ δύο γραμμῶν ἀπτομένων ἀλλήλων καὶ μὴ ἐπ' εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις.
- θ'. Ὄταν δὲ αἱ περιέχουσιν τὴν γωνίαν γραμμαὶ εὐθεῖαι ὦσιν, εὐθύγραμμος καλεῖται ἡ γωνία.
- ι'. Ὄταν δὲ εὐθεῖα ἐπ' εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστί, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται, ἐφ' ἣν ἐφέστηκεν.
- ια'. Ἀμβλεία γωνία ἐστίν ἡ μείζων ὀρθῆς.
- ιβ'. Ὄξεϊα δὲ ἡ ἐλάσσων ὀρθῆς.
- ιγ'. Ὅρος ἐστίν, ὃ τίνος ἐστί πέρας.
- ιδ'. Σχήμα ἐστὶ τὸ ὑπὸ τινος ἢ τινων ὄρων περιεχόμενον.
- ιε'. Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον [ἢ καλεῖται περιφέρεια], πρὸς ἣν ἀφ' ἐνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσιν εὐθεῖαι [πρὸς τὴν τοῦ κύκλου περιφέρειαν] ἴσαι ἀλλήλαις εἰσίν.
- ις'. Κέντρον δὲ τοῦ κύκλου τὸ σημεῖον καλεῖται.
- ιζ'. Διάμετρος δὲ τοῦ κύκλου ἐστὶν εὐθεῖα τις διὰ τοῦ κέντρου ἡγμένη καὶ περατουμένη ἐφ' ἑκάτερα τὰ μέρη ὑπὸ τῆς τοῦ κύκλου περιφερείας, ἥτις καὶ δίχα τέμνει τὸν κύκλον.
- ιη'. Ἡμικύκλιον δὲ ἐστὶ τὸ περιεχόμενον σχῆμα ὑπὸ τε τῆς διαμέτρου καὶ τῆς ἀπολαμβανομένης ὑπ' αὐτῆς περιφερείας. κέντρον δὲ τοῦ ἡμικυκλίου τὸ αὐτό, ὃ καὶ τοῦ κύκλου ἐστίν.
- ιθ'. Σχήματα εὐθύγραμμά ἐστί τὰ ὑπὸ εὐθειῶν περιεχόμενα, τρίπλευρα μὲν τὰ ὑπὸ τριῶν, τετράπλευρα δὲ τὰ ὑπὸ τεσσάρων, πολὺπλευρα δὲ τὰ ὑπὸ πλείονων ἢ τεσσάρων εὐθειῶν περιεχόμενα.
- κ'. Τῶν δὲ τριπλεύρων σχημάτων ἰσόπλευρον μὲν τρίγωνόν ἐστι τὸ τὰς τρεῖς ἴσας ἔχον πλευράς, ἰσοσκελὲς δὲ τὸ τὰς δύο μόνας ἴσας ἔχον πλευράς, σκαληνὸν δὲ τὸ τὰς τρεῖς ἀνίσους ἔχον πλευράς.
- κα' Ἐτι δὲ τῶν τριπλεύρων σχημάτων ὀρθογώνιον μὲν τρίγωνόν ἐστι τὸ ἔχον ὀρθὴν γωνίαν, ἀμβλυγώνιον δὲ τὸ ἔχον ἀμβλείαν γωνίαν, ὀξυγώνιον δὲ τὸ τὰς τρεῖς ὀξείας ἔχον γωνίας.

Definitions

1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points.
4. A straight-line is (any) one which lies evenly with points on itself.
5. And a surface is that which has length and breadth only.
6. And the extremities of a surface are lines.
7. A plane surface is (any) one which lies evenly with the straight-lines on itself.
8. And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight-line.
9. And when the lines containing the angle are straight then the angle is called rectilinear.
10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands.
11. An obtuse angle is one greater than a right-angle.
12. And an acute angle (is) one less than a right-angle.
13. A boundary is that which is the extremity of something.
14. A figure is that which is contained by some boundary or boundaries.
15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.
16. And the point is called the center of the circle.
17. And a diameter of the circle is any straight-line, being drawn through the center, and terminated in each direction by the circumference of the circle. (And) any such (straight-line) also cuts the circle in half.[†]
18. And a semi-circle is the figure contained by the diameter and the circumference cuts off by it. And the center of the semi-circle is the same (point) as (the center of) the circle.
19. Rectilinear figures are those (figures) contained by straight-lines: trilateral figures being those contained by three straight-lines, quadrilateral by four, and multilateral by more than four.
20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.

κβ'. Τῶν δὲ τετραπλεύρων σχημάτων τετράγωνον μὲν ἐστίν, ὃ ἰσόπλευρόν τε ἐστὶ καὶ ὀρθογώνιον, ἑτερόμηκες δέ, ὃ ὀρθογώνιον μὲν, οὐκ ἰσόπλευρον δέ, ῥόμβος δέ, ὃ ἰσόπλευρον μὲν, οὐκ ὀρθογώνιον δέ, ῥομβοειδὲς δὲ τὸ τὰς ἀπεναντίον πλευράς τε καὶ γωνίας ἴσας ἀλλήλαις ἔχον, ὃ οὔτε ἰσόπλευρόν ἐστίν οὔτε ὀρθογώνιον· τὰ δὲ παρὰ ταῦτα τετράπλευρα τραπέζια καλεῖσθω.

κγ'. Παράλληλοι εἰσιν εὐθεῖαι, αἵτινες ἐν τῷ αὐτῷ ἐπιπέδῳ οὔσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ' ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις.

21. And further of the trilateral figures: a right-angled triangle is that having a right-angle, an obtuse-angled (triangle) that having an obtuse angle, and an acute-angled (triangle) that having three acute angles.

22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.

23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).

† This should really be counted as a postulate, rather than as part of a definition.

Αἰτήματα.

α'. Ἡτῆσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

β'. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ' εὐθείας ἐκβαλεῖν.

γ'. Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

δ'. Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

ε'. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ' ἄπειρον συμπίπτειν, ἐφ' ᾧ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

Postulates

1. Let it have been postulated[†] to draw a straight-line from any point to any point.

2. And to produce a finite straight-line continuously in a straight-line.

3. And to draw a circle with any center and radius.

4. And that all right-angles are equal to one another.

5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).[‡]

† The Greek present perfect tense indicates a past action with present significance. Hence, the 3rd-person present perfect imperative Ἡτῆσθω could be translated as “let it be postulated”, in the sense “let it stand as postulated”, but not “let the postulate be now brought forward”. The literal translation “let it have been postulated” sounds awkward in English, but more accurately captures the meaning of the Greek.

‡ This postulate effectively specifies that we are dealing with the geometry of *flat*, rather than curved, space.

Κοινὰ ἔννοιαι.

α'. Τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.

β'. Καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἴσα.

γ'. Καὶ ἐὰν ἀπὸ ἴσων ἴσα ἀφαιρεθῇ, τὰ καταλειπόμενά ἐστιν ἴσα.

δ'. Καὶ τὰ ἐφαρμόζοντα ἐπ' ἀλλήλα ἴσα ἀλλήλοις ἐστὶν.

ε'. Καὶ τὸ ὅλον τοῦ μέρους μεῖζόν [ἐστίν].

Common Notions

1. Things equal to the same thing are also equal to one another.

2. And if equal things are added to equal things then the wholes are equal.

3. And if equal things are subtracted from equal things then the remainders are equal.[†]

4. And things coinciding with one another are equal to one another.

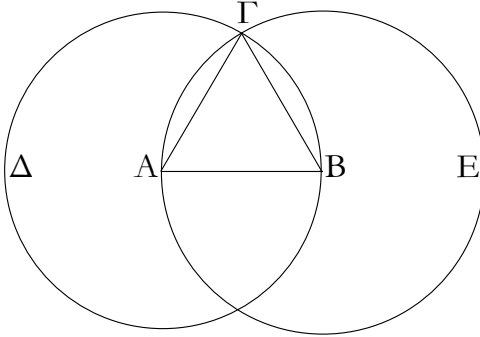
5. And the whole [is] greater than the part.

† As an obvious extension of C.N.s 2 & 3—if equal things are added or subtracted from the two sides of an inequality then the inequality remains

an inequality of the same type.

α'.

Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τριγώνον ἰσόπλευρον συστήσασθαι.



Ἐστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ AB.

Δεῖ δὴ ἐπὶ τῆς AB εὐθείας τριγώνον ἰσόπλευρον συστήσασθαι.

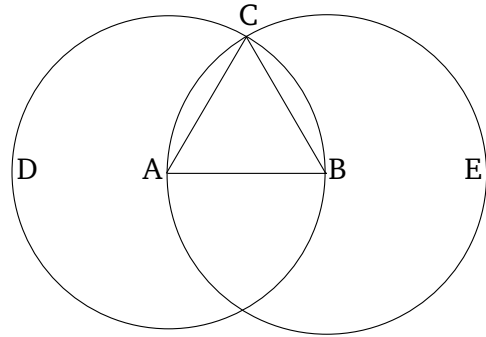
Κέντρῳ μὲν τῷ A διαστήματι δὲ τῷ AB κύκλος γεγράφθω ὁ BΓΔ, καὶ πάλιν κέντρῳ μὲν τῷ B διαστήματι δὲ τῷ BA κύκλος γεγράφθω ὁ AΓΕ, καὶ ἀπὸ τοῦ Γ σημείου, καθ' ὃ τέμνουσιν ἀλλήλους οἱ κύκλοι, ἐπὶ τὰ A, B σημεῖα ἐπεζεύχθωσαν εὐθεῖαι αἱ ΓΑ, ΓΒ.

Καὶ ἐπεὶ τὸ A σημεῖον κέντρον ἐστὶ τοῦ ΓΔΒ κύκλου, ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ· πάλιν, ἐπεὶ τὸ B σημεῖον κέντρον ἐστὶ τοῦ ΓΑΕ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΑ. ἐδείχθη δὲ καὶ ἡ ΓΑ τῇ ΑΒ ἴση· ἑκάτερα ἄρα τῶν ΓΑ, ΓΒ τῇ ΑΒ ἐστὶν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΓΑ ἄρα τῇ ΓΒ ἐστὶν ἴση· αἱ τρεῖς ἄρα αἱ ΓΑ, ΑΒ, ΒΓ ἴσαι ἀλλήλαις εἰσὶν.

Ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον. καὶ συνέσταται ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τῆς ΑΒ. ὅπερ ἔδει ποιῆσαι.

Proposition 1

To construct an equilateral triangle on a given finite straight-line.



Let AB be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line AB .

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C , where the circles cut one another,[†] to the points A and B (respectively) [Post. 1].

And since the point A is the center of the circle CDB , AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE , BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB . Thus, CA and CB are each equal to AB . But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB . Thus, the three (straight-lines) CA , AB , and BC are equal to one another.

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB . (Which is) the very thing it was required to do.

[†] The assumption that the circles do indeed cut one another should be counted as an additional postulate. There is also an implicit assumption that two straight-lines cannot share a common segment.

β'.

Πρὸς τῷ δοθέντι σημείῳ τῇ δοθείσῃ εὐθείᾳ ἴσην εὐθεῖαν θέσθαι.

Ἐστω τὸ μὲν δοθὲν σημεῖον τὸ A, ἡ δὲ δοθεῖσα εὐθεῖα ἡ ΒΓ· δεῖ δὴ πρὸς τῷ A σημείῳ τῇ δοθείσῃ εὐθείᾳ τῇ ΒΓ ἴσην εὐθεῖαν θέσθαι.

Ἐπεζεύχθω γὰρ ἀπὸ τοῦ A σημείου ἐπὶ τὸ B σημεῖον εὐθεῖα ἡ ΑΒ, καὶ συνεστάτω ἐπ' αὐτῆς τρίγωνον ἰσόπλευρον τὸ ΔΑΒ, καὶ ἐκβεβλήσθωσαν ἐπ' εὐθείας ταῖς ΔΑ, ΔΒ

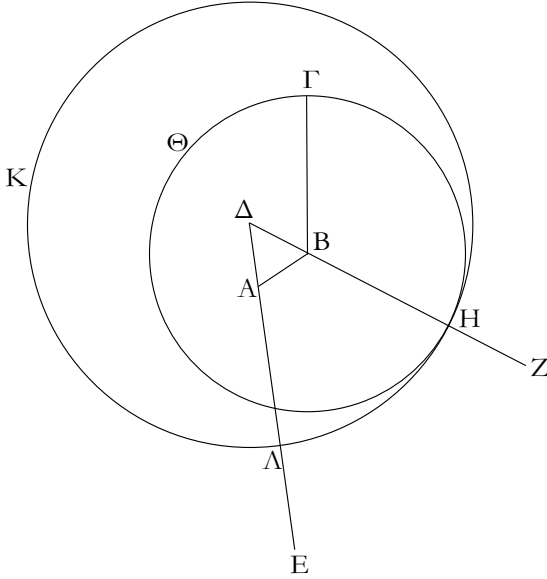
Proposition 2[†]

To place a straight-line equal to a given straight-line at a given point (as an extremity).

Let A be the given point, and BC the given straight-line. So it is required to place a straight-line at point A equal to the given straight-line BC .

For let the straight-line AB have been joined from point A to point B [Post. 1], and let the equilateral triangle DAB have been constructed upon it [Prop. 1.1].

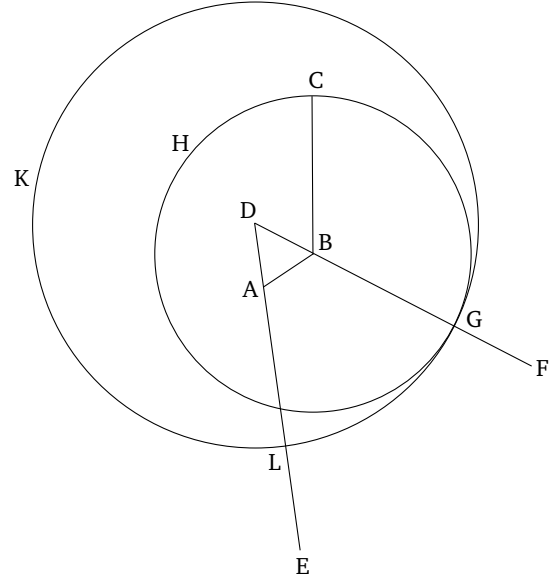
εὐθεΐαι αἱ AE , BZ , καὶ κέντρῳ μὲν τῷ B διαστήματι δὲ τῷ $B\Gamma$ κύκλος γεγράφθω ὁ $\Gamma H\Theta$, καὶ πάλιν κέντρῳ τῷ Δ καὶ διαστήματι τῷ ΔH κύκλος γεγράφθω ὁ $H\kappa\Lambda$.



Ἐπεὶ οὖν τὸ B σημεῖον κέντρον ἐστὶ τοῦ $\Gamma H\Theta$, ἴση ἐστὶν ἡ $B\Gamma$ τῇ BH . πάλιν, ἐπεὶ τὸ Δ σημεῖον κέντρον ἐστὶ τοῦ $H\kappa\Lambda$ κύκλου, ἴση ἐστὶν ἡ $\Delta\Lambda$ τῇ ΔH , ὥν ἡ ΔA τῇ ΔB ἴση ἐστὶν. λοιπὴ ἄρα ἡ AL λοιπῇ τῇ BH ἐστὶν ἴση. ἐδείχθη δὲ καὶ ἡ $B\Gamma$ τῇ BH ἴση· ἑκατέρα ἄρα τῶν AL , $B\Gamma$ τῇ BH ἐστὶν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ AL ἄρα τῇ $B\Gamma$ ἐστὶν ἴση.

Πρὸς ἄρα τῷ δοθέντι σημείῳ τῷ A τῇ δοθείσῃ εὐθείᾳ τῇ $B\Gamma$ ἴση εὐθεΐα κεῖται ἡ AL · ὅπερ ἔδει ποιῆσαι.

And let the straight-lines AE and BZ have been produced in a straight-line with DA and DB (respectively) [Post. 2]. And let the circle CGH with center B and radius BC have been drawn [Post. 3], and again let the circle GKL with center D and radius DG have been drawn [Post. 3].



Therefore, since the point B is the center of (the circle) CGH , BC is equal to BG [Def. 1.15]. Again, since the point D is the center of the circle GKL , DL is equal to DG [Def. 1.15]. And within these, DA is equal to DB . Thus, the remainder AL is equal to the remainder BG [C.N. 3]. But BC was also shown (to be) equal to BG . Thus, AL and BC are each equal to BG . But things equal to the same thing are also equal to one another [C.N. 1]. Thus, AL is also equal to BC .

Thus, the straight-line AL , equal to the given straight-line BC , has been placed at the given point A . (Which is) the very thing it was required to do.

† This proposition admits of a number of different cases, depending on the relative positions of the point A and the line BC . In such situations, Euclid invariably only considers one particular case—usually, the most difficult—and leaves the remaining cases as exercises for the reader.

γ'.

Δύο δοθεισῶν εὐθειῶν ἀνίσων ἀπὸ τῆς μείζονος τῇ ἐλάσσονι ἴσην εὐθεΐαν ἀφελεῖν.

Ἐστωσαν αἱ δοθεῖσαι δύο εὐθεΐαι ἄνισοι αἱ AB , Γ , ὧν μείζων ἔστω ἡ AB · δεῖ δὴ ἀπὸ τῆς μείζονος τῆς AB τῇ ἐλάσσονι τῇ Γ ἴσην εὐθεΐαν ἀφελεῖν.

Κείσθω πρὸς τῷ A σημείῳ τῇ Γ εὐθείᾳ ἴση ἡ AD · καὶ κέντρῳ μὲν τῷ A διαστήματι δὲ τῷ AD κύκλος γεγράφθω ὁ ΔEZ .

Καὶ ἐπεὶ τὸ A σημεῖον κέντρον ἐστὶ τοῦ ΔEZ κύκλου,

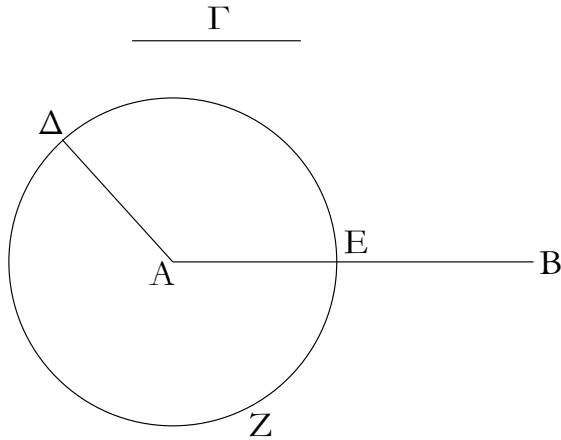
Proposition 3

For two given unequal straight-lines, to cut off from the greater a straight-line equal to the lesser.

Let AB and C be the two given unequal straight-lines, of which let the greater be AB . So it is required to cut off a straight-line equal to the lesser C from the greater AB .

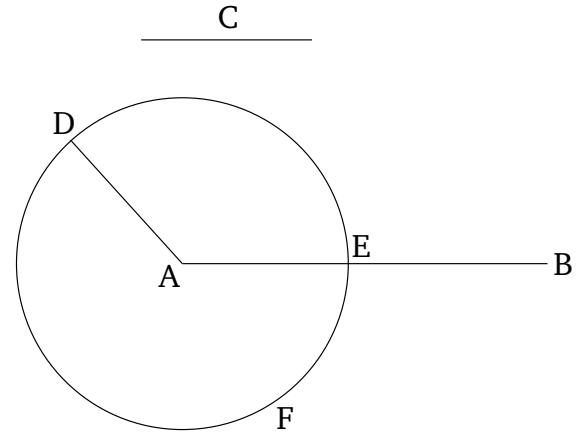
Let the line AD , equal to the straight-line C , have been placed at point A [Prop. 1.2]. And let the circle DEF have been drawn with center A and radius AD [Post. 3].

ἴση ἐστὶν ἡ AE τῇ AD · ἀλλὰ καὶ ἡ Γ τῇ AD ἐστὶν ἴση· ἑκατέρα ἄρα τῶν AE , Γ τῇ AD ἐστὶν ἴση· ὥστε καὶ ἡ AE τῇ Γ ἐστὶν ἴση.



Δύο ἄρα δοθεισῶν εὐθειῶν ἀνίσων τῶν AB , Γ ἀπὸ τῆς μείζονος τῆς AB τῇ ἐλάσσονι τῇ Γ ἴση ἀφῆρηται ἡ AE · ὅπερ ἔδει ποιῆσαι.

And since point A is the center of circle DEF , AE is equal to AD [Def. 1.15]. But, C is also equal to AD . Thus, AE and C are each equal to AD . So AE is also equal to C [C.N. 1].

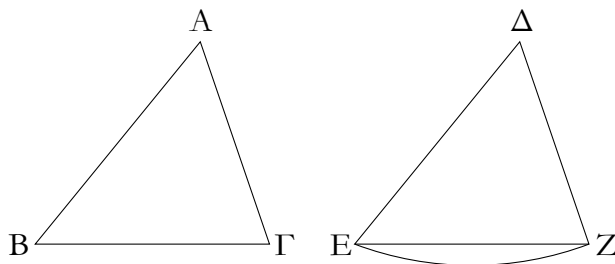


Thus, for two given unequal straight-lines, AB and C , the (straight-line) AE , equal to the lesser C , has been cut off from the greater AB . (Which is) the very thing it was required to do.

δ'.

Proposition 4

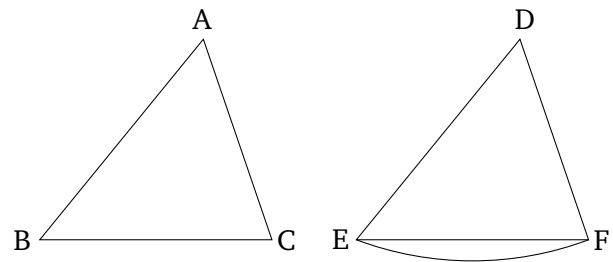
Ἐὰν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δυοὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῇ βάσει ἴσην ἔξει, καὶ τὸ τρίγωνον τῷ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν.



Ἐστω δύο τρίγωνα τὰ $AB\Gamma$, ΔEZ τὰς δύο πλευρὰς τὰς AB , $A\Gamma$ ταῖς δυοὶ πλευραῖς ταῖς ΔE , ΔZ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ τὴν μὲν AB τῇ ΔE τὴν δὲ $A\Gamma$ τῇ ΔZ καὶ γωνίαν τὴν ὑπὸ BAG γωνίᾳ τῇ ὑπὸ $E\Delta Z$ ἴσην. λέγω, ὅτι καὶ βάσις ἡ $B\Gamma$ βάσει τῇ EZ ἴση ἐστὶν, καὶ τὸ $AB\Gamma$ τρίγωνον τῷ ΔEZ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρα ἑκατέρᾳ, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν, ἡ μὲν ὑπὸ $AB\Gamma$ τῇ ὑπὸ ΔEZ , ἡ δὲ ὑπὸ $A\Gamma B$ τῇ ὑπὸ $\Delta Z E$.

Ἐφαρμοζομένου γὰρ τοῦ $AB\Gamma$ τριγώνου ἐπὶ τὸ ΔEZ τρίγωνον καὶ τιθεμένου τοῦ μὲν A σημείου ἐπὶ τὸ Δ σημεῖον

If two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-lines equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles.



Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF , respectively. (That is) AB to DE , and AC to DF . And (let) the angle BAC (be) equal to the angle EDF . I say that the base BC is also equal to the base EF , and triangle ABC will be equal to triangle DEF , and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. (That is) ABC to DEF , and ACB to DFE .

For if triangle ABC is applied to triangle DEF ,[†] the point A being placed on the point D , and the straight-line

τῆς δὲ AB εὐθείας ἐπὶ τὴν DE , ἐφαρμόσει καὶ τὸ B σημεῖον ἐπὶ τὸ E διὰ τὸ ἴσιν εἶναι τὴν AB τῇ DE . ἐφαρμοσάσης δὴ τῆς AB ἐπὶ τὴν DE ἐφαρμόσει καὶ ἡ AG εὐθεῖα ἐπὶ τὴν DZ διὰ τὸ ἴσιν εἶναι τὴν ὑπὸ BAG γωνίαν τῇ ὑπὸ EDZ . ὥστε καὶ τὸ Γ σημεῖον ἐπὶ τὸ Z σημεῖον ἐφαρμόσει διὰ τὸ ἴσιν πάλιν εἶναι τὴν AG τῇ DZ . ἀλλὰ μὴν καὶ τὸ B ἐπὶ τὸ E ἐφαρμόσκει· ὥστε βάσις ἡ BG ἐπὶ βάσιν τὴν EZ ἐφαρμόσει. εἰ γὰρ τοῦ μὲν B ἐπὶ τὸ E ἐφαρμόσαντος τοῦ δὲ Γ ἐπὶ τὸ Z ἡ BG βάσις ἐπὶ τὴν EZ οὐκ ἐφαρμόσει, δύο εὐθεῖαι χωρίον περιέξουσιν· ὅπερ ἐστὶν ἀδύνατον. ἐφαρμόσει ἄρα ἡ BG βάσις ἐπὶ τὴν EZ καὶ ἴση αὐτῇ ἔσται· ὥστε καὶ ὅλον τὸ ABG τρίγωνον ἐπὶ ὅλον τὸ DEZ τρίγωνον ἐφαρμόσει καὶ ἴσον αὐτῷ ἔσται, καὶ αἱ λοιπαὶ γωνίαι ἐπὶ τὰς λοιπὰς γωνίας ἐφαρμόσουσι καὶ ἴσαι αὐταῖς ἔσονται, ἡ μὲν ὑπὸ ABG τῇ ὑπὸ DEZ ἡ δὲ ὑπὸ AGB τῇ ὑπὸ DZE .

Ἐάν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσιν ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῇ βάσει ἴσιν ἔξει, καὶ τὸ τρίγωνον τῷ τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρᾳ ἑκατέρᾳ, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

AB on DE , then the point B will also coincide with E , on account of AB being equal to DE . So (because of) AB coinciding with DE , the straight-line AC will also coincide with DF , on account of the angle BAC being equal to EDF . So the point C will also coincide with the point F , again on account of AC being equal to DF . But, point B certainly also coincided with point E , so that the base BC will coincide with the base EF . For if B coincides with E , and C with F , and the base BC does not coincide with EF , then two straight-lines will encompass an area. The very thing is impossible [Post. 1].[†] Thus, the base BC will coincide with EF , and will be equal to it [C.N. 4]. So the whole triangle ABC will coincide with the whole triangle DEF , and will be equal to it [C.N. 4]. And the remaining angles will coincide with the remaining angles, and will be equal to them [C.N. 4]. (That is) ABC to DEF , and ACB to DFE [C.N. 4].

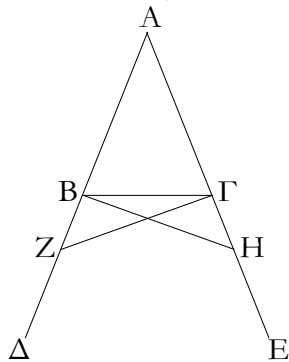
Thus, if two triangles have two sides equal to two sides, respectively, and have the angle(s) enclosed by the equal straight-line equal, then they will also have the base equal to the base, and the triangle will be equal to the triangle, and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles. (Which is) the very thing it was required to show.

[†] The application of one figure to another should be counted as an additional postulate.

[‡] Since Post. 1 implicitly assumes that the straight-line joining two given points is unique.

ε'.

Τῶν ἰσοσκελῶν τριγώνων αἱ τρὸς τῇ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ προσεκβληθεῖσιν τῶν ἴσων εὐθειῶν αἱ ὑπὸ τὴν βάσιν γωνίαι ἴσαι ἀλλήλαις ἔσονται.

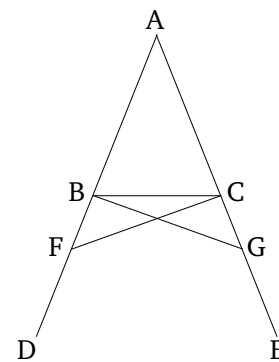


Ἐστω τρίγωνον ἰσοσκελὲς τὸ ABG ἴσιν ἔχον τὴν AB πλευρὰν τῇ AG πλευρᾷ, καὶ προσεκβεβλήσθωσαν ἐπ' εὐθείας ταῖς AB , AG εὐθεῖαι αἱ BD , GE . λέγω, ὅτι ἡ μὲν ὑπὸ ABG γωνία τῇ ὑπὸ AGB ἴση ἐστίν, ἡ δὲ ὑπὸ GBD τῇ ὑπὸ BGE .

Εἰληφθῶ γὰρ ἐπὶ τῆς BD τυχὸν σημεῖον τὸ Z , καὶ ἀφωρήσθω ἀπὸ τῆς μείζονος τῆς AE τῇ ἐλάσσονι τῇ AZ

Proposition 5

For isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another.



Let ABC be an isosceles triangle having the side AB equal to the side AC , and let the straight-lines BD and CE have been produced in a straight-line with AB and AC (respectively) [Post. 2]. I say that the angle ABC is equal to ACB , and (angle) CBD to BCE .

For let the point F have been taken at random on BD , and let AG have been cut off from the greater AE , equal

ἴση ἢ AH , καὶ ἐπεξεύχθησαν αἱ ZG , HB εὐθεῖαι.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν AZ τῇ AH ἢ δὲ AB τῇ AG , δύο δὲ αἱ ZA , AG δυοὶ ταῖς HA , AB ἴσαι εἰσὶν ἑκατέρω καὶ γωνίαν κοινὴν περιέχουσι τὴν ὑπὸ ZAH · βάσεις ἄρα ἡ ZG βάσει τῇ HB ἴση ἐστίν, καὶ τὸ AZG τρίγωνον τῷ AHB τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρω κατέρω, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν, ἡ μὲν ὑπὸ AGZ τῇ ὑπὸ ABH , ἡ δὲ ὑπὸ AZG τῇ ὑπὸ AHB . καὶ ἐπεὶ ὅλη ἡ AZ ὅλη τῇ AH ἐστὶν ἴση, ὦν ἡ AB τῇ AG ἐστὶν ἴση, λοιπὴ ἄρα ἡ BZ λοιπῇ τῇ GH ἐστὶν ἴση. ἐδείχθη δὲ καὶ ἡ ZG τῇ HB ἴση· δύο δὲ αἱ BZ , ZG δυοὶ ταῖς GH , HB ἴσαι εἰσὶν ἑκατέρω κατέρω· καὶ γωνία ἡ ὑπὸ BZG γωνία τῇ ὑπὸ GHB ἴση, καὶ βάσεις αὐτῶν κοινὴ ἡ BG · καὶ τὸ BZG ἄρα τρίγωνον τῷ GHB τριγώνῳ ἴσον ἔσται, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρω κατέρω, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ZBG τῇ ὑπὸ HGB ἢ δὲ ὑπὸ BGZ τῇ ὑπὸ GBH . ἐπεὶ οὖν ὅλη ἡ ὑπὸ ABH γωνία ὅλη τῇ ὑπὸ AGZ γωνίᾳ ἐδείχθη ἴση, ὦν ἡ ὑπὸ GBH τῇ ὑπὸ BGZ ἴση, λοιπὴ ἄρα ἡ ὑπὸ ABG λοιπῇ τῇ ὑπὸ AGB ἐστὶν ἴση· καὶ εἰσι πρὸς τῇ βάσει τοῦ ABG τριγώνου. ἐδείχθη δὲ καὶ ἡ ὑπὸ ZBG τῇ ὑπὸ HGB ἴση· καὶ εἰσὶν ὑπὸ τὴν βάσιν.

Τῶν ἄρα ἰσοσκελῶν τριγώνων αἱ πρὸς τῇ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσὶν, καὶ προσεκβληθεισῶν τῶν ἴσων εὐθειῶν αἱ ὑπὸ τὴν βάσιν γωνίαι ἴσαι ἀλλήλαις ἔσονται· ὅπερ ἔδει δεῖξαι.

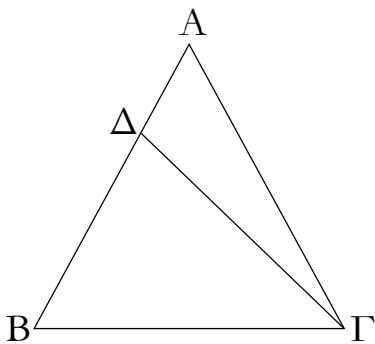
to the lesser AF [Prop. 1.3]. Also, let the straight-lines FC and GB have been joined [Post. 1].

In fact, since AF is equal to AG , and AB to AC , the two (straight-lines) FA , AC are equal to the two (straight-lines) GA , AB , respectively. They also encompass a common angle, FAG . Thus, the base FC is equal to the base GB , and the triangle AFC will be equal to the triangle AGB , and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. (That is) ACF to ABG , and AFC to AGB . And since the whole of AF is equal to the whole of AG , within which AB is equal to AC , the remainder BF is thus equal to the remainder CG [C.N. 3]. But FC was also shown (to be) equal to GB . So the two (straight-lines) BF , FC are equal to the two (straight-lines) CG , GB , respectively, and the angle BFC (is) equal to the angle CGB , and the base BC is common to them. Thus, the triangle BFC will be equal to the triangle CGB , and the remaining angles subtended by the equal sides will be equal to the corresponding remaining angles [Prop. 1.4]. Thus, FBC is equal to GCB , and BCF to CBG . Therefore, since the whole angle ABG was shown (to be) equal to the whole angle ACF , within which CBG is equal to BCF , the remainder ABC is thus equal to the remainder ACB [C.N. 3]. And they are at the base of triangle ABC . And FBC was also shown (to be) equal to GCB . And they are under the base.

Thus, for isosceles triangles, the angles at the base are equal to one another, and if the equal sides are produced then the angles under the base will be equal to one another. (Which is) the very thing it was required to show.

ε'.

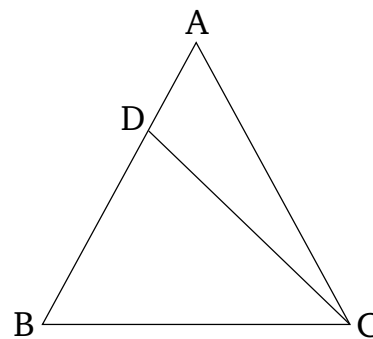
Ἐὰν τριγώνου αἱ δύο γωνίαι ἴσαι ἀλλήλαις ᾦσιν, καὶ αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσιν πλευραὶ ἴσαι ἀλλήλαις ἔσονται.



Ἐστω τρίγωνον τὸ ABG ἴσην ἔχον τὴν ὑπὸ ABG γωνίαν τῇ ὑπὸ AGB γωνίᾳ· λέγω, ὅτι καὶ πλευρὰ ἡ AB πλευρᾷ τῇ AG ἐστὶν ἴση.

Proposition 6

If a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another.



Let ABC be a triangle having the angle ABC equal to the angle ACB . I say that side AB is also equal to side AC .

Εἰ γὰρ ἄνισός ἐστιν ἡ AB τῇ AC , ἡ ἐτέρα αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ AB , καὶ ἀφρηθήσθω ἀπὸ τῆς μείζονος τῆς AB τῇ ἐλάττωι τῇ AC ἴση ἡ DB , καὶ ἐπεζεύχθω ἡ DC .

Ἐπεὶ οὖν ἴση ἐστὶν ἡ DB τῇ AC κοινὴ δὲ ἡ BC , δύο δὲ αἱ DB , BC δύο ταῖς AC , CB ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ γωνία ἡ ὑπὸ DBC γωνία τῇ ὑπὸ ACB ἐστὶν ἴση· βάσεις ἄρα ἡ DC βάσει τῇ AB ἴση ἐστίν, καὶ τὸ DBC τρίγωνον τῷ ACB τριγώνῳ ἴσον ἔσται, τὸ ἑλασσον τῷ μείζονι· ὅπερ ἄτοπον· οὐκ ἄρα ἄνισός ἐστιν ἡ AB τῇ AC · ἴση ἄρα.

Ἐάν ἄρα τριγώνου αἱ δύο γωνίαι ἴσαι ἀλλήλαις ὦσιν, καὶ αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι πλευραὶ ἴσαι ἀλλήλαις ἔσονται· ὅπερ ἔδει δεῖξαι.

For if AB is unequal to AC then one of them is greater. Let AB be greater. And let DB , equal to the lesser AC , have been cut off from the greater AB [Prop. 1.3]. And let DC have been joined [Post. 1].

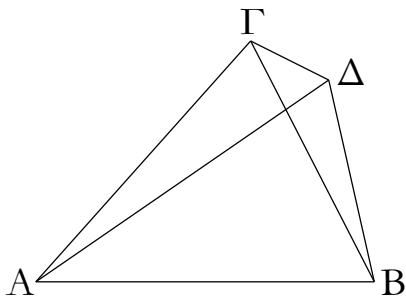
Therefore, since DB is equal to AC , and BC (is) common, the two sides DB , BC are equal to the two sides AC , CB , respectively, and the angle DBC is equal to the angle ACB . Thus, the base DC is equal to the base AB , and the triangle DBC will be equal to the triangle ACB [Prop. 1.4], the lesser to the greater. The very notion (is) absurd [C.N. 5]. Thus, AB is not unequal to AC . Thus, (it is) equal.[†]

Thus, if a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another. (Which is) the very thing it was required to show.

[†] Here, use is made of the previously unmentioned common notion that if two quantities are not unequal then they must be equal. Later on, use is made of the closely related common notion that if two quantities are not greater than or less than one another, respectively, then they must be equal to one another.

ζ'.

Ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρᾳ οὐ συσταθήσονται πρὸς ἄλλω καὶ ἄλλω σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις.



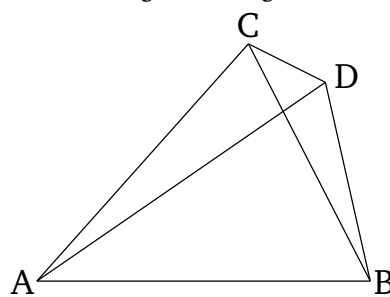
Εἰ γὰρ δυνατόν, ἐπὶ τῆς αὐτῆς εὐθείας τῆς AB δύο ταῖς αὐταῖς εὐθείαις ταῖς AC , CB ἄλλαι δύο εὐθεῖαι αἱ AD , DB ἴσαι ἑκατέρα ἑκατέρᾳ συνεστάτωσαν πρὸς ἄλλω καὶ ἄλλω σημείῳ τῷ τε Γ καὶ Δ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι, ὥστε ἴσην εἶναι τὴν μὲν ΓA τῇ ΔA τὸ αὐτὸ πέρασ ἔχουσιν αὐτῇ τὸ A , τὴν δὲ ΓB τῇ ΔB τὸ αὐτὸ πέρασ ἔχουσιν αὐτῇ τὸ B , καὶ ἐπεζεύχθω ἡ CD .

Ἐπεὶ οὖν ἴση ἐστὶν ἡ AC τῇ AD , ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ACB τῇ ὑπὸ ADB · μείζων ἄρα ἡ ὑπὸ ACB τῆς ὑπὸ ADB · πολλῶν ἄρα ἡ ὑπὸ ACB μείζων ἐστὶ τῆς ὑπὸ ADB . πάλιν ἐπεὶ ἴση ἐστὶν ἡ CB τῇ DB , ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ACB γωνία τῇ ὑπὸ ADB . ἐδείχθη δὲ αὐτῆς καὶ πολλῶν μείζων· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις

Proposition 7

On the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side (of the straight-line), but having the same ends as the given straight-lines.



For, if possible, let the two straight-lines AC , CB , equal to two other straight-lines AD , DB , respectively, have been constructed on the same straight-line AB , meeting at different points, C and D , on the same side (of AB), and having the same ends (on AB). So CA is equal to DA , having the same end A as it, and CB is equal to DB , having the same end B as it. And let CD have been joined [Post. 1].

Therefore, since AC is equal to AD , the angle ACD is also equal to angle ADC [Prop. 1.5]. Thus, ADC (is) greater than DCB [C.N. 5]. Thus, CDB is much greater than DCB [C.N. 5]. Again, since CB is equal to DB , the angle CDB is also equal to angle DCB [Prop. 1.5]. But it was shown that the former (angle) is also much greater

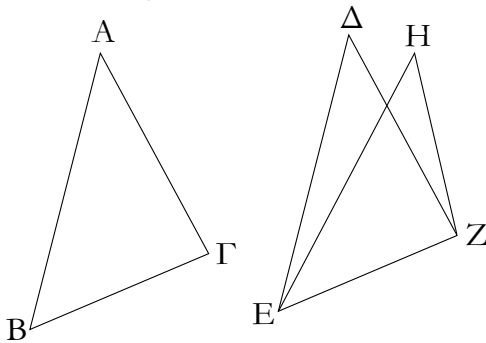
ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρᾳ συσταθήσονται πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις· ὅπερ ἔδει δεῖξαι.

(than the latter). The very thing is impossible.

Thus, on the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines (which meet) cannot be constructed (meeting) at a different point on the same side (of the straight-line), but having the same ends as the given straight-lines. (Which is) the very thing it was required to show.

η'.

Ἐάν δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ, ἔχῃ δὲ καὶ τὴν βάσιν τῇ βάσει ἴσην, καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἔξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην.



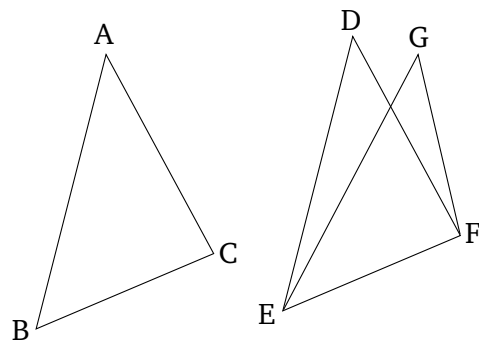
Ἐστω δύο τρίγωνα τὰ ABG , DEZ τὰς δύο πλευρὰς τὰς AB , AG ταῖς δύο πλευραῖς ταῖς DE , DZ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ, τὴν μὲν AB τῇ DE τὴν δὲ AG τῇ DZ · ἐχέτω δὲ καὶ βάσιν τὴν BG βάσει τῇ EZ ἴσην· λέγω, ὅτι καὶ γωνία ἡ ὑπὸ BAG γωνία τῇ ὑπὸ EDZ ἐστὶν ἴση.

Ἐφαρμοζομένου γὰρ τοῦ ABG τριγώνου ἐπὶ τὸ DEZ τρίγωνον καὶ τιθεμένου τοῦ μὲν B σημείου ἐπὶ τὸ E σημεῖον τῆς δὲ BG εὐθείας ἐπὶ τὴν EZ ἐφαρμόσει καὶ τὸ G σημεῖον ἐπὶ τὸ Z διὰ τὸ ἴσην εἶναι τὴν BG τῇ EZ · ἐφαρμοσάσης δὲ τῆς BG ἐπὶ τὴν EZ ἐφαρμόσουσι καὶ αἱ BA , GA ἐπὶ τὰς ED , DZ . εἰ γὰρ βάσις μὲν ἡ BG ἐπὶ βάσιν τὴν EZ ἐφαρμόσει, αἱ δὲ BA , AG πλευραὶ ἐπὶ τὰς ED , DZ οὐκ ἐφαρμόσουσιν ἀλλὰ παραλλάξουσιν ὥς αἱ EH , HZ , συσταθήσονται ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρᾳ πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι. οὐ συνίστανται δὲ οὐκ ἄρα ἐφαρμοζομένης τῆς BG βάσεως ἐπὶ τὴν EZ βάσιν οὐκ ἐφαρμόσουσι καὶ αἱ BA , AG πλευραὶ ἐπὶ τὰς ED , DZ . ἐφαρμόσουσιν ἄρα· ὥστε καὶ γωνία ἡ ὑπὸ BAG ἐπὶ γωνίαν τὴν ὑπὸ EDZ ἐφαρμόσει καὶ ἴση αὐτῇ ἔσται.

Ἐάν ἄρα δύο τρίγωνα τὰς δύο πλευρὰς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ τὴν βάσιν τῇ βάσει ἴσην ἔχῃ, καὶ τὴν γωνίαν τῇ γωνίᾳ ἴσην ἔξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην· ὅπερ ἔδει δεῖξαι.

Proposition 8

If two triangles have two sides equal to two sides, respectively, and also have the base equal to the base, then they will also have equal the angles encompassed by the equal straight-lines.



Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF , respectively. (That is) AB to DE , and AC to DF . Let them also have the base BC equal to the base EF . I say that the angle BAC is also equal to the angle EDF .

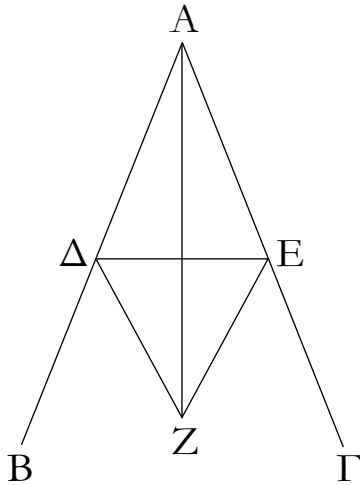
For if triangle ABC is applied to triangle DEF , the point B being placed on point E , and the straight-line BC on EF , then point C will also coincide with F , on account of BC being equal to EF . So (because of) BC coinciding with EF , (the sides) BA and CA will also coincide with ED and DF (respectively). For if base BC coincides with base EF , but the sides AB and AC do not coincide with ED and DF (respectively), but miss like EG and GF (in the above figure), then we will have constructed upon the same straight-line, two other straight-lines equal, respectively, to two (given) straight-lines, and (meeting) at a different point on the same side (of the straight-line), but having the same ends. But (such straight-lines) cannot be constructed [Prop. 1.7]. Thus, the base BC being applied to the base EF , the sides BA and AC cannot not coincide with ED and DF (respectively). Thus, they will coincide. So the angle BAC will also coincide with angle EDF , and will be equal to it [C.N. 4].

Thus, if two triangles have two sides equal to two side, respectively, and have the base equal to the base,

then they will also have equal the angles encompassed by the equal straight-lines. (Which is) the very thing it was required to show.

θ'.

Τὴν δοθεῖσαν γωνίαν εὐθύγραμμον δίχα τεμεῖν.



Ἐστω ἡ δοθεῖσα γωνία εὐθύγραμμος ἡ ὑπὸ ΒΑΓ. δεῖ δὴ αὐτὴν δίχα τεμεῖν.

Εἰλήφθω ἐπὶ τῆς ΑΒ τυχὸν σημεῖον τὸ Δ, καὶ ἀφηρήσθω ἀπὸ τῆς ΑΓ τῇ ΑΔ ἴση ἡ ΑΕ, καὶ ἐπεζεύχθω ἡ ΔΕ, καὶ συνεστάτω ἐπὶ τῆς ΔΕ τρίγωνον ἰσόπλευρον τὸ ΔΕΖ, καὶ ἐπεζεύχθω ἡ ΑΖ· λέγω, ὅτι ἡ ὑπὸ ΒΑΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΑΖ εὐθείας.

Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΑΔ τῇ ΑΕ, κοινὴ δὲ ἡ ΑΖ, δύο δὲ αἱ ΔΑ, ΑΖ δυσὶ ταῖς ΕΑ, ΑΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ. καὶ βάσις ἡ ΔΖ βάσει τῇ ΕΖ ἴση ἐστίν· γωνία ἄρα ἡ ὑπὸ ΔΑΖ γωνία τῇ ὑπὸ ΕΑΖ ἴση ἐστίν.

Ἡ ἄρα δοθεῖσα γωνία εὐθύγραμμος ἡ ὑπὸ ΒΑΓ δίχα τέτμηται ὑπὸ τῆς ΑΖ εὐθείας· ὅπερ ἔδει ποιῆσαι.

ι'.

Τὴν δοθεῖσαν εὐθεῖαν πεπερασμένην δίχα τεμεῖν.

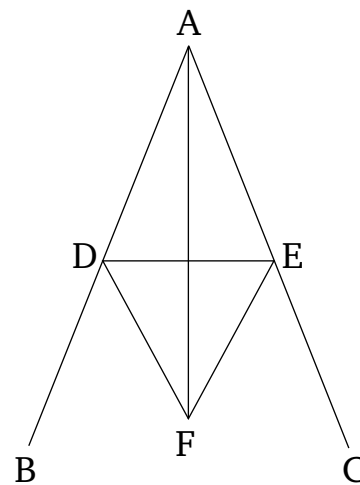
Ἐστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ· δεῖ δὴ τὴν ΑΒ εὐθεῖαν πεπερασμένην δίχα τεμεῖν.

Συνεστάτω ἐπ' αὐτῆς τρίγωνον ἰσόπλευρον τὸ ΑΒΓ, καὶ τετμήσθω ἡ ὑπὸ ΑΓΒ γωνία δίχα τῇ ΓΔ εὐθείᾳ· λέγω, ὅτι ἡ ΑΒ εὐθεῖα δίχα τέτμηται κατὰ τὸ Δ σημεῖον.

Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, κοινὴ δὲ ἡ ΓΔ, δύο δὲ αἱ ΑΓ, ΓΔ δύο ταῖς ΒΓ, ΓΔ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΑΓΔ γωνία τῇ ὑπὸ ΒΓΔ ἴση ἐστίν· βάσις ἄρα

Proposition 9

To cut a given rectilinear angle in half.



Let BAC be the given rectilinear angle. So it is required to cut it in half.

Let the point D have been taken at random on AB , and let AE , equal to AD , have been cut off from AC [Prop. 1.3], and let DE have been joined. And let the equilateral triangle DEF have been constructed upon DE [Prop. 1.1], and let AF have been joined. I say that the angle BAC has been cut in half by the straight-line AF .

For since AD is equal to AE , and AF is common, the two (straight-lines) DA , AF are equal to the two (straight-lines) EA , AF , respectively. And the base DF is equal to the base EF . Thus, angle DAF is equal to angle EAF [Prop. 1.8].

Thus, the given rectilinear angle BAC has been cut in half by the straight-line AF . (Which is) the very thing it was required to do.

Proposition 10

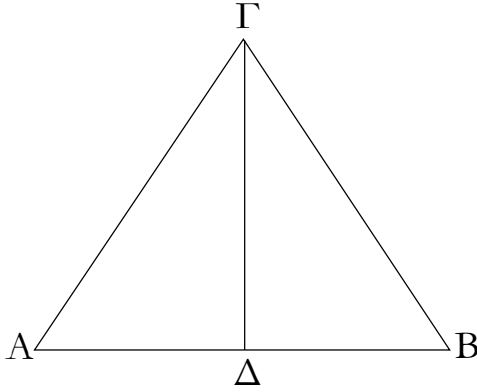
To cut a given finite straight-line in half.

Let AB be the given finite straight-line. So it is required to cut the finite straight-line AB in half.

Let the equilateral triangle ABC have been constructed upon (AB) [Prop. 1.1], and let the angle ACB have been cut in half by the straight-line CD [Prop. 1.9]. I say that the straight-line AB has been cut in half at point D .

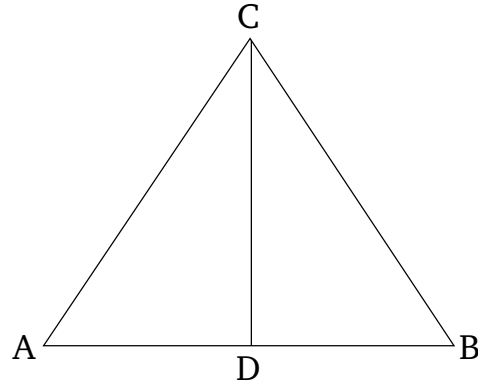
For since AC is equal to CB , and CD (is) common,

ἡ AD βάσει τῇ BD ἴση ἐστίν.



Ἡ ἄρα δοθεῖσα εὐθεῖα πεπερασμένη ἡ AB δίχα τέτμηται κατὰ τὸ Δ ὅπερ ἔδει ποιῆσαι.

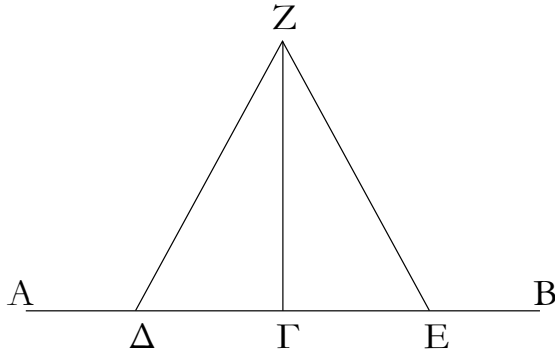
the two (straight-lines) AC , CD are equal to the two (straight-lines) BC , CD , respectively. And the angle ACD is equal to the angle BCD . Thus, the base AD is equal to the base BD [Prop. 1.4].



Thus, the given finite straight-line AB has been cut in half at (point) D . (Which is) the very thing it was required to do.

ια'.

Τῇ δοθείσῃ εὐθείᾳ ἀπὸ τοῦ πρὸς αὐτῇ δοθέντος σημείου πρὸς ὀρθὰς γωνίας εὐθεῖαν γραμμὴν ἀγαγεῖν.



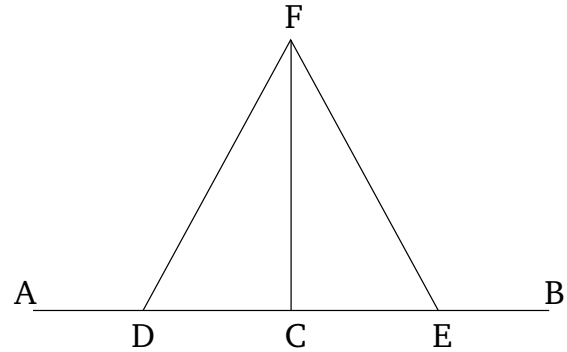
Ἐστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ AB τὸ δὲ δοθὲν σημεῖον ἐπ' αὐτῆς τὸ Γ . δεῖ δὴ ἀπὸ τοῦ Γ σημείου τῇ AB εὐθείᾳ πρὸς ὀρθὰς γωνίας εὐθεῖαν γραμμὴν ἀγαγεῖν.

Εἰλήφθω ἐπὶ τῆς AG τυχὸν σημεῖον τὸ Δ , καὶ κείσθω τῇ $\Gamma\Delta$ ἴση ἡ ΓE , καὶ συνεστάτω ἐπὶ τῆς ΔE τρίγωνον ἰσόπλευρον τὸ $Z\Delta E$, καὶ ἐπεζεύχθω ἡ $Z\Gamma$. λέγω, ὅτι τῇ δοθείσῃ εὐθείᾳ τῇ AB ἀπὸ τοῦ πρὸς αὐτῇ δοθέντος σημείου τοῦ Γ πρὸς ὀρθὰς γωνίας εὐθεῖα γραμμὴ ἦκται ἡ $Z\Gamma$.

Ἐπεὶ γὰρ ἴση ἐστίν ἡ $\Delta\Gamma$ τῇ ΓE , κοινὴ δὲ ἡ ΓZ , δύο δὴ αἱ $\Delta\Gamma$, ΓZ δυσὶ ταῖς $E\Gamma$, ΓZ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ βάσις ἡ ΔZ βάσει τῇ ZE ἴση ἐστίν· γωνία ἄρα ἡ ὑπὸ $\Delta\Gamma Z$ γωνία τῇ ὑπὸ $E\Gamma Z$ ἴση ἐστίν· καὶ εἰσιν ἐφεξῆς. ὅταν δὲ εὐθεῖα ἐπ' εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστίν· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ $\Delta\Gamma Z$, $Z\Gamma E$.

Proposition 11

To draw a straight-line at right-angles to a given straight-line from a given point on it.



Let AB be the given straight-line, and C the given point on it. So it is required to draw a straight-line from the point C at right-angles to the straight-line AB .

Let the point D be have been taken at random on AC , and let CE be made equal to CD [Prop. 1.3], and let the equilateral triangle FDE have been constructed on DE [Prop. 1.1], and let FC have been joined. I say that the straight-line FC has been drawn at right-angles to the given straight-line AB from the given point C on it.

For since DC is equal to CE , and CF is common, the two (straight-lines) DC , CF are equal to the two (straight-lines), EC , CF , respectively. And the base DF is equal to the base FE . Thus, the angle DCF is equal to the angle ECF [Prop. 1.8], and they are adjacent. But when a straight-line stood on a(nother) straight-line

Τῇ ἄρα δοθείσῃ εὐθείᾳ τῇ AB ἀπὸ τοῦ πρὸς αὐτῇ δοθέντος σημείου τοῦ Γ πρὸς ὀρθὰς γωνίας εὐθεῖα γραμμὴ ῥχται ἡ ΓZ · ὅπερ ἔδει ποιῆσαι.

makes the adjacent angles equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, each of the (angles) DCF and FCE is a right-angle.

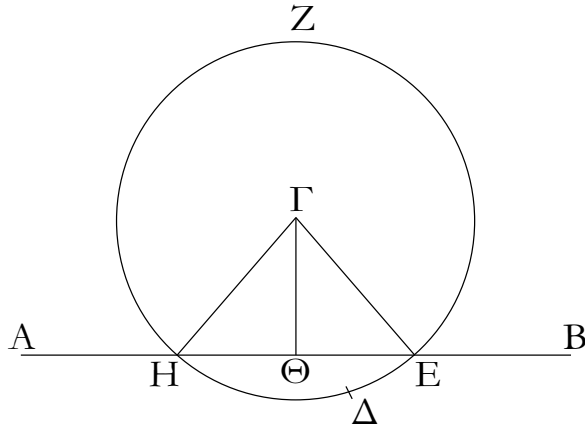
Thus, the straight-line CF has been drawn at right-angles to the given straight-line AB from the given point C on it. (Which is) the very thing it was required to do.

ιβ'.

Ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον ἀπὸ τοῦ δοθέντος σημείου, ὃ μὴ ἐστὶν ἐπ' αὐτῆς, κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Proposition 12

To draw a straight-line perpendicular to a given infinite straight-line from a given point which is not on it.

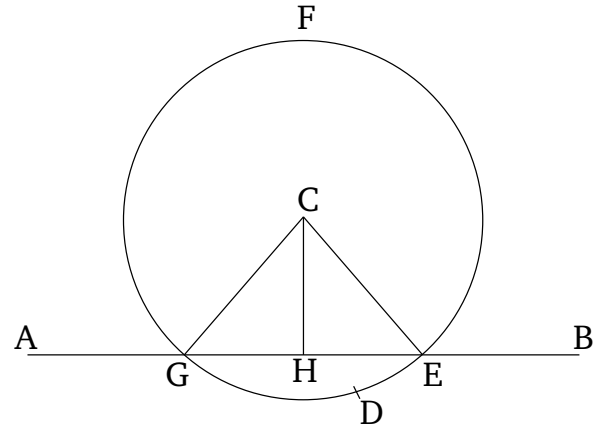


Ἐστω ἡ μὲν δοθεῖσα εὐθεῖα ἄπειρος ἡ AB τὸ δὲ δοθέν σημείον, ὃ μὴ ἐστὶν ἐπ' αὐτῆς, τὸ Γ · δεῖ δὲ ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον τὴν AB ἀπὸ τοῦ δοθέντος σημείου τοῦ Γ , ὃ μὴ ἐστὶν ἐπ' αὐτῆς, κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Εἰλήφθω γὰρ ἐπὶ τὰ ἕτερα μέρη τῆς AB εὐθείας τυχὸν σημείον τὸ Δ , καὶ κέντρῳ μὲν τῷ Γ διαστήματι δὲ τῷ $\Gamma\Delta$ κύκλος γεγράφθω ὁ EZH , καὶ τετμήσθω ἡ EH εὐθεῖα δίχᾳ κατὰ τὸ Θ , καὶ ἐπεζυχθῶσαν αἱ ΓH , $\Gamma\Theta$, ΓE εὐθεῖαι· λέγω, ὅτι ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον τὴν AB ἀπὸ τοῦ δοθέντος σημείου τοῦ Γ , ὃ μὴ ἐστὶν ἐπ' αὐτῆς, κάθετος ῥχται ἡ $\Gamma\Theta$.

Ἐπεὶ γὰρ ἴση ἐστὶν ἡ $H\Theta$ τῇ ΘE , κοινὴ δὲ ἡ $\Theta\Gamma$, δύο δὲ αἱ $H\Theta$, $\Theta\Gamma$ δύο ταῖς $E\Theta$, $\Theta\Gamma$ ἴσαι εἰσὶν ἑκατέρᾳ ἑκατέρᾳ· καὶ βάσις ἡ ΓH βάσει τῇ ΓE ἐστὶν ἴση· γωνία ἄρα ἡ ὑπὸ $\Gamma\Theta H$ γωνία τῇ ὑπὸ $E\Theta\Gamma$ ἐστὶν ἴση. καὶ εἰσὶν ἐφεξῆς. ὅταν δὲ εὐθεῖα ἐπ' εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστὶν, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται ἐφ' ἣν ἐφέστηκεν.

Ἐπὶ τὴν δοθεῖσαν ἄρα εὐθεῖαν ἄπειρον τὴν AB ἀπὸ τοῦ δοθέντος σημείου τοῦ Γ , ὃ μὴ ἐστὶν ἐπ' αὐτῆς, κάθετος ῥχται ἡ $\Gamma\Theta$ · ὅπερ ἔδει ποιῆσαι.



Let AB be the given infinite straight-line and C the given point, which is not on (AB). So it is required to draw a straight-line perpendicular to the given infinite straight-line AB from the given point C , which is not on (AB).

For let point D have been taken at random on the other side (to C) of the straight-line AB , and let the circle EFG have been drawn with center C and radius CD [Post. 3], and let the straight-line EG have been cut in half at (point) H [Prop. 1.10], and let the straight-lines CG , CH , and CE have been joined. I say that the (straight-line) CH has been drawn perpendicular to the given infinite straight-line AB from the given point C , which is not on (AB).

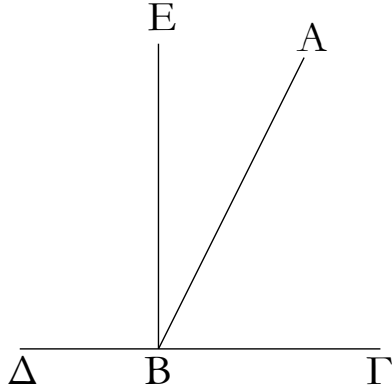
For since GH is equal to HE , and HC (is) common, the two (straight-lines) GH , HC are equal to the two (straight-lines) EH , HC , respectively, and the base CG is equal to the base CE . Thus, the angle CHG is equal to the angle EHC [Prop. 1.8], and they are adjacent. But when a straight-line stood on a(nother) straight-line makes the adjacent angles equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands [Def. 1.10].

Thus, the (straight-line) CH has been drawn perpendicular to the given infinite straight-line AB from the

given point C , which is not on (AB) . (Which is) the very thing it was required to do.

ιγ'.

Ἐάν εὐθεΐα ἐπ' εὐθεΐαν σταθεῖσα γωνίας ποιῇ, ἤτοι δύο ὀρθὰς ἢ δυσὶν ὀρθαῖς ἴσας ποιήσῃ.



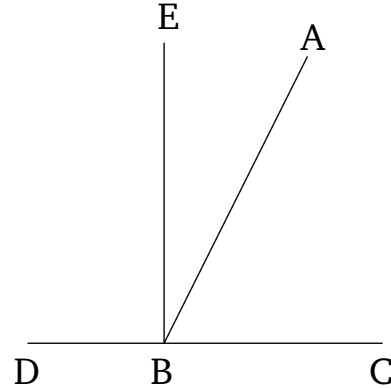
Εὐθεΐα γάρ τις ἡ AB ἐπ' εὐθεΐαν τὴν $\Gamma\Delta$ σταθεῖσα γωνίας ποιεῖται τὰς ὑπὸ ΓBA , $AB\Delta$. λέγω, ὅτι αἱ ὑπὸ ΓBA , $AB\Delta$ γωνίαι ἤτοι δύο ὀρθαὶ εἰσιν ἢ δυσὶν ὀρθαῖς ἴσαι.

Εἰ μὲν οὖν ἴση ἐστὶν ἡ ὑπὸ ΓBA τῇ ὑπὸ $AB\Delta$, δύο ὀρθαὶ εἰσιν. εἰ δὲ οὐ, ἤχθω ἀπὸ τοῦ B σημείου τῇ $\Gamma\Delta$ [εὐθείᾳ] πρὸς ὀρθὰς ἡ BE . αἱ ἄρα ὑπὸ ΓBE , $EB\Delta$ δύο ὀρθαὶ εἰσιν· καὶ ἐπεὶ ἡ ὑπὸ ΓBE δυσὶ ταῖς ὑπὸ ΓBA , ABE ἴση ἐστίν, κοινὴ προσκείσθω ἡ ὑπὸ $EB\Delta$. αἱ ἄρα ὑπὸ ΓBE , $EB\Delta$ τρισὶ ταῖς ὑπὸ ΓBA , ABE , $EB\Delta$ ἴσαι εἰσιν. πάλιν, ἐπεὶ ἡ ὑπὸ ΔBA δυσὶ ταῖς ὑπὸ ΔBE , EBA ἴση ἐστίν, κοινὴ προσκείσθω ἡ ὑπὸ $AB\Gamma$. αἱ ἄρα ὑπὸ ΔBA , $AB\Gamma$ τρισὶ ταῖς ὑπὸ ΔBE , EBA , $AB\Gamma$ ἴσαι εἰσιν. ἐδείχθησαν δὲ καὶ αἱ ὑπὸ ΓBE , $EB\Delta$ τρισὶ ταῖς αὐταῖς ἴσαι· τὰ δὲ τῶ αὐτῶ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ αἱ ὑπὸ ΓBE , $EB\Delta$ ἄρα ταῖς ὑπὸ ΔBA , $AB\Gamma$ ἴσαι εἰσιν· ἀλλὰ αἱ ὑπὸ ΓBE , $EB\Delta$ δύο ὀρθαὶ εἰσιν· καὶ αἱ ὑπὸ ΔBA , $AB\Gamma$ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσιν.

Ἐάν ἄρα εὐθεΐα ἐπ' εὐθεΐαν σταθεῖσα γωνίας ποιῇ, ἤτοι δύο ὀρθὰς ἢ δυσὶν ὀρθαῖς ἴσας ποιήσῃ· ὅπερ εἶδει δεῖξαι.

Proposition 13

If a straight-line stood on a(nother) straight-line makes angles, it will certainly either make two right-angles, or (angles whose sum is) equal to two right-angles.



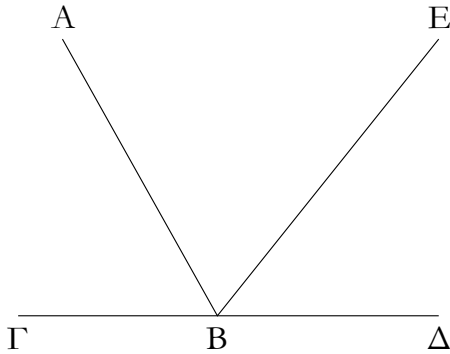
For let some straight-line AB stood on the straight-line CD make the angles CBA and ABD . I say that the angles CBA and ABD are certainly either two right-angles, or (have a sum) equal to two right-angles.

In fact, if CBA is equal to ABD then they are two right-angles [Def. 1.10]. But, if not, let BE have been drawn from the point B at right-angles to [the straight-line] CD [Prop. 1.11]. Thus, CBE and EBD are two right-angles. And since CBE is equal to the two (angles) CBA and ABE , let EBD have been added to both. Thus, the (sum of the angles) CBE and EBD is equal to the (sum of the) three (angles) CBA , ABE , and EBD [C.N. 2]. Again, since DBA is equal to the two (angles) DBE and EBA , let ABC have been added to both. Thus, the (sum of the angles) DBA and ABC is equal to the (sum of the) three (angles) DBE , EBA , and ABC [C.N. 2]. But (the sum of) CBE and EBD was also shown (to be) equal to the (sum of the) same three (angles). And things equal to the same thing are also equal to one another [C.N. 1]. Therefore, (the sum of) CBE and EBD is also equal to (the sum of) DBA and ABC . But, (the sum of) CBE and EBD is two right-angles. Thus, (the sum of) ABD and ABC is also equal to two right-angles.

Thus, if a straight-line stood on a(nother) straight-line makes angles, it will certainly either make two right-angles, or (angles whose sum is) equal to two right-angles. (Which is) the very thing it was required to show.

ιδ'.

Ἐάν πρὸς τινι εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ δύο εὐθεῖαι μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιῶσιν, ἐπ' εὐθείας ἔσσονται ἀλλήλαις αἱ εὐθεῖαι.



Πρὸς γάρ τινι εὐθείᾳ τῇ ΑΒ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Β δύο εὐθεῖαι αἱ ΒΓ, ΒΔ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας τὰς ὑπὸ ΑΒΓ, ΑΒΔ δύο ὀρθαῖς ἴσας ποιείτωσαν· λέγω, ὅτι ἐπ' εὐθείας ἔστί τῇ ΓΒ ἡ ΒΔ.

Εἰ γὰρ μὴ ἔστι τῇ ΒΓ ἐπ' εὐθείας ἡ ΒΔ, ἔστω τῇ ΓΒ ἐπ' εὐθείας ἡ ΒΕ.

Ἐπεὶ οὖν εὐθεῖα ἡ ΑΒ ἐπ' εὐθεῖαν τὴν ΓΒΕ ἐφέστηκεν, αἱ ἄρα ὑπὸ ΑΒΓ, ΑΒΕ γωνίαι δύο ὀρθαῖς ἴσαι εἰσὶν· εἰσὶ δὲ καὶ αἱ ὑπὸ ΑΒΓ, ΑΒΔ δύο ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΓΒΑ, ΑΒΕ ταῖς ὑπὸ ΓΒΑ, ΑΒΔ ἴσαι εἰσὶν. κοινὴ ἀφηρήσθω ἡ ὑπὸ ΓΒΑ· λοιπὴ ἄρα ἡ ὑπὸ ΑΒΕ λοιπὴ τῇ ὑπὸ ΑΒΔ ἔστιν ἴση, ἡ ἐλάσσων τῇ μείζονι· ὅπερ ἔστιν ἀδύνατον. οὐκ ἄρα ἐπ' εὐθείας ἔστιν ἡ ΒΕ τῇ ΓΒ. ὁμοίως δὲ δείξομεν, ὅτι οὐδὲ ἄλλη τις πλὴν τῆς ΒΔ· ἐπ' εὐθείας ἄρα ἔστιν ἡ ΓΒ τῇ ΒΔ.

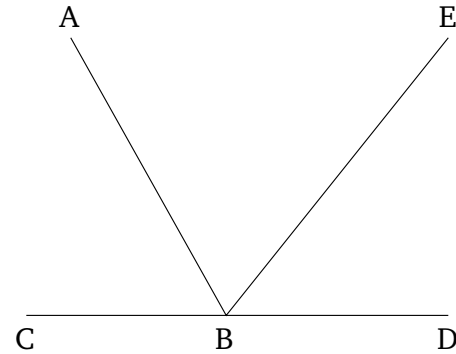
Ἐάν ἄρα πρὸς τινι εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ δύο εὐθεῖαι μὴ ἐπὶ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιῶσιν, ἐπ' εὐθείας ἔσσονται ἀλλήλαις αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

ιε'.

Ἐάν δύο εὐθεῖαι τέμνωσιν ἀλλήλας, τὰς κατὰ κορυφὴν γωνίας ἴσας ἀλλήλαις ποιοῦσιν.

Proposition 14

If two straight-lines, not lying on the same side, make adjacent angles (whose sum is) equal to two right-angles with some straight-line, at a point on it, then the two straight-lines will be straight-on (with respect) to one another.



For let two straight-lines BC and BD , not lying on the same side, make adjacent angles ABC and ABD (whose sum is) equal to two right-angles with some straight-line AB , at the point B on it. I say that BD is straight-on with respect to CB .

For if BD is not straight-on to BC then let BE be straight-on to CB .

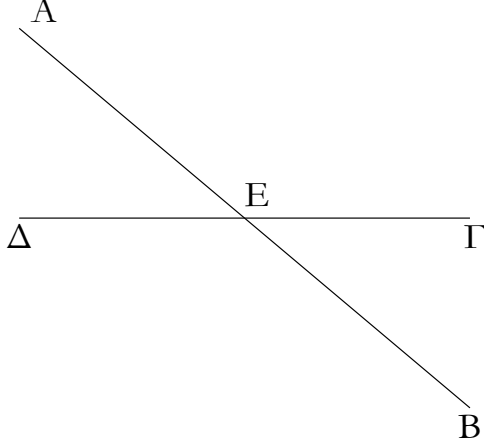
Therefore, since the straight-line AB stands on the straight-line CBE , the (sum of the) angles ABC and ABE is thus equal to two right-angles [Prop. 1.13]. But (the sum of) ABC and ABD is also equal to two right-angles. Thus, (the sum of angles) CBA and ABE is equal to (the sum of angles) CBA and ABD [C.N. 1]. Let (angle) CBA have been subtracted from both. Thus, the remainder ABE is equal to the remainder ABD [C.N. 3], the lesser to the greater. The very thing is impossible. Thus, BE is not straight-on with respect to CB . Similarly, we can show that neither (is) any other (straight-line) than BD . Thus, CB is straight-on with respect to BD .

Thus, if two straight-lines, not lying on the same side, make adjacent angles (whose sum is) equal to two right-angles with some straight-line, at a point on it, then the two straight-lines will be straight-on (with respect) to one another. (Which is) the very thing it was required to show.

Proposition 15

If two straight-lines cut one another then they make the vertically opposite angles equal to one another.

Δύο γὰρ εὐθεῖαι αἱ AB , $\Gamma\Delta$ τεμνέτωσαν ἀλλήλας κατὰ τὸ E σημεῖον· λέγω, ὅτι ἴση ἐστὶν ἡ μὲν ὑπὸ $AE\Gamma$ γωνία τῇ ὑπὸ ΔEB , ἡ δὲ ὑπὸ ΓEB τῇ ὑπὸ $AE\Delta$.



Ἐπεὶ γὰρ εὐθεῖα ἡ AE ἐπ' εὐθεῖαν τὴν $\Gamma\Delta$ ἐφέστηκε γωνίας ποιοῦσα τὰς ὑπὸ ΓEA , $AE\Delta$, αἱ ἄρα ὑπὸ ΓEA , $AE\Delta$ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσὶν. πάλιν, ἐπεὶ εὐθεῖα ἡ ΔE ἐπ' εὐθεῖαν τὴν AB ἐφέστηκε γωνίας ποιοῦσα τὰς ὑπὸ $AE\Delta$, ΔEB , αἱ ἄρα ὑπὸ $AE\Delta$, ΔEB γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσὶν. ἐδείχθησαν δὲ καὶ αἱ ὑπὸ ΓEA , $AE\Delta$ δυσὶν ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΓEA , $AE\Delta$ ταῖς ὑπὸ $AE\Delta$, ΔEB ἴσαι εἰσὶν. κοινὴ ἀφρηθήσθω ἡ ὑπὸ $AE\Delta$ · λοιπὴ ἄρα ἡ ὑπὸ ΓEA λοιπὴ τῇ ὑπὸ ΔEB ἴση ἐστίν· ὁμοίως δὲ δεῖχθήσεται, ὅτι καὶ αἱ ὑπὸ ΓEB , ΔEA ἴσαι εἰσὶν.

Ἐὰν ἄρα δύο εὐθεῖαι τέμνωσιν ἀλλήλας, τὰς κατὰ κορυφὴν γωνίας ἴσας ἀλλήλαις ποιοῦσιν· ὅπερ ἔδει δεῖξαι.

ιζ'.

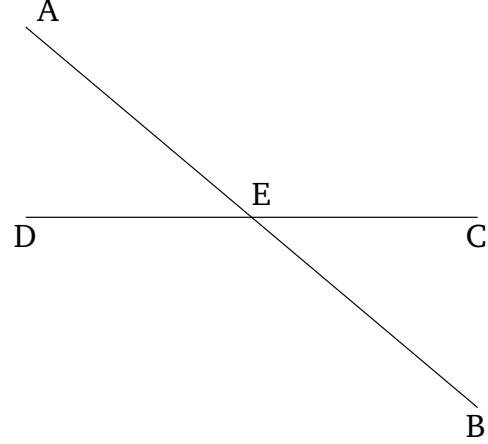
Παντὸς τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία ἑκατέρας τῶν ἐντὸς καὶ ἀπεναντίων γωνιῶν μείζων ἐστίν.

Ἐστω τρίγωνον τὸ $AB\Gamma$, καὶ προσεκβεβλήσθω αὐτοῦ μία πλευρὰ ἡ $B\Gamma$ ἐπὶ τὸ Δ · λέγω, ὅτι ἡ ἐκτὸς γωνία ἡ ὑπὸ $ΑΓΔ$ μείζων ἐστὶν ἑκατέρας τῶν ἐντὸς καὶ ἀπεναντίων τῶν ὑπὸ ΓBA , $BA\Gamma$ γωνιῶν.

Τετμήσθω ἡ $ΑΓ$ δίχα κατὰ τὸ E , καὶ ἐπιζευχθεῖσα ἡ BE ἐκβεβλήσθω ἐπ' εὐθείας ἐπὶ τὸ Z , καὶ κείσθω τῇ BE ἴση ἡ EZ , καὶ ἐπεζεύχθω ἡ $Z\Gamma$, καὶ διήχθω ἡ $ΑΓ$ ἐπὶ τὸ H .

Ἐπεὶ οὖν ἴση ἐστὶν ἡ μὲν AE τῇ $ΕΓ$, ἡ δὲ BE τῇ EZ , δύο δὲ αἱ AE , EB δυσὶ ταῖς ΓE , EZ ἴσαι εἰσὶν ἑκατέρα ἑκατέρῃ· καὶ γωνία ἡ ὑπὸ AEB γωνία τῇ ὑπὸ $ZΕΓ$ ἴση ἐστίν· κατὰ κορυφὴν γάρ· βάσει ἄρα ἡ AB βάσει τῇ $Z\Gamma$ ἴση ἐστίν, καὶ τὸ ABE τρίγωνον τῷ $ZΕΓ$ τριγώνῳ ἐστὶν ἴσον, καὶ αἱ λοιπαὶ

For let the two straight-lines AB and CD cut one another at the point E . I say that angle AEC is equal to (angle) DEB , and (angle) CEB to (angle) AED .



For since the straight-line AE stands on the straight-line CD , making the angles CEA and AED , the (sum of the) angles CEA and AED is thus equal to two right-angles [Prop. 1.13]. Again, since the straight-line DE stands on the straight-line AB , making the angles AED and DEB , the (sum of the) angles AED and DEB is thus equal to two right-angles [Prop. 1.13]. But (the sum of) CEA and AED was also shown (to be) equal to two right-angles. Thus, (the sum of) CEA and AED is equal to (the sum of) AED and DEB [C.N. 1]. Let AED have been subtracted from both. Thus, the remainder CEA is equal to the remainder DEB [C.N. 3]. Similarly, it can be shown that CEB and DEA are also equal.

Thus, if two straight-lines cut one another then they make the vertically opposite angles equal to one another. (Which is) the very thing it was required to show.

Proposition 16

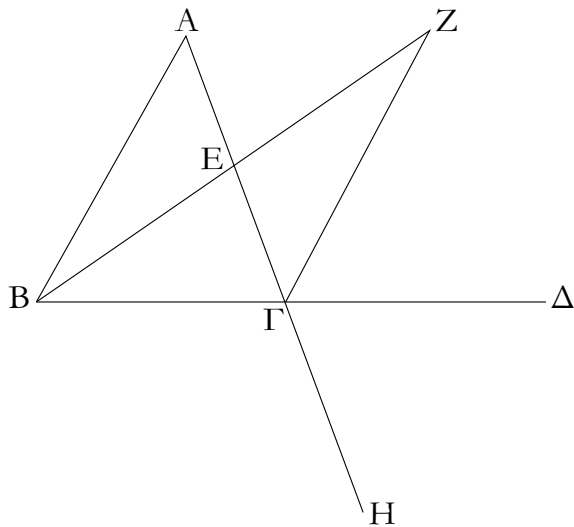
For any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles.

Let ABC be a triangle, and let one of its sides BC have been produced to D . I say that the external angle ACD is greater than each of the internal and opposite angles, CBA and BAC .

Let the (straight-line) AC have been cut in half at (point) E [Prop. 1.10]. And BE being joined, let it have been produced in a straight-line to (point) F .[†] And let EF be made equal to BE [Prop. 1.3], and let FC have been joined, and let AC have been drawn through to (point) G .

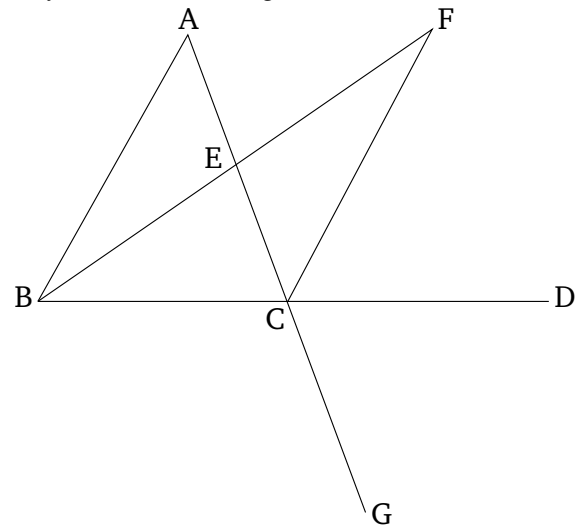
Therefore, since AE is equal to EC , and BE to EF , the two (straight-lines) AE , EB are equal to the two

γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι εἰσὶν ἑκατέρα ἑκατέρα, ὅφ' ἄς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστὶν ἡ ὑπὸ BAE τῇ ὑπὸ EGZ . μείζων δέ ἐστιν ἡ ὑπὸ $EG\Delta$ τῆς ὑπὸ EGZ · μείζων ἄρα ἡ ὑπὸ $AG\Delta$ τῆς ὑπὸ BAE . Ὅμοίως δὲ τῆς $B\Gamma$ τετμημένης δίχα δειχθήσεται καὶ ἡ ὑπὸ $B\Gamma H$, τοῦτέστιν ἡ ὑπὸ $AG\Delta$, μείζων καὶ τῆς ὑπὸ $AB\Gamma$.



Παντός ἄρα τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία ἑκατέρας τῶν ἐντὸς καὶ ἀπεναντίον γωνιῶν μείζων ἐστίν· ὅπερ εἶδει δεῖξαι.

(straight-lines) CE , EF , respectively. Also, angle AEB is equal to angle FEC , for (they are) vertically opposite [Prop. 1.15]. Thus, the base AB is equal to the base FC , and the triangle ABE is equal to the triangle FEC , and the remaining angles subtended by the equal sides are equal to the corresponding remaining angles [Prop. 1.4]. Thus, BAE is equal to ECF . But ECD is greater than ECF . Thus, ACD is greater than BAE . Similarly, by having cut BC in half, it can be shown (that) BCG —that is to say, ACD —(is) also greater than ABC .

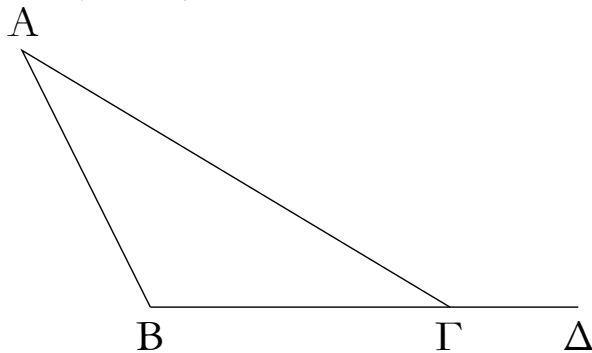


Thus, for any triangle, when one of the sides is produced, the external angle is greater than each of the internal and opposite angles. (Which is) the very thing it was required to show.

† The implicit assumption that the point F lies in the interior of the angle ABC should be counted as an additional postulate.

ιζ'.

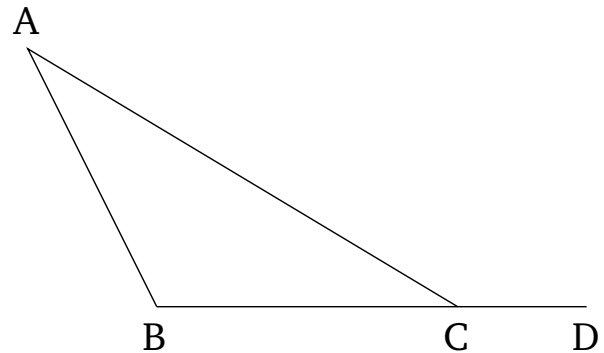
Παντός τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάσσονες εἰσι πάντῃ μεταλαμβανόμεναι.



Ἐστω τρίγωνον τὸ $AB\Gamma$ · λέγω, ὅτι τοῦ $AB\Gamma$ τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάττονες εἰσι πάντῃ μεταλαμβανόμεναι.

Proposition 17

For any triangle, (the sum of) two angles taken together in any (possible way) is less than two right-angles.



Let ABC be a triangle. I say that (the sum of) two angles of triangle ABC taken together in any (possible way) is less than two right-angles.

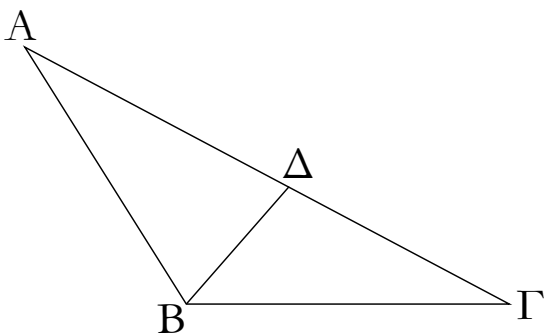
Ἐκβεβλήσθω γὰρ ἡ ΒΓ ἐπὶ τὸ Δ.

Καὶ ἐπεὶ τριγώνου τοῦ ΑΒΓ ἐκτός ἐστι γωνία ἡ ὑπὸ ΑΓΔ, μείζων ἐστὶ τῆς ἐντός καὶ ἀπεναντίον τῆς ὑπὸ ΑΒΓ. κοινὴ προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΔ, ΑΓΒ τῶν ὑπὸ ΑΒΓ, ΒΓΑ μείζονες εἰσιν. ἀλλ' αἱ ὑπὸ ΑΓΔ, ΑΓΒ δύο ὀρθαῖς ἴσαι εἰσιν· αἱ ἄρα ὑπὸ ΑΒΓ, ΒΓΑ δύο ὀρθῶν ἐλάσσονες εἰσιν. ὁμοίως δὲ δείξομεν, ὅτι καὶ αἱ ὑπὸ ΒΑΓ, ΑΓΒ δύο ὀρθῶν ἐλάσσονες εἰσι καὶ ἔτι αἱ ὑπὸ ΓΑΒ, ΑΒΓ.

Παντὸς ἄρα τριγώνου αἱ δύο γωνίαι δύο ὀρθῶν ἐλάσσονες εἰσι πάντῃ μεταλαμβανόμεναι· ὅπερ ἔδει δείξαι.

ιη'.

Παντὸς τριγώνου ἡ μείζων πλευρὰ τὴν μείζονα γωνίαν ὑποτείνει.



Ἐστω γὰρ τρίγωνον τὸ ΑΒΓ μείζονα ἔχον τὴν ΑΓ πλευρὰν τῆς ΑΒ· λέγω, ὅτι καὶ γωνία ἡ ὑπὸ ΑΒΓ μείζων ἐστὶ τῆς ὑπὸ ΒΓΑ.

Ἐπεὶ γὰρ μείζων ἐστὶν ἡ ΑΓ τῆς ΑΒ, κείσθω τῇ ΑΒ ἴση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΒΔ.

Καὶ ἐπεὶ τριγώνου τοῦ ΒΓΔ ἐκτός ἐστι γωνία ἡ ὑπὸ ΑΔΒ, μείζων ἐστὶ τῆς ἐντός καὶ ἀπεναντίον τῆς ὑπὸ ΔΓΒ· ἴση δὲ ἡ ὑπὸ ΑΔΒ τῇ ὑπὸ ΑΒΔ, ἐπεὶ καὶ πλευρὰ ἡ ΑΒ τῇ ΑΔ ἐστὶν ἴση· μείζων ἄρα καὶ ἡ ὑπὸ ΑΒΔ τῆς ὑπὸ ΑΓΒ· πολλῶ ἄρα ἡ ὑπὸ ΑΒΓ μείζων ἐστὶ τῆς ὑπὸ ΑΓΒ.

Παντὸς ἄρα τριγώνου ἡ μείζων πλευρὰ τὴν μείζονα γωνίαν ὑποτείνει· ὅπερ ἔδει δείξαι.

ιθ'.

Παντὸς τριγώνου ὑπὸ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει.

Ἐστω τρίγωνον τὸ ΑΒΓ μείζονα ἔχον τὴν ὑπὸ ΑΒΓ γωνίαν τῆς ὑπὸ ΒΓΑ· λέγω, ὅτι καὶ πλευρὰ ἡ ΑΓ πλευρᾶς τῆς ΑΒ μείζων ἐστὶν.

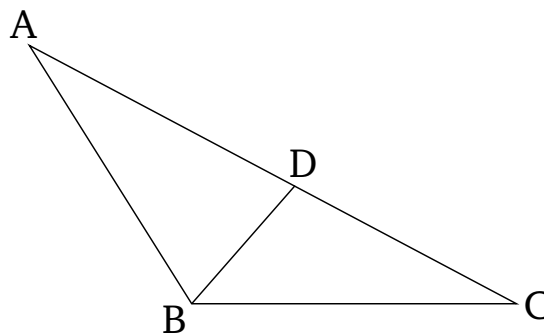
For let BC have been produced to D .

And since the angle ACD is external to triangle ABC , it is greater than the internal and opposite angle ABC [Prop. 1.16]. Let ACB have been added to both. Thus, the (sum of the angles) ACD and ACB is greater than the (sum of the angles) ABC and BCA . But, (the sum of) ACD and ACB is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) ABC and BCA is less than two right-angles. Similarly, we can show that (the sum of) BAC and ACB is also less than two right-angles, and further (that the sum of) CAB and ABC (is less than two right-angles).

Thus, for any triangle, (the sum of) two angles taken together in any (possible way) is less than two right-angles. (Which is) the very thing it was required to show.

Proposition 18

In any triangle, the greater side subtends the greater angle.



For let ABC be a triangle having side AC greater than AB . I say that angle ABC is also greater than BCA .

For since AC is greater than AB , let AD be made equal to AB [Prop. 1.3], and let BD have been joined.

And since angle ADB is external to triangle BCD , it is greater than the internal and opposite (angle) DCB [Prop. 1.16]. But ADB (is) equal to ABD , since side AB is also equal to side AD [Prop. 1.5]. Thus, ABD is also greater than ACB . Thus, ABC is much greater than ACB .

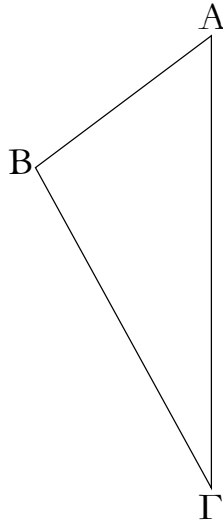
Thus, in any triangle, the greater side subtends the greater angle. (Which is) the very thing it was required to show.

Proposition 19

In any triangle, the greater angle is subtended by the greater side.

Let ABC be a triangle having the angle ABC greater than BCA . I say that side AC is also greater than side AB .

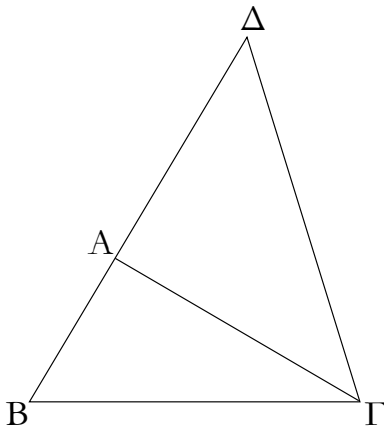
Εἰ γὰρ μή, ἦτοι ἴση ἐστὶν ἡ $ΑΓ$ τῇ $ΑΒ$ ἢ ἐλάσσων· ἴση μὲν οὖν οὐκ ἔστιν ἡ $ΑΓ$ τῇ $ΑΒ$ · ἴση γὰρ ἂν ἦν καὶ γωνία ἡ ὑπὸ $ΑΒΓ$ τῇ ὑπὸ $ΑΓΒ$ · οὐκ ἔστι δέ· οὐκ ἄρα ἴση ἐστὶν ἡ $ΑΓ$ τῇ $ΑΒ$. οὐδὲ μὴν ἐλάσσων ἐστὶν ἡ $ΑΓ$ τῆς $ΑΒ$ · ἐλάσσων γὰρ ἂν ἦν καὶ γωνία ἡ ὑπὸ $ΑΒΓ$ τῆς ὑπὸ $ΑΓΒ$ · οὐκ ἔστι δέ· οὐκ ἄρα ἐλάσσων ἐστὶν ἡ $ΑΓ$ τῆς $ΑΒ$. ἐδείχθη δέ, ὅτι οὐδὲ ἴση ἐστὶν. μείζων ἄρα ἐστὶν ἡ $ΑΓ$ τῆς $ΑΒ$.



Παντὸς ἄρα τριγώνου ὑπὸ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει· ὅπερ εἶδει δεῖξαι.

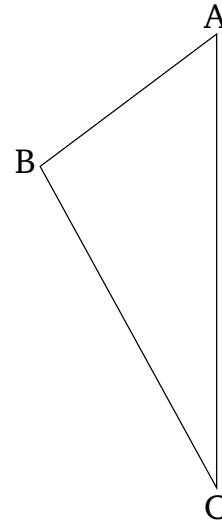
κ'.

Παντὸς τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονες εἰσι πάντῃ μεταλαμβανόμεναι.



Ἐστω γὰρ τρίγωνον τὸ $ΑΒΓ$ · λέγω, ὅτι τοῦ $ΑΒΓ$ τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονες εἰσι πάντῃ μεταλαμβανόμεναι, αἱ μὲν $ΒΑ$, $ΑΓ$ τῆς $ΒΓ$, αἱ δὲ $ΑΒ$, $ΒΓ$ τῆς $ΑΓ$, αἱ δὲ $ΒΓ$, $ΓΑ$ τῆς $ΑΒ$.

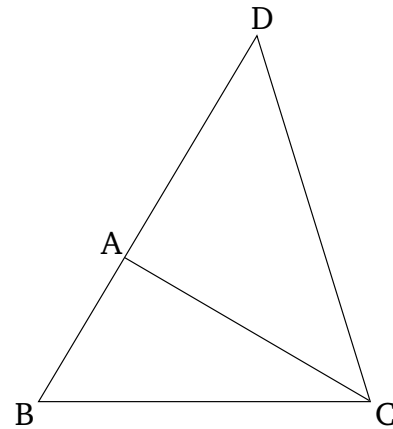
For if not, AC is certainly either equal to, or less than, AB . In fact, AC is not equal to AB . For then angle ABC would also have been equal to ACB [Prop. 1.5]. But it is not. Thus, AC is not equal to AB . Neither, indeed, is AC less than AB . For then angle ABC would also have been less than ACB [Prop. 1.18]. But it is not. Thus, AC is not less than AB . But it was shown that (AC) is not equal (to AB) either. Thus, AC is greater than AB .



Thus, in any triangle, the greater angle is subtended by the greater side. (Which is) the very thing it was required to show.

Proposition 20

In any triangle, (the sum of) two sides taken together in any (possible way) is greater than the remaining (side).



For let ABC be a triangle. I say that in triangle ABC (the sum of) two sides taken together in any (possible way) is greater than the remaining (side). (So), (the sum of) BA and AC (is greater) than BC , (the sum of) AB

Διήχθω γὰρ ἡ BA ἐπὶ τὸ Δ σημεῖον, καὶ κείσθω τῇ GA ἴση ἡ AD , καὶ ἐπεζεύχθω ἡ $\Delta\Gamma$.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΔA τῇ $A\Gamma$, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ $\Delta A\Gamma$ τῇ ὑπὸ $A\Gamma\Delta$. μείζων ἄρα ἡ ὑπὸ $B\Gamma\Delta$ τῆς ὑπὸ $\Delta A\Gamma$ καὶ ἐπεὶ τρίγωνόν ἐστι τὸ $\Delta\Gamma B$ μείζονα ἔχον τὴν ὑπὸ $B\Gamma\Delta$ γωνίαν τῆς ὑπὸ $B\Delta\Gamma$, ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει, ἡ ΔB ἄρα τῆς $B\Gamma$ ἐστὶ μείζων. ἴση δὲ ἡ ΔA τῇ $A\Gamma$. μείζονες ἄρα αἱ BA , $A\Gamma$ τῆς $B\Gamma$. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ μὲν AB , $B\Gamma$ τῆς GA μείζονες εἰσιν, αἱ δὲ $B\Gamma$, GA τῆς AB .

Παντὸς ἄρα τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονες εἰσι πάντῃ μεταλαμβανόμεναι· ὅπερ ἔδει δεῖξαι.

and BC than AC , and (the sum of) BC and CA than AB .

For let BA have been drawn through to point D , and let AD be made equal to CA [Prop. 1.3], and let DC have been joined.

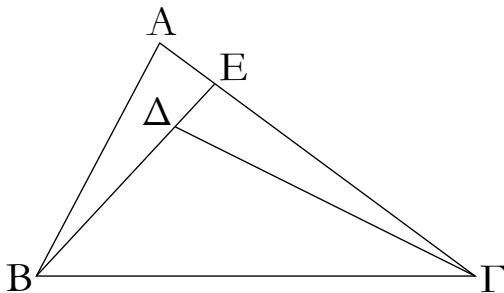
Therefore, since DA is equal to AC , the angle ADC is also equal to ACD [Prop. 1.5]. Thus, BCD is greater than ADC . And since DCB is a triangle having the angle BCD greater than BDC , and the greater angle subtends the greater side [Prop. 1.19], DB is thus greater than BC . But DA is equal to AC . Thus, (the sum of) BA and AC is greater than BC . Similarly, we can show that (the sum of) AB and BC is also greater than CA , and (the sum of) BC and CA than AB .

Thus, in any triangle, (the sum of) two sides taken together in any (possible way) is greater than the remaining (side). (Which is) the very thing it was required to show.

κα'.

Proposition 21

Ἐὰν τριγώνου ἐπὶ μιᾷ τῶν πλευρῶν ἀπὸ τῶν περάτων δύο εὐθεῖαι ἐντὸς συσταθῶσιν, αἱ συσταθεῖσαι τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ἐλάττονες μὲν ἔσονται, μείζονα δὲ γωνίαν περιέχουσιν.

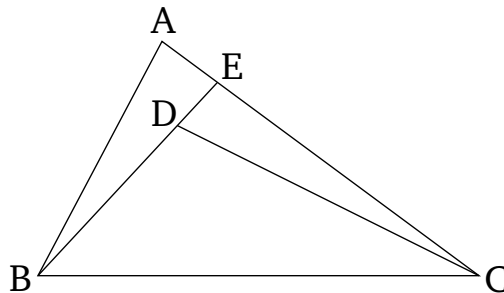


Τριγώνου γὰρ τοῦ $AB\Gamma$ ἐπὶ μιᾷ τῶν πλευρῶν τῆς $B\Gamma$ ἀπὸ τῶν περάτων τῶν B , Γ δύο εὐθεῖαι ἐντὸς συνεστάτωσαν αἱ $B\Delta$, $\Delta\Gamma$. λέγω, ὅτι αἱ $B\Delta$, $\Delta\Gamma$ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν τῶν BA , $A\Gamma$ ἐλάσσονες μὲν εἰσιν, μείζονα δὲ γωνίαν περιέχουσι τὴν ὑπὸ $B\Delta\Gamma$ τῆς ὑπὸ $BA\Gamma$.

Διήχθω γὰρ ἡ $B\Delta$ ἐπὶ τὸ E . καὶ ἐπεὶ παντὸς τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονες εἰσιν, τοῦ ABE ἄρα τριγώνου αἱ δύο πλευραὶ αἱ AB , AE τῆς BE μείζονες εἰσιν· κοινὴ προσκείσθω ἡ $ΕΓ$. αἱ ἄρα BA , $A\Gamma$ τῶν BE , $E\Gamma$ μείζονες εἰσιν. πάλιν, ἐπεὶ τοῦ $\Gamma E\Delta$ τριγώνου αἱ δύο πλευραὶ αἱ ΓE , $E\Delta$ τῆς $\Gamma\Delta$ μείζονες εἰσιν, κοινὴ προσκείσθω ἡ ΔB . αἱ ΓE , EB ἄρα τῶν $\Gamma\Delta$, ΔB μείζονες εἰσιν. ἀλλὰ τῶν BE , $E\Gamma$ μείζονες ἐδείχθησαν αἱ BA , $A\Gamma$. πολλὰ ἄρα αἱ BA , $A\Gamma$ τῶν $B\Delta$, $\Delta\Gamma$ μείζονες εἰσιν.

Πάλιν, ἐπεὶ παντὸς τριγώνου ἡ ἐκτὸς γωνία τῆς ἐντὸς καὶ ἀπεναντίον μείζων ἐστίν, τοῦ $\Gamma\Delta E$ ἄρα τριγώνου ἡ ἐκτὸς γωνία ἡ ὑπὸ $B\Delta\Gamma$ μείζων ἐστὶ τῆς ὑπὸ $\Gamma E\Delta$. διὰ ταῦτά τοίνυν καὶ τοῦ ABE τριγώνου ἡ ἐκτὸς γωνία ἡ ὑπὸ

If two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) will be less than the two remaining sides of the triangle, but will encompass a greater angle.



For let the two internal straight-lines BD and DC have been constructed on one of the sides BC of the triangle ABC , from its ends B and C (respectively). I say that BD and DC are less than the (sum of the) two remaining sides of the triangle BA and AC , but encompass an angle BDC greater than BAC .

For let BD have been drawn through to E . And since in any triangle (the sum of any) two sides is greater than the remaining (side) [Prop. 1.20], in triangle ABE the (sum of the) two sides AB and AE is thus greater than BE . Let EC have been added to both. Thus, (the sum of) BA and AC is greater than (the sum of) BE and EC . Again, since in triangle CED the (sum of the) two sides CE and ED is greater than CD , let DB have been added to both. Thus, (the sum of) CE and EB is greater than (the sum of) CD and DB . But, (the sum of) BA and AC was shown (to be) greater than (the sum of) BE and EC . Thus, (the sum of) BA and AC is much greater than

ΓΕΒ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ. ἀλλὰ τῆς ὑπὸ ΓΕΒ μείζων ἐδείχθη ἡ ὑπὸ ΒΔΓ. πολλῶ ἄρα ἡ ὑπὸ ΒΔΓ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ.

Ἐὰν ἄρα τριγώνου ἐπὶ μιᾷς τῶν πλευρῶν ἀπὸ τῶν περάτων δύο εὐθείαι ἐντὸς συσταθῶσιν, αἱ συσταθεῖσαι τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ἐλάττονες μὲν εἰσιν, μείζονα δὲ γωνίαν περιέχουσιν· ὅπερ ἔδει δεῖξαι.

(the sum of) BD and DC .

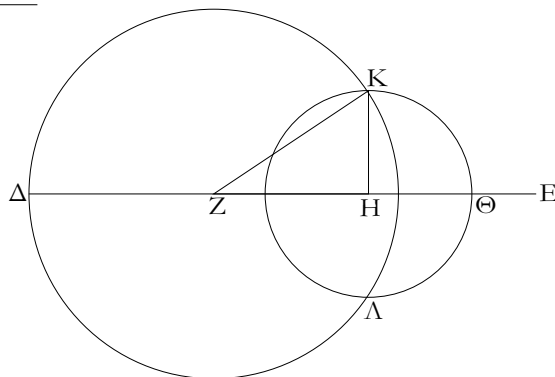
Again, since in any triangle the external angle is greater than the internal and opposite (angles) [Prop. 1.16], in triangle CDE the external angle BDC is thus greater than CED . Accordingly, for the same (reason), the external angle CEB of the triangle ABE is also greater than BAC . But, BDC was shown (to be) greater than CEB . Thus, BDC is much greater than BAC .

Thus, if two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) are less than the two remaining sides of the triangle, but encompass a greater angle. (Which is) the very thing it was required to show.

 $\kappa\beta'$.

Ἐκ τριῶν εὐθειῶν, αἱ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις [εὐθείαις], τρίγωνον συστήσασθαι· δεῖ δὲ τὰς δύο τῆς λοιπῆς μείζοντας εἶναι πάντῃ μεταλαμβανομένας [διὰ τὸ καὶ παντὸς τριγώνου τὰς δύο πλευρὰς τῆς λοιπῆς μείζοντας εἶναι πάντῃ μεταλαμβανομένας].

A _____
B _____
Γ _____



Ἐστωσαν αἱ δοθεῖσαι τρεῖς εὐθεῖαι αἱ Α, Β, Γ, ὧν αἱ δύο τῆς λοιπῆς μείζονες ἕστωσαν πάντῃ μεταλαμβανόμεναι, αἱ μὲν Α, Β τῆς Γ, αἱ δὲ Α, Γ τῆς Β, καὶ ἔτι αἱ Β, Γ τῆς Α· δεῖ δὴ ἐκ τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συστήσασθαι.

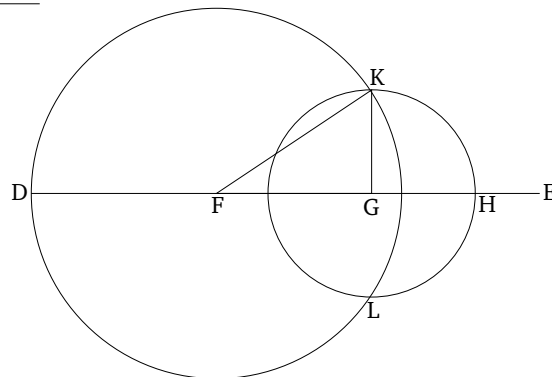
Ἐκκείσθω τις εὐθεία ἡ ΔΕ πεπερασμένη μὲν κατὰ τὸ Δ ἄπειρος δὲ κατὰ τὸ Ε, καὶ κείσθω τῇ μὲν Α ἴση ἡ ΔΖ, τῇ δὲ Β ἴση ἡ ΖΗ, τῇ δὲ Γ ἴση ἡ ΗΘ· καὶ κέντρῳ μὲν τῷ Ζ, διαστήματι δὲ τῷ ΖΔ κύκλος γεγράφθω ὁ ΔΚΛ· πάλιν κέντρῳ μὲν τῷ Η, διαστήματι δὲ τῷ ΗΘ κύκλος γεγράφθω ὁ ΚΛΘ, καὶ ἐπεζεύχθωσαν αἱ ΚΖ, ΚΗ· λέγω, ὅτι ἐκ τριῶν εὐθειῶν τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συνέσταται τὸ ΚΖΗ.

Ἐπεὶ γὰρ τὸ Z σημεῖον κέντρον ἐστὶ τοῦ ΔΚΛ κύκλου,
ἴση ἐστὶν ἡ ΖΔ τῇ ΖΚ· ἀλλὰ ἡ ΖΔ τῇ Α ἐστὶν ἴση. καὶ ἡ

Proposition 22

To construct a triangle from three straight-lines which are equal to three given [straight-lines]. It is necessary for (the sum of) two (of the straight-lines) taken together in any (possible way) to be greater than the remaining (one), [on account of the (fact that) in any triangle (the sum of) two sides taken together in any (possible way) is greater than the remaining (one) [Prop. 1.20]].

A _____
B _____
C _____



Let A , B , and C be the three given straight-lines, of which let (the sum of) two taken together in any (possible way) be greater than the remaining (one). (Thus), (the sum of) A and B (is greater) than C , (the sum of) A and C than B , and also (the sum of) B and C than A . So it is required to construct a triangle from (straight-lines) equal to A , B , and C .

Let some straight-line DE be set out, terminated at D , and infinite in the direction of E . And let DF made equal to A , and FG equal to B , and GH equal to C [Prop. 1.3]. And let the circle DKL have been drawn with center F and radius FD . Again, let the circle KLH have been drawn with center G and radius GH . And let KF and KG have been joined. I say that the triangle KFG has

KZ ἄρα τῇ A ἐστὶν ἴση. πάλιν, ἐπεὶ τὸ H σημεῖον κέντρον ἐστὶ τοῦ ΛΚΘ κύκλου, ἴση ἐστὶν ἡ ΗΘ τῇ ΗΚ· ἀλλὰ ἡ ΗΘ τῇ Γ ἐστὶν ἴση· καὶ ἡ ΚΗ ἄρα τῇ Γ ἐστὶν ἴση. ἐστὶ δὲ καὶ ἡ ΖΗ τῇ Β ἴση· αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΚΖ, ΖΗ, ΗΚ τρισὶ ταῖς Α, Β, Γ ἴσαι εἰσὶν.

Ἐκ τριῶν ἄρα εὐθειῶν τῶν ΚΖ, ΖΗ, ΗΚ, αἱ εἰσὶν ἴσαι τρισὶ ταῖς δοθείσαις εὐθείαις ταῖς Α, Β, Γ, τρίγωνον συνέσταται τὸ ΚΖΗ· ὅπερ ἔδει ποιῆσαι.

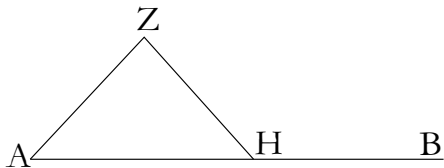
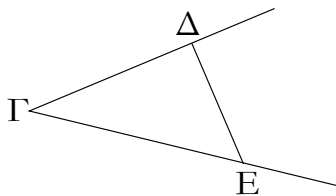
been constructed from three straight-lines equal to A , B , and C .

For since point F is the center of the circle DKL , FD is equal to FK . But, FD is equal to A . Thus, KF is also equal to A . Again, since point G is the center of the circle LKH , GH is equal to GK . But, GH is equal to C . Thus, KG is also equal to C . And FG is also equal to B . Thus, the three straight-lines KF , FG , and GK are equal to A , B , and C (respectively).

Thus, the triangle KFG has been constructed from the three straight-lines KF , FG , and GK , which are equal to the three given straight-lines A , B , and C (respectively). (Which is) the very thing it was required to do.

κγ'.

Πρὸς τῇ δοθείσῃ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ ἴσην γωνίαν εὐθύγραμμον συστήσασθαι.



Ἐστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ πρὸς αὐτῇ σημεῖον τὸ Α, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ ὑπὸ ΔΓΕ· δεῖ δὲ πρὸς τῇ δοθείσῃ εὐθείᾳ τῇ ΑΒ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ τῇ ὑπὸ ΔΓΕ ἴσην γωνίαν εὐθύγραμμον συστήσασθαι.

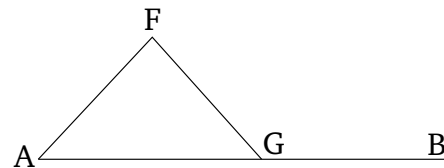
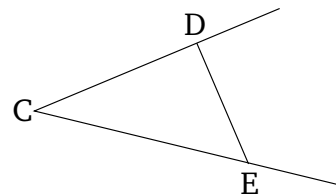
Εἰλήφθω ἐφ' ἑκατέρας τῶν ΓΔ, ΓΕ τυχόντα σημεία τὰ Δ, Ε, καὶ ἐπεζεύχθω ἡ ΔΕ· καὶ ἐκ τριῶν εὐθειῶν, αἱ εἰσὶν ἴσαι τρισὶ ταῖς ΓΔ, ΔΕ, ΓΕ, τρίγωνον συνεστάτω τὸ ΑΖΗ, ὥστε ἴσην εἶναι τὴν μὲν ΓΔ τῇ ΑΖ, τὴν δὲ ΓΕ τῇ ΑΗ, καὶ ἔτι τὴν ΔΕ τῇ ΖΗ.

Ἐπεὶ οὖν δύο αἱ ΔΓ, ΓΕ δύο ταῖς ΖΑ, ΑΗ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ βάσις ἡ ΔΕ βάσει τῇ ΖΗ ἴση, γωνία ἄρα ἡ ὑπὸ ΔΓΕ γωνία τῇ ὑπὸ ΖΑΗ ἐστὶν ἴση.

Πρὸς ἄρα τῇ δοθείσῃ εὐθείᾳ τῇ ΑΒ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ τῇ ὑπὸ ΔΓΕ ἴση γωνία εὐθύγραμμος συνέσταται ἡ ὑπὸ ΖΑΗ· ὅπερ ἔδει ποιῆσαι.

Proposition 23

To construct a rectilinear angle equal to a given rectilinear angle at a (given) point on a given straight-line.



Let AB be the given straight-line, A the (given) point on it, and DCE the given rectilinear angle. So it is required to construct a rectilinear angle equal to the given rectilinear angle DCE at the (given) point A on the given straight-line AB .

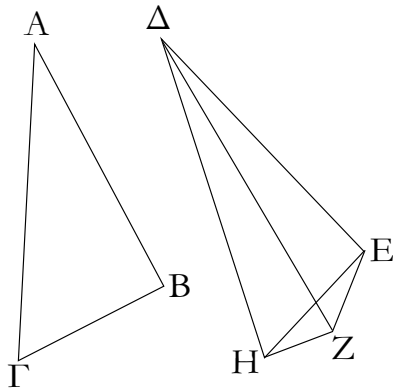
Let the points D and E have been taken at random on each of the (straight-lines) CD and CE (respectively), and let DE have been joined. And let the triangle AFG have been constructed from three straight-lines which are equal to CD , DE , and CE , such that CD is equal to AF , CE to AG , and further DE to FG [Prop. 1.22].

Therefore, since the two (straight-lines) DC , CE are equal to the two (straight-lines) FA , AG , respectively, and the base DE is equal to the base FG , the angle DCE is thus equal to the angle FAG [Prop. 1.8].

Thus, the rectilinear angle FAG , equal to the given rectilinear angle DCE , has been constructed at the (given) point A on the given straight-line AB . (Which

κδ'.

Ἐάν δύο τρίγωνα τὰς δύο πλευράς [ταῖς] δύο πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ, τὴν δὲ γωνίαν τῆς γωνίας μείζονα ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῆς βάσεως μείζονα ἔξει.



Ἐστω δύο τρίγωνα τὰ $AB\Gamma$, ΔEZ τὰς δύο πλευράς τὰς AB , $A\Gamma$ ταῖς δύο πλευραῖς ταῖς ΔE , ΔZ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ, τὴν μὲν AB τῇ ΔE τὴν δὲ $A\Gamma$ τῇ ΔZ , ἡ δὲ πρὸς τῷ A γωνία τῆς πρὸς τῷ Δ γωνίας μείζων ἔστω· λέγω, ὅτι καὶ βάσις ἡ $B\Gamma$ βάσεως τῆς EZ μείζων ἔστί.

Ἐπεὶ γὰρ μείζων ἡ ὑπὸ $BA\Gamma$ γωνία τῆς ὑπὸ $E\Delta Z$ γωνίας, συνεστιάτω πρὸς τῇ ΔE εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Δ τῇ ὑπὸ $BA\Gamma$ γωνίᾳ ἴση ἡ ὑπὸ $E\Delta H$, καὶ κείσθω ὁποτέρᾳ τῶν $A\Gamma$, ΔZ ἴση ἡ ΔH , καὶ ἐπεξεύχθωσαν αἱ EH , ZH .

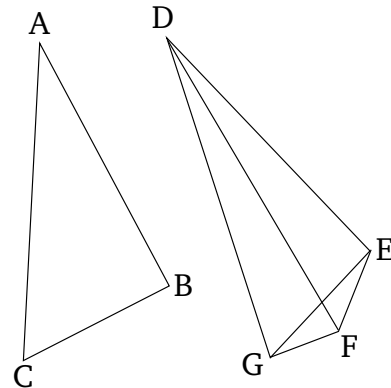
Ἐπεὶ οὖν ἴση ἔστιν ἡ μὲν AB τῇ ΔE , ἡ δὲ $A\Gamma$ τῇ ΔH , δύο δὲ αἱ BA , $A\Gamma$ δυσὶ ταῖς $E\Delta$, ΔH ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ $BA\Gamma$ γωνία τῇ ὑπὸ $E\Delta H$ ἴση· βάσις ἄρα ἡ $B\Gamma$ βάσει τῇ EH ἔστιν ἴση. πάλιν, ἐπεὶ ἴση ἔστιν ἡ ΔZ τῇ ΔH , ἴση ἔστί καὶ ἡ ὑπὸ ΔHZ γωνία τῇ ὑπὸ ΔZH · μείζων ἄρα ἡ ὑπὸ ΔZH τῆς ὑπὸ EHZ · πολλῶ ἄρα μείζων ἔστιν ἡ ὑπὸ EZH τῆς ὑπὸ EHZ . καὶ ἐπεὶ τρίγωνόν ἐστι τὸ EZH μείζονα ἔχον τὴν ὑπὸ EZH γωνίαν τῆς ὑπὸ EHZ , ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει, μείζων ἄρα καὶ πλευρὰ ἡ EH τῆς EZ . ἴση δὲ ἡ EH τῇ $B\Gamma$ · μείζων ἄρα καὶ ἡ $B\Gamma$ τῆς EZ .

Ἐάν ἄρα δύο τρίγωνα τὰς δύο πλευράς δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ, τὴν δὲ γωνίαν τῆς γωνίας μείζονα ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην, καὶ τὴν βάσιν τῆς βάσεως μείζονα ἔξει· ὁπερ ἔδει δεῖξαι.

is) the very thing it was required to do.

Proposition 24

If two triangles have two sides equal to two sides, respectively, but (one) has the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the other), then (the former triangle) will also have a base greater than the base (of the latter).



Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF , respectively. (That is), AB (equal) to DE , and AC to DF . Let them also have the angle at A greater than the angle at D . I say that the base BC is also greater than the base EF .

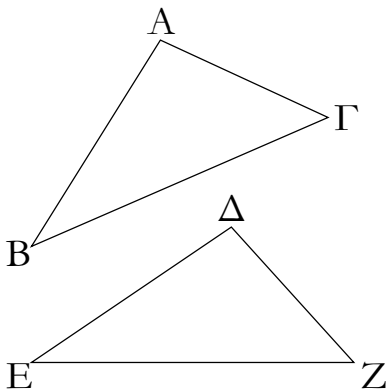
For since angle BAC is greater than angle EDF , let (angle) EDG , equal to angle BAC , have been constructed at the point D on the straight-line DE [Prop. 1.23]. And let DG be made equal to either of AC or DF [Prop. 1.3], and let EG and FG have been joined.

Therefore, since AB is equal to DE and AC to DG , the two (straight-lines) BA , AC are equal to the two (straight-lines) ED , DG , respectively. Also the angle BAC is equal to the angle EDG . Thus, the base BC is equal to the base EG [Prop. 1.4]. Again, since DF is equal to DG , angle DGF is also equal to angle DFG [Prop. 1.5]. Thus, DFG (is) greater than EGF . Thus, EFG is much greater than EGF . And since triangle EFG has angle EFG greater than EGF , and the greater angle is subtended by the greater side [Prop. 1.19], side EG (is) thus also greater than EF . But EG (is) equal to BC . Thus, BC (is) also greater than EF .

Thus, if two triangles have two sides equal to two sides, respectively, but (one) has the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the other), then (the former triangle) will also have a base greater than the base (of the latter).

κε'.

Ἐάν δύο τρίγωνα τὰς δύο πλευράς δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρῃ, τὴν δὲ βάσιν τῆς βάσεως μείζονα ἔχῃ, καὶ τὴν γωνίαν τῆς γωνίας μείζονα ἔξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην.



Ἐστω δύο τρίγωνα τὰ $AB\Gamma$, ΔEZ τὰς δύο πλευράς τὰς AB , $A\Gamma$ ταῖς δύο πλευραῖς ταῖς ΔE , ΔZ ἴσας ἔχοντα ἑκατέραν ἑκατέρῃ, τὴν μὲν AB τῇ ΔE , τὴν δὲ $A\Gamma$ τῇ ΔZ . βάσις δὲ ἡ $B\Gamma$ βάσεως τῆς EZ μείζων ἔστω· λέγω, ὅτι καὶ γωνία ἡ ὑπὸ BAG γωνίας τῆς ὑπὸ EDZ μείζων ἐστίν.

Εἰ γὰρ μή, ἦτοι ἴση ἐστὶν αὐτῇ ἢ ἐλάσσων· ἴση μὲν οὖν οὐκ ἔστιν ἡ ὑπὸ BAG τῇ ὑπὸ EDZ · ἴση γὰρ ἂν ἦν καὶ βάσις ἡ $B\Gamma$ βάσει τῇ EZ · οὐκ ἔστι δέ· οὐκ ἄρα ἴση ἐστὶ γωνία ἡ ὑπὸ BAG τῇ ὑπὸ EDZ · οὐδὲ μὴν ἐλάσσων ἐστὶν ἡ ὑπὸ BAG τῆς ὑπὸ EDZ · ἐλάσσων γὰρ ἂν ἦν καὶ βάσις ἡ $B\Gamma$ βάσεως τῆς EZ · οὐκ ἔστι δέ· οὐκ ἄρα ἐλάσσων ἐστὶν ἡ ὑπὸ BAG γωνία τῆς ὑπὸ EDZ . ἐδείχθη δέ, ὅτι οὐδὲ ἴση· μείζων ἄρα ἐστὶν ἡ ὑπὸ BAG τῆς ὑπὸ EDZ .

Ἐάν ἄρα δύο τρίγωνα τὰς δύο πλευράς δυσὶ πλευραῖς ἴσας ἔχῃ ἑκατέραν ἑκατέρῃ, τὴν δὲ βάσιν τῆς βάσεως μείζονα ἔχῃ, καὶ τὴν γωνίαν τῆς γωνίας μείζονα ἔξει τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην· ὅπερ εἶδει δεῖξαι.

κε'.

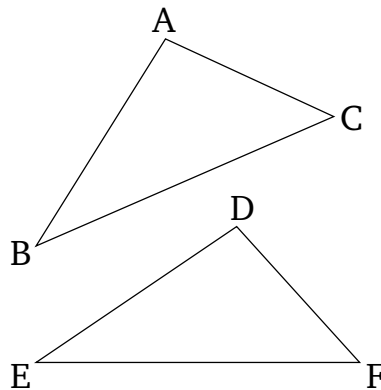
Ἐάν δύο τρίγωνα τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχῃ ἑκατέραν ἑκατέρῃ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην ἦτοι τὴν πρὸς ταῖς ἴσαις γωνίαις ἢ τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν, καὶ τὰς λοιπὰς πλευράς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξει [ἑκατέραν ἑκατέρῃ] καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ.

Ἐστω δύο τρίγωνα τὰ $AB\Gamma$, ΔEZ τὰς δύο γωνίας τὰς

(Which is) the very thing it was required to show.

Proposition 25

If two triangles have two sides equal to two sides, respectively, but (one) has a base greater than the base (of the other), then (the former triangle) will also have the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the latter).



Let ABC and DEF be two triangles having the two sides AB and AC equal to the two sides DE and DF , respectively (That is), AB (equal) to DE , and AC to DF . And let the base BC be greater than the base EF . I say that angle BAC is also greater than EDF .

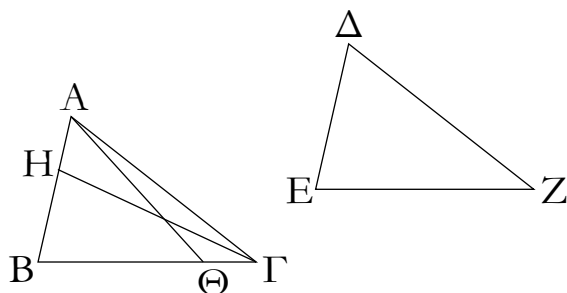
For if not, (BAC) is certainly either equal to, or less than, (EDF). In fact, BAC is not equal to EDF . For then the base BC would also have been equal to the base EF [Prop. 1.4]. But it is not. Thus, angle BAC is not equal to EDF . Neither, indeed, is BAC less than EDF . For then the base BC would also have been less than the base EF [Prop. 1.24]. But it is not. Thus, angle BAC is not less than EDF . But it was shown that (BAC is) not equal (to EDF) either. Thus, BAC is greater than EDF .

Thus, if two triangles have two sides equal to two sides, respectively, but (one) has a base greater than the base (of the other), then (the former triangle) will also have the angle encompassed by the equal straight-lines greater than the (corresponding) angle (in the latter). (Which is) the very thing it was required to show.

Proposition 26

If two triangles have two angles equal to two angles, respectively, and one side equal to one side—in fact, either that by the equal angles, or that subtending one of the equal angles—then (the triangles) will also have the remaining sides equal to the [corresponding] remaining sides, and the remaining angle (equal) to the remaining angle.

ὑπὸ $AB\Gamma$, $B\Gamma A$ δυσὶ ταῖς ὑπὸ ΔEZ , $EZ\Delta$ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ, τὴν μὲν ὑπὸ $AB\Gamma$ τῇ ὑπὸ ΔEZ , τὴν δὲ ὑπὸ $B\Gamma A$ τῇ ὑπὸ $EZ\Delta$. ἐχέτω δὲ καὶ μίαν πλευρὰν μιᾶ πλευρᾷ ἴσην, πρότερον τὴν πρὸς ταῖς ἴσας γωνίαις τὴν $B\Gamma$ τῇ EZ . λέγω, ὅτι καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξει ἑκατέραν ἑκατέρᾳ, τὴν μὲν AB τῇ ΔE τὴν δὲ $A\Gamma$ τῇ ΔZ , καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ, τὴν ὑπὸ BAG τῇ ὑπὸ $E\Delta Z$.



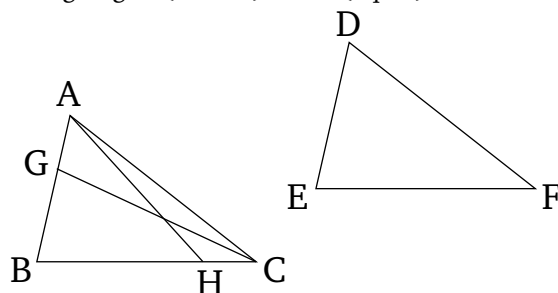
Εἰ γὰρ ἄνισός ἐστιν ἡ AB τῇ ΔE , μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ AB , καὶ κείσθω τῇ ΔE ἴση ἡ BH , καὶ ἐπεζεύχθω ἡ $H\Gamma$.

Ἐπεὶ οὖν ἴση ἐστίν ἡ μὲν BH τῇ ΔE , ἡ δὲ $B\Gamma$ τῇ EZ , δύο δὲ αἱ BH , $B\Gamma$ δυσὶ ταῖς ΔE , EZ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ $H\Gamma B$ γωνία τῇ ὑπὸ ΔEZ ἴση ἐστίν· βάσις ἄρα ἡ $H\Gamma$ βάσει τῇ ΔZ ἴση ἐστίν, καὶ τὸ $H\Gamma B$ τρίγωνον τῷ ΔEZ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἡ ὑπὸ $H\Gamma B$ γωνία τῇ ὑπὸ ΔZE . ἀλλὰ ἡ ὑπὸ ΔZE τῇ ὑπὸ $B\Gamma A$ ὑπόκειται ἴση· καὶ ἡ ὑπὸ $B\Gamma H$ ἄρα τῇ ὑπὸ $B\Gamma A$ ἴση ἐστίν, ἡ ἐλάσσων τῇ μείζονι· ὅπερ ἀδύνατον. οὐκ ἄρα ἄνισός ἐστιν ἡ AB τῇ ΔE . ἴση ἄρα. ἔστι δὲ καὶ ἡ $B\Gamma$ τῇ EZ ἴση· δύο δὲ αἱ AB , $B\Gamma$ δυσὶ ταῖς ΔE , EZ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ $AB\Gamma$ γωνία τῇ ὑπὸ ΔEZ ἐστίν ἴση· βάσις ἄρα ἡ $A\Gamma$ βάσει τῇ ΔZ ἴση ἐστίν, καὶ λοιπὴ γωνία ἡ ὑπὸ BAG τῇ λοιπῇ γωνίᾳ τῇ ὑπὸ $E\Delta Z$ ἴση ἐστίν.

Ἀλλὰ δὴ πάλιν ἔστωσαν αἱ ὑπὸ τὰς ἴσας γωνίας πλευραὶ ὑποτείνουσαι ἴσαι, ὥς ἡ AB τῇ ΔE . λέγω πάλιν, ὅτι καὶ αἱ λοιπαὶ πλευραὶ ταῖς λοιπαῖς πλευραῖς ἴσαι ἔσονται, ἡ μὲν $A\Gamma$ τῇ ΔZ , ἡ δὲ $B\Gamma$ τῇ EZ καὶ ἔτι ἡ λοιπὴ γωνία ἡ ὑπὸ BAG τῇ λοιπῇ γωνίᾳ τῇ ὑπὸ $E\Delta Z$ ἴση ἐστίν.

Εἰ γὰρ ἄνισός ἐστιν ἡ $B\Gamma$ τῇ EZ , μία αὐτῶν μείζων ἐστίν. ἔστω μείζων, εἰ δυνατόν, ἡ $B\Gamma$, καὶ κείσθω τῇ EZ ἴση ἡ $B\Theta$, καὶ ἐπεζεύχθω ἡ $A\Theta$. καὶ ἐπεὶ ἴση ἐστίν ἡ μὲν $B\Theta$ τῇ EZ ἡ δὲ AB τῇ ΔE , δύο δὲ αἱ AB , $B\Theta$ δυσὶ ταῖς ΔE , EZ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνίας ἴσας περιέχουσιν· βάσις ἄρα ἡ $A\Theta$ βάσει τῇ ΔZ ἴση ἐστίν, καὶ τὸ $AB\Theta$ τρίγωνον τῷ ΔEZ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὅφ' ἂς αἱ ἴσας πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἐστίν ἡ ὑπὸ $B\Theta A$ γωνία τῇ ὑπὸ $EZ\Delta$. ἀλλὰ ἡ ὑπὸ

Let ABC and DEF be two triangles having the two angles ABC and BCA equal to the two (angles) DEF and EFD , respectively. (That is) ABC (equal) to DEF , and BCA to EFD . And let them also have one side equal to one side. First of all, the (side) by the equal angles. (That is) BC (equal) to EF . I say that they will have the remaining sides equal to the corresponding remaining sides. (That is) AB (equal) to DE , and AC to DF . And (they will have) the remaining angle (equal) to the remaining angle. (That is) BAC (equal) to EDF .



For if AB is unequal to DE then one of them is greater. Let AB be greater, and let BG be made equal to DE [Prop. 1.3], and let GC have been joined.

Therefore, since BG is equal to DE , and BC to EF , the two (straight-lines) GB , BC are equal to the two (straight-lines) DE , EF , respectively. And angle GBC is equal to angle DEF . Thus, the base GC is equal to the base DF , and triangle GBC is equal to triangle DEF , and the remaining angles subtended by the equal sides will be equal to the (corresponding) remaining angles [Prop. 1.4]. Thus, GCB (is equal) to DFE . But, DFE was assumed (to be) equal to BCA . Thus, BCG is also equal to BCA , the lesser to the greater. The very thing (is) impossible. Thus, AB is not unequal to DE . Thus, (it is) equal. And BC is also equal to EF . So the two (straight-lines) AB , BC are equal to the two (straight-lines) DE , EF , respectively. And angle ABC is equal to angle DEF . Thus, the base AC is equal to the base DF , and the remaining angle BAC is equal to the remaining angle EDF [Prop. 1.4].

But, again, let the sides subtending the equal angles be equal: for instance, (let) AB (be equal) to DE . Again, I say that the remaining sides will be equal to the remaining sides. (That is) AC (equal) to DF , and BC to EF . Furthermore, the remaining angle BAC is equal to the remaining angle EDF .

For if BC is unequal to EF then one of them is greater. If possible, let BC be greater. And let BH be made equal to EF [Prop. 1.3], and let AH have been joined. And since BH is equal to EF , and AB to DE , the two (straight-lines) AB , BH are equal to the two

ΕΖΔ τῇ ὑπὸ ΒΓΑ ἔστιν ἴση· τριγώνου δὴ τοῦ ΑΘΓ ἡ ἐκτὸς γωνία ἡ ὑπὸ ΒΘΑ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΒΓΑ· ὅπερ ἀδύνατον. οὐκ ἄρα ἀνισός ἐστιν ἡ ΒΓ τῇ ΕΖ· ἴση ἄρα. ἐστὶ δὲ καὶ ἡ ΑΒ τῇ ΔΕ ἴση. δύο δὲ αἱ ΑΒ, ΒΓ δύο ταῖς ΔΕ, ΕΖ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνίας ἴσας περιέχουσιν· βάσεις ἄρα ἡ ΑΓ βάσει τῇ ΔΖ ἴση ἐστίν, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ ἴσον καὶ λοιπὴ γωνία ἡ ὑπὸ ΒΑΓ τῇ λοιπῇ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἴση.

Ἐάν ἄρα δύο τρίγωνα τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχῃ ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην ᾗτοι τὴν πρὸς ταῖς ἴσαις γωνίαις, ἢ τὴν ὑποτείνουσιν ὑπὸ μίαν τῶν ἴσων γωνιῶν, καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξει καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ· ὅπερ ἔδει δεῖξαι.

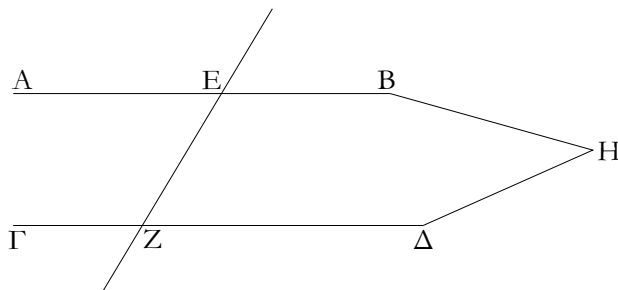
(straight-lines) DE , EF , respectively. And the angles they encompass (are also equal). Thus, the base AH is equal to the base DF , and the triangle ABH is equal to the triangle DEF , and the remaining angles subtended by the equal sides will be equal to the (corresponding) remaining angles [Prop. 1.4]. Thus, angle BHA is equal to EFD . But, EFD is equal to BCA . So, in triangle AHC , the external angle BHA is equal to the internal and opposite angle BCA . The very thing (is) impossible [Prop. 1.16]. Thus, BC is not unequal to EF . Thus, (it is) equal. And AB is also equal to DE . So the two (straight-lines) AB , BC are equal to the two (straight-lines) DE , EF , respectively. And they encompass equal angles. Thus, the base AC is equal to the base DF , and triangle ABC (is) equal to triangle DEF , and the remaining angle BAC (is) equal to the remaining angle EDF [Prop. 1.4].

Thus, if two triangles have two angles equal to two angles, respectively, and one side equal to one side—in fact, either that by the equal angles, or that subtending one of the equal angles—then (the triangles) will also have the remaining sides equal to the (corresponding) remaining sides, and the remaining angle (equal) to the remaining angle. (Which is) the very thing it was required to show.

† The Greek text has “ BG , BC ”, which is obviously a mistake.

κζ'.

Ἐάν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιῇ, παράλληλοι ἔσονται ἀλλήλαις αἱ εὐθεῖαι.

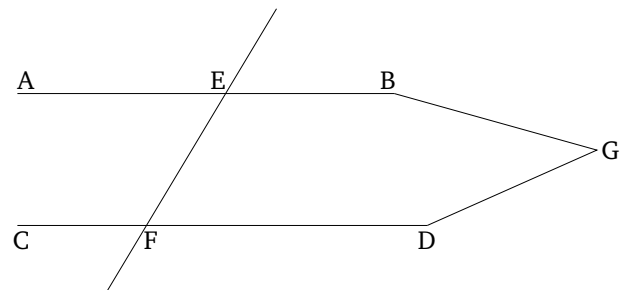


Εἰς γὰρ δύο εὐθείας τὰς ΑΒ, ΓΔ εὐθεῖα ἐμπίπτουσα ἡ ΕΖ τὰς ἐναλλὰξ γωνίας τὰς ὑπὸ ΑΕΖ, ΕΖΔ ἴσας ἀλλήλαις ποιεῖτω· λέγω, ὅτι παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ.

Εἰ γὰρ μὴ, ἐκβαλλόμεναι αἱ ΑΒ, ΓΔ συμπεσοῦνται ᾗτοι ἐπὶ τὰ Β, Δ μέρη ἢ ἐπὶ τὰ Α, Γ. ἐκβεβλήσθωσαν καὶ συμπίπτωσαν ἐπὶ τὰ Β, Δ μέρη κατὰ τὸ Η. τριγώνου δὴ τοῦ ΗΕΖ ἡ ἐκτὸς γωνία ἡ ὑπὸ ΑΕΖ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ΕΖΗ· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα αἱ ΑΒ, ΓΔ ἐκβαλλόμεναι συμπεσοῦνται ἐπὶ τὰ Β, Δ μέρη. ὁμοίως

Proposition 27

If a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel to one another.



For let the straight-line EF , falling across the two straight-lines AB and CD , make the alternate angles AEF and EFD equal to one another. I say that AB and CD are parallel.

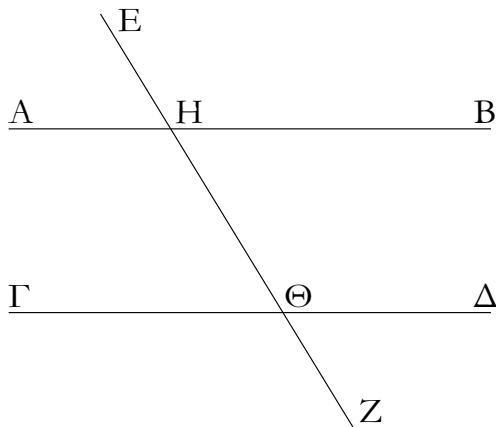
For if not, being produced, AB and CD will certainly meet together: either in the direction of B and D , or (in the direction) of A and C [Def. 1.23]. Let them have been produced, and let them meet together in the direction of B and D at (point) G . So, for the triangle

δη δειχθήσεται, ὅτι οὐδὲ ἐπὶ τὰ A, Γ αἱ δὲ ἐπὶ μηδέτερα τὰ μέρη συμπίπτουσαι παράλληλοί εἰσιν· παράλληλος ἄρα ἐστὶν ἡ AB τῇ $\Gamma\Delta$.

Ἐάν ἄρα εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐναλλάξ γωνίας ἴσας ἀλλήλαις ποιῇ, παράλληλοι ἔσονται αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

κη'.

Ἐάν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὴν ἐκτὸς γωνίαν τῇ ἐντὸς καὶ ἀπεναντίον καὶ ἐπὶ τὰ αὐτὰ μέρη ἴσην ποιῇ ἢ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας, παράλληλοι ἔσονται ἀλλήλαις αἱ εὐθεῖαι.



Εἰς γὰρ δύο εὐθείας τὰς $AB, \Gamma\Delta$ εὐθεῖα ἐμπίπτουσα ἡ EZ τὴν ἐκτὸς γωνίαν τὴν ὑπὸ EHB τῇ ἐντὸς καὶ ἀπεναντίον γωνίᾳ τῇ ὑπὸ $H\Theta\Delta$ ἴσην ποιείτω ἢ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη τὰς ὑπὸ $BH\Theta, H\Theta\Delta$ δυσὶν ὀρθαῖς ἴσας· λέγω, ὅτι παράλληλός ἐστιν ἡ AB τῇ $\Gamma\Delta$.

Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ὑπὸ EHB τῇ ὑπὸ $H\Theta\Delta$, ἀλλὰ ἡ ὑπὸ EHB τῇ ὑπὸ $AH\Theta$ ἐστὶν ἴση, καὶ ἡ ὑπὸ $AH\Theta$ ἄρα τῇ ὑπὸ $H\Theta\Delta$ ἐστὶν ἴση· καὶ εἰσιν ἐναλλάξ· παράλληλος ἄρα ἐστὶν ἡ AB τῇ $\Gamma\Delta$.

Πάλιν, ἐπεὶ αἱ ὑπὸ $BH\Theta, H\Theta\Delta$ δύο ὀρθαῖς ἴσαι εἰσίν, εἰσὶ δὲ καὶ αἱ ὑπὸ $AH\Theta, BH\Theta$ δυσὶν ὀρθαῖς ἴσαι, αἱ ἄρα ὑπὸ $AH\Theta, BH\Theta$ ταῖς ὑπὸ $BH\Theta, H\Theta\Delta$ ἴσαι εἰσίν· κοινὴ ἀφρηθήσθω ἡ ὑπὸ $BH\Theta$ · λοιπὴ ἄρα ἡ ὑπὸ $AH\Theta$ λοιπῇ τῇ ὑπὸ $H\Theta\Delta$ ἐστὶν ἴση· καὶ εἰσιν ἐναλλάξ· παράλληλος ἄρα ἐστὶν ἡ AB τῇ $\Gamma\Delta$.

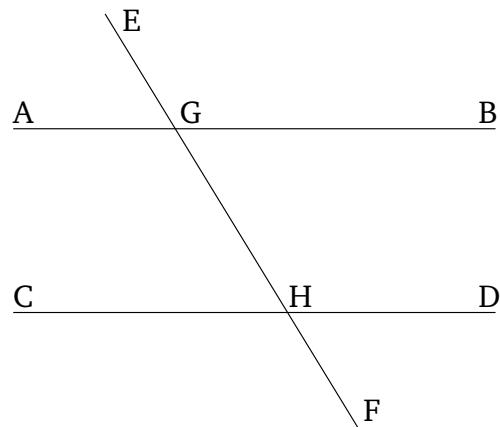
Ἐάν ἄρα εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὴν ἐκτὸς γωνίαν τῇ ἐντὸς καὶ ἀπεναντίον καὶ ἐπὶ τὰ αὐτὰ μέρη ἴσην

GEF , the external angle AEF is equal to the interior and opposite (angle) EFG . The very thing is impossible [Prop. 1.16]. Thus, being produced, AB and CD will not meet together in the direction of B and D . Similarly, it can be shown that neither (will they meet together) in (the direction of) A and C . But (straight-lines) meeting in neither direction are parallel [Def. 1.23]. Thus, AB and CD are parallel.

Thus, if a straight-line falling across two straight-lines makes the alternate angles equal to one another then the (two) straight-lines will be parallel (to one another). (Which is) the very thing it was required to show.

Proposition 28

If a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, or (makes) the (sum of the) internal (angles) on the same side equal to two right-angles, then the (two) straight-lines will be parallel to one another.



For let EF , falling across the two straight-lines AB and CD , make the external angle EGB equal to the internal and opposite angle GHD , or the (sum of the) internal (angles) on the same side, BGH and GHD , equal to two right-angles. I say that AB is parallel to CD .

For since (in the first case) EGB is equal to GHD , but EGB is equal to AGH [Prop. 1.15], AGH is thus also equal to GHD . And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

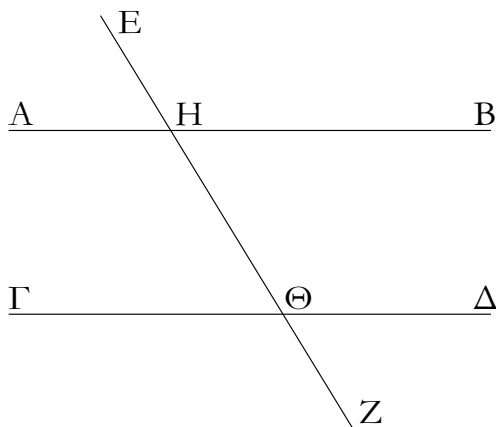
Again, since (in the second case, the sum of) BGH and GHD is equal to two right-angles, and (the sum of) AGH and BGH is also equal to two right-angles [Prop. 1.13], (the sum of) AGH and BGH is thus equal to (the sum of) BGH and GHD . Let BGH have been subtracted from both. Thus, the remainder AGH is equal to the remainder GHD . And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

ποιῇ ἢ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας, παράλληλοι ἔσονται αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

Thus, if a straight-line falling across two straight-lines makes the external angle equal to the internal and opposite angle on the same side, or (makes) the (sum of the) internal (angles) on the same side equal to two right-angles, then the (two) straight-lines will be parallel (to one another). (Which is) the very thing it was required to show.

κθ'.

Ἐὰν εἰς τὰς παραλλήλους εὐθείας εὐθεῖα ἐμπίπτουσα τὰς τε ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιεῖ καὶ τὴν ἐκτὸς τῇ ἐντὸς καὶ ἀπεναντίον ἴσην καὶ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη δυσὶν ὀρθαῖς ἴσας.



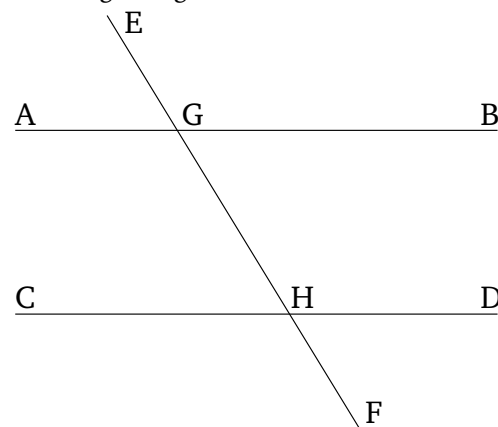
Εἰς γὰρ παραλλήλους εὐθείας τὰς AB, ΓΔ εὐθεῖα ἐμπίπτετω ἡ EZ· λέγω, ὅτι τὰς ἐναλλὰξ γωνίας τὰς ὑπὸ AHΘ, HΘΔ ἴσας ποιεῖ καὶ τὴν ἐκτὸς γωνίαν τὴν ὑπὸ EHB τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ HΘΔ ἴσην καὶ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη τὰς ὑπὸ BHΘ, HΘΔ δυσὶν ὀρθαῖς ἴσας.

Εἰ γὰρ ἄνισός ἐστιν ἡ ὑπὸ AHΘ τῇ ὑπὸ HΘΔ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ὑπὸ AHΘ· κοινὴ προσκείσθω ἡ ὑπὸ BHΘ· αἱ ἄρα ὑπὸ AHΘ, BHΘ τῶν ὑπὸ BHΘ, HΘΔ μείζονές εἰσιν. ἀλλὰ αἱ ὑπὸ AHΘ, BHΘ δυσὶν ὀρθαῖς ἴσαι εἰσίν. [καὶ] αἱ ἄρα ὑπὸ BHΘ, HΘΔ δύο ὀρθῶν ἐλάσσονές εἰσιν. αἱ δὲ ἀπ' ἐλασσόνων ἢ δύο ὀρθῶν ἐκβαλλόμεναι εἰς ἄπειρον συμπίπτουσιν· αἱ ἄρα AB, ΓΔ ἐκβαλλόμεναι εἰς ἄπειρον συμπεσοῦνται· οὐ συμπίπτουσι δὲ διὰ τὸ παραλλήλους αὐτὰς ὑποκεῖσθαι· οὐκ ἄρα ἄνισός ἐστιν ἡ ὑπὸ AHΘ τῇ ὑπὸ HΘΔ· ἴση ἄρα. ἀλλὰ ἡ ὑπὸ AHΘ τῇ ὑπὸ EHB ἐστὶν ἴση· καὶ ἡ ὑπὸ EHB ἄρα τῇ ὑπὸ HΘΔ ἐστὶν ἴση· κοινὴ προσκείσθω ἡ ὑπὸ BHΘ· αἱ ἄρα ὑπὸ EHB, BHΘ ταῖς ὑπὸ BHΘ, HΘΔ ἴσαι εἰσίν. ἀλλὰ αἱ ὑπὸ EHB, BHΘ δύο ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ BHΘ, HΘΔ ἄρα δύο ὀρθαῖς ἴσαι εἰσίν.

Ἐὰν εἰς τὰς παραλλήλους εὐθείας εὐθεῖα ἐμπίπτουσα τὰς τε ἐναλλὰξ γωνίας ἴσας ἀλλήλαις ποιεῖ καὶ τὴν ἐκτὸς τῇ ἐντὸς καὶ ἀπεναντίον ἴσην καὶ τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ

Proposition 29

A straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles.



For let the straight-line *EF* fall across the parallel straight-lines *AB* and *CD*. I say that it makes the alternate angles, *AGH* and *GHD*, equal, the external angle *EGB* equal to the internal and opposite (angle) *GHD*, and the (sum of the) internal (angles) on the same side, *BGH* and *GHD*, equal to two right-angles.

For if *AGH* is unequal to *GHD* then one of them is greater. Let *AGH* be greater. Let *BGH* have been added to both. Thus, (the sum of) *AGH* and *BGH* is greater than (the sum of) *BGH* and *GHD*. But, (the sum of) *AGH* and *BGH* is equal to two right-angles [Prop 1.13]. Thus, (the sum of) *BGH* and *GHD* is [also] less than two right-angles. But (straight-lines) being produced to infinity from (internal angles whose sum is) less than two right-angles meet together [Post. 5]. Thus, *AB* and *CD*, being produced to infinity, will meet together. But they do not meet, on account of them (initially) being assumed parallel (to one another) [Def. 1.23]. Thus, *AGH* is not unequal to *GHD*. Thus, (it is) equal. But, *AGH* is equal to *EGB* [Prop. 1.15]. And *EGB* is thus also equal to *GHD*. Let *BGH* be added to both. Thus, (the sum of) *EGB* and *BGH* is equal to (the sum of) *BGH* and *GHD*. But, (the sum of) *EGB* and *BGH* is equal to two right-

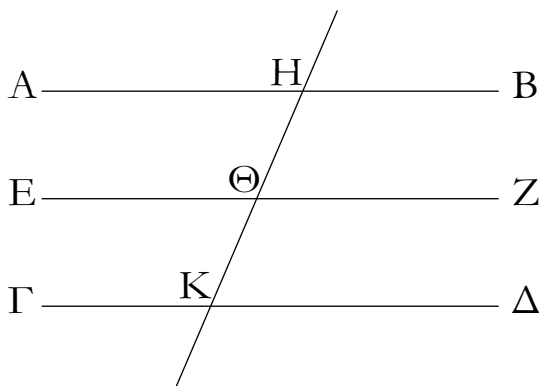
μέρη δυσὶν ὀρθαῖς ἴσας· ὅπερ ἔδει δεῖξαι.

angles [Prop. 1.13]. Thus, (the sum of) BGH and GHD is also equal to two right-angles.

Thus, a straight-line falling across parallel straight-lines makes the alternate angles equal to one another, the external (angle) equal to the internal and opposite (angle), and the (sum of the) internal (angles) on the same side equal to two right-angles. (Which is) the very thing it was required to show.

λ'.

Αἱ τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ ἀλλήλαις εἰσὶ παράλληλοι.



Ἐστω ἑκατέρα τῶν AB , $\Gamma\Delta$ τῇ EZ παράλληλος· λέγω, ὅτι καὶ ἡ AB τῇ $\Gamma\Delta$ ἐστὶ παράλληλος.

Ἐμπίπττω γὰρ εἰς αὐτὰς εὐθεῖα ἡ HK .

Καὶ ἐπεὶ εἰς παραλλήλους εὐθείας τὰς AB , EZ εὐθεῖα ἐμπίπτωκεν ἡ HK , ἴση ἄρα ἡ ὑπὸ AHK τῇ ὑπὸ $H\Theta Z$. πάλιν, ἐπεὶ εἰς παραλλήλους εὐθείας τὰς EZ , $\Gamma\Delta$ εὐθεῖα ἐμπίπτωκεν ἡ HK , ἴση ἐστὶν ἡ ὑπὸ $H\Theta Z$ τῇ ὑπὸ $HK\Delta$. ἐδείχθη δὲ καὶ ἡ ὑπὸ AHK τῇ ὑπὸ $H\Theta Z$ ἴση. καὶ ἡ ὑπὸ AHK ἄρα τῇ ὑπὸ $HK\Delta$ ἐστὶν ἴση· καὶ εἰσιν ἐναλλάξ. παράλληλος ἄρα ἐστὶν ἡ AB τῇ $\Gamma\Delta$.

[Αἱ ἄρα τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ ἀλλήλαις εἰσὶ παράλληλοι·] ὅπερ ἔδει δεῖξαι.

λα'.

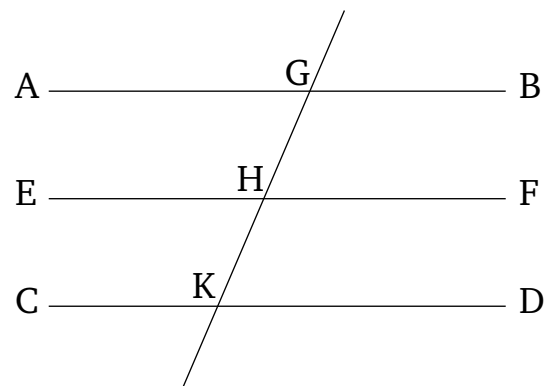
Διὰ τοῦ δοθέντος σημείου τῇ δοθείσῃ εὐθείᾳ παράλληλον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Ἐστω τὸ μὲν δοθὲν σημεῖον τὸ A , ἡ δὲ δοθεῖσα εὐθεῖα ἡ $B\Gamma$ · δεῖ δὴ διὰ τοῦ A σημείου τῇ $B\Gamma$ εὐθείᾳ παράλληλον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Εἰλήφθω ἐπὶ τῆς $B\Gamma$ τυχὸν σημεῖον τὸ Δ , καὶ ἐπεζεύχθω ἡ $A\Delta$ · καὶ συνεστάτω πρὸς τῇ ΔA εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ A τῇ ὑπὸ $A\Delta\Gamma$ γωνίᾳ ἴση ἡ ὑπὸ $\Delta A E$ · καὶ

Proposition 30

(Straight-lines) parallel to the same straight-line are also parallel to one another.



Let each of the (straight-lines) AB and CD be parallel to EF . I say that AB is also parallel to CD .

For let the straight-line GK fall across (AB , CD , and EF).

And since the straight-line GK has fallen across the parallel straight-lines AB and EF , (angle) AGK (is) thus equal to GHE [Prop. 1.29]. Again, since the straight-line GK has fallen across the parallel straight-lines EF and CD , (angle) GHE is equal to GKD [Prop. 1.29]. But AGK was also shown (to be) equal to GHE . Thus, AGK is also equal to GKD . And they are alternate (angles). Thus, AB is parallel to CD [Prop. 1.27].

[Thus, (straight-lines) parallel to the same straight-line are also parallel to one another.] (Which is) the very thing it was required to show.

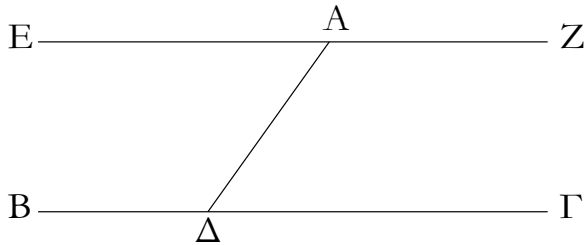
Proposition 31

To draw a straight-line parallel to a given straight-line, through a given point.

Let A be the given point, and BC the given straight-line. So it is required to draw a straight-line parallel to the straight-line BC , through the point A .

Let the point D have been taken a random on BC , and let AD have been joined. And let (angle) DAE , equal to angle ADC , have been constructed on the straight-line

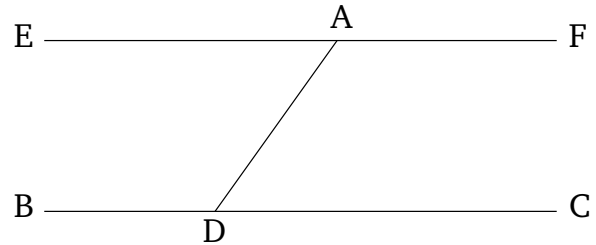
ἐκβεβλήσθω ἐπ' εὐθείας τῇ EA εὐθεΐα ἡ AZ .



Καὶ ἐπεὶ εἰς δύο εὐθείας τὰς BG , EZ εὐθεΐα ἐμπίπτουσα ἡ AD τὰς ἐναλλὰξ γωνίας τὰς ὑπὸ EAD , ADG ἴσας ἀλλήλαις πεποίηκεν, παράλληλος ἄρα ἐστὶν ἡ EAZ τῇ BG .

Διὰ τοῦ δοθέντος ἄρα σημείου τοῦ A τῇ δοθείσῃ εὐθείᾳ τῇ BG παράλληλος εὐθεΐα γραμμὴ ἤκται ἡ EAZ · ὅπερ ἔδει ποιῆσαι.

DA at the point A on it [Prop. 1.23]. And let the straight-line AF have been produced in a straight-line with EA .

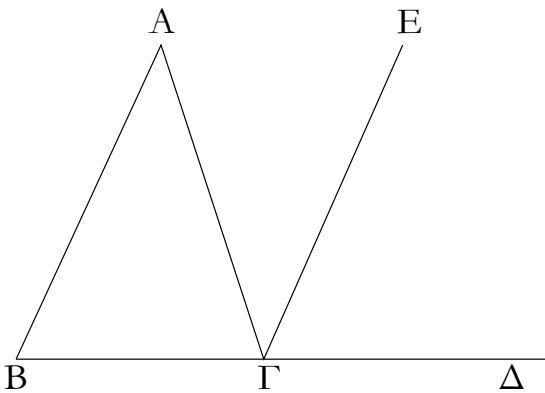


And since the straight-line AD , (in) falling across the two straight-lines BC and EF , has made the alternate angles EAD and ADC equal to one another, EAF is thus parallel to BC [Prop. 1.27].

Thus, the straight-line EAF has been drawn parallel to the given straight-line BC , through the given point A . (Which is) the very thing it was required to do.

λβ'.

Παντὸς τριγώνου μιᾷ τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ἴση ἐστίν, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν.



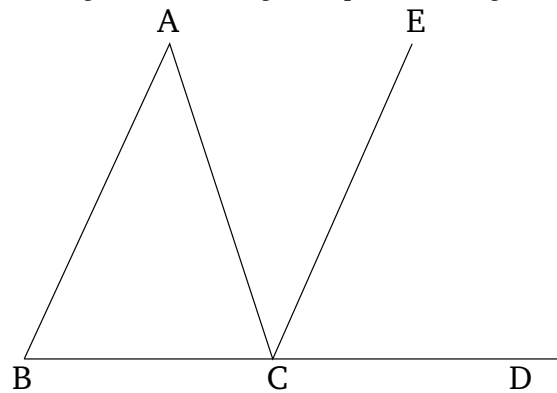
Ἐστω τρίγωνον τὸ ABG , καὶ προσεκβεβλήσθω αὐτοῦ μία πλευρὰ ἡ BG ἐπὶ τὸ Δ · λέγω, ὅτι ἡ ἐκτὸς γωνία ἡ ὑπὸ AGD ἴση ἐστὶ δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ταῖς ὑπὸ GAB , ABG , καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι αἱ ὑπὸ ABG , BGA , GAB δυσὶν ὀρθαῖς ἴσαι εἰσίν.

Ἦχθω γὰρ διὰ τοῦ G σημείου τῇ AB εὐθείᾳ παράλληλος ἡ GE .

Καὶ ἐπεὶ παράλληλός ἐστιν ἡ AB τῇ GE , καὶ εἰς αὐτὰς ἐμπίπτωκεν ἡ AG , αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ BAG , AGE ἴσαι ἀλλήλαις εἰσίν. πάλιν, ἐπεὶ παράλληλός ἐστιν ἡ AB τῇ GE , καὶ εἰς αὐτὰς ἐμπίπτωκεν εὐθεΐα ἡ BD , ἡ ἐκτὸς γωνία ἡ ὑπὸ EGD ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ ABG . ἐδείχθη δὲ καὶ ἡ ὑπὸ AGE τῇ ὑπὸ BAG ἴση· ὅλη ἄρα ἡ ὑπὸ AGD γωνία ἴση ἐστὶ δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ταῖς ὑπὸ BAG , ABG .

Proposition 32

In any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles.



Let ABC be a triangle, and let one of its sides BC have been produced to D . I say that the external angle ACD is equal to the (sum of the) two internal and opposite angles CAB and ABC , and the (sum of the) three internal angles of the triangle— ABC , BCA , and CAB —is equal to two right-angles.

For let CE have been drawn through point C parallel to the straight-line AB [Prop. 1.31].

And since AB is parallel to CE , and AC has fallen across them, the alternate angles BAC and ACE are equal to one another [Prop. 1.29]. Again, since AB is parallel to CE , and the straight-line BD has fallen across them, the external angle ECD is equal to the internal and opposite (angle) ABC [Prop. 1.29]. But ACE was also shown (to be) equal to BAC . Thus, the whole an-

Κοινή προσκείσθω ἡ ὑπὸ ΑΓΒ· αἱ ἄρα ὑπὸ ΑΓΔ, ΑΓΒ τρισὶ ταῖς ὑπὸ ΑΒΓ, ΒΓΑ, ΓΑΒ ἴσαι εἰσὶν. ἀλλ' αἱ ὑπὸ ΑΓΔ, ΑΓΒ δυσὶν ὀρθαῖς ἴσαι εἰσὶν· καὶ αἱ ὑπὸ ΑΓΒ, ΓΒΑ, ΓΑΒ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσὶν.

Παντὸς ἄρα τριγώνου μιᾶς τῶν πλευρῶν προσεκβληθείσης ἡ ἐκτὸς γωνία δυσὶ ταῖς ἐντὸς καὶ ἀπεναντίον ἴση ἐστίν, καὶ αἱ ἐντὸς τοῦ τριγώνου τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσὶν· ὅπερ ἔδει δεῖξαι.

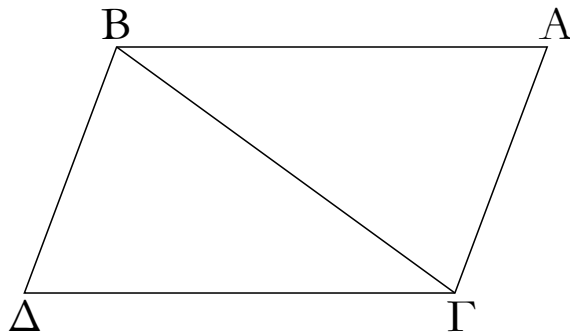
gle ACD is equal to the (sum of the) two internal and opposite (angles) BAC and ABC .

Let ACB have been added to both. Thus, (the sum of) ACD and ACB is equal to the (sum of the) three (angles) ABC , BCA , and CAB . But, (the sum of) ACD and ACB is equal to two right-angles [Prop. 1.13]. Thus, (the sum of) ACB , CBA , and CAB is also equal to two right-angles.

Thus, in any triangle, (if) one of the sides (is) produced (then) the external angle is equal to the (sum of the) two internal and opposite (angles), and the (sum of the) three internal angles of the triangle is equal to two right-angles. (Which is) the very thing it was required to show.

λγ'.

Αἱ τὰς ἴσας τε καὶ παραλλήλους ἐπὶ τὰ αὐτὰ μέρη ἐπιζευγνύουσαι εὐθεῖαι καὶ αὐταὶ ἴσαι τε καὶ παράλληλοί εἰσιν.



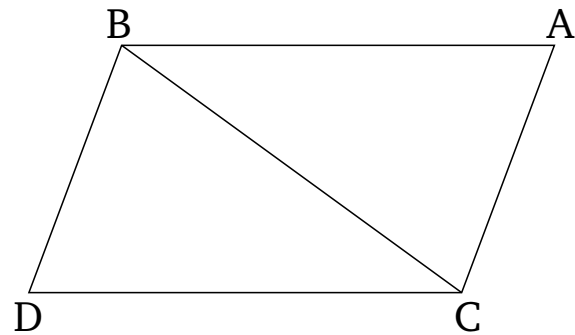
Ἐστωσαν ἴσαι τε καὶ παράλληλοι αἱ ΑΒ, ΓΔ, καὶ ἐπιζευγνύτωσαν αὐτὰς ἐπὶ τὰ αὐτὰ μέρη εὐθεῖαι αἱ ΑΓ, ΒΔ· λέγω, ὅτι καὶ αἱ ΑΓ, ΒΔ ἴσαι τε καὶ παράλληλοί εἰσιν.

Ἐπεζύχθω ἡ ΒΓ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ, καὶ εἰς αὐτὰς ἐμπίπτωκεν ἡ ΒΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΓΔ ἴσαι ἀλλήλαις εἰσὶν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΒ τῇ ΓΔ κοινὴ δὲ ἡ ΒΓ, δύο δὴ αἱ ΑΒ, ΒΓ δύο ταῖς ΒΓ, ΓΔ ἴσαι εἰσὶν· καὶ γωνία ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΒΓΔ ἴση· βάσεις ἄρα ἡ ΑΓ βάσει τῇ ΒΔ ἐστὶν ἴση, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΒΓΔ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκατέρω ἑκατέρω, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἡ ὑπὸ ΑΓΒ γωνία τῇ ὑπὸ ΓΒΔ. καὶ ἐπεὶ εἰς δύο εὐθείας τὰς ΑΓ, ΒΔ εὐθεῖα ἐμπίπτουσα ἡ ΒΓ τὰς ἐναλλὰξ γωνίας ἴσας ἀλλήλαις πεποίηκεν, παράλληλος ἄρα ἐστὶν ἡ ΑΓ τῇ ΒΔ. ἐδείχθη δὲ αὐτῇ καὶ ἴση.

Αἱ ἄρα τὰς ἴσας τε καὶ παραλλήλους ἐπὶ τὰ αὐτὰ μέρη ἐπιζευγνύουσαι εὐθεῖαι καὶ αὐταὶ ἴσαι τε καὶ παράλληλοί εἰσιν· ὅπερ ἔδει δεῖξαι.

Proposition 33

Straight-lines joining equal and parallel (straight-lines) on the same sides are themselves also equal and parallel.



Let AB and CD be equal and parallel (straight-lines), and let the straight-lines AC and BD join them on the same sides. I say that AC and BD are also equal and parallel.

Let BC have been joined. And since AB is parallel to CD , and BC has fallen across them, the alternate angles ABC and BCD are equal to one another [Prop. 1.29]. And since AB is equal to CD , and BC is common, the two (straight-lines) AB , BC are equal to the two (straight-lines) DC , CB .[†] And the angle ABC is equal to the angle BCD . Thus, the base AC is equal to the base BD , and triangle ABC is equal to triangle DCB ,[‡] and the remaining angles will be equal to the corresponding remaining angles subtended by the equal sides [Prop. 1.4]. Thus, angle ACB is equal to CBD . Also, since the straight-line BC , (in) falling across the two straight-lines AC and BD , has made the alternate angles (ACB and CBD) equal to one another, AC is thus parallel to BD [Prop. 1.27]. And (AC) was also shown (to be) equal to (BD).

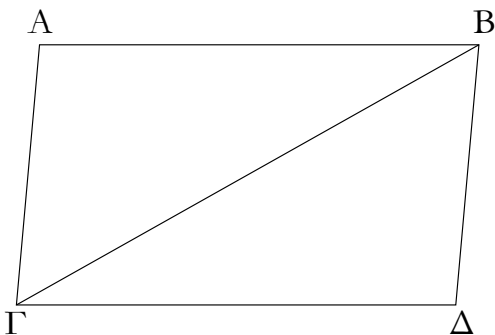
Thus, straight-lines joining equal and parallel (straight-

† The Greek text has “ BC, CD ”, which is obviously a mistake.

‡ The Greek text has “ DCB ”, which is obviously a mistake.

λδ'.

Τῶν παραλληλογράμμων χωρίων αἱ ἀπεναντίον πλευραὶ τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ ἡ διάμετρος αὐτὰ διχα τέμνει.



Ἐστω παραλληλόγραμμον χωρίον τὸ ΑΓΔΒ, διάμετρος δὲ αὐτοῦ ἡ ΒΓ· λέγω, ὅτι τοῦ ΑΓΔΒ παραλληλογράμμου αἱ ἀπεναντίον πλευραὶ τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν, καὶ ἡ ΒΓ διάμετρος αὐτὸ διχα τέμνει.

Ἐπεὶ γὰρ παράλληλός ἐστιν ἡ ΑΒ τῇ ΓΔ, καὶ εἰς αὐτὰς ἐμπίπτωκεν εὐθεῖα ἡ ΒΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΑΒΓ, ΒΓΔ ἴσαι ἀλλήλαις εἰσίν. πάλιν ἐπεὶ παράλληλός ἐστιν ἡ ΑΓ τῇ ΒΔ, καὶ εἰς αὐτὰς ἐμπίπτωκεν ἡ ΒΓ, αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ ΑΓΒ, ΓΒΔ ἴσαι ἀλλήλαις εἰσίν. δύο δὲ τρίγωνά ἐστι τὰ ΑΒΓ, ΒΓΔ τὰς δύο γωνίας τὰς ὑπὸ ΑΒΓ, ΒΓΔ δυσὶ ταῖς ὑπὸ ΒΓΔ, ΓΒΔ ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν πρὸς ταῖς ἴσαις γωνίαις κοινὴν αὐτῶν τὴν ΒΓ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς ἴσας ἔξει ἑκατέραν ἑκατέρᾳ καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ· ἴση ἄρα ἡ μὲν ΑΒ πλευρὰ τῇ ΓΔ, ἡ δὲ ΑΓ τῇ ΒΔ, καὶ ἔτι ἴση ἐστὶν ἡ ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΓΔΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΒΓΔ, ἡ δὲ ὑπὸ ΓΒΔ τῇ ὑπὸ ΑΓΒ, ὅλη ἄρα ἡ ὑπὸ ΑΒΔ ὅλη τῇ ὑπὸ ΑΓΔ ἐστὶν ἴση. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΓΔΒ ἴση.

Τῶν ἄρα παραλληλογράμμων χωρίων αἱ ἀπεναντίον πλευραὶ τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

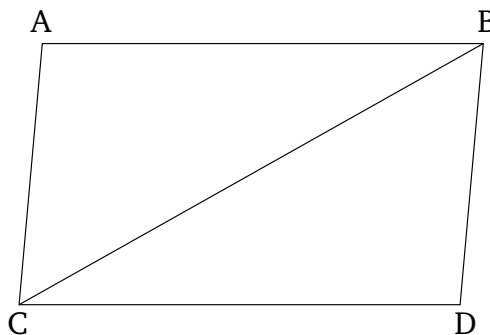
Λέγω δὴ, ὅτι καὶ ἡ διάμετρος αὐτὰ διχα τέμνει. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ΑΒ τῇ ΓΔ, κοινὴ δὲ ἡ ΒΓ, δύο δὲ αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΓΔ, ΒΓ ἴσαι εἰσὶν ἑκατέρᾳ ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΒΓΔ ἴση. καὶ βάσεις ἄρα ἡ ΑΓ τῇ ΔΒ ἴση. καὶ τὸ ΑΒΓ [ἄρα] τρίγωνον τῷ ΒΓΔ τριγώνῳ ἴσον ἐστίν.

Ἡ ἄρα ΒΓ διάμετρος διχα τέμνει τὸ ΑΒΓΔ παραλληλόγραμμον· ὅπερ ἔδει δεῖξαι.

lines) on the same sides are themselves also equal and parallel. (Which is) the very thing it was required to show.

Proposition 34

In parallelogrammic figures the opposite sides and angles are equal to one another, and a diagonal cuts them in half.



Let $ACDB$ be a parallelogrammic figure, and BC its diagonal. I say that for parallelogram $ACDB$, the opposite sides and angles are equal to one another, and the diagonal BC cuts it in half.

For since AB is parallel to CD , and the straight-line BC has fallen across them, the alternate angles ABC and BCD are equal to one another [Prop. 1.29]. Again, since AC is parallel to BD , and BC has fallen across them, the alternate angles ACB and CBD are equal to one another [Prop. 1.29]. So ABC and BCD are two triangles having the two angles ABC and BCA equal to the two (angles) BCD and CBD , respectively, and one side equal to one side—the (one) by the equal angles and common to them, (namely) BC . Thus, they will also have the remaining sides equal to the corresponding remaining (sides), and the remaining angle (equal) to the remaining angle [Prop. 1.26]. Thus, side AB is equal to CD , and AC to BD . Furthermore, angle BAC is equal to CDB . And since angle ABC is equal to BCD , and CBD to ACB , the whole (angle) ABD is thus equal to the whole (angle) ACD . And BAC was also shown (to be) equal to CDB .

Thus, in parallelogrammic figures the opposite sides and angles are equal to one another.

And, I also say that a diagonal cuts them in half. For since AB is equal to CD , and BC (is) common, the two (straight-lines) AB, BC are equal to the two (straight-lines) DC, CB [†], respectively. And angle ABC is equal to angle BCD . Thus, the base AC (is) also equal to DB ,

and triangle ABC is equal to triangle BCD [Prop. 1.4].

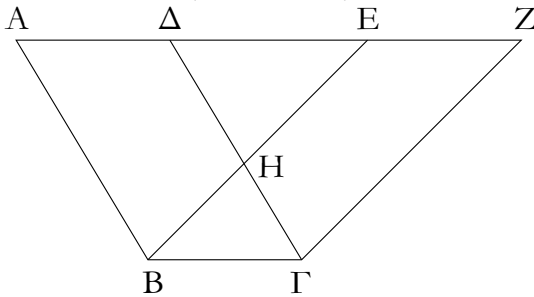
Thus, the diagonal BC cuts the parallelogram $ACDB$ [†] in half. (Which is) the very thing it was required to show.

[†] The Greek text has " CD, BC ", which is obviously a mistake.

[‡] The Greek text has " $ABCD$ ", which is obviously a mistake.

λε'.

Τὰ παραλληλόγραμμα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.



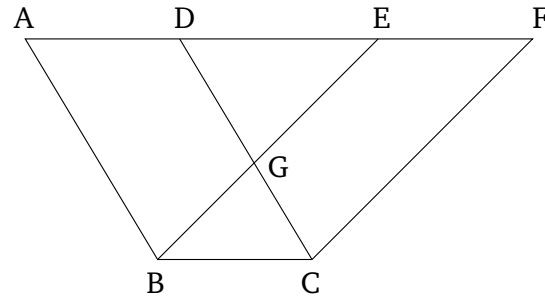
Ἐστω παραλληλόγραμμα τὰ $AB\Gamma\Delta$, $EB\Gamma Z$ ἐπὶ τῆς αὐτῆς βάσεως τῆς $B\Gamma$ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς AZ , $B\Gamma$ · λέγω, ὅτι ἴσον ἐστὶ τὸ $AB\Gamma\Delta$ τῷ $EB\Gamma Z$ παραλληλογράμμῳ.

Ἐπεὶ γὰρ παραλληλόγραμμὸν ἐστὶ τὸ $AB\Gamma\Delta$, ἴση ἐστὶν ἡ $A\Delta$ τῇ $B\Gamma$. διὰ τὰ αὐτὰ δὴ καὶ ἡ EZ τῇ $B\Gamma$ ἐστὶν ἴση· ὥστε καὶ ἡ $A\Delta$ τῇ EZ ἐστὶν ἴση· καὶ κοινὴ ἡ ΔE · ὅλη ἄρα ἡ AE ὅλη τῇ ΔZ ἐστὶν ἴση. ἐστὶ δὲ καὶ ἡ AB τῇ $\Delta\Gamma$ ἴση· δύο δὲ αἱ EA , AB δύο ταῖς $Z\Delta$, $\Delta\Gamma$ ἴσαι εἰσὶν ἑκατέρωθεν· καὶ γωνία ἡ ὑπὸ $Z\Delta\Gamma$ γωνία τῇ ὑπὸ EAB ἐστὶν ἴση ἡ ἐκτὸς τῇ ἐντὸς· βάσις ἄρα ἡ EB βάσει τῇ $Z\Gamma$ ἴση ἐστίν, καὶ τὸ EAB τρίγωνον τῷ $\Delta Z\Gamma$ τριγώνῳ ἴσον ἔσται· κοινὸν ἀφρηθήσθω τὸ ΔHE · λοιπὸν ἄρα τὸ $ABH\Delta$ τραπέζιον λοιπῶν τῷ $EH\Gamma Z$ τραπέζιῳ ἐστὶν ἴσον· κοινὸν προσκεῖσθω τὸ $HBF\Gamma$ τριγώνον· ὅλον ἄρα τὸ $AB\Gamma\Delta$ παραλληλόγραμμον ὅλῳ τῷ $EB\Gamma Z$ παραλληλογράμμῳ ἴσον ἐστίν.

Τὰ ἄρα παραλληλόγραμμα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

Proposition 35

Parallelograms which are on the same base and between the same parallels are equal[†] to one another.



Let $ABCD$ and $EBCF$ be parallelograms on the same base BC , and between the same parallels AF and BC . I say that $ABCD$ is equal to parallelogram $EBCF$.

For since $ABCD$ is a parallelogram, AD is equal to BC [Prop. 1.34]. So, for the same (reasons), EF is also equal to BC . So AD is also equal to EF . And DE is common. Thus, the whole (straight-line) AE is equal to the whole (straight-line) DF . And AB is also equal to DC . So the two (straight-lines) EA , AB are equal to the two (straight-lines) FD , DC , respectively. And angle FDC is equal to angle EAB , the external to the internal [Prop. 1.29]. Thus, the base EB is equal to the base FC , and triangle EAB will be equal to triangle DFC [Prop. 1.4]. Let DGE have been taken away from both. Thus, the remaining trapezium $ABGD$ is equal to the remaining trapezium $EGCF$. Let triangle GBC have been added to both. Thus, the whole parallelogram $ABCD$ is equal to the whole parallelogram $EBCF$.

Thus, parallelograms which are on the same base and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

[†] Here, for the first time, "equal" means "equal in area", rather than "congruent".

λζ'.

Τὰ παραλληλόγραμμα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.

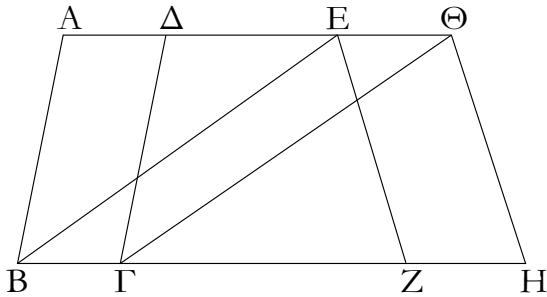
Ἐστω παραλληλόγραμμα τὰ $AB\Gamma\Delta$, $EZH\Theta$ ἐπὶ ἴσων βάσεων ὄντα τῶν $B\Gamma$, ZH καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς $A\Theta$, BH · λέγω, ὅτι ἴσον ἐστὶ τὸ $AB\Gamma\Delta$ παραλ-

Proposition 36

Parallelograms which are on equal bases and between the same parallels are equal to one another.

Let $ABCD$ and $EFGH$ be parallelograms which are on the equal bases BC and FG , and (are) between the same parallels AH and BG . I say that the parallelogram

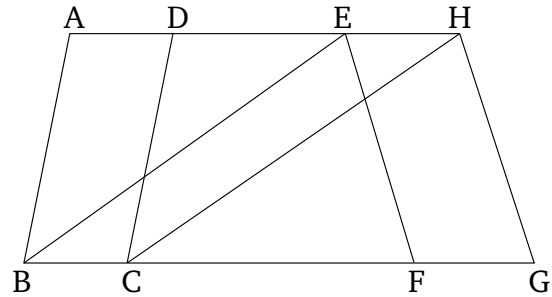
ληλόγραμμον τῷ $EZH\Theta$.



Ἐπεξεύχθωσαν γὰρ αἱ BE , $G\Theta$. καὶ ἐπεὶ ἴση ἐστὶν ἡ BG τῇ ZH , ἀλλὰ ἡ ZH τῇ $E\Theta$ ἐστὶν ἴση, καὶ ἡ BG ἄρα τῇ $E\Theta$ ἐστὶν ἴση. εἰσὶ δὲ καὶ παράλληλοι. καὶ ἐπιzeugνύουσιν αὐτάς αἱ EB , $\Theta\Gamma$. αἱ δὲ τὰς ἴσας τε καὶ παράλληλους ἐπὶ τὰ αὐτὰ μέρη ἐπιzeugνύουσαι ἴσαι τε καὶ παράλληλοί εἰσι [καὶ αἱ EB , $\Theta\Gamma$ ἄρα ἴσαι τέ εἰσι καὶ παράλληλοι]. παραλληλόγραμμον ἄρα ἐστὶ τὸ $EBG\Theta$. καὶ ἐστὶν ἴσον τῷ $ABG\Delta$. βάσιν τε γὰρ αὐτῷ τὴν αὐτὴν ἔχει τὴν BG , καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστὶν αὐτῷ ταῖς BG , $A\Theta$. διὰ τὰ αὐτὰ δὴ καὶ τὸ $EZH\Theta$ τῷ αὐτῷ τῷ $EBG\Theta$ ἐστὶν ἴσον· ὥστε καὶ τὸ $ABG\Delta$ παραλληλόγραμμον τῷ $EZH\Theta$ ἐστὶν ἴσον.

Τὰ ἄρα παραλληλόγραμμα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

$ABCD$ is equal to $EFGH$.

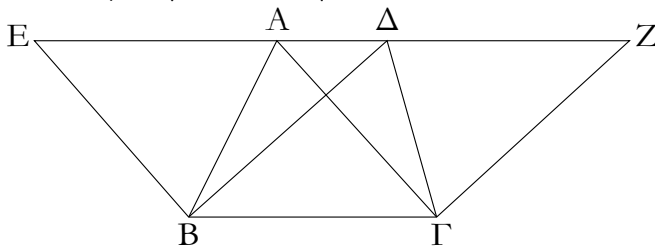


For let BE and CH have been joined. And since BC is equal to FG , but FG is equal to EH [Prop. 1.34], BC is thus equal to EH . And they are also parallel, and EB and HC join them. But (straight-lines) joining equal and parallel (straight-lines) on the same sides are (themselves) equal and parallel [Prop. 1.33] [thus, EB and HC are also equal and parallel]. Thus, $EBCH$ is a parallelogram [Prop. 1.34], and is equal to $ABCD$. For it has the same base, BC , as ($ABCD$), and is between the same parallels, BC and AH , as ($ABCD$) [Prop. 1.35]. So, for the same (reasons), $EFGH$ is also equal to the same (parallelogram) $EBCH$ [Prop. 1.34]. So that the parallelogram $ABCD$ is also equal to $EFGH$.

Thus, parallelograms which are on equal bases and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

λζ'.

Τὰ τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.

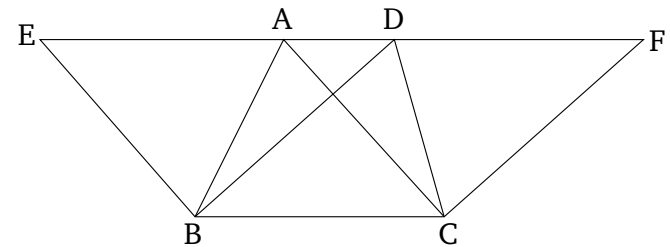


Ἐστω τρίγωνα τὰ ABG , ΔBG ἐπὶ τῆς αὐτῆς βάσεως τῆς BG καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς AD , $B\Gamma$. λέγω, ὅτι ἴσον ἐστὶ τὸ ABG τρίγωνον τῷ ΔBG τριγώνῳ.

Ἐκβεβλήσθω ἡ AD ἐφ' ἐκάτερα τὰ μέρη ἐπὶ τὰ E , Z , καὶ διὰ μὲν τοῦ B τῇ GA παράλληλος ἦχθω ἡ BE , διὰ δὲ τοῦ Γ τῇ BD παράλληλος ἦχθω ἡ GZ . παραλληλόγραμμον ἄρα ἐστὶν ἐκάτερον τῶν $EBGA$, ΔBGZ . καὶ εἰσιν ἴσα· ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως εἰσι τῆς BG καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς BG , EZ . καὶ ἐστὶ τοῦ μὲν $EBGA$ παραλληλογράμμου ἡμισυ τὸ ABG τρίγωνον· ἡ γὰρ AB διάμετρος αὐτὸ διχα τέμνει· τοῦ δὲ ΔBGZ παραλληλογράμμου ἡμισυ τὸ ΔBG τρίγωνον· ἡ γὰρ ΔG διάμετρος αὐτὸ διχα τέμνει. [τὰ δὲ

Proposition 37

Triangles which are on the same base and between the same parallels are equal to one another.



Let ABC and DBC be triangles on the same base BC , and between the same parallels AD and BC . I say that triangle ABC is equal to triangle DBC .

Let AD have been produced in both directions to E and F , and let the (straight-line) BE have been drawn through B parallel to CA [Prop. 1.31], and let the (straight-line) CF have been drawn through C parallel to BD [Prop. 1.31]. Thus, $EBCA$ and $DBCF$ are both parallelograms, and are equal. For they are on the same base BC , and between the same parallels BC and EF [Prop. 1.35]. And the triangle ABC is half of the parallelogram $EBCA$. For the diagonal AB cuts the latter in

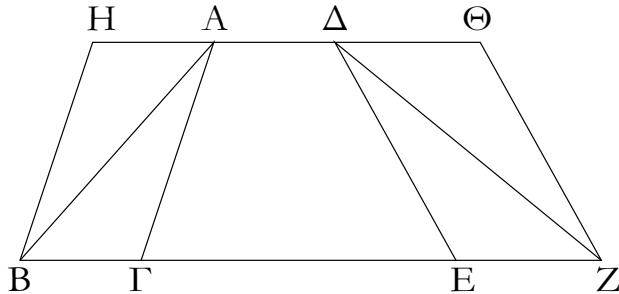
τῶν ἴσων ἡμίση ἴσα ἀλλήλοις ἐστίν]. ἴσον ἄρα ἐστὶ τὸ $AB\Gamma$ τρίγωνον τῷ $\Delta B\Gamma$ τριγώνῳ.

Τὰ ἄρα τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

† This is an additional common notion.

λη'.

Τὰ τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν.



Ἐστω τρίγωνα τὰ $AB\Gamma$, ΔEZ ἐπὶ ἴσων βάσεων τῶν $B\Gamma$, EZ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς BZ , AD . λέγω, ὅτι ἴσον ἐστὶ τὸ $AB\Gamma$ τρίγωνον τῷ ΔEZ τριγώνῳ.

Ἐκβεβλήσθω γὰρ ἡ AD ἐφ' ἐκάτερα τὰ μέρη ἐπὶ τὰ H , Θ , καὶ διὰ τοῦ B τῇ GA παράλληλος ἦχθω ἡ BH , διὰ δὲ τοῦ Z τῇ ΔE παράλληλος ἦχθω ἡ $Z\Theta$. παραλληλόγραμμον ἄρα ἐστὶν ἐκάτερον τῶν $HBGA$, $\Delta EZ\Theta$ · καὶ ἴσον τὸ $HBGA$ τῷ $\Delta EZ\Theta$ · ἐπὶ τε γὰρ ἴσων βάσεων εἰσι τῶν $B\Gamma$, EZ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς BZ , $H\Theta$ · καὶ ἐστὶ τοῦ μὲν $HBGA$ παραλληλογράμμου ἡμισυ τὸ $AB\Gamma$ τρίγωνον. ἡ γὰρ AB διάμετρος αὐτὸ δίχα τέμνει· τοῦ δὲ $\Delta EZ\Theta$ παραλληλογράμμου ἡμισυ τὸ $ZE\Delta$ τρίγωνον· ἡ γὰρ ΔZ διάμετρος αὐτὸ δίχα τέμνει [τὰ δὲ τῶν ἴσων ἡμίση ἴσα ἀλλήλοις ἐστίν]. ἴσον ἄρα ἐστὶ τὸ $AB\Gamma$ τρίγωνον τῷ ΔEZ τριγώνῳ.

Τὰ ἄρα τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

λθ'.

Τὰ ἴσα τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.

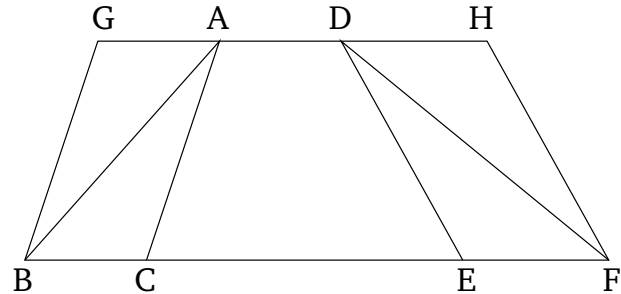
Ἐστω ἴσα τρίγωνα τὰ $AB\Gamma$, $\Delta B\Gamma$ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη τῆς $B\Gamma$. λέγω, ὅτι καὶ ἐν ταῖς

half [Prop. 1.34]. And the triangle DBC (is) half of the parallelogram $DBCF$. For the diagonal DC cuts the latter in half [Prop. 1.34]. [And the halves of equal things are equal to one another.]† Thus, triangle ABC is equal to triangle DBC .

Thus, triangles which are on the same base and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

Proposition 38

Triangles which are on equal bases and between the same parallels are equal to one another.



Let ABC and DEF be triangles on the equal bases BC and EF , and between the same parallels BF and AD . I say that triangle ABC is equal to triangle DEF .

For let AD have been produced in both directions to G and H , and let the (straight-line) BG have been drawn through B parallel to CA [Prop. 1.31], and let the (straight-line) FH have been drawn through F parallel to DE [Prop. 1.31]. Thus, $GBCA$ and $DEFH$ are each parallelograms. And $GBCA$ is equal to $DEFH$. For they are on the equal bases BC and EF , and between the same parallels BF and GH [Prop. 1.36]. And triangle ABC is half of the parallelogram $GBCA$. For the diagonal AB cuts the latter in half [Prop. 1.34]. And triangle FED (is) half of parallelogram $DEFH$. For the diagonal DF cuts the latter in half. [And the halves of equal things are equal to one another.] Thus, triangle ABC is equal to triangle DEF .

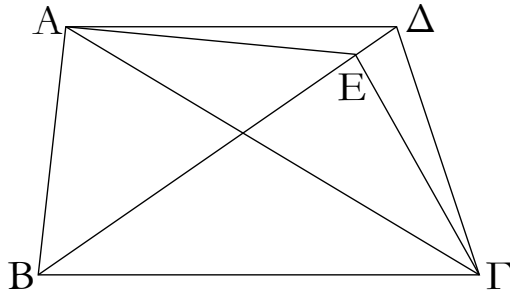
Thus, triangles which are on equal bases and between the same parallels are equal to one another. (Which is) the very thing it was required to show.

Proposition 39

Equal triangles which are on the same base, and on the same side, are also between the same parallels.

Let ABC and DBC be equal triangles which are on the same base BC , and on the same side (of it). I say that

αὐταῖς παραλλήλοις ἐστίν.



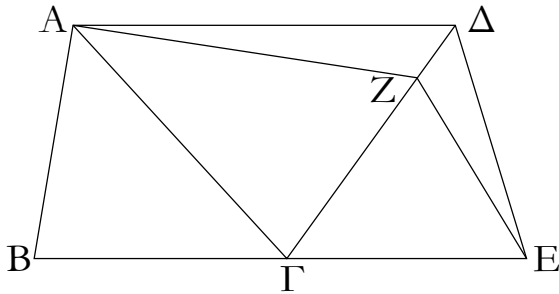
Ἐπεζεύχθω γὰρ ἡ AD · λέγω, ὅτι παράλληλός ἐστιν ἡ AD τῇ BG .

Εἰ γὰρ μή, ἤχθω διὰ τοῦ A σημείου τῇ BG εὐθείᾳ παράλληλος ἡ AE , καὶ ἐπεζεύχθω ἡ EG . ἴσον ἄρα ἐστὶ τὸ ABG τρίγωνον τῷ EBG τριγώνῳ· ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως ἐστὶν αὐτῷ τῆς BG καὶ ἐν ταῖς αὐταῖς παραλλήλοις. ἀλλὰ τὸ ABG τῷ $ΔBG$ ἐστὶν ἴσον· καὶ τὸ $ΔBG$ ἄρα τῷ EBG ἴσον ἐστὶ τὸ μείζον τῷ ἐλάσσονι· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα παράλληλός ἐστιν ἡ AE τῇ BG . ὁμοίως δὲ δείξομεν, ὅτι οὐδ' ἄλλη τις πλὴν τῆς AD · ἡ AD ἄρα τῇ BG ἐστὶ παράλληλος.

Τὰ ἄρα ἴσα τρίγωνα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

μ'.

Τὰ ἴσα τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.

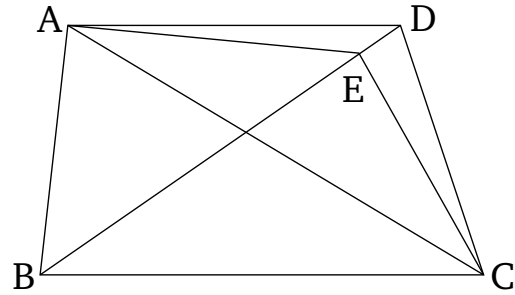


Ἐστω ἴσα τρίγωνα τὰ ABG , $ΓΔE$ ἐπὶ ἴσων βάσεων τῶν BG , $ΓE$ καὶ ἐπὶ τὰ αὐτὰ μέρη. λέγω, ὅτι καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν.

Ἐπεζεύχθω γὰρ ἡ AD · λέγω, ὅτι παράλληλός ἐστιν ἡ AD τῇ BE .

Εἰ γὰρ μή, ἤχθω διὰ τοῦ A τῇ BE παράλληλος ἡ AZ , καὶ ἐπεζεύχθω ἡ ZE . ἴσον ἄρα ἐστὶ τὸ ABG τρίγωνον τῷ $ZΓE$ τριγώνῳ· ἐπὶ τε γὰρ ἴσων βάσεων εἰσι τῶν BG , $ΓE$ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς BE , AZ . ἀλλὰ τὸ ABG τρίγωνον ἴσον ἐστὶ τῷ $ΔΓE$ [τρίγωνον]· καὶ τὸ $ΔΓE$ ἄρα [τρίγωνον] ἴσον ἐστὶ τῷ $ZΓE$ τριγώνῳ τὸ μείζον τῷ

they are also between the same parallels.



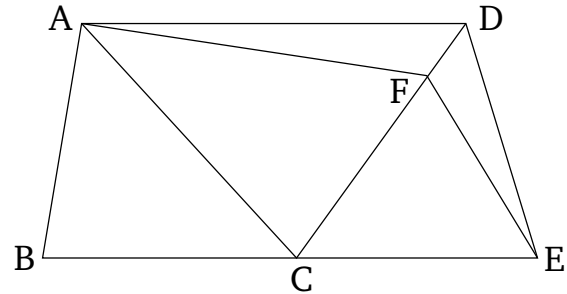
For let AD have been joined. I say that AD and BC are parallel.

For, if not, let AE have been drawn through point A parallel to the straight-line BC [Prop. 1.31], and let EC have been joined. Thus, triangle ABC is equal to triangle EBC . For it is on the same base as it, BC , and between the same parallels [Prop. 1.37]. But ABC is equal to DBC . Thus, DBC is also equal to EBC , the greater to the lesser. The very thing is impossible. Thus, AE is not parallel to BC . Similarly, we can show that neither (is) any other (straight-line) than AD . Thus, AD is parallel to BC .

Thus, equal triangles which are on the same base, and on the same side, are also between the same parallels. (Which is) the very thing it was required to show.

Proposition 40[†]

Equal triangles which are on equal bases, and on the same side, are also between the same parallels.



Let ABC and CDE be equal triangles on the equal bases BC and CE (respectively), and on the same side (of BE). I say that they are also between the same parallels.

For let AD have been joined. I say that AD is parallel to BE .

For if not, let AF have been drawn through A parallel to BE [Prop. 1.31], and let FE have been joined. Thus, triangle ABC is equal to triangle FCE . For they are on equal bases, BC and CE , and between the same parallels, BE and AF [Prop. 1.38]. But, triangle ABC is equal

ἐλάσσονι· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα παράλληλος ἡ AZ τῇ BE . ὁμοίως δὲ δείξομεν, ὅτι οὐδ' ἄλλη τις πλὴν τῆς AD · ἡ AD ἄρα τῇ BE ἐστὶ παράλληλος.

Τὰ ἄρα ἴσα τρίγωνα τὰ ἐπὶ ἴσων βάσεων ὄντα καὶ ἐπὶ τὰ αὐτὰ μέρη καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

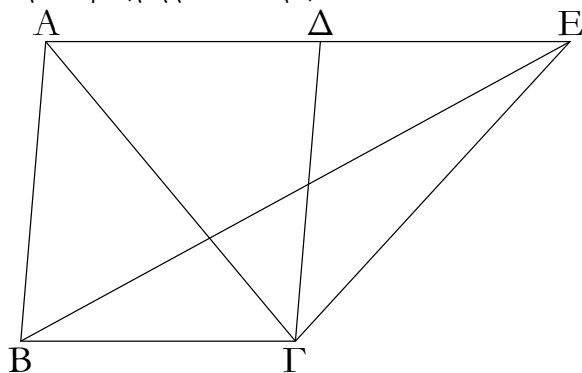
to [triangle] DCE . Thus, [triangle] DCE is also equal to triangle FCE , the greater to the lesser. The very thing is impossible. Thus, AF is not parallel to BE . Similarly, we can show that neither (is) any other (straight-line) than AD . Thus, AD is parallel to BE .

Thus, equal triangles which are on equal bases, and on the same side, are also between the same parallels. (Which is) the very thing it was required to show.

† This whole proposition is regarded by Heiberg as a relatively early interpolation to the original text.

μα'.

Ἐὰν παραλληλόγραμμον τριγώνω βάσιν τε ἔχη τὴν αὐτὴν καὶ ἐν ταῖς αὐταῖς παραλλήλοις ᾗ, διπλάσιόν ἐστὶ τὸ παραλληλόγραμμον τοῦ τριγώνου.



Παραλληλόγραμμον γάρ τὸ $ABΓΔ$ τριγώνω τῷ $EBΓ$ βάσιν τε ἔχεται τὴν αὐτὴν τὴν $BΓ$ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἔστω ταῖς $BΓ$, AE · λέγω, ὅτι διπλάσιόν ἐστὶ τὸ $ABΓΔ$ παραλληλόγραμμον τοῦ $EBΓ$ τριγώνου.

Ἐπεζεύχθω γὰρ ἡ $ΑΓ$. ἴσον δὲ ἐστὶ τὸ $ABΓ$ τρίγωνον τῷ $EBΓ$ τριγώνω· ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως ἐστὶν αὐτῷ τῆς $BΓ$ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς $BΓ$, AE . ἀλλὰ τὸ $ABΓΔ$ παραλληλόγραμμον διπλάσιόν ἐστὶ τοῦ $ABΓ$ τριγώνου· ἡ γὰρ $ΑΓ$ διάμετρος αὐτὸ δίχα τέμνει· ὥστε τὸ $ABΓΔ$ παραλληλόγραμμον καὶ τοῦ $EBΓ$ τριγώνου ἐστὶ διπλάσιον.

Ἐὰν ἄρα παραλληλόγραμμον τριγώνω βάσιν τε ἔχη τὴν αὐτὴν καὶ ἐν ταῖς αὐταῖς παραλλήλοις ᾗ, διπλάσιόν ἐστὶ τὸ παραλληλόγραμμον τοῦ τριγώνου· ὅπερ ἔδει δεῖξαι.

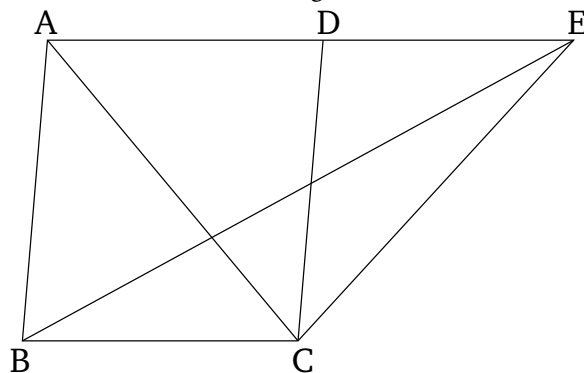
μβ'.

Τῷ δοθέντι τριγώνω ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῇ δοθείσῃ γωνίᾳ εὐθυγράμμω.

Ἐστω τὸ μὲν δοθέν τρίγωνον τὸ $ABΓ$, ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ $Δ$ · δεῖ δὲ τῷ $ABΓ$ τριγώνω ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῇ $Δ$ γωνίᾳ εὐθυγράμμω.

Proposition 41

If a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle.



For let parallelogram $ABCD$ have the same base BC as triangle EBC , and let it be between the same parallels, BC and AE . I say that parallelogram $ABCD$ is double (the area) of triangle EBC .

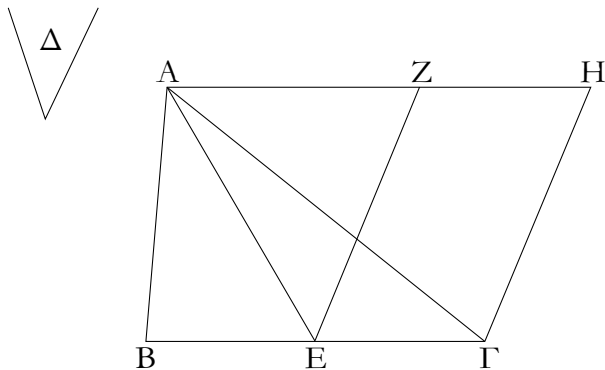
For let AC have been joined. So triangle ABC is equal to triangle EBC . For it is on the same base, BC , as (EBC), and between the same parallels, BC and AE [Prop. 1.37]. But, parallelogram $ABCD$ is double (the area) of triangle ABC . For the diagonal AC cuts the former in half [Prop. 1.34]. So parallelogram $ABCD$ is also double (the area) of triangle EBC .

Thus, if a parallelogram has the same base as a triangle, and is between the same parallels, then the parallelogram is double (the area) of the triangle. (Which is) the very thing it was required to show.

Proposition 42

To construct a parallelogram equal to a given triangle in a given rectilinear angle.

Let ABC be the given triangle, and D the given rectilinear angle. So it is required to construct a parallelogram equal to triangle ABC in the rectilinear angle D .



Τετμήσθω ἡ ΒΓ δίχα κατὰ τὸ Ε, καὶ ἐπεζεύχθω ἡ ΑΕ, καὶ συνεστάτω πρὸς τῇ ΕΓ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Ε τῇ Δ γωνία ἴση ἡ ὑπὸ ΓΕΖ, καὶ διὰ μέν τοῦ Α τῇ ΕΓ παράλληλος ἤχθω ἡ ΑΗ, διὰ δὲ τοῦ Γ τῇ ΕΖ παράλληλος ἤχθω ἡ ΓΗ· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΖΕΓΗ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΕ τῇ ΕΓ, ἴσον ἐστὶ καὶ τὸ ΑΒΕ τρίγωνον τῷ ΑΕΓ τριγώνῳ· ἐπὶ τε γὰρ ἴσων βάσεων εἰσι τῶν ΒΕ, ΕΓ καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΒΓ, ΑΗ· διπλάσιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τοῦ ΑΕΓ τριγώνου. ἔστι δὲ καὶ τὸ ΖΕΓΗ παραλληλόγραμμον διπλάσιον τοῦ ΑΕΓ τριγώνου· βάσιν τε γὰρ αὐτῶ τὴν αὐτὴν ἔχει καὶ ἐν ταῖς αὐταῖς ἐστὶν αὐτῶ παραλλήλοις· ἴσον ἄρα ἐστὶ τὸ ΖΕΓΗ παραλληλόγραμμον τῷ ΑΒΓ τριγώνῳ. καὶ ἔχει τὴν ὑπὸ ΓΕΖ γωνίαν ἴσην τῇ δοθείσῃ τῇ Δ.

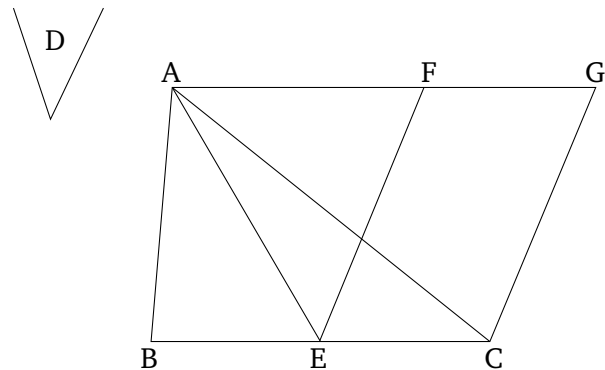
Τῷ ἄρα δοθέντι τριγώνῳ τῷ ΑΒΓ ἴσον παραλληλόγραμμον συνέσσεται τὸ ΖΕΓΗ ἐν γωνίᾳ τῇ ὑπὸ ΓΕΖ, ἥτις ἐστὶν ἴση τῇ Δ· ὅπερ ἔδει ποιῆσαι.

μγ'.

Παντὸς παραλληλογράμμου τῶν περὶ τὴν διάμετρον παραλληλογράμμων τὰ παραπληρώματα ἴσα ἀλλήλοις ἐστίν.

Ἐστω παραλληλόγραμμον τὸ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἡ ΑΓ, περὶ δὲ τὴν ΑΓ παραλληλόγραμμα μὲν ἔστω τὰ ΕΘ, ΖΗ, τὰ δὲ λεγόμενα παραπληρώματα τὰ ΒΚ, ΚΔ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΒΚ παραπλήρωμα τῷ ΚΔ παραπληρώματι.

Ἐπεὶ γὰρ παραλληλόγραμμον ἐστὶ τὸ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἡ ΑΓ, ἴσον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΓΔ τριγώνῳ. πάλιν, ἐπεὶ παραλληλόγραμμον ἐστὶ τὸ ΕΘ, διάμετρος δὲ αὐτοῦ ἐστὶν ἡ ΑΚ, ἴσον ἐστὶ τὸ ΑΕΚ τρίγωνον τῷ ΑΘΚ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΚΖΓ τρίγωνον τῷ ΚΗΓ ἐστὶν ἴσον. ἐπεὶ οὖν τὸ μὲν ΑΕΚ τρίγωνον τῷ ΑΘΚ τριγώνῳ ἐστὶν ἴσον, τὸ δὲ ΚΖΓ τῷ ΚΗΓ, τὸ ΑΕΚ τρίγωνον μετὰ τοῦ ΚΗΓ ἴσον ἐστὶ τῷ ΑΘΚ τριγώνῳ μετὰ τοῦ ΚΖΓ· ἔστι δὲ καὶ ὅλον τὸ ΑΒΓ τρίγωνον ὅλῳ τῷ ΑΔΓ ἴσον· λοιπὸν ἄρα τὸ ΒΚ παραπλήρωμα λοιπῷ τῷ ΚΔ παρα-



Let BC have been cut in half at E [Prop. 1.10], and let AE have been joined. And let (angle) CEF , equal to angle D , have been constructed at the point E on the straight-line EC [Prop. 1.23]. And let AG have been drawn through A parallel to EC [Prop. 1.31], and let CG have been drawn through C parallel to EF [Prop. 1.31]. Thus, $FECG$ is a parallelogram. And since BE is equal to EC , triangle ABE is also equal to triangle AEC . For they are on the equal bases, BE and EC , and between the same parallels, BC and AG [Prop. 1.38]. Thus, triangle ABC is double (the area) of triangle AEC . And parallelogram $FECG$ is also double (the area) of triangle AEC . For it has the same base as (AEC), and is between the same parallels as (AEC) [Prop. 1.41]. Thus, parallelogram $FECG$ is equal to triangle ABC . ($FECG$) also has the angle CEF equal to the given (angle) D .

Thus, parallelogram $FECG$, equal to the given triangle ABC , has been constructed in the angle CEF , which is equal to D . (Which is) the very thing it was required to do.

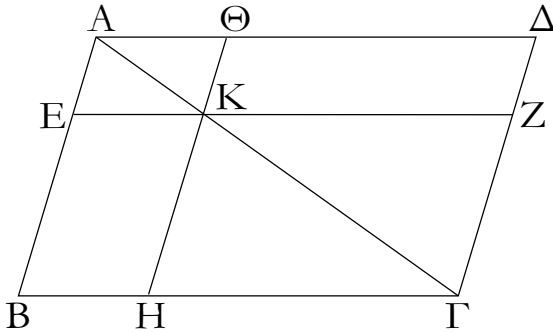
Proposition 43

For any parallelogram, the complements of the parallelograms about the diagonal are equal to one another.

Let $ABCD$ be a parallelogram, and AC its diagonal. And let EH and FG be the parallelograms about AC , and BK and KD the so-called complements (about AC). I say that the complement BK is equal to the complement KD .

For since $ABCD$ is a parallelogram, and AC its diagonal, triangle ABC is equal to triangle ACD [Prop. 1.34]. Again, since EH is a parallelogram, and AK is its diagonal, triangle AEK is equal to triangle AHK [Prop. 1.34]. So, for the same (reasons), triangle KFC is also equal to (triangle) KGC . Therefore, since triangle AEK is equal to triangle AHK , and KFC to KGC , triangle AEK plus KGC is equal to triangle AHK plus KFC . And the whole triangle ABC is also equal to the whole (triangle) ADC . Thus, the remaining complement BK is equal to

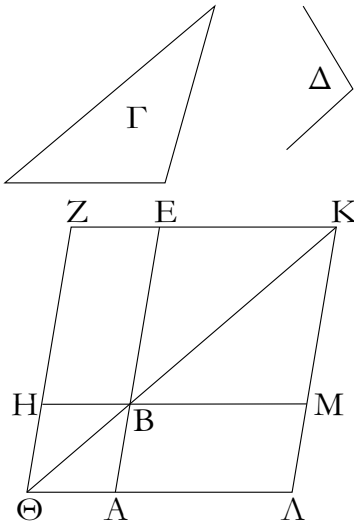
πληρώματί ἐστιν ἴσον.



Παντὸς ἄρα παραλληλογράμμου χωρίου τῶν περὶ τὴν διάμετρον παραλληλογράμμων τὰ παραπληρώματα ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

μδ'.

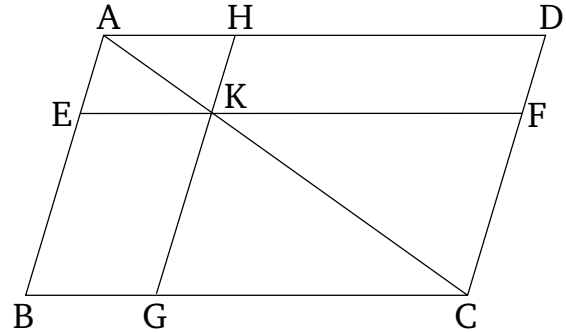
Παρὰ τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι τριγώνῳ ἴσον παραλληλόγραμμον παραβαλεῖν ἐν τῇ δοθείσῃ γωνίᾳ εὐθύγραμμῳ.



Ἐστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ AB , τὸ δὲ δοθὲν τρίγωνον τὸ Γ , ἡ δὲ δοθεῖσα γωνία εὐθύγραμμος ἡ Δ . δεῖ δὴ παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν AB τῷ δοθέντι τριγώνῳ τῷ Γ ἴσον παραλληλόγραμμον παραβαλεῖν ἐν ἰσῇ τῇ Δ γωνίᾳ.

Συνεστάτω τῷ Γ τριγώνῳ ἴσον παραλληλόγραμμον τὸ $BEZH$ ἐν γωνίᾳ τῇ ὑπὸ EBH , ἣ ἐστὶν ἰση τῇ Δ · καὶ κείσθω ὥστε ἐπ' εὐθείας εἶναι τὴν BE τῇ AB , καὶ διήχθω ἡ ZH ἐπὶ τὸ Θ , καὶ διὰ τοῦ A ὁποτέρᾳ τῶν BH , EZ παράλληλος ἦχθω ἡ $A\Theta$, καὶ ἐπεξεύχθω ἡ ΘB . καὶ ἐπεὶ εἰς παραλλήλους τὰς $A\Theta$, EZ εὐθεῖα ἐνέπεσεν ἡ ΘZ , αἱ ἄρα ὑπὸ $A\Theta Z$, ΘZE γωνίαι δυσὶν ὀρθαῖς εἰσιν ἴσαι. αἱ ἄρα ὑπὸ $B\Theta H$, HZE δύο ὀρθῶν ἐλάσσονές εἰσιν· αἱ δὲ ἀπὸ ἐλασσόνων ἡ δύο ὀρθῶν εἰς ἄπειρον ἐκβαλλόμεναι συμπίπτουσιν· αἱ ΘB , ZE

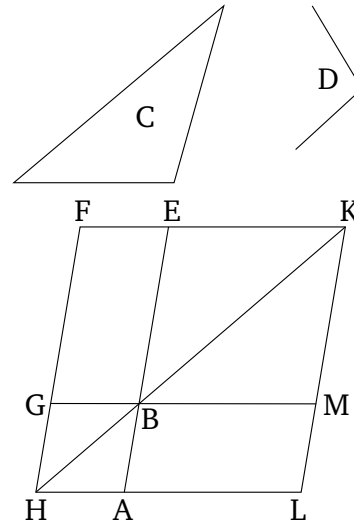
the remaining complement KD .



Thus, for any parallelogramic figure, the complements of the parallelograms about the diagonal are equal to one another. (Which is) the very thing it was required to show.

Proposition 44

To apply a parallelogram equal to a given triangle to a given straight-line in a given rectilinear angle.



Let AB be the given straight-line, C the given triangle, and D the given rectilinear angle. So it is required to apply a parallelogram equal to the given triangle C to the given straight-line AB in an angle equal to (angle) D .

Let the parallelogram $BEFG$, equal to the triangle C , have been constructed in the angle EBG , which is equal to D [Prop. 1.42]. And let it have been placed so that BE is straight-on to AB .[†] And let FG have been drawn through to H , and let AH have been drawn through A parallel to either of BG or EF [Prop. 1.31], and let HB have been joined. And since the straight-line HF falls across the parallels AH and EF , the (sum of the) angles AHF and HFE is thus equal to two right-angles

ἄρα ἐκβαλλόμεναι συμπεσοῦνται. ἐκβεβλήσθωσαν καὶ συμ-
πιπτέωσαν κατὰ τὸ K , καὶ διὰ τοῦ K σημείου ὁποτέρᾳ
τῶν EA , $Z\Theta$ παράλληλος ἦχθω ἡ KL , καὶ ἐκβεβλήσθωσαν
αἱ ΘA , HB ἐπὶ τὰ Λ , M σημεία. παραλληλόγραμμον ἄρα
ἐστὶ τὸ ΘAKZ , διάμετρος δὲ αὐτοῦ ἡ ΘK , περὶ δὲ τὴν ΘK
παραλληλόγραμμοι μὲν τὰ AH , ME , τὰ δὲ λεγόμενα παρα-
πληρώματα τὰ AB , BZ ἴσον ἄρα ἐστὶ τὸ AB τῷ BZ . ἀλλὰ
τὸ BZ τῷ Γ τριγώνῳ ἐστὶν ἴσον· καὶ τὸ AB ἄρα τῷ Γ ἐστὶν
ἴσον. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ HBE γωνία τῇ ὑπὸ ABM ,
ἀλλὰ ἡ ὑπὸ HBE τῇ Δ ἐστὶν ἴση, καὶ ἡ ὑπὸ ABM ἄρα τῇ Δ
γωνίᾳ ἐστὶν ἴση.

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν AB τῷ δοθέντι
τριγώνῳ τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ AB
ἐν γωνίᾳ τῇ ὑπὸ ABM , ἥ ἐστὶν ἴση τῇ Δ · ὅπερ ἔδει ποιῆσαι.

[Prop. 1.29]. Thus, (the sum of) BHG and GFE is less
than two right-angles. And (straight-lines) produced to
infinity from (internal angles whose sum is) less than two
right-angles meet together [Post. 5]. Thus, being pro-
duced, HB and FE will meet together. Let them have
been produced, and let them meet together at K . And let
 KL have been drawn through point K parallel to either
of EA or FH [Prop. 1.31]. And let HA and GB have
been produced to points L and M (respectively). Thus,
 $HLKF$ is a parallelogram, and HK its diagonal. And
 AG and ME (are) parallelograms, and LB and BF the
so-called complements, about HK . Thus, LB is equal to
 BF [Prop. 1.43]. But, BF is equal to triangle C . Thus,
 LB is also equal to C . Also, since angle GBE is equal to
 ABM [Prop. 1.15], but GBE is equal to D , ABM is thus
also equal to angle D .

Thus, the parallelogram LB , equal to the given trian-
gle C , has been applied to the given straight-line AB in
the angle ABM , which is equal to D . (Which is) the very
thing it was required to do.

† This can be achieved using Props. 1.3, 1.23, and 1.31.

με'.

Τῷ δοθέντι εὐθυγράμμῳ ἴσον παραλληλόγραμμον συστ-
ῆσασθαι ἐν τῇ δοθείσῃ γωνίᾳ εὐθυγράμμῳ.

Ἐστω τὸ μὲν δοθὲν εὐθύγραμμοι τὸ $AB\Gamma\Delta$, ἡ δὲ
δοθεῖσα γωνία εὐθύγραμμος ἡ E · δεῖ δὴ τῷ $AB\Gamma\Delta$ εὐθυ-
γράμμῳ ἴσον παραλληλόγραμμον συστήσασθαι ἐν τῇ δοθείσῃ
γωνίᾳ τῇ E .

Ἐπεζεύχθω ἡ ΔB , καὶ συνεστάτω τῷ $AB\Delta$ τριγώνῳ
ἴσον παραλληλόγραμμον τὸ $Z\Theta$ ἐν τῇ ὑπὸ ΘKZ γωνίᾳ, ἥ
ἐστὶν ἴση τῇ E · καὶ παραβέβλησθω παρὰ τὴν $H\Theta$ εὐθεῖαν τῷ
 $\Delta B\Gamma$ τριγώνῳ ἴσον παραλληλόγραμμον τὸ HM ἐν τῇ ὑπὸ
 $H\Theta M$ γωνίᾳ, ἥ ἐστὶν ἴση τῇ E . καὶ ἐπεὶ ἡ E γωνία ἐκατέρᾳ
τῶν ὑπὸ ΘKZ , $H\Theta M$ ἐστὶν ἴση, καὶ ἡ ὑπὸ ΘKZ ἄρα τῇ ὑπὸ
 $H\Theta M$ ἐστὶν ἴση. κοινὴ προσκείσθω ἡ ὑπὸ $K\Theta H$ · αἱ ἄρα
ὑπὸ $ZK\Theta$, $K\Theta H$ ταῖς ὑπὸ $K\Theta H$, $H\Theta M$ ἴσαι εἰσὶν. ἀλλ' αἱ
ὑπὸ $ZK\Theta$, $K\Theta H$ δυσὶν ὀρθαῖς ἴσαι εἰσὶν· καὶ αἱ ὑπὸ $K\Theta H$,
 $H\Theta M$ ἄρα δύο ὀρθαῖς ἴσαι εἰσὶν. πρὸς δὲ τινι εὐθεΐᾳ τῇ $H\Theta$
καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Θ δύο εὐθεῖαι αἱ $K\Theta$, ΘM μὴ
ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δύο ὀρθαῖς
ἴσας ποιοῦσιν· ἐπ' εὐθείας ἄρα ἐστὶν ἡ $K\Theta$ τῇ ΘM · καὶ
ἐπεὶ εἰς παραλλήλους τὰς KM , ZH εὐθεῖα ἐνέπεσεν ἡ ΘH ,
αἱ ἐναλλὰξ γωνίαι αἱ ὑπὸ $M\Theta H$, ΘHZ ἴσαι ἀλλήλαις εἰσὶν.
κοινὴ προσκείσθω ἡ ὑπὸ $\Theta H\Lambda$ · αἱ ἄρα ὑπὸ $M\Theta H$, $\Theta H\Lambda$ ταῖς
ὑπὸ ΘHZ , $\Theta H\Lambda$ ἴσαι εἰσὶν. ἀλλ' αἱ ὑπὸ $M\Theta H$, $\Theta H\Lambda$ δύο
ὀρθαῖς ἴσαι εἰσὶν· καὶ αἱ ὑπὸ ΘHZ , $\Theta H\Lambda$ ἄρα δύο ὀρθαῖς
ἴσαι εἰσὶν· ἐπ' εὐθείας ἄρα ἐστὶν ἡ ZH τῇ $H\Lambda$. καὶ ἐπεὶ ἡ
 ZK τῇ ΘH ἴση τε καὶ παράλληλός ἐστιν, ἀλλὰ καὶ ἡ ΘH τῇ
 $M\Lambda$, καὶ ἡ KZ ἄρα τῇ $M\Lambda$ ἴση τε καὶ παράλληλός ἐστιν· καὶ

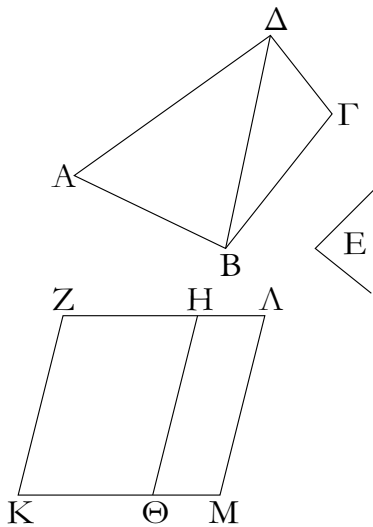
Proposition 45

To construct a parallelogram equal to a given rectilin-
ear figure in a given rectilinear angle.

Let $ABCD$ be the given rectilinear figure,[†] and E the
given rectilinear angle. So it is required to construct a
parallelogram equal to the rectilinear figure $ABCD$ in
the given angle E .

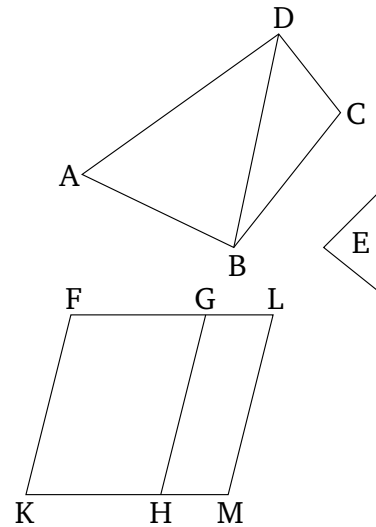
Let DB have been joined, and let the parallelogram
 FH , equal to the triangle ABD , have been constructed
in the angle HKF , which is equal to E [Prop. 1.42]. And
let the parallelogram GM , equal to the triangle DBC ,
have been applied to the straight-line GH in the angle
 GHM , which is equal to E [Prop. 1.44]. And since angle
 E is equal to each of (angles) HKF and GHM , (an-
gle) HKF is thus also equal to GHM . Let KHG have
been added to both. Thus, (the sum of) FKH and KHG
is equal to (the sum of) KHG and GHM . But, (the
sum of) FKH and KHG is equal to two right-angles
[Prop. 1.29]. Thus, (the sum of) KHG and GHM is
also equal to two right-angles. So two straight-lines, KH
and HM , not lying on the same side, make adjacent an-
gles with some straight-line GH , at the point H on it,
(whose sum is) equal to two right-angles. Thus, KH is
straight-on to HM [Prop. 1.14]. And since the straight-
line HG falls across the parallels KM and FG , the al-
ternate angles MHG and HGF are equal to one another
[Prop. 1.29]. Let HGL have been added to both. Thus,
(the sum of) MHG and HGL is equal to (the sum of)

ἐπιζευγνύουσιν αὐτὰς εὐθεῖαι αἱ KM , $Z\Lambda$ · καὶ αἱ KM , $Z\Lambda$ ἄρα ἴσαι τε καὶ παράλληλοί εἰσιν· παραλληλόγραμμον ἄρα ἐστὶ τὸ $KZ\Lambda M$. καὶ ἐπεὶ ἴσον ἐστὶ τὸ μὲν $AB\Delta$ τρίγωνον τῷ $Z\Theta$ παραλληλογράμῳ, τὸ δὲ $\Delta B\Gamma$ τῷ HM , ὅλον ἄρα τὸ $AB\Gamma\Delta$ εὐθύγραμμον ὅλῳ τῷ $KZ\Lambda M$ παραλληλογράμῳ ἐστὶν ἴσον.



Τῷ ἄρα δοθέντι εὐθυγράμῳ τῷ $AB\Gamma\Delta$ ἴσον παραλληλόγραμμον συνέσταιται τὸ $KZ\Lambda M$ ἐν γωνίᾳ τῇ ὑπὸ ZKM , ἥ ἐστὶν ἴση τῇ δοθείσῃ τῇ E · ὅπερ ἔδει ποιῆσαι.

HGF and HGL . But, (the sum of) MHG and HGL is equal to two right-angles [Prop. 1.29]. Thus, (the sum of) HGF and HGL is also equal to two right-angles. Thus, FG is straight-on to GL [Prop. 1.14]. And since FK is equal and parallel to HG [Prop. 1.34], but also HG to ML [Prop. 1.34], KF is thus also equal and parallel to ML [Prop. 1.30]. And the straight-lines KM and FL join them. Thus, KM and FL are equal and parallel as well [Prop. 1.33]. Thus, $KFLM$ is a parallelogram. And since triangle ABD is equal to parallelogram FH , and DBC to GM , the whole rectilinear figure $ABCD$ is thus equal to the whole parallelogram $KFLM$.



Thus, the parallelogram $KFLM$, equal to the given rectilinear figure $ABCD$, has been constructed in the angle FKM , which is equal to the given (angle) E . (Which is) the very thing it was required to do.

† The proof is only given for a four-sided figure. However, the extension to many-sided figures is trivial.

μζ'.

Ἀπὸ τῆς δοθείσης εὐθείας τετράγωνον ἀναγράψαι.

Ἐστω ἡ δοθεῖσα εὐθεῖα ἡ AB · δεῖ δὴ ἀπὸ τῆς AB εὐθείας τετράγωνον ἀναγράψαι.

Ἦχθω τῇ AB εὐθείᾳ ἀπὸ τοῦ πρὸς αὐτῇ σημείου τοῦ A πρὸς ὀρθὰς ἡ AG , καὶ κείσθω τῇ AB ἴση ἡ AD · καὶ διὰ μὲν τοῦ Δ σημείου τῇ AB παράλληλος ἦχθω ἡ DE , διὰ δὲ τοῦ B σημείου τῇ AD παράλληλος ἦχθω ἡ BE . παραλληλόγραμμον ἄρα ἐστὶ τὸ $ADEB$ · ἴση ἄρα ἐστὶν ἡ μὲν AB τῇ DE , ἡ δὲ AD τῇ BE . ἀλλὰ ἡ AB τῇ AD ἐστὶν ἴση· αἱ τέσσαρες ἄρα αἱ BA , AD , DE , EB ἴσαι ἀλλήλαις εἰσὶν· ἰσόπλευρον ἄρα ἐστὶ τὸ $ADEB$ παραλληλόγραμμον. λέγω δὴ, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ εἰς παραλλήλους τὰς AB , DE εὐθεῖα ἐνέπεσεν ἡ AD , αἱ ἄρα ὑπὸ BAD , ADE γωνίαι δύο ὀρθαῖς ἴσαι εἰσὶν. ὀρθὴ δὲ ἡ ὑπὸ BAD · ὀρθὴ ἄρα καὶ

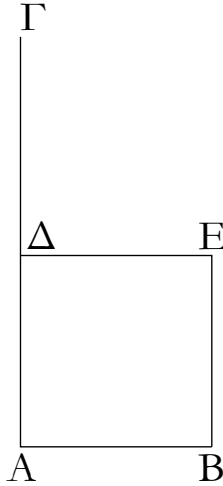
Proposition 46

To describe a square on a given straight-line.

Let AB be the given straight-line. So it is required to describe a square on the straight-line AB .

Let AC have been drawn at right-angles to the straight-line AB from the point A on it [Prop. 1.11], and let AD have been made equal to AB [Prop. 1.3]. And let DE have been drawn through point D parallel to AB [Prop. 1.31], and let BE have been drawn through point B parallel to AD [Prop. 1.31]. Thus, $ADEB$ is a parallelogram. Therefore, AB is equal to DE , and AD to BE [Prop. 1.34]. But, AB is equal to AD . Thus, the four (sides) BA , AD , DE , and EB are equal to one another. Thus, the parallelogram $ADEB$ is equilateral. So I say that (it is) also right-angled. For since the straight-line

ἡ ὑπὸ $\Delta\Delta\epsilon$. τῶν δὲ παραλληλογράμμων χωρίων αἱ ἀπεναντίον πλευραὶ τε καὶ γωνίαι ἴσαι ἀλλήλαις εἰσὶν· ὀρθὴ ἄρα καὶ ἑκατέρα τῶν ἀπεναντίον τῶν ὑπὸ $\Delta\Delta\epsilon$, $\Delta\epsilon\Delta$ γωνιῶν· ὀρθογώνιον ἄρα ἐστὶ τὸ $\Delta\Delta\epsilon\Delta$. ἐδείχθη δὲ καὶ ἰσόπλευρον.



Τετράγωνον ἄρα ἐστὶν· καὶ ἐστὶν ἀπὸ τῆς $\Delta\Delta$ εὐθείας ἀναγεγραμμένον· ὅπερ ἔδει ποιῆσαι.

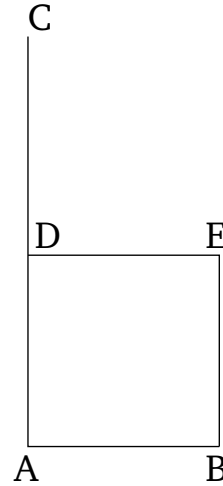
μζ'.

Ἐν τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτείνουσας πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν τετραγώνοις.

Ἐστω τρίγωνον ὀρθογώνιον τὸ $\Delta\Delta\Gamma$ ὀρθὴν ἔχον τὴν ὑπὸ $\Delta\Delta\Gamma$ γωνίαν· λέγω, ὅτι τὸ ἀπὸ τῆς $\Delta\Gamma$ τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν $\Delta\Delta$, $\Delta\Gamma$ τετραγώνοις.

Ἀναγεγράφθω γὰρ ἀπὸ μὲν τῆς $\Delta\Gamma$ τετράγωνον τὸ $\Delta\Delta\epsilon\Delta$, ἀπὸ δὲ τῶν $\Delta\Delta$, $\Delta\Gamma$ τὰ $\Delta\Delta$, $\Theta\Gamma$, καὶ διὰ τοῦ Δ ὁποτέρᾳ τῶν $\Delta\Delta$, $\Gamma\epsilon$ παράλληλος ἦχθω ἡ $\Delta\Delta$ · καὶ ἐπεξέχθωσαν αἱ $\Delta\Delta$, $\Delta\Gamma$. καὶ ἐπεὶ ὀρθὴ ἐστὶν ἑκατέρα τῶν ὑπὸ $\Delta\Delta\Gamma$, $\Delta\Delta\epsilon$ γωνιῶν, πρὸς δὴ τινὶ εὐθείᾳ τῇ $\Delta\Delta$ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Δ δύο εὐθεῖαι αἱ $\Delta\Gamma$, $\Delta\epsilon$ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας δυσὶν ὀρθαῖς ἴσας ποιοῦσιν· ἐπ' εὐθείας ἄρα ἐστὶν ἡ $\Delta\Delta$ τῇ $\Delta\epsilon$. διὰ τὰ αὐτὰ δὲ καὶ ἡ $\Delta\Delta$ τῇ $\Delta\Theta$ ἐστὶν ἐπ' εὐθείας. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ $\Delta\Delta\Gamma$ γωνία τῇ ὑπὸ $\Delta\Delta\epsilon$ · ὀρθὴ γὰρ ἑκατέρα· κοινὴ προσκείσθω ἡ ὑπὸ $\Delta\Delta\Gamma$ · ὅλη ἄρα ἡ ὑπὸ $\Delta\Delta\epsilon$ ὅλη τῇ ὑπὸ $\Delta\Delta\Gamma$ ἐστὶν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν $\Delta\Delta$ τῇ $\Delta\Gamma$, ἡ δὲ $\Delta\Delta$ τῇ $\Delta\epsilon$, δύο δὴ αἱ $\Delta\Delta$, $\Delta\epsilon$ δύο ταῖς $\Delta\Delta$, $\Delta\Gamma$ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ $\Delta\Delta\epsilon$ γωνία τῇ ὑπὸ $\Delta\Delta\Gamma$ ἴση· βάσις ἄρα ἡ $\Delta\Delta$ βάσει τῇ $\Delta\Gamma$ [ἐστὶν] ἴση, καὶ τὸ $\Delta\Delta\epsilon$

$\Delta\Delta$ falls across the parallels $\Delta\Delta$ and $\Delta\epsilon$, the (sum of the) angles $\Delta\Delta\epsilon$ and $\Delta\Delta\Gamma$ is equal to two right-angles [Prop. 1.29]. But $\Delta\Delta\epsilon$ (is a) right-angle. Thus, $\Delta\Delta\Gamma$ (is) also a right-angle. And for parallelogrammic figures, the opposite sides and angles are equal to one another [Prop. 1.34]. Thus, each of the opposite angles $\Delta\Delta\epsilon$ and $\Delta\Delta\Gamma$ (are) also right-angles. Thus, $\Delta\Delta\epsilon\Delta$ is right-angled. And it was also shown (to be) equilateral.



Thus, ($\Delta\Delta\epsilon\Delta$) is a square [Def. 1.22]. And it is described on the straight-line $\Delta\Delta$. (Which is) the very thing it was required to do.

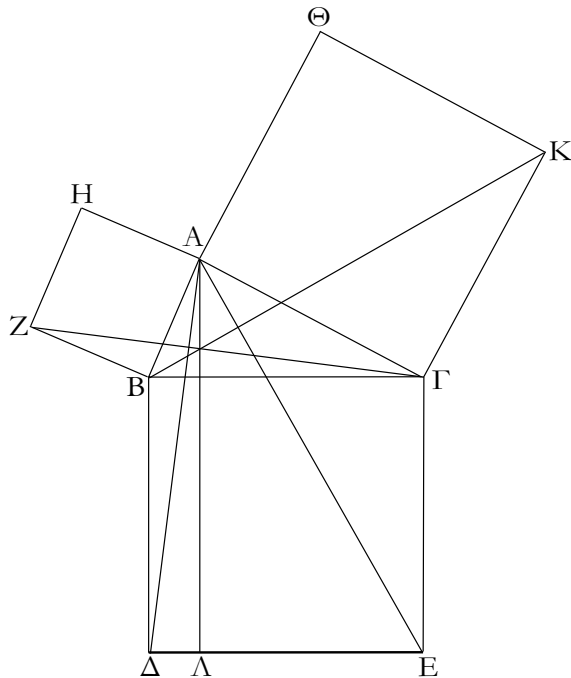
Proposition 47

In right-angled triangles, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides containing the right-angle.

Let $\Delta\Delta\Gamma$ be a right-angled triangle having the angle $\Delta\Delta\Gamma$ a right-angle. I say that the square on $\Delta\Gamma$ is equal to the (sum of the) squares on $\Delta\Delta$ and $\Delta\epsilon$.

For let the square $\Delta\Delta\epsilon\Delta$ have been described on $\Delta\Delta$, and (the squares) $\Delta\Delta$ and $\Delta\epsilon$ on $\Delta\Delta$ and $\Delta\epsilon$ (respectively) [Prop. 1.46]. And let $\Delta\Delta$ have been drawn through point Δ parallel to either of $\Delta\Delta$ or $\Delta\epsilon$ [Prop. 1.31]. And let $\Delta\Delta$ and $\Delta\epsilon$ have been joined. And since angles $\Delta\Delta\Gamma$ and $\Delta\Delta\epsilon$ are each right-angles, then two straight-lines $\Delta\Delta$ and $\Delta\epsilon$, not lying on the same side, make the adjacent angles with some straight-line $\Delta\Delta$, at the point Δ on it, (whose sum is) equal to two right-angles. Thus, $\Delta\Delta$ is straight-on to $\Delta\epsilon$ [Prop. 1.14]. So, for the same (reasons), $\Delta\Delta$ is also straight-on to $\Delta\epsilon$. And since angle $\Delta\Delta\Gamma$ is equal to $\Delta\Delta\epsilon$, for (they are) both right-angles, let $\Delta\Delta\Gamma$ have been added to both. Thus, the whole (angle) $\Delta\Delta\epsilon$ is equal to the whole (angle) $\Delta\Delta\Gamma$. And since $\Delta\Delta$ is equal to $\Delta\epsilon$, and $\Delta\Delta$ to $\Delta\epsilon$, the two (straight-lines) $\Delta\Delta$, $\Delta\epsilon$ are equal to the

τρίγωνον τῷ ZBF τριγώνῳ ἐστὶν ἴσον· καὶ [ἐστι] τοῦ μὲν ABD τριγώνου διπλάσιον τὸ BA παραλληλόγραμμον· βάσιν τε γὰρ τὴν αὐτὴν ἔχουσι τὴν BD καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς BD , AL · τοῦ δὲ ZBF τριγώνου διπλάσιον τὸ HB τετράγωνον· βάσιν τε γὰρ πάλιν τὴν αὐτὴν ἔχουσι τὴν ZB καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς ZB , $HΓ$. [τὰ δὲ τῶν ἴσων διπλάσια ἴσα ἀλλήλοις ἐστίν·] ἴσον ἄρα ἐστὶ καὶ τὸ BA παραλληλόγραμμον τῷ HB τετραγώνῳ. ὁμοίως δὴ ἐπιζευγνυμένων τῶν AE , BK δειχθήσεται καὶ τὸ $ΓA$ παραλληλόγραμμον ἴσον τῷ $ΘΓ$ τετραγώνῳ· ὅλον ἄρα τὸ $BΔΕΓ$ τετράγωνον δυοῖ τοῖς HB , $ΘΓ$ τετραγώνοις ἴσον ἐστίν. καὶ ἐστὶ τὸ μὲν $BΔΕΓ$ τετράγωνον ἀπὸ τῆς $BΓ$ ἀναγραφέν, τὰ δὲ HB , $ΘΓ$ ἀπὸ τῶν BA , AG . τὸ ἄρα ἀπὸ τῆς $BΓ$ πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν BA , AG πλευρῶν τετραγώνοις.

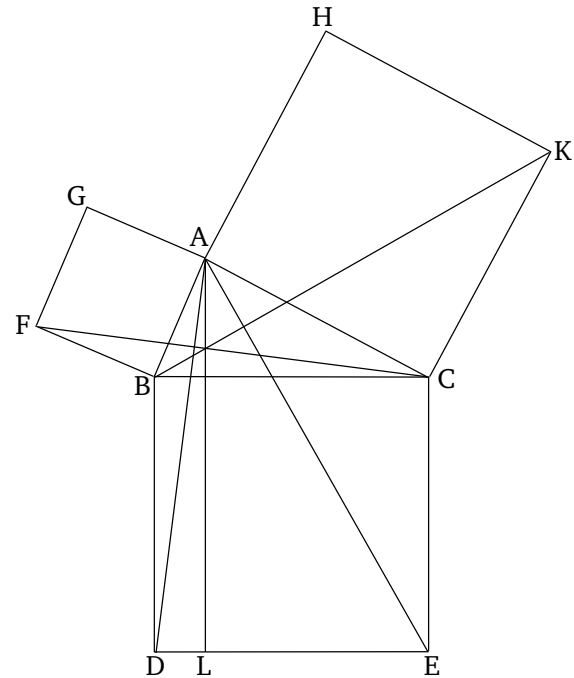


Ἐν ἄρα τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτείνουσας πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν [γωνίαν] περιεχουσῶν πλευρῶν τετραγώνοις· ὅπερ ἔδει δεῖξαι.

† The Greek text has " FB , BC ", which is obviously a mistake.

‡ This is an additional common notion.

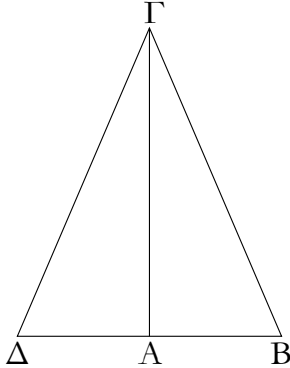
two (straight-lines) CB , BF ,[†] respectively. And angle DBA (is) equal to angle FBC . Thus, the base AD [is] equal to the base FC , and the triangle ABD is equal to the triangle FBC [Prop. 1.4]. And parallelogram BL [is] double (the area) of triangle ABD . For they have the same base, BD , and are between the same parallels, BD and AL [Prop. 1.41]. And square GB is double (the area) of triangle FBC . For again they have the same base, FB , and are between the same parallels, FB and GC [Prop. 1.41]. [And the doubles of equal things are equal to one another.][‡] Thus, the parallelogram BL is also equal to the square GB . So, similarly, AE and BK being joined, the parallelogram CL can be shown (to be) equal to the square HC . Thus, the whole square $BDEC$ is equal to the (sum of the) two squares GB and HC . And the square $BDEC$ is described on BC , and the (squares) GB and HC on BA and AC (respectively). Thus, the square on the side BC is equal to the (sum of the) squares on the sides BA and AC .



Thus, in right-angled triangles, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides surrounding the right-[angle]. (Which is) the very thing it was required to show.

μη'.

Ἐάν τριγώνου τὸ ἀπὸ μιᾶς τῶν πλευρῶν τετράγωνον ἴσον ᾗ τοῖς ἀπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν τετραγώνοις, ἡ περιεχομένη γωνία ὑπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ὀρθή ἐστιν.



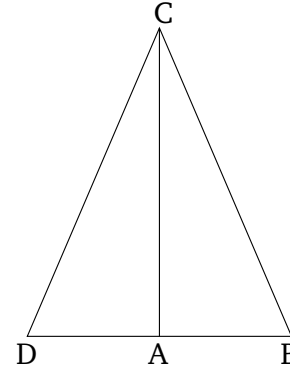
Τριγώνου γάρ τοῦ ΑΒΓ τὸ ἀπὸ μιᾶς τῆς ΒΓ πλευρᾶς τετράγωνον ἴσον ἔστω τοῖς ἀπὸ τῶν ΒΑ, ΑΓ πλευρῶν τετραγώνοις· λέγω, ὅτι ὀρθή ἐστιν ἡ ὑπὸ ΒΑΓ γωνία.

Ἦχθω γάρ ἀπὸ τοῦ Α σημείου τῆ ΑΓ εὐθείας πρὸς ὀρθὰς ἡ ΑΔ καὶ κείσθω τῇ ΒΑ ἴση ἡ ΑΔ, καὶ ἐπεζεύχθω ἡ ΔΓ. ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΒ, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς ΔΑ τετράγωνον τῷ ἀπὸ τῆς ΑΒ τετραγώνῳ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΑΓ τετράγωνον· τὰ ἄρα ἀπὸ τῶν ΔΑ, ΑΓ τετράγωνα ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ τετραγώνοις. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΔΑ, ΑΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΔΓ· ὀρθή γάρ ἐστιν ἡ ὑπὸ ΔΑΓ γωνία· τοῖς δὲ ἀπὸ τῶν ΒΑ, ΑΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΒΓ· ὑπόκειται γάρ· τὸ ἄρα ἀπὸ τῆς ΔΓ τετράγωνον ἴσον ἐστὶ τῷ ἀπὸ τῆς ΒΓ τετραγώνῳ· ὥστε καὶ πλευρὰ ἡ ΔΓ τῇ ΒΓ ἐστὶν ἴση· καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΒ, κοινὴ δὲ ἡ ΑΓ, δύο δὴ αἱ ΔΑ, ΑΓ δύο ταῖς ΒΑ, ΑΓ ἴσαι εἰσὶν· καὶ βάσις ἡ ΔΓ βάσει τῇ ΒΓ ἴση· γωνία ἄρα ἡ ὑπὸ ΔΑΓ γωνία τῇ ὑπὸ ΒΑΓ [ἐστὶν] ἴση. ὀρθή δὲ ἡ ὑπὸ ΔΑΓ· ὀρθή ἄρα καὶ ἡ ὑπὸ ΒΑΓ.

Ἐάν ἄρα τριγώνου τὸ ἀπὸ μιᾶς τῶν πλευρῶν τετράγωνον ἴσον ᾗ τοῖς ἀπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν τετραγώνοις, ἡ περιεχομένη γωνία ὑπὸ τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ὀρθή ἐστιν· ὅπερ ἔδει δεῖξαι.

Proposition 48

If the square on one of the sides of a triangle is equal to the (sum of the) squares on the two remaining sides of the triangle then the angle contained by the two remaining sides of the triangle is a right-angle.



For let the square on one of the sides, BC , of triangle ABC be equal to the (sum of the) squares on the sides BA and AC . I say that angle BAC is a right-angle.

For let AD have been drawn from point A at right-angles to the straight-line BC [Prop. 1.11], and let AD have been made equal to BA [Prop. 1.3], and let DC have been joined. Since DA is equal to AB , the square on DA is thus also equal to the square on AB .[†] Let the square on AC have been added to both. Thus, the (sum of the) squares on DA and AC is equal to the (sum of the) squares on BA and AC . But, the (square) on DC is equal to the (sum of the squares) on DA and AC . For angle DAC is a right-angle [Prop. 1.47]. But, the (square) on BC is equal to (sum of the squares) on BA and AC . For (that) was assumed. Thus, the square on DC is equal to the square on BC . So side DC is also equal to (side) BC . And since DA is equal to AB , and AC (is) common, the two (straight-lines) DA , AC are equal to the two (straight-lines) BA , AC . And the base DC is equal to the base BC . Thus, angle DAC [is] equal to angle BAC [Prop. 1.8]. But DAC is a right-angle. Thus, BAC is also a right-angle.

Thus, if the square on one of the sides of a triangle is equal to the (sum of the) squares on the remaining two sides of the triangle then the angle contained by the remaining two sides of the triangle is a right-angle. (Which is) the very thing it was required to show.

[†] Here, use is made of the additional common notion that the squares of equal things are themselves equal. Later on, the inverse notion is used.

ELEMENTS BOOK 2

Fundamentals of Geometric Algebra

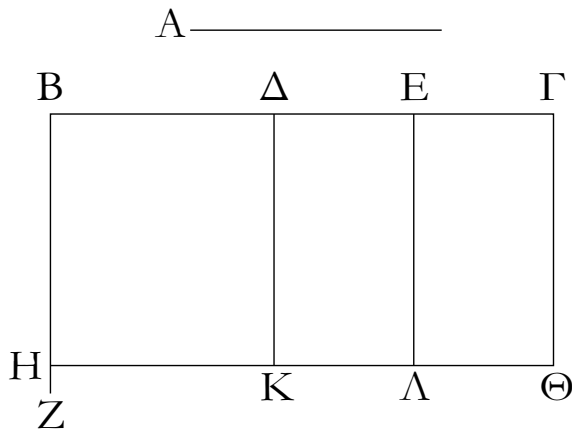
Ὅροι.

α'. Πᾶν παραλληλόγραμμον ὀρθογώνιον περιέχεσθαι λέγεται ὑπὸ δύο τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν εὐθειῶν.

β'. Παντὸς δὲ παραλληλογράμμου χωρίου τῶν περὶ τὴν διάμετρον αὐτοῦ παραλληλογράμμων ἐν ὁποιοῦν σὺν τοῖς δυοῖ παραπληρώμασι γνῶμων καλεῖσθω.

α'.

Ἐὰν ᾧσι δύο εὐθεῖαι, τμηθῇ δὲ ἡ ἑτέρα αὐτῶν εἰς ὅσα δηποτοῦν τμήματα, τὸ περιεχόμενον ὀρθογώνιον ὑπὸ τῶν δύο εὐθειῶν ἴσον ἐστὶ τοῖς ὑπὸ τε τῆς ἀτμήτου καὶ ἐκάστου τῶν τμημάτων περιεχομένοις ὀρθογωνίοις.



Ἐστωσαν δύο εὐθεῖαι αἱ Α, ΒΓ, καὶ τετμήσθω ἡ ΒΓ, ὡς ἔτυχεν, κατὰ τὰ Δ, Ε σημεία· λέγω, ὅτι τὸ ὑπὸ τῶν Α, ΒΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν Α, ΒΔ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ὑπὸ τῶν Α, ΔΕ καὶ ἔτι τῷ ὑπὸ τῶν Α, ΕΓ.

Ἦχθω γὰρ ἀπὸ τοῦ Β τῇ ΒΓ πρὸς ὀρθὰς ἡ ΒΖ, καὶ κείσθω τῇ Α ἴση ἡ ΒΗ, καὶ διὰ μὲν τοῦ Η τῇ ΒΓ παράλληλος ἡ ΗΘ, διὰ δὲ τῶν Δ, Ε, Γ τῇ ΒΗ παράλληλοι ἡχθωσαν αἱ ΔΚ, ΕΛ, ΓΘ.

Ἰσον δὴ ἐστὶ τὸ ΒΘ τοῖς ΒΚ, ΔΛ, ΕΘ. καὶ ἐστὶ τὸ μὲν ΒΘ τὸ ὑπὸ τῶν Α, ΒΓ· περιέχεται μὲν γὰρ ὑπὸ τῶν ΗΒ, ΒΓ, ἴση δὲ ἡ ΒΗ τῇ Α· τὸ δὲ ΒΚ τὸ ὑπὸ τῶν Α, ΒΔ· περιέχεται μὲν γὰρ ὑπὸ τῶν ΗΒ, ΒΔ, ἴση δὲ ἡ ΒΗ τῇ Α. τὸ δὲ ΔΛ τὸ ὑπὸ τῶν Α, ΔΕ· ἴση γὰρ ἡ ΔΚ, τουτέστιν ἡ ΒΗ, τῇ Α. καὶ ἔτι ὁμοίως τὸ ΕΘ τὸ ὑπὸ τῶν Α, ΕΓ· τὸ ἄρα ὑπὸ τῶν Α, ΒΓ ἴσον ἐστὶ τῷ τε ὑπὸ Α, ΒΔ καὶ τῷ ὑπὸ Α, ΔΕ καὶ ἔτι τῷ ὑπὸ Α, ΕΓ.

Ἐὰν ἄρα ᾧσι δύο εὐθεῖαι, τμηθῇ δὲ ἡ ἑτέρα αὐτῶν εἰς ὅσα δηποτοῦν τμήματα, τὸ περιεχόμενον ὀρθογώνιον ὑπὸ τῶν δύο εὐθειῶν ἴσον ἐστὶ τοῖς ὑπὸ τε τῆς ἀτμήτου καὶ ἐκάστου τῶν τμημάτων περιεχομένοις ὀρθογωνίοις· ὅπερ

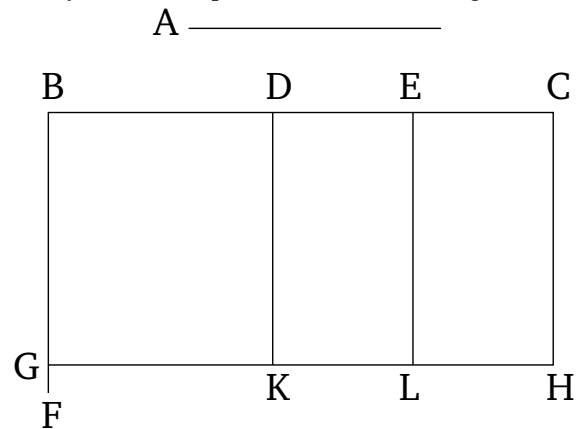
Definitions

1. Any rectangular parallelogram is said to be contained by the two straight-lines containing the right-angle.

2. And in any parallelogrammic figure, let any one whatsoever of the parallelograms about its diagonal, (taken) with its two complements, be called a gnomon.

Proposition 1[†]

If there are two straight-lines, and one of them is cut into any number of pieces whatsoever, then the rectangle contained by the two straight-lines is equal to the (sum of the) rectangles contained by the uncut (straight-line), and every one of the pieces (of the cut straight-line).



Let A and BC be the two straight-lines, and let BC be cut, at random, at points D and E . I say that the rectangle contained by A and BC is equal to the rectangle(s) contained by A and BD , by A and DE , and, finally, by A and EC .

For let BF have been drawn from point B , at right-angles to BC [Prop. 1.11], and let BG be made equal to A [Prop. 1.3], and let GH have been drawn through (point) G , parallel to BC [Prop. 1.31], and let DK , EL , and CH have been drawn through (points) D , E , and C (respectively), parallel to BG [Prop. 1.31].

So the (rectangle) BH is equal to the (rectangles) BK , DL , and EH . And BH is the (rectangle contained) by A and BC . For it is contained by GB and BC , and BG (is) equal to A . And BK (is) the (rectangle contained) by A and BD . For it is contained by GB and BD , and BG (is) equal to A . And DL (is) the (rectangle contained) by A and DE . For DK , that is to say BG [Prop. 1.34], (is) equal to A . Similarly, EH (is) also the (rectangle contained) by A and EC . Thus, the (rectangle contained) by A and BC is equal to the (rectangles contained) by A

ἔδει δεῖξαι.

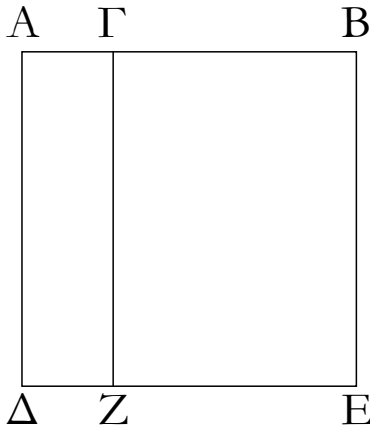
and BD , by A and DE , and, finally, by A and EC .

Thus, if there are two straight-lines, and one of them is cut into any number of pieces whatsoever, then the rectangle contained by the two straight-lines is equal to the (sum of the) rectangles contained by the uncut (straight-line), and every one of the pieces (of the cut straight-line). (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $a(b + c + d + \dots) = ab + ac + ad + \dots$.

β'.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ, ὥς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἑκατέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ὅλης τετραγώνῳ.



Εὐθεῖα γὰρ ἡ AB τετμήσθω, ὥς ἔτυχεν, κατὰ τὸ Γ σημεῖον· λέγω, ὅτι τὸ ὑπὸ τῶν AB , $B\Gamma$ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ὑπὸ BA , $A\Gamma$ περιεχομένου ὀρθογώνιου ἴσον ἐστὶ τῷ ἀπὸ τῆς AB τετραγώνῳ.

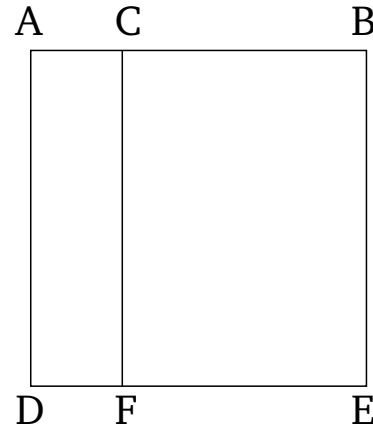
Ἀναγεγράφθω γὰρ ἀπὸ τῆς AB τετράγωνον τὸ $ADEB$, καὶ ἦχθω διὰ τοῦ Γ ὁποτέρῃ τῶν AD , BE παράλληλος ἡ ΓZ .

Ἰσον δὲ ἐστὶ τὸ AE τοῖς AZ , ΓE . καὶ ἐστὶ τὸ μὲν AE τὸ ἀπὸ τῆς AB τετράγωνον, τὸ δὲ AZ τὸ ὑπὸ τῶν BA , $A\Gamma$ περιεχόμενον ὀρθογώνιον· περιέχεται μὲν γὰρ ὑπὸ τῶν ΔA , $A\Gamma$, ἴση δὲ ἡ $A\Delta$ τῇ AB · τὸ δὲ ΓE τὸ ὑπὸ τῶν AB , $B\Gamma$ · ἴση γὰρ ἡ BE τῇ AB . τὸ ἄρα ὑπὸ τῶν BA , $A\Gamma$ μετὰ τοῦ ὑπὸ τῶν AB , $B\Gamma$ ἴσον ἐστὶ τῷ ἀπὸ τῆς AB τετραγώνῳ.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὥς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἑκατέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς ὅλης τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

Proposition 2†

If a straight-line is cut at random then the (sum of the) rectangle(s) contained by the whole (straight-line), and each of the pieces (of the straight-line), is equal to the square on the whole.



For let the straight-line AB have been cut, at random, at point C . I say that the rectangle contained by AB and BC , plus the rectangle contained by BA and AC , is equal to the square on AB .

For let the square $ADEB$ have been described on AB [Prop. 1.46], and let CF have been drawn through C , parallel to either of AD or BE [Prop. 1.31].

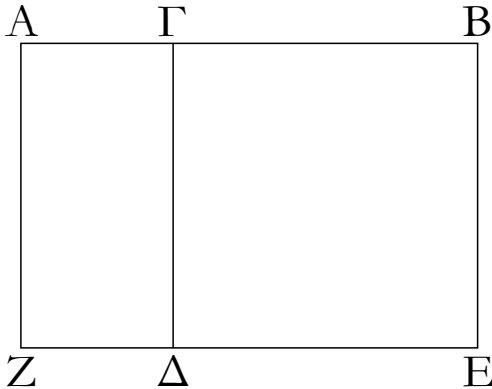
So the (square) AE is equal to the (rectangles) AF and CE . And AE is the square on AB . And AF (is) the rectangle contained by the (straight-lines) BA and AC . For it is contained by DA and AC , and AD (is) equal to AB . And CE (is) the (rectangle contained) by AB and BC . For BE (is) equal to AB . Thus, the (rectangle contained) by BA and AC , plus the (rectangle contained) by AB and BC , is equal to the square on AB .

Thus, if a straight-line is cut at random then the (sum of the) rectangle(s) contained by the whole (straight-line), and each of the pieces (of the straight-line), is equal to the square on the whole. (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $ab + ac = a^2$ if $a = b + c$.

γ'.

Ἐάν εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἐνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθογώνιῳ καὶ τῷ ἀπὸ τοῦ προειρημένου τμήματος τετραγώνῳ.



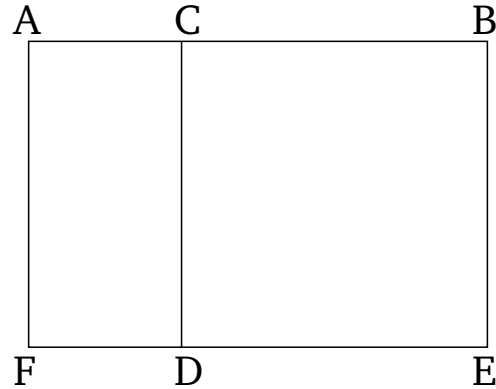
Εὐθεῖα γὰρ ἡ AB τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ· λέγω, ὅτι τὸ ὑπὸ τῶν AB, BΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν AΓ, ΓB περιεχομένῳ ὀρθογώνιῳ μετὰ τοῦ ἀπὸ τῆς BΓ τετραγώνου.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΓB τετράγωνον τὸ ΓΔEB, καὶ διήχθω ἡ EΔ ἐπὶ τὸ Z, καὶ διὰ τοῦ A ὁποτέρῃ τῶν ΓΔ, BE παράλληλος ἦχθω ἡ AZ. ἴσον δὲ ἐστὶ τὸ AE τοῖς AΔ, ΓE· καὶ ἐστὶ τὸ μὲν AE τὸ ὑπὸ τῶν AB, BΓ περιεχόμενον ὀρθογώνιον· περιέχεται μὲν γὰρ ὑπὸ τῶν AB, BE, ἴση δὲ ἡ BE τῇ BΓ· τὸ δὲ AΔ τὸ ὑπὸ τῶν AΓ, ΓB· ἴση γὰρ ἡ ΔΓ τῇ ΓB· τὸ δὲ ΔB τὸ ἀπὸ τῆς ΓB τετράγωνον· τὸ ἄρα ὑπὸ τῶν AB, BΓ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν AΓ, ΓB περιεχομένῳ ὀρθογώνιῳ μετὰ τοῦ ἀπὸ τῆς BΓ τετραγώνου.

Ἐάν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ὑπὸ τῆς ὅλης καὶ ἐνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ τε ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθογώνιῳ καὶ τῷ ἀπὸ τοῦ προειρημένου τμήματος τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

Proposition 3†

If a straight-line is cut at random then the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the rectangle contained by (both of) the pieces, and the square on the aforementioned piece.



For let the straight-line AB have been cut, at random, at (point) C. I say that the rectangle contained by AB and BC is equal to the rectangle contained by AC and CB, plus the square on BC.

For let the square CDEB have been described on CB [Prop. 1.46], and let ED have been drawn through to F, and let AF have been drawn through A, parallel to either of CD or BE [Prop. 1.31]. So the (rectangle) AE is equal to the (rectangle) AD and the (square) CE. And AE is the rectangle contained by AB and BC. For it is contained by AB and BE, and BE (is) equal to BC. And AD (is) the (rectangle contained) by AC and CB. For DC (is) equal to CB. And DB (is) the square on CB. Thus, the rectangle contained by AB and BC is equal to the rectangle contained by AC and CB, plus the square on BC.

Thus, if a straight-line is cut at random then the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the rectangle contained by (both of) the pieces, and the square on the aforementioned piece. (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $(a + b)a = ab + a^2$.

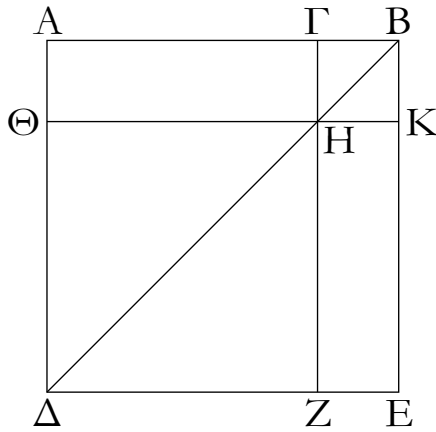
δ'.

Ἐάν εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν τμημάτων τετραγώνοις καὶ τῷ δις ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθο-

Proposition 4†

If a straight-line is cut at random then the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the

γωνίῳ.

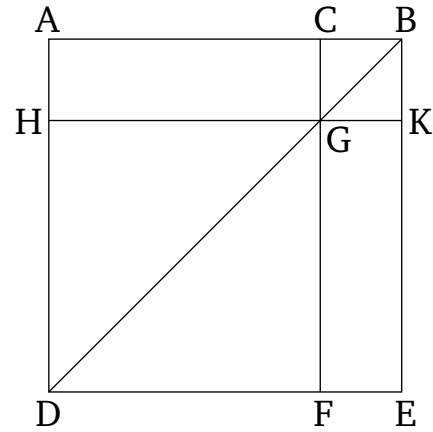


Εὐθεῖα γὰρ γραμμὴ ἡ AB τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ . λέγω, ὅτι τὸ ἀπὸ τῆς AB τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν AG , GB τετραγώνοις καὶ τῷ δις ὑπὸ τῶν AG , GB περιεχομένῳ ὀρθογωνίῳ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς AB τετράγωνον τὸ $ADEB$, καὶ ἐπεζεύχθω ἡ BD , καὶ διὰ μὲν τοῦ Γ ὁποτέρῃ τῶν AD , EB παράλληλος ἦχθω ἡ ΓZ , διὰ δὲ τοῦ H ὁποτέρῃ τῶν AB , DE παράλληλος ἦχθω ἡ ΘK . καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΓZ τῇ AD , καὶ εἰς αὐτὰς ἐμπίπτωκεν ἡ BD , ἡ ἐκτὸς γωνία ἡ ὑπὸ $ΓHB$ ἴση ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ $AΔB$. ἀλλ' ἡ ὑπὸ $AΔB$ τῇ ὑπὸ $ABΔ$ ἐστὶν ἴση, ἐπεὶ καὶ πλευρὰ ἡ BA τῇ AD ἐστὶν ἴση· καὶ ἡ ὑπὸ $ΓHB$ ἄρα γωνία τῇ ὑπὸ HBF ἐστὶν ἴση· ὥστε καὶ πλευρὰ ἡ BF πλευρᾷ τῇ GH ἐστὶν ἴση· ἀλλ' ἡ μὲν GB τῇ HK ἐστὶν ἴση. ἡ δὲ GH τῇ KB · καὶ ἡ HK ἄρα τῇ KB ἐστὶν ἴση· ἰσόπλευρον ἄρα ἐστὶ τὸ $ΓHKB$. λέγω δὴ, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ παράλληλός ἐστιν ἡ GH τῇ BK [καὶ εἰς αὐτὰς ἐμπίπτωκεν εὐθεῖα ἡ GB], αἱ ἄρα ὑπὸ KBF , HGF γωνίαι δύο ὀρθαῖς εἰσιν ἴσαι. ὀρθὴ δὲ ἡ ὑπὸ KBF · ὀρθὴ ἄρα καὶ ἡ ὑπὸ BGF · ὥστε καὶ αἱ ἀπεναντίον αἱ ὑπὸ $ΓHK$, HKB ὀρθαῖς εἰσιν. ὀρθογώνιον ἄρα ἐστὶ τὸ $ΓHKB$ · ἐδείχθη δὲ καὶ ἰσόπλευρον· τετράγωνον ἄρα ἐστίν· καὶ ἐστὶν ἀπὸ τῆς GB . διὰ τὰ αὐτὰ δὴ καὶ τὸ ΘZ τετράγωνόν ἐστιν· καὶ ἐστὶν ἀπὸ τῆς ΘH , τουτέστιν [ἀπὸ] τῆς AG · τὰ ἄρα ΘZ , KF τετράγωνα ἀπὸ τῶν AG , GB εἰσιν. καὶ ἐπεὶ ἴσον ἐστὶ τὸ AH τῷ HE , καὶ ἐστὶ τὸ AH τὸ ὑπὸ τῶν AG , GB · ἴση γὰρ ἡ $HΓ$ τῇ GB · καὶ τὸ HE ἄρα ἴσον ἐστὶ τῷ ὑπὸ AG , GB · τὰ ἄρα AH , HE ἴσα ἐστὶ τῷ δις ὑπὸ τῶν AG , GB . ἔστι δὲ καὶ τὰ ΘZ , KF τετράγωνα ἀπὸ τῶν AG , GB · τὰ ἄρα τέσσαρα τὰ ΘZ , KF , AH , HE ἴσα ἐστὶ τοῖς τε ἀπὸ τῶν AG , GB τετραγώνοις καὶ τῷ δις ὑπὸ τῶν AG , GB περιεχομένῳ ὀρθογωνίῳ. ἀλλὰ τὰ ΘZ , KF , AH , HE ὅλον ἐστὶ τὸ $ADEB$, ὃ ἐστὶν ἀπὸ τῆς AB τετράγωνον· τὸ ἄρα ἀπὸ τῆς AB τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν AG , GB τετραγώνοις καὶ τῷ δις ὑπὸ τῶν AG , GB περιεχομένῳ ὀρθογωνίῳ.

Ἐάν ἄρα εὐθεῖα γραμμὴ τηρῇ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς

rectangle contained by the pieces.



For let the straight-line AB have been cut, at random, at (point) C . I say that the square on AB is equal to the (sum of the) squares on AC and CB , and twice the rectangle contained by AC and CB .

For let the square $ADEB$ have been described on AB [Prop. 1.46], and let BD have been joined, and let CF have been drawn through C , parallel to either of AD or EB [Prop. 1.31], and let HG have been drawn through G , parallel to either of AB or DE [Prop. 1.31]. And since CF is parallel to AD , and BD has fallen across them, the external angle CGB is equal to the internal and opposite (angle) ADB [Prop. 1.29]. But, ADB is equal to ABD , since the side BA is also equal to AD [Prop. 1.5]. Thus, angle CGB is also equal to GBC . So the side BC is equal to the side CG [Prop. 1.6]. But, CB is equal to GK , and CG to KB [Prop. 1.34]. Thus, GK is also equal to KB . Thus, $CGKB$ is equilateral. So I say that (it is) also right-angled. For since CG is parallel to BK [and the straight-line CB has fallen across them], the angles KBC and GCB are thus equal to two right-angles [Prop. 1.29]. But KBC (is) a right-angle. Thus, BCG (is) also a right-angle. So the opposite (angles) CGK and GKB are also right-angles [Prop. 1.34]. Thus, $CGKB$ is right-angled. And it was also shown (to be) equilateral. Thus, it is a square. And it is on CB . So, for the same (reasons), HF is also a square. And it is on HG , that is to say [on] AC [Prop. 1.34]. Thus, the squares HF and KC are on AC and CB (respectively). And the (rectangle) AG is equal to the (rectangle) GE [Prop. 1.43]. And AG is the (rectangle contained) by AC and CB . For GC (is) equal to CB . Thus, GE is also equal to the (rectangle contained) by AC and CB . Thus, the (rectangles) AG and GE are equal to twice the (rectangle contained) by AC and CB . And HF and CK are the squares on AC and CB (respectively). Thus, the four (figures) HF , CK , AG , and GE are equal to the (sum of the) squares on

ὅλης τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν τμημάτων τετραγώνοις καὶ τῷ δις ὑπὸ τῶν τμημάτων περιεχομένῳ ὀρθογώνῳ· ὅπερ ἔδει δεῖξαι.

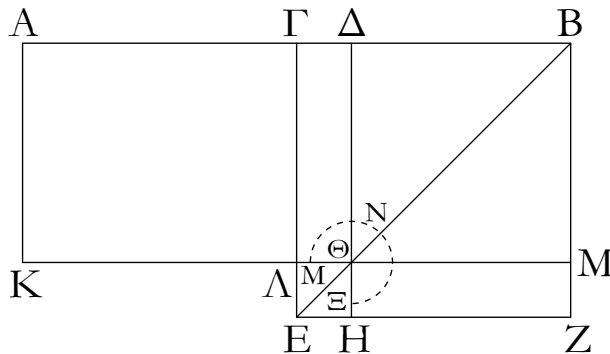
AC and BC , and twice the rectangle contained by AC and CB . But, the (figures) HF , CK , AG , and GE are (equivalent to) the whole of $ADEB$, which is the square on AB . Thus, the square on AB is equal to the (sum of the) squares on AC and CB , and twice the rectangle contained by AC and CB .

Thus, if a straight-line is cut at random then the square on the whole (straight-line) is equal to the (sum of the) squares on the pieces (of the straight-line), and twice the rectangle contained by the pieces. (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $(a + b)^2 = a^2 + b^2 + 2ab$.

ε'.

Ἐάν εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὸ ὑπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ἡμισείας τετραγώνῳ.

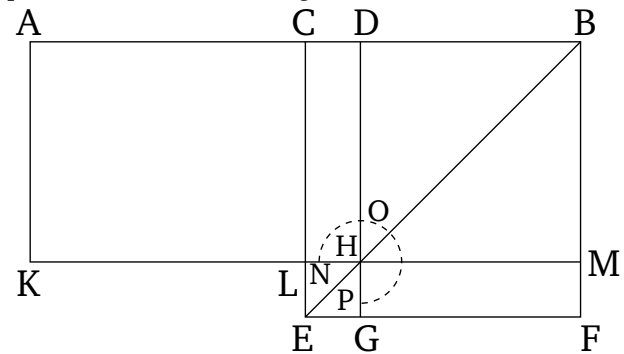


Εὐθεῖα γάρ τις ἡ AB τετμήσθω εἰς μὲν ἴσα κατὰ τὸ Γ , εἰς δὲ ἄνισα κατὰ τὸ Δ · λέγω, ὅτι τὸ ὑπὸ τῶν $A\Delta$, ΔB περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς $\Gamma\Delta$ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς GB τετραγώνῳ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς GB τετράγωνον τὸ $ΓΕΖΒ$, καὶ ἐπεζεύχθω ἡ BE , καὶ διὰ μὲν τοῦ Δ ὁποτέρῃ τῶν $ΓΕ$, BZ παράλληλος ἦχθω ἡ ΔH , διὰ δὲ τοῦ Θ ὁποτέρῃ τῶν AB , EZ παράλληλος πάλιν ἦχθω ἡ KM , καὶ πάλιν διὰ τοῦ A ὁποτέρῃ τῶν $ΓΛ$, BM παράλληλος ἦχθω ἡ AK . καὶ ἐπεὶ ἴσον ἐστὶ τὸ $\Gamma\Theta$ παραπλήρωμα τῷ ΘZ παραπληρώματι, κοινὸν προσκείσθω τὸ ΔM · ὅλον ἄρα τὸ ΓM ὅλῳ τῷ ΔZ ἴσον ἐστίν. ἀλλὰ τὸ ΓM τῷ $ΑΛ$ ἴσον ἐστίν, ἐπεὶ καὶ ἡ $ΑΓ$ τῇ $ΓΒ$ ἐστὶν ἴση· καὶ τὸ $ΑΛ$ ἄρα τῷ ΔZ ἴσον ἐστίν. κοινὸν προσκείσθω τὸ $\Gamma\Theta$ · ὅλον ἄρα τὸ $A\Theta$ τῷ $MN\Xi$ † γνῶμονι ἴσον ἐστίν. ἀλλὰ τὸ $A\Theta$ τὸ ὑπὸ τῶν $A\Delta$, ΔB ἐστίν· ἴση γὰρ ἡ $\Delta\Theta$ τῇ ΔB · καὶ ὁ $MN\Xi$ ἄρα γνῶμων ἴσος ἐστὶ τῷ ὑπὸ $A\Delta$, ΔB . κοινὸν προσκείσθω τὸ ΛH , ὃ ἐστὶν ἴσον τῷ ἀπὸ τῆς $\Gamma\Delta$ · ὁ ἄρα $MN\Xi$ γνῶμων καὶ τὸ ΛH ἴσα ἐστὶ τῷ ὑπὸ τῶν $A\Delta$, ΔB περιεχομένῳ ὀρθογώνῳ καὶ τῷ ἀπὸ τῆς

Proposition 5‡

If a straight-line is cut into equal and unequal (pieces) then the rectangle contained by the unequal pieces of the whole (straight-line), plus the square on the (difference) between the (equal and unequal) pieces, is equal to the square on half (of the straight-line).



For let any straight-line AB have been cut—equally at C , and unequally at D . I say that the rectangle contained by AD and DB , plus the square on CD , is equal to the square on CB .

For let the square $CEFB$ have been described on CB [Prop. 1.46], and let BE have been joined, and let DG have been drawn through D , parallel to either of CE or BF [Prop. 1.31], and again let KM have been drawn through H , parallel to either of AB or EF [Prop. 1.31], and again let AK have been drawn through A , parallel to either of CL or BM [Prop. 1.31]. And since the complement CH is equal to the complement HF [Prop. 1.43], let the (square) DM have been added to both. Thus, the whole (rectangle) CM is equal to the whole (rectangle) DF . But, (rectangle) CM is equal to (rectangle) AL , since AC is also equal to CB [Prop. 1.36]. Thus, (rectangle) AL is also equal to (rectangle) DF . Let (rectangle) CH have been added to both. Thus, the whole (rectangle) AH is equal to the gnomon NOP . But, AH

ΓΔ τετραγώνω. ἀλλὰ ὁ ΜΝΞ γνώμων καὶ τὸ ΛΗ ὅλον ἐστὶ τὸ ΓΕΖΒ τετράγωνον, ὃ ἐστὶν ἀπὸ τῆς ΓΒ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΔ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΒ τετραγώνω.

Ἐὰν ἄρα εὐθεία γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὸ ὑπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ἡμισείας τετραγώνω. ὅπερ ἔδει δεῖξαι.

is the (rectangle contained) by AD and DB . For DH (is) equal to DB . Thus, the gnomon NOP is also equal to the (rectangle contained) by AD and DB . Let LG , which is equal to the (square) on CD , have been added to both. Thus, the gnomon NOP and the (square) LG are equal to the rectangle contained by AD and DB , and the square on CD . But, the gnomon NOP and the (square) LG is (equivalent to) the whole square $CEFB$, which is on CB . Thus, the rectangle contained by AD and DB , plus the square on CD , is equal to the square on CB .

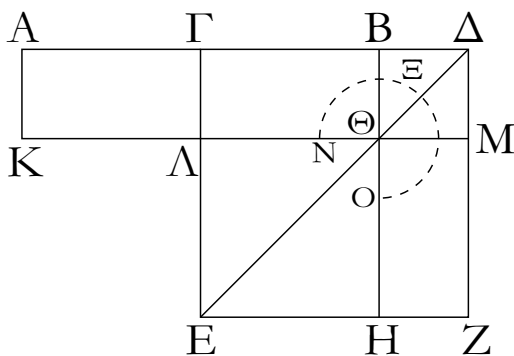
Thus, if a straight-line is cut into equal and unequal (pieces) then the rectangle contained by the unequal pieces of the whole (straight-line), plus the square on the (difference) between the (equal and unequal) pieces, is equal to the square on half (of the straight-line). (Which is) the very thing it was required to show.

† Note the (presumably mistaken) double use of the label M in the Greek text.

‡ This proposition is a geometric version of the algebraic identity: $ab + [(a+b)/2 - b]^2 = [(a+b)/2]^2$.

ζ'.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ δίχα, προστεθῇ δὲ τις αὐτῇ εὐθεῖα ἐπ' εὐθείας, τὸ ὑπὸ τῆς ὅλης σὺν τῇ προσκειμένῃ καὶ τῆς προσκειμένης περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς συγκεκλιμένης ἔκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης τετραγώνω.



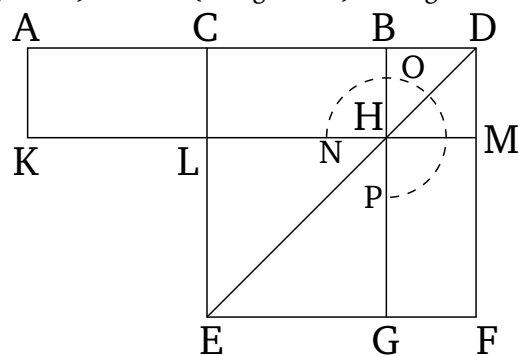
Εὐθεῖα γάρ τις ἡ ΑΒ τετμήσθω δίχα κατὰ τὸ Γ σημεῖον, προσκείσθω δὲ τις αὐτῇ εὐθεῖα ἐπ' εὐθείας ἡ ΒΔ· λέγω, ὅτι τὸ ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΒ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΔ τετραγώνω.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΓΔ τετράγωνον τὸ ΓΕΖΔ, καὶ ἐπεζεύχθω ἡ ΔΕ, καὶ διὰ μὲν τοῦ Β σημείου ὁποτέρᾳ τῶν ΕΓ, ΔΖ παράλληλος ᾗχθω ἡ ΒΗ, διὰ δὲ τοῦ Θ σημείου ὁποτέρᾳ τῶν ΑΒ, ΕΖ παράλληλος ᾗχθω ἡ ΚΜ, καὶ ἔτι διὰ τοῦ Α ὁποτέρᾳ τῶν ΓΛ, ΔΜ παράλληλος ᾗχθω ἡ ΑΚ.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΑΓ τῇ ΓΒ, ἴσον ἐστὶ καὶ τὸ ΑΛ

Proposition 6[†]

If a straight-line is cut in half, and any straight-line added to it straight-on, then the rectangle contained by the whole (straight-line) with the (straight-line) having been added, and the (straight-line) having been added, plus the square on half (of the original straight-line), is equal to the square on the sum of half (of the original straight-line) and the (straight-line) having been added.



For let any straight-line AB have been cut in half at point C , and let any straight-line BD have been added to it straight-on. I say that the rectangle contained by AD and DB , plus the square on CB , is equal to the square on CD .

For let the square $CEFD$ have been described on CD [Prop. 1.46], and let DE have been joined, and let BG have been drawn through point B , parallel to either of EC or DF [Prop. 1.31], and let KM have been drawn through point H , parallel to either of AB or EF [Prop. 1.31], and finally let AK have been drawn

τῷ ΓΘ. ἀλλὰ τὸ ΓΘ τῷ ΘΖ ἴσον ἐστίν. καὶ τὸ ΑΛ ἄρα τῷ ΘΖ ἐστὶν ἴσον. κοινὸν προσκείσθω τὸ ΓΜ· ὅλον ἄρα τὸ ΑΜ τῷ ΝΞΟ γνῶμονί ἐστιν ἴσον. ἀλλὰ τὸ ΑΜ ἐστὶ τὸ ὑπὸ τῶν ΑΔ, ΔΒ· ἴση γάρ ἐστιν ἡ ΔΜ τῇ ΔΒ· καὶ ὁ ΝΞΟ ἄρα γνῶμων ἴσος ἐστὶ τῷ ὑπὸ τῶν ΑΔ, ΔΒ [περιεχομένῳ ὀρθογωνίῳ]. κοινὸν προσκείσθω τὸ ΛΗ, ὃ ἐστὶν ἴσον τῷ ἀπὸ τῆς ΒΓ τετραγώνῳ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΒ τετραγώνου ἴσον ἐστὶ τῷ ΝΞΟ γνῶμονι καὶ τῷ ΛΗ. ἀλλὰ ὁ ΝΞΟ γνῶμων καὶ τὸ ΛΗ ὅλον ἐστὶ τὸ ΓΕΖΔ τετράγωνον, ὃ ἐστὶν ἀπὸ τῆς ΓΔ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΒ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΓΒ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΔ τετραγώνῳ.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ δίχα, προστεθῇ δέ τις αὐτῇ εὐθεῖα ἐπ' εὐθείας, τὸ ὑπὸ τῆς ὅλης σὺν τῇ προσκειμένη καὶ τῆς προσκειμένης περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς συγκεκλιμένης ἕκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

through A, parallel to either of CL or DM [Prop. 1.31].

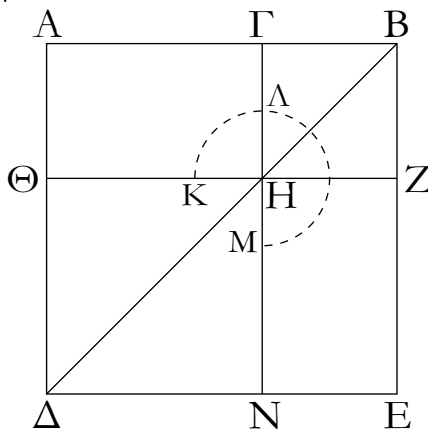
Therefore, since AC is equal to CB , (rectangle) AL is also equal to (rectangle) CH [Prop. 1.36]. But, (rectangle) CH is equal to (rectangle) HF [Prop. 1.43]. Thus, (rectangle) AL is also equal to (rectangle) HF . Let (rectangle) CM have been added to both. Thus, the whole (rectangle) AM is equal to the gnomon NOP . But, AM is the (rectangle contained) by AD and DB . For DM is equal to DB . Thus, gnomon NOP is also equal to the [rectangle contained] by AD and DB . Let LG , which is equal to the square on BC , have been added to both. Thus, the rectangle contained by AD and DB , plus the square on CB , is equal to the gnomon NOP and the (square) LG . But the gnomon NOP and the (square) LG is (equivalent to) the whole square $CEFD$, which is on CD . Thus, the rectangle contained by AD and DB , plus the square on CB , is equal to the square on CD .

Thus, if a straight-line is cut in half, and any straight-line added to it straight-on, then the rectangle contained by the whole (straight-line) with the (straight-line) having being added, and the (straight-line) having being added, plus the square on half (of the original straight-line), is equal to the square on the sum of half (of the original straight-line) and the (straight-line) having been added. (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $(2a + b)b + a^2 = (a + b)^2$.

ζ'.

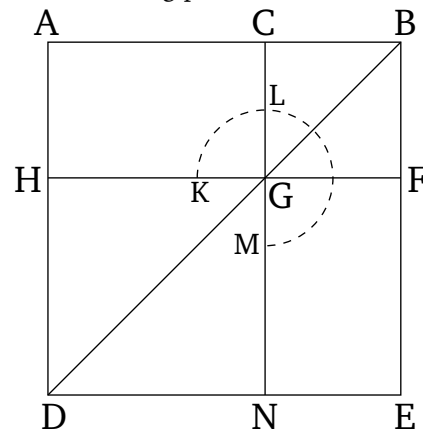
Ἐὰν εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης καὶ τὸ ἄφ' ἐνὸς τῶν τμημάτων τὰ συναμφοτέρα τετράγωνα ἴσα ἐστὶ τῷ τε δις ὑπὸ τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ.



Εὐθεῖα γάρ τις ἡ AB τετμήσθω, ὡς ἔτυχεν, κατὰ τὸ Γ σημεῖον· λέγω, ὅτι τὰ ἀπὸ τῶν AB , $B\Gamma$ τετράγωνα ἴσα ἐστὶ τῷ τε δις ὑπὸ τῶν AB , $B\Gamma$ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ

Proposition 7†

If a straight-line is cut at random then the sum of the squares on the whole (straight-line), and one of the pieces (of the straight-line), is equal to twice the rectangle contained by the whole, and the said piece, and the square on the remaining piece.



For let any straight-line AB have been cut, at random, at point C . I say that the (sum of the) squares on AB and BC is equal to twice the rectangle contained by AB and

ἀπὸ τῆς ΓΑ τετραγώνω.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔΕΒ· καὶ καταγεγράφθω τὸ σχῆμα.

Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΑΗ τῷ ΗΕ, κοινὸν προσκείσθω τὸ ΓΖ· ὅλον ἄρα τὸ ΑΖ ὅλῳ τῷ ΓΕ ἴσον ἐστίν· τὰ ἄρα ΑΖ, ΓΕ διπλάσιά ἐστι τοῦ ΑΖ. ἀλλὰ τὰ ΑΖ, ΓΕ ὁ ΚΑΜ ἐστὶ γνῶμων καὶ τὸ ΓΖ τετράγωνον· ὁ ΚΑΜ ἄρα γνῶμων καὶ τὸ ΓΖ διπλάσιά ἐστι τοῦ ΑΖ. ἐστὶ δὲ τοῦ ΑΖ διπλάσιον καὶ τὸ δις ὑπὸ τῶν ΑΒ, ΒΓ· ἴση γὰρ ἡ ΒΖ τῇ ΒΓ· ὁ ἄρα ΚΑΜ γνῶμων καὶ τὸ ΓΖ τετράγωνον ἴσον ἐστὶ τῷ δις ὑπὸ τῶν ΑΒ, ΒΓ. κοινὸν προσκείσθω τὸ ΔΗ, ὃ ἐστὶν ἀπὸ τῆς ΑΓ τετράγωνον· ὁ ἄρα ΚΑΜ γνῶμων καὶ τὰ ΒΗ, ΗΔ τετράγωνα ἴσα ἐστὶ τῷ τε δις ὑπὸ τῶν ΑΒ, ΒΓ περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τῆς ΑΓ τετραγώνῳ. ἀλλὰ ὁ ΚΑΜ γνῶμων καὶ τὰ ΒΗ, ΗΔ τετράγωνα ὅλον ἐστὶ τὸ ΑΔΕΒ καὶ τὸ ΓΖ, ἃ ἐστὶν ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα· τὰ ἄρα ἀπὸ τῶν ΑΒ, ΒΓ τετράγωνα ἴσα ἐστὶ τῷ [τε] δις ὑπὸ τῶν ΑΒ, ΒΓ περιεχομένῳ ὀρθογωνίῳ μετὰ τοῦ ἀπὸ τῆς ΑΓ τετραγώνου.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὥς ἔτυχεν, τὸ ἀπὸ τῆς ὅλης καὶ τὸ ἀφ' ἑνὸς τῶν τμημάτων τὰ συναμφότερα τετράγωνα ἴσα ἐστὶ τῷ τε δις ὑπὸ τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος περιεχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

BC, and the square on *CA*.

For let the square *ADEB* have been described on *AB* [Prop. 1.46], and let the (rest of) the figure have been drawn.

Therefore, since (rectangle) *AG* is equal to (rectangle) *GE* [Prop. 1.43], let the (square) *CF* have been added to both. Thus, the whole (rectangle) *AF* is equal to the whole (rectangle) *CE*. Thus, (rectangle) *AF* plus (rectangle) *CE* is double (rectangle) *AF*. But, (rectangle) *AF* plus (rectangle) *CE* is the gnomon *KLM*, and the square *CF*. Thus, the gnomon *KLM*, and the square *CF*, is double the (rectangle) *AF*. But double the (rectangle) *AF* is also twice the (rectangle contained) by *AB* and *BC*. For *BF* (is) equal to *BC*. Thus, the gnomon *KLM*, and the square *CF*, are equal to twice the (rectangle contained) by *AB* and *BC*. Let *DG*, which is the square on *AC*, have been added to both. Thus, the gnomon *KLM*, and the squares *BG* and *GD*, are equal to twice the rectangle contained by *AB* and *BC*, and the square on *AC*. But, the gnomon *KLM* and the squares *BG* and *GD* is (equivalent to) the whole of *ADEB* and *CF*, which are the squares on *AB* and *BC* (respectively). Thus, the (sum of the) squares on *AB* and *BC* is equal to twice the rectangle contained by *AB* and *BC*, and the square on *AC*.

Thus, if a straight-line is cut at random then the sum of the squares on the whole (straight-line), and one of the pieces (of the straight-line), is equal to twice the rectangle contained by the whole, and the said piece, and the square on the remaining piece. (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $(a + b)^2 + a^2 = 2(a + b)a + b^2$.

η'.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ, ὥς ἔτυχεν, τὸ τετράκις ὑπὸ τῆς ὅλης καὶ ἑνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος ὥς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῳ.

Εὐθεῖα γὰρ τις ἡ ΑΒ τετμήσθω, ὥς ἔτυχεν, κατὰ τὸ Γ σημεῖον· λέγω, ὅτι τὸ τετράκις ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΑΓ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ, ΒΓ ὥς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῳ.

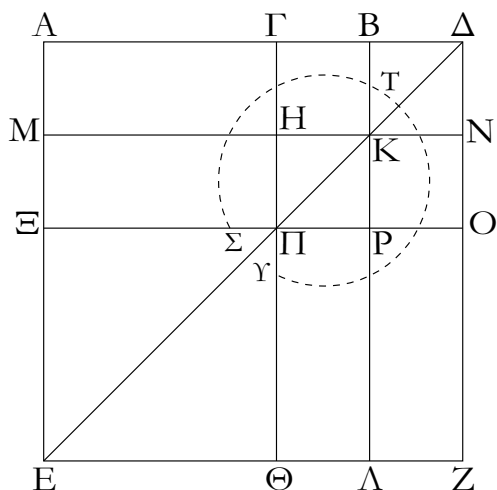
Ἐκβεβλήσθω γὰρ ἐπ' εὐθείας [τῇ ΑΒ εὐθείᾳ] ἡ ΒΔ, καὶ κείσθω τῇ ΓΒ ἴση ἡ ΒΔ, καὶ ἀναγεγράφθω ἀπὸ τῆς ΑΔ τετράγωνον τὸ ΑΕΖΔ, καὶ καταγεγράφθω διπλοῦν τὸ σχῆμα.

Proposition 8[†]

If a straight-line is cut at random then four times the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), plus the square on the remaining piece, is equal to the square described on the whole and the former piece, as on one (complete straight-line).

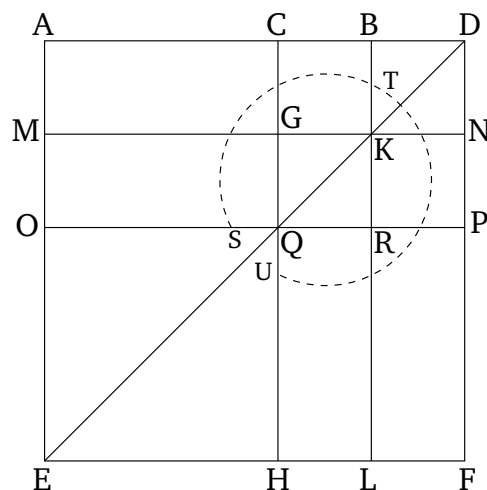
For let any straight-line *AB* have been cut, at random, at point *C*. I say that four times the rectangle contained by *AB* and *BC*, plus the square on *AC*, is equal to the square described on *AB* and *BC*, as on one (complete straight-line).

For let *BD* have been produced in a straight-line [with the straight-line *AB*], and let *BD* be made equal to *CB* [Prop. 1.3], and let the square *AEFD* have been described on *AD* [Prop. 1.46], and let the (rest of the) figure have been drawn double.



Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΓΒ τῇ ΒΔ, ἀλλὰ ἡ μὲν ΓΒ τῇ ΗΚ ἐστὶν ἴση, ἡ δὲ ΒΔ τῇ ΚΝ, καὶ ἡ ΗΚ ἄρα τῇ ΚΝ ἐστὶν ἴση. διὰ τὰ αὐτὰ δὲ καὶ ἡ ΠΡ τῇ ΡΟ ἐστὶν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΓ τῇ ΒΔ, ἡ δὲ ΗΚ τῇ ΚΝ, ἴσον ἄρα ἐστὶ καὶ τὸ μὲν ΓΚ τῷ ΚΔ, τὸ δὲ ΗΡ τῷ ΡΝ. ἀλλὰ τὸ ΓΚ τῷ ΡΝ ἐστὶν ἴσον· παραπληρώματα γὰρ τοῦ ΓΟ παραλληλογράμμου· καὶ τὸ ΚΔ ἄρα τῷ ΗΡ ἴσον ἐστίν· τὰ τέσσαρα ἄρα τὰ ΔΚ, ΓΚ, ΗΡ, ΡΝ ἴσα ἀλλήλοις ἐστίν. τὰ τέσσαρα ἄρα τετραπλάσιά ἐστι τοῦ ΓΚ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΓΒ τῇ ΒΔ, ἀλλὰ ἡ μὲν ΒΔ τῇ ΒΚ, τουτέστι τῇ ΓΗ ἴση, ἡ δὲ ΓΒ τῇ ΗΚ, τουτέστι τῇ ΗΠ, ἐστὶν ἴση, καὶ ἡ ΓΗ ἄρα τῇ ΗΠ ἴση ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΓΗ τῇ ΗΠ, ἡ δὲ ΠΡ τῇ ΡΟ, ἴσον ἐστὶ καὶ τὸ μὲν ΑΗ τῷ ΜΠ, τὸ δὲ ΠΛ τῷ ΡΖ. ἀλλὰ τὸ ΜΠ τῷ ΠΛ ἐστὶν ἴσον· παραπληρώματα γὰρ τοῦ ΜΛ παραλληλογράμμου· καὶ τὸ ΑΗ ἄρα τῷ ΡΖ ἴσον ἐστίν· τὰ τέσσαρα ἄρα τὰ ΑΗ, ΜΠ, ΠΛ, ΡΖ ἴσα ἀλλήλοις ἐστίν· τὰ τέσσαρα ἄρα τοῦ ΑΗ ἐστὶ τετραπλάσια. ἐδείχθη δὲ καὶ τὰ τέσσαρα τὰ ΓΚ, ΚΔ, ΗΡ, ΡΝ τοῦ ΓΚ τετραπλάσια· τὰ ἄρα ὀκτώ, ἃ περιέχει τὸν ΣΤΥ γνῶμονα, τετραπλάσιά ἐστι τοῦ ΑΚ. καὶ ἐπεὶ τὸ ΑΚ τὸ ὑπὸ τῶν ΑΒ, ΒΔ ἐστίν· ἴση γὰρ ἡ ΒΚ τῇ ΒΔ· τὸ ἄρα τετράχως ὑπὸ τῶν ΑΒ, ΒΔ τετραπλάσιόν ἐστι τοῦ ΑΚ. ἐδείχθη δὲ τοῦ ΑΚ τετραπλάσιος καὶ ὁ ΣΤΥ γνῶμων· τὸ ἄρα τετράχως ὑπὸ τῶν ΑΒ, ΒΔ ἴσον ἐστὶ τῷ ΣΤΥ γνῶμονι. κοινὸν προσκείσθω τὸ ΞΘ, ὃ ἐστὶν ἴσον τῷ ἀπὸ τῆς ΑΓ τετραγώνῳ· τὸ ἄρα τετράχως ὑπὸ τῶν ΑΒ, ΒΔ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ ΑΓ τετραγώνου ἴσον ἐστὶ τῷ ΣΤΥ γνῶμονι καὶ τῷ ΞΘ. ἀλλὰ ὁ ΣΤΥ γνῶμων καὶ τὸ ΞΘ ὅλον ἐστὶ τὸ ΑΕΖΔ τετράγωνον, ὃ ἐστὶν ἀπὸ τῆς ΑΔ· τὸ ἄρα τετράχως ὑπὸ τῶν ΑΒ, ΒΔ μετὰ τοῦ ἀπὸ ΑΓ ἴσον ἐστὶ τῷ ἀπὸ ΑΔ τετραγώνῳ· ἴση δὲ ἡ ΒΔ τῇ ΒΓ. τὸ ἄρα τετράχως ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ ΑΓ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΔ, τουτέστι τῷ ἀπὸ τῆς ΑΒ καὶ ΒΓ ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῳ.

Ἐάν ἄρα εὐθεῖα γραμμὴ τμηθῇ, ὡς ἔτυχεν, τὸ τετράχως ὑπὸ τῆς ὅλης καὶ ἐνὸς τῶν τμημάτων περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνου ἴσου



Therefore, since CB is equal to BD , but CB is equal to GK [Prop. 1.34], and BD to KN [Prop. 1.34], GK is thus also equal to KN . So, for the same (reasons), QR is equal to RP . And since BC is equal to BD , and GK to KN , (square) CK is thus also equal to (square) KD , and (square) GR to (square) RN [Prop. 1.36]. But, (square) CK is equal to (square) RN . For (they are) complements in the parallelogram CP [Prop. 1.43]. Thus, (square) KD is also equal to (square) GR . Thus, the four (squares) DK , CK , GR , and RN are equal to one another. Thus, the four (taken together) are quadruple (square) CK . Again, since CB is equal to BD , but BD (is) equal to BK —that is to say, CG —and CB is equal to GK —that is to say, GQ — CG is thus also equal to GQ . And since CG is equal to GQ , and QR to RP , (rectangle) AG is also equal to (rectangle) MQ , and (rectangle) QL to (rectangle) RF [Prop. 1.36]. But, (rectangle) MQ is equal to (rectangle) QL . For (they are) complements in the parallelogram ML [Prop. 1.43]. Thus, (rectangle) AG is also equal to (rectangle) RF . Thus, the four (rectangles) AG , MQ , QL , and RF are equal to one another. Thus, the four (taken together) are quadruple (rectangle) AG . And it was also shown that the four (squares) CK , KD , GR , and RN (taken together are) quadruple (square) CK . Thus, the eight (figures taken together), which comprise the gnomon STU , are quadruple (rectangle) AK . And since AK is the (rectangle contained) by AB and BD , for BK (is) equal to BD , four times the (rectangle contained) by AB and BD is quadruple (rectangle) AK . But the gnomon STU was also shown (to be equal to) quadruple (rectangle) AK . Thus, four times the (rectangle contained) by AB and BD is equal to the gnomon STU . Let OH , which is equal to the square on AC , have been added to both. Thus, four times the rectangle contained by AB and BD , plus the square on AC , is equal to the gnomon STU , and the (square) OH . But,

ἐστὶ τῷ ἀπὸ τε τῆς ὅλης καὶ τοῦ εἰρημένου τμήματος ὡς ἀπὸ μιᾶς ἀναγραφέντι τετραγώνῳ· ὅπερ εἶδει δεῖξαι.

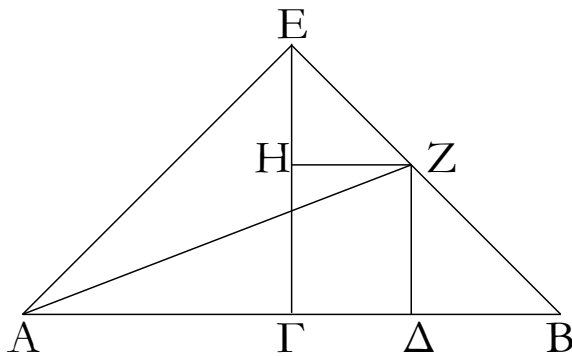
the gnomon STU and the (square) OH is (equivalent to) the whole square $AEFD$, which is on AD . Thus, four times the (rectangle contained) by AB and BD , plus the (square) on AC , is equal to the square on AD . And BD (is) equal to BC . Thus, four times the rectangle contained by AB and BC , plus the square on AC , is equal to the (square) on AD , that is to say the square described on AB and BC , as on one (complete straight-line).

Thus, if a straight-line is cut at random then four times the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), plus the square on the remaining piece, is equal to the square described on the whole and the former piece, as on one (complete straight-line). (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $4(a+b)a + b^2 = [(a+b) + a]^2$.

θ'.

Ἐὰν εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμίσειας καὶ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου.

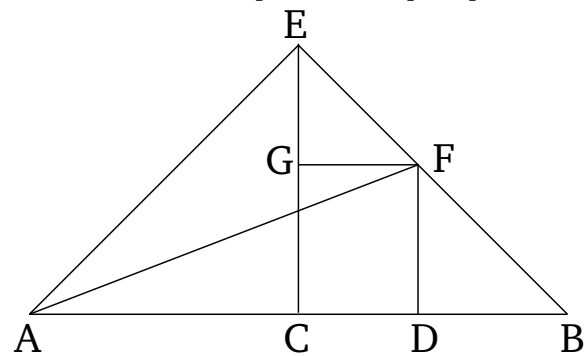


Εὐθεῖα γὰρ τις ἡ AB τετμήσθω εἰς μὲν ἴσα κατὰ τὸ Γ , εἰς δὲ ἄνισα κατὰ τὸ Δ . λέγω, ὅτι τὰ ἀπὸ τῶν AD , DB τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν AG , GD τετραγώνων.

Ἦχθω γὰρ ἀπὸ τοῦ Γ τῇ AB πρὸς ὀρθὰς ἡ GE , καὶ κείσθω ἴση ἑκατέρᾳ τῶν AG , GB , καὶ ἐπεζεύχθωσαν αἱ EA , EB , καὶ διὰ μὲν τοῦ Δ τῇ EG παράλληλος ἦχθω ἡ DZ , διὰ δὲ τοῦ Z τῇ AB ἡ ZH , καὶ ἐπεζεύχθω ἡ AZ . καὶ ἐπεὶ ἴση ἐστὶν ἡ AG τῇ GE , ἴση ἐστὶ καὶ ἡ ὑπὸ EAG γωνία τῇ ὑπὸ AEG . καὶ ἐπεὶ ὀρθὴ ἐστὶν ἡ πρὸς τῷ Γ , λοιπαὶ ἄρα αἱ ὑπὸ EAG , AEG μιᾷ ὀρθῇ ἴσαι εἰσὶν· καὶ εἰσιν ἴσαι· ἡμίσεια ἄρα ὀρθῆς ἐστὶν ἑκατέρᾳ τῶν ὑπὸ GEA , GAE . διὰ τὰ αὐτὰ δὲ καὶ ἑκατέρᾳ τῶν ὑπὸ GEB , EBG ἡμίσειά ἐστιν ὀρθῆς· ὅλη ἄρα ἡ ὑπὸ AEB ὀρθὴ ἐστὶν. καὶ ἐπεὶ ἡ ὑπὸ HEZ ἡμίσειά ἐστὶν ὀρθῆς, ὀρθὴ δὲ ἡ ὑπὸ EHZ · ἴση γὰρ ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ EGB · λοιπὴ ἄρα ἡ ὑπὸ EZH ἡμίσειά ἐστιν

Proposition 9†

If a straight-line is cut into equal and unequal (pieces) then the (sum of the) squares on the unequal pieces of the whole (straight-line) is double the (sum of the) square on half (the straight-line) and (the square) on the (difference) between the (equal and unequal) pieces.



For let any straight-line AB have been cut—equally at C , and unequally at D . I say that the (sum of the) squares on AD and DB is double the (sum of the squares) on AC and CD .

For let CE have been drawn from (point) C , at right-angles to AB [Prop. 1.11], and let it be made equal to each of AC and CB [Prop. 1.3], and let EA and EB have been joined. And let DF have been drawn through (point) D , parallel to EC [Prop. 1.31], and (let) FG (have been drawn) through (point) F , (parallel) to AB [Prop. 1.31]. And let AF have been joined. And since AC is equal to CE , the angle EAC is also equal to the (angle) AEC [Prop. 1.5]. And since the (angle) at C is a right-angle, the (sum of the) remaining angles (of triangle AEC), EAC and AEC , is thus equal to one right-

ὀρθῆς· ἴση ἄρα [ἐστίν] ἡ ὑπὸ HEZ γωνία τῇ ὑπὸ EZH· ὥστε καὶ πλευρὰ ἡ EH τῇ HZ ἐστὶν ἴση. πάλιν ἐπεὶ ἡ πρὸς τῷ B γωνία ἡμίσειά ἐστιν ὀρθῆς, ὀρθὴ δὲ ἡ ὑπὸ ZΔB· ἴση γὰρ πάλιν ἐστὶ τῇ ἐντὸς καὶ ἀπεναντίον τῇ ὑπὸ EΓB· λοιπὴ ἄρα ἡ ὑπὸ BZΔ ἡμίσειά ἐστιν ὀρθῆς· ἴση ἄρα ἡ πρὸς τῷ B γωνία τῇ ὑπὸ ΔZB· ὥστε καὶ πλευρὰ ἡ ZΔ πλευρᾷ τῇ ΔB ἐστὶν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΓ τῇ ΓΕ, ἴσον ἐστὶ καὶ τὸ ἀπὸ ΑΓ τῷ ἀπὸ ΓΕ· τὰ ἄρα ἀπὸ τῶν ΑΓ, ΓΕ τετράγωνα διπλάσιά ἐστι τοῦ ἀπὸ ΑΓ. τοῖς δὲ ἀπὸ τῶν ΑΓ, ΓΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΑ τετράγωνον· ὀρθὴ γάρ ἡ ὑπὸ ΑΓΕ γωνία· τὸ ἄρα ἀπὸ τῆς ΕΑ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΑΓ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ EH τῇ HZ, ἴσον καὶ τὸ ἀπὸ τῆς EH τῷ ἀπὸ τῆς HZ· τὰ ἄρα ἀπὸ τῶν EH, HZ τετράγωνα διπλάσιά ἐστι τοῦ ἀπὸ τῆς HZ τετραγώνου. τοῖς δὲ ἀπὸ τῶν EH, HZ τετραγώνοις ἴσον ἐστὶ τὸ ἀπὸ τῆς EZ τετράγωνον· τὸ ἄρα ἀπὸ τῆς EZ τετράγωνον διπλάσιόν ἐστι τοῦ ἀπὸ τῆς HZ. ἴση δὲ ἡ HZ τῇ ΓΔ· τὸ ἄρα ἀπὸ τῆς EZ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓΔ. ἐστὶ δὲ καὶ τὸ ἀπὸ τῆς ΕΑ διπλάσιον τοῦ ἀπὸ τῆς ΑΓ· τὰ ἄρα ἀπὸ τῶν ΑΕ, ΕΖ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. τοῖς δὲ ἀπὸ τῶν ΑΕ, ΕΖ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΑΖ τετράγωνον· ὀρθὴ γάρ ἐστὶν ἡ ὑπὸ ΑΕΖ γωνία· τὸ ἄρα ἀπὸ τῆς ΑΖ τετράγωνον διπλάσιόν ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ. τῷ δὲ ἀπὸ τῆς ΑΖ ἴσα τὰ ἀπὸ τῶν ΑΔ, ΔΖ· ὀρθὴ γάρ ἡ πρὸς τῷ Δ γωνία· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΖ διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. ἴση δὲ ἡ ΔΖ τῇ ΔB· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔB τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ εἰς ἴσα καὶ ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τῆς ὅλης τμημάτων τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς μεταξὺ τῶν τομῶν τετραγώνου· ὅπερ ἔδει δεῖξαι.

angle [Prop. 1.32]. And they are equal. Thus, (angles) CEA and CAE are each half a right-angle. So, for the same (reasons), (angles) CEB and EBC are also each half a right-angle. Thus, the whole (angle) AEB is a right-angle. And since GEF is half a right-angle, and EGF (is) a right-angle—for it is equal to the internal and opposite (angle) ECB [Prop. 1.29]—the remaining (angle) EFG is thus half a right-angle [Prop. 1.32]. Thus, angle GEF [is] equal to EFG . So the side EG is also equal to the (side) GF [Prop. 1.6]. Again, since the angle at B is half a right-angle, and (angle) FDB (is) a right-angle—for again it is equal to the internal and opposite (angle) ECB [Prop. 1.29]—the remaining (angle) BFD is half a right-angle [Prop. 1.32]. Thus, the angle at B (is) equal to DFB . So the side FD is also equal to the side DB [Prop. 1.6]. And since AC is equal to CE , the (square) on AC (is) also equal to the (square) on CE . Thus, the (sum of the) squares on AC and CE is double the (square) on AC . And the square on EA is equal to the (sum of the) squares on AC and CE . For angle ACE (is) a right-angle [Prop. 1.47]. Thus, the (square) on EA is double the (square) on AC . Again, since EG is equal to GF , the (square) on EG (is) also equal to the (square) on GF . Thus, the (sum of the squares) on EG and GF is double the square on GF . And the square on EF is equal to the (sum of the) squares on EG and GF [Prop. 1.47]. Thus, the square on EF is double the (square) on GF . And GF (is) equal to CD [Prop. 1.34]. Thus, the (square) on EF is double the (square) on CD . And the (square) on EA is also double the (square) on AC . Thus, the (sum of the) squares on AE and EF is double the (sum of the) squares on AC and CD . And the square on AF is equal to the (sum of the squares) on AE and EF . For the angle AEF is a right-angle [Prop. 1.47]. Thus, the square on AF is double the (sum of the squares) on AC and CD . And the (sum of the squares) on AD and DF (is) equal to the (square) on AF . For the angle at D is a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on AD and DF is double the (sum of the) squares on AC and CD . And DF (is) equal to DB . Thus, the (sum of the) squares on AD and DB is double the (sum of the) squares on AC and CD .

Thus, if a straight-line is cut into equal and unequal (pieces) then the (sum of the) squares on the unequal pieces of the whole (straight-line) is double the (sum of the) square on half (the straight-line) and (the square) on the (difference) between the (equal and unequal) pieces. (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $a^2 + b^2 = 2[(a+b)/2]^2 + [(a+b)/2 - b]^2$.

τετραγώνω· τὰ ἄρα ἀπὸ τῶν ΕΓ, ΓΑ τετράγωνα διπλάσιά ἐστι τοῦ ἀπὸ τῆς ΓΑ τετραγώνου. τοῖς δὲ ἀπὸ τῶν ΕΓ, ΓΑ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΑ· τὸ ἄρα ἀπὸ τῆς ΕΑ τετράγωνον διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΑΓ τετραγώνου. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΖΗ τῇ ΕΖ, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΖΕ· τὰ ἄρα ἀπὸ τῶν ΗΖ, ΖΕ διπλάσιά ἐστι τοῦ ἀπὸ τῆς ΕΖ. τοῖς δὲ ἀπὸ τῶν ΗΖ, ΖΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΗ· τὸ ἄρα ἀπὸ τῆς ΕΗ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΖ. ἴση δὲ ἡ ΕΖ τῇ ΓΔ· τὸ ἄρα ἀπὸ τῆς ΕΗ τετράγωνον διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΕΑ διπλάσιον τοῦ ἀπὸ τῆς ΑΓ· τὰ ἄρα ἀπὸ τῶν ΑΕ, ΕΗ τετράγωνα διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων. τοῖς δὲ ἀπὸ τῶν ΑΕ, ΕΗ τετραγώνοις ἴσον ἐστὶ τὸ ἀπὸ τῆς ΑΗ τετράγωνον· τὸ ἄρα ἀπὸ τῆς ΑΗ διπλάσιόν ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ. τῷ δὲ ἀπὸ τῆς ΑΗ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΑΔ, ΔΗ· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΗ [τετράγωνα] διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ [τετραγώνων]. ἴση δὲ ἡ ΔΗ τῇ ΔΒ· τὰ ἄρα ἀπὸ τῶν ΑΔ, ΔΒ [τετράγωνα] διπλάσιά ἐστι τῶν ἀπὸ τῶν ΑΓ, ΓΔ τετραγώνων.

Ἐὰν ἄρα εὐθεῖα γραμμὴ τμηθῇ δίχα, προστεθῇ δὲ τις αὐτῇ εὐθεῖα ἐπ' εὐθείας, τὸ ἀπὸ τῆς ὅλης σὺν τῇ προσκειμένῃ καὶ τὸ ἀπὸ τῆς προσκειμένης τὰ συναμφότερα τετράγωνα διπλάσιά ἐστι τοῦ τε ἀπὸ τῆς ἡμισείας καὶ τοῦ ἀπὸ τῆς συγκειμένης ἕκ τε τῆς ἡμισείας καὶ τῆς προσκειμένης ὡς ἀπὸ μιᾶς ἀναγραφέντος τετραγώνου· ὅπερ ἔδει δείξαι.

[Prop. 1.29]. Thus, the remaining (angle) DGB is half a right-angle. Thus, DGB is equal to DBG . So side BD is also equal to side GD [Prop. 1.6]. Again, since EGF is half a right-angle, and the (angle) at F (is) a right-angle, for it is equal to the opposite (angle) at C [Prop. 1.34], the remaining (angle) FEG is thus half a right-angle. Thus, angle EGF (is) equal to FEG . So the side GF is also equal to the side EF [Prop. 1.6]. And since [EC is equal to CA] the square on EC is [also] equal to the square on CA . Thus, the (sum of the) squares on EC and CA is double the square on CA . And the (square) on EA is equal to the (sum of the squares) on EC and CA [Prop. 1.47]. Thus, the square on EA is double the square on AC . Again, since FG is equal to EF , the (square) on FG is also equal to the (square) on FE . Thus, the (sum of the squares) on GF and FE is double the (square) on EF . And the (square) on EG is equal to the (sum of the squares) on GF and FE [Prop. 1.47]. Thus, the (square) on EG is double the (square) on EF . And EF (is) equal to CD [Prop. 1.34]. Thus, the square on EG is double the (square) on CD . But it was also shown that the (square) on EA (is) double the (square) on AC . Thus, the (sum of the) squares on AE and EG is double the (sum of the) squares on AC and CD . And the square on AG is equal to the (sum of the) squares on AE and EG [Prop. 1.47]. Thus, the (square) on AG is double the (sum of the squares) on AC and CD . And the (sum of the squares) on AD and DG is equal to the (square) on AG [Prop. 1.47]. Thus, the (sum of the) [squares] on AD and DG is double the (sum of the) [squares] on AC and CD . And DG (is) equal to DB . Thus, the (sum of the) [squares] on AD and DB is double the (sum of the) squares on AC and CD .

Thus, if a straight-line is cut in half, and any straight-line added to it straight-on, then the sum of the square on the whole (straight-line) with the (straight-line) having been added, and the (square) on the (straight-line) having been added, is double the (sum of the square) on half (the straight-line), and the square described on the sum of half (the straight-line) and (straight-line) having been added, as on one (complete straight-line). (Which is) the very thing it was required to show.

† This proposition is a geometric version of the algebraic identity: $(2a + b)^2 + b^2 = 2[a^2 + (a + b)^2]$.

ια'.

Proposition 11[†]

Τὴν δοθεῖσαν εὐθεῖαν τεμεῖν ὥστε τὸ ὑπὸ τῆς ὅλης καὶ τοῦ ἐτέρου τῶν τμημάτων περιεχόμενον ὀρθογώνιον ἴσον εἶναι τῷ ἀπὸ τοῦ λοιποῦ τμήματος τετραγώνω.

To cut a given straight-line such that the rectangle contained by the whole (straight-line), and one of the pieces (of the straight-line), is equal to the square on the remaining piece.

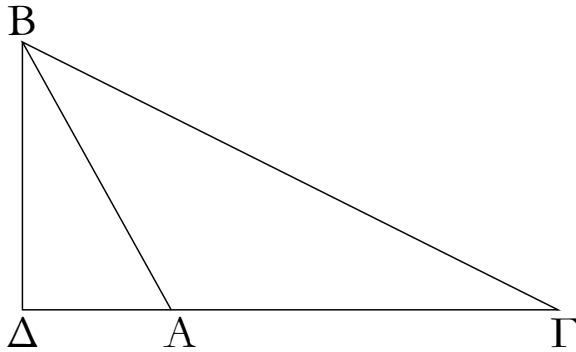
and BH is equal to the square on HA .

Thus, the given straight-line AB has been cut at (point) H such as to make the rectangle contained by AB and BH equal to the square on HA . (Which is) the very thing it was required to do.

† This manner of cutting a straight-line—so that the ratio of the whole to the larger piece is equal to the ratio of the larger to the smaller piece—is sometimes called the “Golden Section”.

ιβ'.

Ἐν τοῖς ἀμβλυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ἀμβλείαν γωνίαν ὑποτείνουσας πλευρᾶς τετράγωνον μεῖζόν ἐστι τῶν ἀπὸ τῶν τὴν ἀμβλείαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιεχομένῳ δις ὑπὸ τε μιᾶς τῶν περὶ τὴν ἀμβλείαν γωνίαν, ἐφ' ἣν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐκτὸς ὑπὸ τῆς καθέτου πρὸς τῇ ἀμβλείᾳ γωνίᾳ.



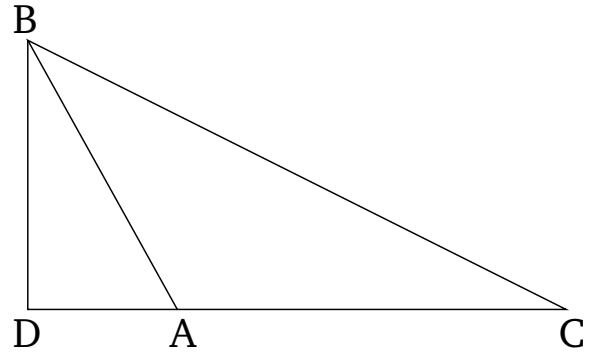
Ἐστω ἀμβλυγώνιον τρίγωνον τὸ $AB\Gamma$ ἀμβλείαν ἔχον τὴν ὑπὸ $BA\Gamma$, καὶ ἤχθω ἀπὸ τοῦ B σημείου ἐπὶ τὴν ΓA ἐκβληθεῖσαν κάθετος ἡ BD . λέγω, ὅτι τὸ ἀπὸ τῆς $B\Gamma$ τετράγωνον μεῖζόν ἐστι τῶν ἀπὸ τῶν BA , $A\Gamma$ τετραγώνων τῷ δις ὑπὸ τῶν ΓA , $A\Delta$ περιεχομένῳ ὀρθογωνίῳ.

Ἐπεὶ γὰρ εὐθεῖα ἡ $\Gamma\Delta$ τέμνεται, ὡς ἔτυχεν, κατὰ τὸ A σημείον, τὸ ἄρα ἀπὸ τῆς $\Delta\Gamma$ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΓA , $A\Delta$ τετραγώνοις καὶ τῷ δις ὑπὸ τῶν ΓA , $A\Delta$ περιεχομένῳ ὀρθογωνίῳ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΔB · τὰ ἄρα ἀπὸ τῶν $\Gamma\Delta$, ΔB ἴσα ἐστὶ τοῖς τε ἀπὸ τῶν ΓA , $A\Delta$, ΔB τετραγώνοις καὶ τῷ δις ὑπὸ τῶν ΓA , $A\Delta$ [περιεχομένῳ ὀρθογωνίῳ]. ἀλλὰ τοῖς μὲν ἀπὸ τῶν $\Gamma\Delta$, ΔB ἴσον ἐστὶ τὸ ἀπὸ τῆς ΓB · ὀρθὴ γὰρ ἡ πρὸς τῷ Δ γωνία· τοῖς δὲ ἀπὸ τῶν $A\Delta$, ΔB ἴσον τὸ ἀπὸ τῆς AB · τὸ ἄρα ἀπὸ τῆς ΓB τετράγωνον ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΓA , AB τετραγώνοις καὶ τῷ δις ὑπὸ τῶν ΓA , $A\Delta$ περιεχομένῳ ὀρθογωνίῳ· ὥστε τὸ ἀπὸ τῆς ΓB τετράγωνον τῶν ἀπὸ τῶν ΓA , AB τετραγώνων μεῖζόν ἐστι τῷ δις ὑπὸ τῶν ΓA , $A\Delta$ περιεχομένῳ ὀρθογωνίῳ.

Ἐν ἄρα τοῖς ἀμβλυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ἀμβλείαν γωνίαν ὑποτείνουσας πλευρᾶς τετράγωνον μεῖζόν ἐστι τῶν ἀπὸ τῶν τὴν ἀμβλείαν γωνίαν περιεχουσῶν

Proposition 12†

In obtuse-angled triangles, the square on the side subtending the obtuse angle is greater than the (sum of the) squares on the sides containing the obtuse angle by twice the (rectangle) contained by one of the sides around the obtuse angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off outside (the triangle) by the perpendicular (straight-line) towards the obtuse angle.



Let ABC be an obtuse-angled triangle, having the angle BAC obtuse. And let BD be drawn from point B , perpendicular to CA produced [Prop. 1.12]. I say that the square on BC is greater than the (sum of the) squares on BA and AC , by twice the rectangle contained by CA and AD .

For since the straight-line CD has been cut, at random, at point A , the (square) on DC is thus equal to the (sum of the) squares on CA and AD , and twice the rectangle contained by CA and AD [Prop. 2.4]. Let the (square) on DB have been added to both. Thus, the (sum of the squares) on CD and DB is equal to the (sum of the) squares on CA , AD , and DB , and twice the [rectangle contained] by CA and AD . But, the (square) on CB is equal to the (sum of the squares) on CD and DB . For the angle at D (is) a right-angle [Prop. 1.47]. And the (square) on AB (is) equal to the (sum of the squares) on AD and DB [Prop. 1.47]. Thus, the square on CB is equal to the (sum of the) squares on CA and AB , and twice the rectangle contained by CA and AD . So the square on CB is greater than the (sum of the) squares on

πλευρῶν τετραγώνων τῷ περιχομένῳ δις ὑπὸ τε μιᾶς τῶν περὶ τὴν ἀμβλείαν γωνίαν, ἐφ' ἣν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐκτὸς ὑπὸ τῆς καθέτου πρὸς τῇ ἀμβλείᾳ γωνίᾳ· ὅπερ ἔδει δεῖξαι.

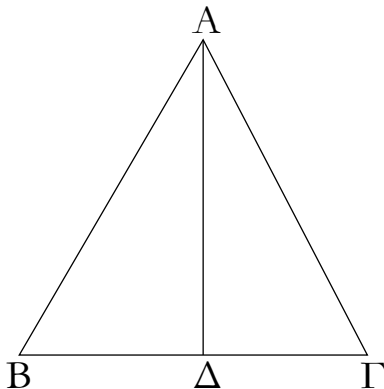
CA and AB by twice the rectangle contained by CA and AD .

Thus, in obtuse-angled triangles, the square on the side subtending the obtuse angle is greater than the (sum of the) squares on the sides containing the obtuse angle by twice the (rectangle) contained by one of the sides around the obtuse angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off outside (the triangle) by the perpendicular (straight-line) towards the obtuse angle. (Which is) the very thing it was required to show.

† This proposition is equivalent to the well-known cosine formula: $BC^2 = AB^2 + AC^2 - 2 AB AC \cos BAC$, since $\cos BAC = -AD/AB$.

ιγ'.

Ἐν τοῖς ὀξυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀξεῖαν γωνίαν ὑποτείνουσας πλευρᾶς τετράγωνον ἑλαττόν ἐστι τῶν ἀπὸ τῶν τὴν ὀξεῖαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιχομένῳ δις ὑπὸ τε μιᾶς τῶν περὶ τὴν ὀξεῖαν γωνίαν, ἐφ' ἣν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐντὸς ὑπὸ τῆς καθέτου πρὸς τῇ ὀξεῖᾳ γωνίᾳ.

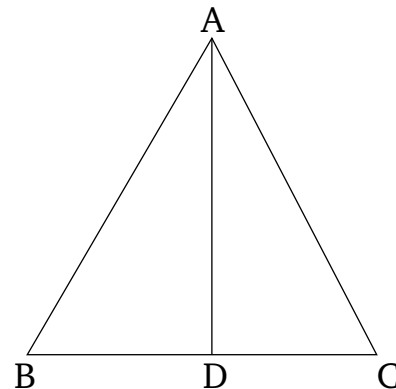


Ἐστω ὀξυγώνιον τρίγωνον τὸ $AB\Gamma$ ὀξεῖαν ἔχον τὴν πρὸς τῷ B γωνίαν, καὶ ἤχθω ἀπὸ τοῦ A σημείου ἐπὶ τὴν $B\Gamma$ κάθετος ἡ AD . λέγω, ὅτι τὸ ἀπὸ τῆς AG τετράγωνον ἑλαττόν ἐστι τῶν ἀπὸ τῶν GB , BA τετραγώνων τῷ δις ὑπὸ τῶν GB , BD περιχομένῳ ὀρθογωνίῳ.

Ἐπεὶ γὰρ εὐθεῖα ἡ GB τέμνεται, ὡς ἔτυχεν, κατὰ τὸ Δ , τὰ ἄρα ἀπὸ τῶν GB , BD τετράγωνα ἴσα ἐστὶ τῷ τε δις ὑπὸ τῶν GB , BD περιχομένῳ ὀρθογωνίῳ καὶ τῷ ἀπὸ τῆς $\Delta\Gamma$ τετραγώνῳ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΔA τετράγωνον· τὰ ἄρα ἀπὸ τῶν GB , BD , ΔA τετράγωνα ἴσα ἐστὶ τῷ τε δις ὑπὸ τῶν GB , BD περιχομένῳ ὀρθογωνίῳ καὶ τοῖς ἀπὸ τῶν ΔA , $\Delta\Gamma$ τετραγώνοις. ἀλλὰ τοῖς μὲν ἀπὸ τῶν BD , ΔA ἴσον τὸ ἀπὸ τῆς AB · ὀρθὴ γὰρ ἡ πρὸς τῷ Δ γωνία· τοῖς δὲ ἀπὸ τῶν ΔA , $\Delta\Gamma$ ἴσον τὸ ἀπὸ τῆς AG · τὰ ἄρα ἀπὸ τῶν GB , BA ἴσα ἐστὶ τῷ τε ἀπὸ τῆς AG καὶ τῷ δις ὑπὸ τῶν GB , BD · ὥστε μόνον τὸ ἀπὸ τῆς AG ἑλαττόν ἐστι

Proposition 13†

In acute-angled triangles, the square on the side subtending the acute angle is less than the (sum of the) squares on the sides containing the acute angle by twice the (rectangle) contained by one of the sides around the acute angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off inside (the triangle) by the perpendicular (straight-line) towards the acute angle.



Let ABC be an acute-angled triangle, having the angle at (point) B acute. And let AD have been drawn from point A , perpendicular to BC [Prop. 1.12]. I say that the square on AC is less than the (sum of the) squares on CB and BA , by twice the rectangle contained by CB and BD .

For since the straight-line CB has been cut, at random, at (point) D , the (sum of the) squares on CB and BD is thus equal to twice the rectangle contained by CB and BD , and the square on DC [Prop. 2.7]. Let the square on DA have been added to both. Thus, the (sum of the) squares on CB , BD , and DA is equal to twice the rectangle contained by CB and BD , and the (sum of the) squares on AD and DC . But, the (square) on AB (is) equal to the (sum of the squares) on BD and DA . For the angle at (point) D is a right-angle [Prop. 1.47].

τῶν ἀπὸ τῶν ΓΒ, ΒΑ τετραγώνων τῷ δις ὑπὸ τῶν ΓΒ, ΒΔ περιεχομένῳ ὀρθογώνιῳ.

Ἐν ἄρα τοῖς ὀξυγωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀξείαν γωνίαν ὑποτείνουσας πλευρᾶς τετραγώνον ἑλαττόν ἐστι τῶν ἀπὸ τῶν τὴν ὀξείαν γωνίαν περιεχουσῶν πλευρῶν τετραγώνων τῷ περιεχομένῳ δις ὑπὸ τε μιᾶς τῶν περὶ τὴν ὀξείαν γωνίαν, ἐφ' ἣν ἡ κάθετος πίπτει, καὶ τῆς ἀπολαμβανομένης ἐντὸς ὑπὸ τῆς καθέτου πρὸς τῇ ὀξείᾳ γωνίᾳ· ὅπερ ἔδει δεῖξαι.

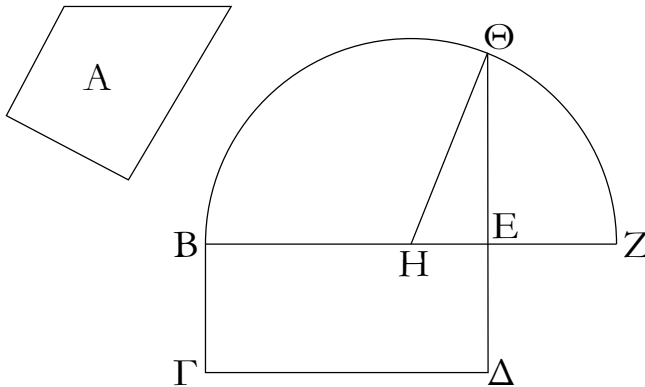
And the (square) on AC (is) equal to the (sum of the squares) on AD and DC [Prop. 1.47]. Thus, the (sum of the squares) on CB and BA is equal to the (square) on AC , and twice the (rectangle contained) by CB and BD . So the (square) on AC alone is less than the (sum of the) squares on CB and BA by twice the rectangle contained by CB and BD .

Thus, in acute-angled triangles, the square on the side subtending the acute angle is less than the (sum of the) squares on the sides containing the acute angle by twice the (rectangle) contained by one of the sides around the acute angle, to which a perpendicular (straight-line) falls, and the (straight-line) cut off inside (the triangle) by the perpendicular (straight-line) towards the acute angle. (Which is) the very thing it was required to show.

† This proposition is equivalent to the well-known cosine formula: $AC^2 = AB^2 + BC^2 - 2 AB BC \cos ABC$, since $\cos ABC = BD/AB$.

ιδ'.

Τῷ δοθέντι εὐθύγραμμῳ ἴσον τετράγωνον συστήσασθαι.



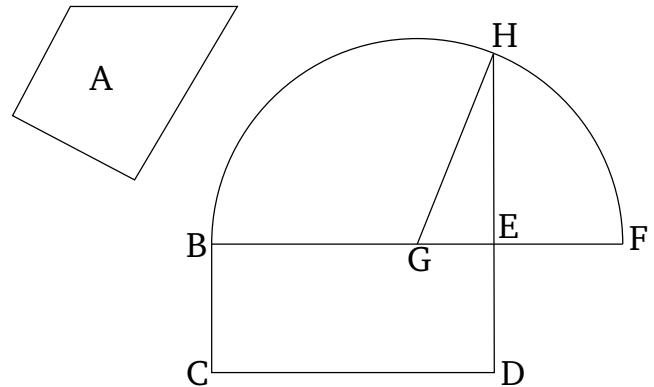
Ἐστω τὸ δοθὲν εὐθύγραμμον τὸ Α· δεῖ δὴ τῷ Α εὐθύγραμμῳ ἴσον τετράγωνον συστήσασθαι.

Συνεστάτω γάρ τῷ Α εὐθύγραμμῳ ἴσον παραλληλόγραμμον ὀρθογώνιον τὸ ΒΔ· εἰ μὲν οὖν ἴση ἐστὶν ἡ ΒΕ τῇ ΕΔ, γεγονός ἂν εἴη τὸ ἐπιταχθέν. συνέσταται γάρ τῷ Α εὐθύγραμμῳ ἴσον τετράγωνον τὸ ΒΔ· εἰ δὲ οὐ, μία τῶν ΒΕ, ΕΔ μείζων ἐστίν. ἔστω μείζων ἡ ΒΕ, καὶ ἐκβεβλήσθω ἐπὶ τὸ Ζ, καὶ κείσθω τῇ ΕΔ ἴση ἡ ΕΖ, καὶ τετμήσθω ἡ ΒΖ δίχα κατὰ τὸ Η, καὶ κέντρῳ τῷ Η, διαστήματι δὲ ἐνὶ τῶν ΗΒ, ΗΖ ἡμικύκλιον γεγράφθω τὸ ΒΘΖ, καὶ ἐκβεβλήσθω ἡ ΔΕ ἐπὶ τὸ Θ, καὶ ἐπεξεύχθω ἡ ΗΘ.

Ἐπεὶ οὖν εὐθεία ἡ ΒΖ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Η, εἰς δὲ ἄνισα κατὰ τὸ Ε, τὸ ἄρα ὑπὸ τῶν ΒΕ, ΕΖ περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΗ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΗΖ τετραγώνῳ. ἴση δὲ ἡ ΗΖ τῇ ΗΘ· τὸ ἄρα ὑπὸ τῶν ΒΕ, ΕΖ μετὰ τοῦ ἀπὸ τῆς ΗΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΗΘ. τῷ δὲ ἀπὸ τῆς ΗΘ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΘΕ, ΕΗ

Proposition 14

To construct a square equal to a given rectilinear figure.



Let A be the given rectilinear figure. So it is required to construct a square equal to the rectilinear figure A .

For let the right-angled parallelogram BD , equal to the rectilinear figure A , have been constructed [Prop. 1.45]. Therefore, if BE is equal to ED then that (which) was prescribed has taken place. For the square BD , equal to the rectilinear figure A , has been constructed. And if not, then one of the (straight-lines) BE or ED is greater (than the other). Let BE be greater, and let it have been produced to F , and let EF be made equal to ED [Prop. 1.3]. And let BF have been cut in half at (point) G [Prop. 1.10]. And, with center G , and radius one of the (straight-lines) GB or GF , let the semi-circle BHF have been drawn. And let DE have been produced to H , and let GH have been joined.

Therefore, since the straight-line BF has been cut—equally at G , and unequally at E —the rectangle con-

τετράγωνα· τὸ ἄρα ὑπὸ τῶν BE , EZ μετὰ τοῦ ἀπὸ HE ἴσα ἐστὶ τοῖς ἀπὸ τῶν $ΘE$, EH . κοινὸν ἀφηρήσθω τὸ ἀπὸ τῆς HE τετράγωνον· λοιπὸν ἄρα τὸ ὑπὸ τῶν BE , EZ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς $EΘ$ τετραγώνῳ. ἀλλὰ τὸ ὑπὸ τῶν BE , EZ τὸ $BΔ$ ἐστίν· ἴση γὰρ ἡ EZ τῇ $ΕΔ$ · τὸ ἄρα $BΔ$ παραλληλόγραμμον ἴσον ἐστὶ τῷ ἀπὸ τῆς $ΘE$ τετραγώνῳ. ἴσον δὲ τὸ $BΔ$ τῷ A εὐθυγράμμῳ. καὶ τὸ A ἄρα εὐθυγράμμον ἴσον ἐστὶ τῷ ἀπὸ τῆς $EΘ$ ἀναγραφησομένῳ τετραγώνῳ.

Τῷ ἄρα δοθέντι εὐθυγράμμῳ τῷ A ἴσον τετράγωνον συνέσταται τὸ ἀπὸ τῆς $EΘ$ ἀναγραφησόμενον· ὅπερ ἔδει ποιῆσαι.

tained by BE and EF , plus the square on EG , is thus equal to the square on GF [Prop. 2.5]. And GF (is) equal to GH . Thus, the (rectangle contained) by BE and EF , plus the (square) on GE , is equal to the (square) on GH . And the (sum of the) squares on HE and EG is equal to the (square) on GH [Prop. 1.47]. Thus, the (rectangle contained) by BE and EF , plus the (square) on GE , is equal to the (sum of the squares) on HE and EG . Let the square on GE have been taken from both. Thus, the remaining rectangle contained by BE and EF is equal to the square on EH . But, BD is the (rectangle contained) by BE and EF . For EF (is) equal to ED . Thus, the parallelogram BD is equal to the square on HE . And BD (is) equal to the rectilinear figure A . Thus, the rectilinear figure A is also equal to the square (which) can be described on EH .

Thus, a square—(namely), that (which) can be described on EH —has been constructed, equal to the given rectilinear figure A . (Which is) the very thing it was required to do.

ELEMENTS BOOK 3

Fundamentals of Plane Geometry Involving Circles

Ὅροι.

α'. Ἰσοὶ κύκλοι εἰσὶν, ὧν αἱ διαμέτροι ἴσαι εἰσὶν, ἢ ὧν αἱ ἐκ τῶν κέντρων ἴσαι εἰσὶν.

β'. Εὐθεῖα κύκλου ἐφάπτεσθαι λέγεται, ἥτις ἀπτομένη τοῦ κύκλου καὶ ἐκβαλλομένη οὐ τέμνει τὸν κύκλον.

γ'. Κύκλοι ἐφάπτεσθαι ἀλλήλων λέγονται οἵτινες ἀπτόμενοι ἀλλήλων οὐ τέμνουσιν ἀλλήλους.

δ'. Ἐν κύκλῳ ἴσον ἀπέχειν ἀπὸ τοῦ κέντρου εὐθεῖαι λέγονται, ὅταν αἱ ἀπὸ τοῦ κέντρου ἐπ' αὐτάς κάθετοι ἀγόμεναι ἴσαι ὦσιν.

ε'. Μείζον δὲ ἀπέχειν λέγεται, ἐφ' ἣν ἡ μείζων κάθετος πίπτει.

ς'. Τμήμα κύκλου ἐστὶ τὸ περιεχόμενον σχῆμα ὑπὸ τε εὐθείας καὶ κύκλου περιφερείας.

ζ'. Τμήματος δὲ γωνία ἐστὶν ἡ περιεχομένη ὑπὸ τε εὐθείας καὶ κύκλου περιφερείας.

η'. Ἐν τμήματι δὲ γωνία ἐστίν, ὅταν ἐπὶ τῆς περιφερείας τοῦ τμήματος ληφθῇ τι σημεῖον καὶ ἀπ' αὐτοῦ ἐπὶ τὰ πέρατα τῆς εὐθείας, ἥ ἐστι βάσις τοῦ τμήματος, ἐπιζευχθῶσιν εὐθεῖαι, ἡ περιεχομένη γωνία ὑπὸ τῶν ἐπιζευχθεισῶν εὐθειῶν.

θ'. Ὅταν δὲ αἱ περιέχουσιν τὴν γωνίαν εὐθεῖαι ἀπολαμβάνωσιν τινὰ περιφέρειαν, ἐπ' ἐκείνης λέγεται βεβηκέναι ἡ γωνία.

ι'. Τομεὺς δὲ κύκλου ἐστίν, ὅταν πρὸς τῷ κέντρῳ τοῦ κύκλου συσταθῇ γωνία, τὸ περιεχόμενον σχῆμα ὑπὸ τε τῶν τὴν γωνίαν περιεχουσῶν εὐθειῶν καὶ τῆς ἀπολαμβανομένης ὑπ' αὐτῶν περιφερείας.

ια'. Ὅμοια τμήματα κύκλων ἐστὶ τὰ δεχόμενα γωνίας ἴσας, ἢ ἐν οἷς αἱ γωνίαι ἴσαι ἀλλήλαις εἰσὶν.

α'.

Τοῦ δοθέντος κύκλου τὸ κέντρον εὑρεῖν.

Ἐστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ· δεῖ δὴ τοῦ ΑΒΓ κύκλου τὸ κέντρον εὑρεῖν.

Διήχθω τις εἰς αὐτόν, ὡς ἔτυχεν, εὐθεῖα ἡ ΑΒ, καὶ τετμήσθω δίχα κατὰ τὸ Δ σημεῖον, καὶ ἀπὸ τοῦ Δ τῇ ΑΒ πρὸς ὀρθὰς ἤχθω ἡ ΔΓ καὶ διήχθω ἐπὶ τὸ Ε, καὶ τετμήσθω ἡ ΓΕ δίχα κατὰ τὸ Ζ· λέγω, ὅτι τὸ Ζ κέντρον ἐστὶ τοῦ ΑΒΓ [κύκλου].

Μὴ γάρ, ἀλλ' εἰ δυνατόν, ἔστω τὸ Η, καὶ ἐπεζεύχθωσαν αἱ ΗΑ, ΗΔ, ΗΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΔΒ, κοινὴ δὲ ἡ ΔΗ, δύο δὲ αἱ ΑΔ, ΔΗ δύο ταῖς ΗΔ, ΔΒ ἴσαι εἰσὶν ἑκατέρᾳ ἑκατέρᾳ· καὶ βάσις ἡ ΗΑ βάσει τῇ ΗΒ ἐστὶν ἴση· ἐκ κέντρου γάρ· γωνία ἄρα ἡ ὑπὸ ΑΔΗ γωνία τῇ ὑπὸ ΗΔΒ ἴση ἐστίν.

Definitions

1. Equal circles are (circles) whose diameters are equal, or whose (distances) from the centers (to the circumferences) are equal (i.e., whose radii are equal).

2. A straight-line said to touch a circle is any (straight-line) which, meeting the circle and being produced, does not cut the circle.

3. Circles said to touch one another are any (circles) which, meeting one another, do not cut one another.

4. In a circle, straight-lines are said to be equally far from the center when the perpendiculars drawn to them from the center are equal.

5. And (that straight-line) is said to be further (from the center) on which the greater perpendicular falls (from the center).

6. A segment of a circle is the figure contained by a straight-line and a circumference of a circle.

7. And the angle of a segment is that contained by a straight-line and a circumference of a circle.

8. And the angle in a segment is the angle contained by the joined straight-lines, when any point is taken on the circumference of a segment, and straight-lines are joined from it to the ends of the straight-line which is the base of the segment.

9. And when the straight-lines containing an angle cut off some circumference, the angle is said to stand upon that (circumference).

10. And a sector of a circle is the figure contained by the straight-lines surrounding an angle, and the circumference cut off by them, when the angle is constructed at the center of a circle.

11. Similar segments of circles are those accepting equal angles, or in which the angles are equal to one another.

Proposition 1

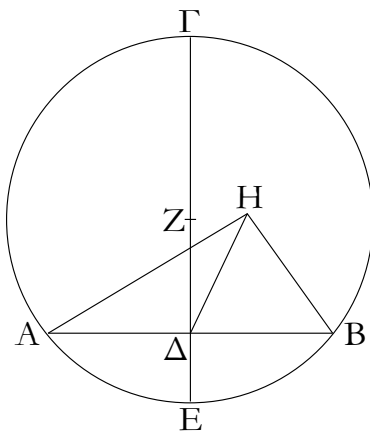
To find the center of a given circle.

Let ABC be the given circle. So it is required to find the center of circle ABC .

Let some straight-line AB have been drawn through (ABC), at random, and let (AB) have been cut in half at point D [Prop. 1.9]. And let DC have been drawn from D , at right-angles to AB [Prop. 1.11]. And let (CD) have been drawn through to E . And let CE have been cut in half at F [Prop. 1.9]. I say that (point) F is the center of the [circle] ABC .

For (if) not then, if possible, let G (be the center of the circle), and let GA , GD , and GB have been joined. And since AD is equal to DB , and DG (is) common, the two

ὅταν δὲ εὐθεῖα ἐπ' εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρω τῶν ἴσων γωνιῶν ἐστίν· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ $H\Delta B$. ἐστὶ δὲ καὶ ἡ ὑπὸ $Z\Delta B$ ὀρθή· ἴση ἄρα ἡ ὑπὸ $Z\Delta B$ τῇ ὑπὸ $H\Delta B$, ἡ μείζων τῇ ἐλάττω· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ H κέντρον ἐστὶ τοῦ $AB\Gamma$ κύκλου. ὁμοίως δὲ δείξομεν, ὅτι οὐδ' ἄλλο τι πλὴν τοῦ Z .



Τὸ Z ἄρα σημεῖον κέντρον ἐστὶ τοῦ $AB\Gamma$ [κύκλου].

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν ἐν κύκλῳ εὐθεῖα τις εὐθεῖαν τινὰ δίχα καὶ πρὸς ὀρθὰς τέμνῃ, ἐπὶ τῆς τεμνούσης ἐστὶ τὸ κέντρον τοῦ κύκλου. — ὅπερ ἔδει ποιῆσαι.

† The Greek text has " GD, DB ", which is obviously a mistake.

β'.

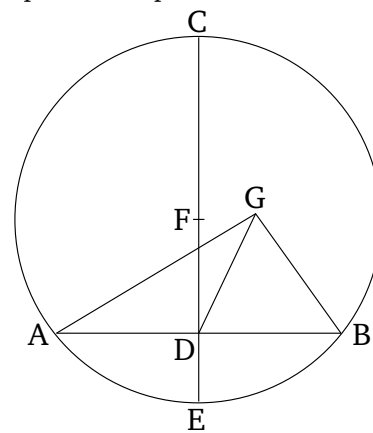
Ἐὰν κύκλου ἐπὶ τῆς περιφερείας ληφθῇ δύο τυχόντα σημεία, ἡ ἐπὶ τὰ σημεία ἐπιζευγνυμένη εὐθεῖα ἐντὸς πεσεῖται τοῦ κύκλου.

Ἐστω κύκλος ὁ $AB\Gamma$, καὶ ἐπὶ τῆς περιφερείας αὐτοῦ εἰληφθῶ δύο τυχόντα σημεία τὰ A, B · λέγω, ὅτι ἡ ἀπὸ τοῦ A ἐπὶ τὸ B ἐπιζευγνυμένη εὐθεῖα ἐντὸς πεσεῖται τοῦ κύκλου.

Μὴ γάρ, ἀλλ' εἰ δυνατόν, πιπτέτω ἐκτὸς ὡς ἡ AEB , καὶ εἰληφθῶ τὸ κέντρον τοῦ $AB\Gamma$ κύκλου, καὶ ἔστω τὸ Δ , καὶ ἐπεζεύχθωσαν αἱ $\Delta A, \Delta B$, καὶ διήχθω ἡ ΔZE .

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΔA τῇ ΔB , ἴση ἄρα καὶ γωνία ἡ ὑπὸ ΔAE τῇ ὑπὸ ΔBE · καὶ ἐπεὶ τριγώνου τοῦ ΔAE μία

(straight-lines) AD, DG are equal to the two (straight-lines) BD, DG ,[†] respectively. And the base GA is equal to the base GB . For (they are both) radii. Thus, angle ADG is equal to angle GDB [Prop. 1.8]. And when a straight-line stood upon (another) straight-line make adjacent angles (which are) equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, GDB is a right-angle. And FDB is also a right-angle. Thus, FDB (is) equal to GDB , the greater to the lesser. The very thing is impossible. Thus, (point) G is not the center of the circle ABC . So, similarly, we can show that neither is any other (point) except F .



Thus, point F is the center of the [circle] ABC .

Corollary

So, from this, (it is) manifest that if any straight-line in a circle cuts any (other) straight-line in half, and at right-angles, then the center of the circle is on the former (straight-line). — (Which is) the very thing it was required to do.

Proposition 2

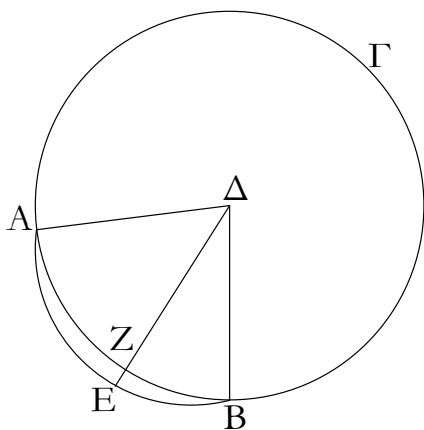
If two points are taken at random on the circumference of a circle then the straight-line joining the points will fall inside the circle.

Let ABC be a circle, and let two points A and B have been taken at random on its circumference. I say that the straight-line joining A to B will fall inside the circle.

For (if) not then, if possible, let it fall outside (the circle), like AEB (in the figure). And let the center of the circle ABC have been found [Prop. 3.1], and let it be (at point) D . And let DA and DB have been joined, and let DFE have been drawn through.

Therefore, since DA is equal to DB , the angle DAE

πλευρὰ προσεχβέβληται ἡ AEB , μείζων ἄρα ἡ ὑπὸ ΔEB γωνία τῆς ὑπὸ ΔAE . ἴση δὲ ἡ ὑπὸ ΔAE τῇ ὑπὸ ΔBE · μείζων ἄρα ἡ ὑπὸ ΔEB τῆς ὑπὸ ΔBE . ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει· μείζων ἄρα ἡ ΔB τῆς ΔE . ἴση δὲ ἡ ΔB τῇ ΔZ . μείζων ἄρα ἡ ΔZ τῆς ΔE ἢ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ A ἐπὶ τὸ B ἐπιζευγνυμένη εὐθεῖα ἐκτὸς πεσεῖται τοῦ κύκλου. ὁμοίως δὲ δείξομεν, ὅτι οὐδὲ ἐπ' αὐτῆς τῆς περιφερείας· ἐντὸς ἄρα.



Ἐάν ἄρα κύκλου ἐπὶ τῆς περιφερείας ληφθῇ δύο τυχόντα σημεία, ἡ ἐπὶ τὰ σημεία ἐπιζευγνυμένη εὐθεῖα ἐντὸς πεσεῖται τοῦ κύκλου· ὅπερ ἔδει δείξαι.

γ'.

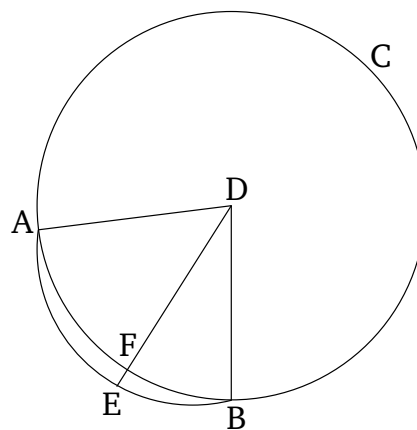
Ἐάν ἐν κύκλῳ εὐθεῖα τις διὰ τοῦ κέντρου εὐθεῖαν τινὰ μὴ διὰ τοῦ κέντρου δίχα τέμνῃ, καὶ πρὸς ὀρθὰς αὐτὴν τέμνῃ· καὶ ἐάν πρὸς ὀρθὰς αὐτὴν τέμνῃ, καὶ δίχα αὐτὴν τέμνῃ.

Ἐστω κύκλος ὁ $AB\Gamma$, καὶ ἐν αὐτῷ εὐθεῖα τις διὰ τοῦ κέντρου ἡ $\Gamma\Delta$ εὐθεῖαν τινὰ μὴ διὰ τοῦ κέντρου τὴν AB δίχα τεμνέτω κατὰ τὸ Z σημεῖον· λέγω, ὅτι καὶ πρὸς ὀρθὰς αὐτὴν τέμνει.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ $AB\Gamma$ κύκλου, καὶ ἔστω τὸ E , καὶ ἐπεξεύχθωσαν αἱ EA , EB .

Καὶ ἐπεὶ ἴση ἐστὶν ἡ AZ τῇ ZB , κοινὴ δὲ ἡ ZE , δύο δυσὶν ἴσαι [εἰσὶν]· καὶ βάσις ἡ EA βάσει τῇ EB ἴση· γωνία ἄρα ἡ ὑπὸ AZE γωνίᾳ τῇ ὑπὸ BZE ἴση ἐστίν. ὅταν δὲ εὐθεῖα ἐπ' εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρω τῶν ἴσων γωνιῶν ἐστίν· ἑκατέρω ἄρα τῶν ὑπὸ AZE , BZE ὀρθὴ ἐστίν. ἡ $\Gamma\Delta$ ἄρα διὰ τοῦ κέντρου οὕσα τὴν AB μὴ διὰ τοῦ κέντρου οὕσαν δίχα τέμνουσα καὶ πρὸς ὀρθὰς τέμνει.

(is) thus also equal to DBE [Prop. 1.5]. And since in triangle DAE the one side, AEB , has been produced, angle DEB (is) thus greater than DAE [Prop. 1.16]. And DAE (is) equal to DBE [Prop. 1.5]. Thus, DEB (is) greater than DBE . And the greater angle is subtended by the greater side [Prop. 1.19]. Thus, DB (is) greater than DE . And DB (is) equal to DF . Thus, DF (is) greater than DE , the lesser than the greater. The very thing is impossible. Thus, the straight-line joining A to B will not fall outside the circle. So, similarly, we can show that neither (will it fall) on the circumference itself. Thus, (it will fall) inside (the circle).



Thus, if two points are taken at random on the circumference of a circle then the straight-line joining the points will fall inside the circle. (Which is) the very thing it was required to show.

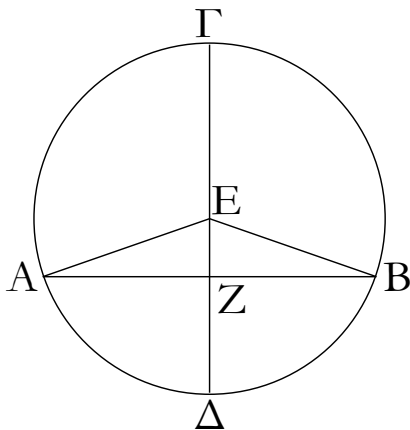
Proposition 3

In a circle, if any straight-line through the center cuts in half any straight-line not through the center then it also cuts it at right-angles. And (conversely) if it cuts it at right-angles then it also cuts it in half.

Let ABC be a circle, and, within it, let some straight-line through the center, CD , cut in half some straight-line not through the center, AB , at the point F . I say that (CD) also cuts (AB) at right-angles.

For let the center of the circle ABC have been found [Prop. 3.1], and let it be (at point) E , and let EA and EB have been joined.

And since AF is equal to FB , and FE (is) common, two (sides of triangle AFE) [are] equal to two (sides of triangle BFE). And the base EA (is) equal to the base EB . Thus, angle AFE is equal to angle BFE [Prop. 1.8]. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle [Def. 1.10]. Thus, AFE and BFE are each right-angles. Thus, the



Ἀλλὰ δὴ ἡ ΓΔ τὴν ΑΒ πρὸς ὀρθὰς τεμνέτω· λέγω, ὅτι καὶ δίχα αὐτὴν τέμνει, τουτέστιν, ὅτι ἴση ἐστὶν ἡ ΑΖ τῇ ΖΒ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἴση ἐστὶν ἡ ΕΑ τῇ ΕΒ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΕΑΖ τῇ ὑπὸ ΕΒΖ. ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ ΑΖΕ ὀρθὴ τῇ ὑπὸ ΒΖΕ ἴση· δύο ἄρα τρίγωνά ἐστι ΕΑΖ, ΕΒΖ τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην κοινὴν αὐτῶν τὴν ΕΖ ὑποτείνουσιν ὑπὸ μίαν τῶν ἴσων γωνιῶν· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξει· ἴση ἄρα ἡ ΑΖ τῇ ΖΒ.

Ἐάν ἄρα ἐν κύκλῳ εὐθεΐα τις διὰ τοῦ κέντρου εὐθεϊάν τινα μὴ διὰ τοῦ κέντρου δίχα τέμνη, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· καὶ ἐάν πρὸς ὀρθὰς αὐτὴν τέμνη, καὶ δίχα αὐτὴν τέμνει· ὅπερ ἔδει δεῖξαι.

δ'.

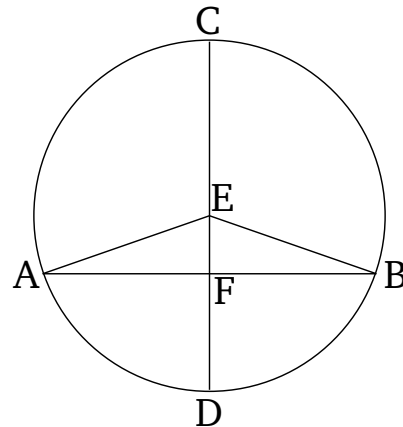
Ἐάν ἐν κύκλῳ δύο εὐθεΐαι τέμνωσιν ἀλλήλας μὴ διὰ τοῦ κέντρου οὔσαι, οὐ τέμνουσιν ἀλλήλας δίχα.

Ἐστω κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ δύο εὐθεΐαι αἱ ΑΓ, ΒΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε μὴ διὰ τοῦ κέντρου οὔσαι· λέγω, ὅτι οὐ τέμνουσιν ἀλλήλας δίχα.

Εἰ γὰρ δυνατόν, τεμνέτωσαν ἀλλήλας δίχα ὥστε ἴσην εἶναι τὴν μὲν ΑΕ τῇ ΕΓ, τὴν δὲ ΒΕ τῇ ΕΔ· καὶ εἰληφθῶ τὸ κέντρον τοῦ ΑΒΓΔ κύκλου, καὶ ἔστω τὸ Ζ, καὶ ἐπεζεύχθω ἡ ΖΕ.

Ἐπεὶ οὖν εὐθεΐα τις διὰ τοῦ κέντρου ἡ ΖΕ εὐθεϊάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΓ δίχα τέμνει, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΖΕΑ· πάλιν, ἐπεὶ εὐθεΐα τις ἡ ΖΕ εὐθεϊάν τινα τὴν ΒΔ δίχα τέμνει, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει· ὀρθὴ ἄρα ἡ ὑπὸ ΖΕΒ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΖΕΑ ὀρθή· ἴση ἄρα ἡ ὑπὸ ΖΕΑ τῇ ὑπὸ ΖΕΒ ἡ ἐλάττων τῇ

(straight-line) CD , which is through the center and cuts in half the (straight-line) AB , which is not through the center, also cuts (AB) at right-angles.



And so let CD cut AB at right-angles. I say that it also cuts (AB) in half. That is to say, that AF is equal to FB .

For, with the same construction, since EA is equal to EB , angle EAF is also equal to EBF [Prop. 1.5]. And the right-angle AFE is also equal to the right-angle BFE . Thus, EAF and EFB are two triangles having two angles equal to two angles, and one side equal to one side—(namely), their common (side) EF , subtending one of the equal angles. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, AF (is) equal to FB .

Thus, in a circle, if any straight-line through the center cuts in half any straight-line not through the center then it also cuts it at right-angles. And (conversely) if it cuts it at right-angles then it also cuts it in half. (Which is) the very thing it was required to show.

Proposition 4

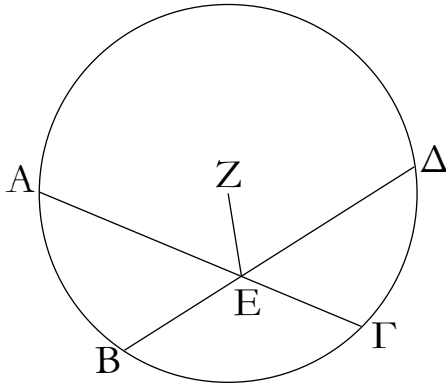
In a circle, if two straight-lines, which are not through the center, cut one another then they do not cut one another in half.

Let $ABCD$ be a circle, and within it, let two straight-lines, AC and BD , which are not through the center, cut one another at (point) E . I say that they do not cut one another in half.

For, if possible, let them cut one another in half, such that AE is equal to EC , and BE to ED . And let the center of the circle $ABCD$ have been found [Prop. 3.1], and let it be (at point) F , and let FE have been joined.

Therefore, since some straight-line through the center, FE , cuts in half some straight-line not through the center, AC , it also cuts it at right-angles [Prop. 3.3]. Thus, FEA is a right-angle. Again, since some straight-line FE

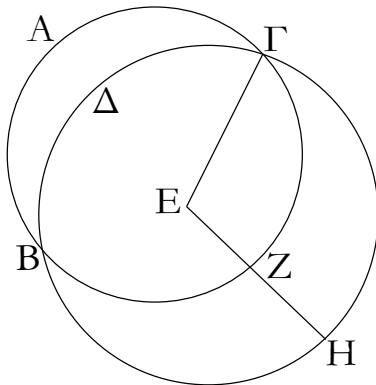
μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα αἱ $ΑΓ$, $ΒΔ$ τέμνουσιν ἀλλήλας δίχα.



Ἐὰν ἄρα ἐν κύκλῳ δύο εὐθεῖαι τέμνωσιν ἀλλήλας μὴ διὰ τοῦ κέντρου οὐσαι, οὐ τέμνουσιν ἀλλήλας δίχα· ὅπερ ἔδει δεῖξαι.

ε'.

Ἐὰν δύο κύκλοι τέμνωσιν ἀλλήλους, οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.

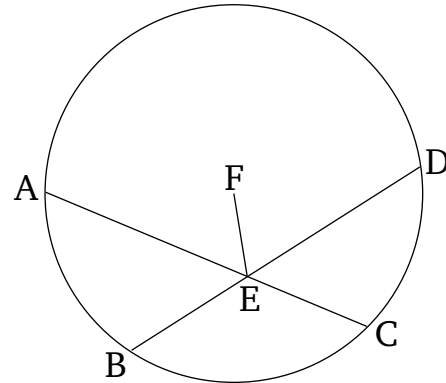


Δύο γὰρ κύκλοι οἱ $ΑΒΓ$, $ΓΔΗ$ τεμνέτωσαν ἀλλήλους κατὰ τὰ $Β$, $Γ$ σημεία. λέγω, ὅτι οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.

Εἰ γὰρ δυνατόν, ἔστω τὸ $Ε$, καὶ ἐπεξεύχθω ἡ $ΕΓ$, καὶ διήχθω ἡ $ΕΖΗ$, ὡς ἔτυχεν. καὶ ἐπεὶ τὸ $Ε$ σημεῖον κέντρον ἐστὶ τοῦ $ΑΒΓ$ κύκλου, ἴση ἐστὶν ἡ $ΕΓ$ τῇ $ΕΖ$. πάλιν, ἐπεὶ τὸ $Ε$ σημεῖον κέντρον ἐστὶ τοῦ $ΓΔΗ$ κύκλου, ἴση ἐστὶν ἡ $ΕΓ$ τῇ $ΕΗ$. ἐδείχθη δὲ ἡ $ΕΓ$ καὶ τῇ $ΕΖ$ ἴση· καὶ ἡ $ΕΖ$ ἄρα τῇ $ΕΗ$ ἐστὶν ἴση ἢ ἐλάσσων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ $Ε$ σημεῖον κέντρον ἐστὶ τῶν $ΑΒΓ$, $ΓΔΗ$ κύκλων.

Ἐὰν ἄρα δύο κύκλοι τέμνωσιν ἀλλήλους, οὐκ ἔστιν

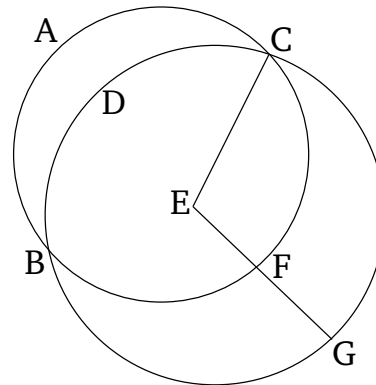
cuts in half some straight-line BD , it also cuts it at right-angles [Prop. 3.3]. Thus, FEB (is) a right-angle. But FEA was also shown (to be) a right-angle. Thus, FEA (is) equal to FEB , the lesser to the greater. The very thing is impossible. Thus, AC and BD do not cut one another in half.



Thus, in a circle, if two straight-lines, which are not through the center, cut one another then they do not cut one another in half. (Which is) the very thing it was required to show.

Proposition 5

If two circles cut one another then they will not have the same center.



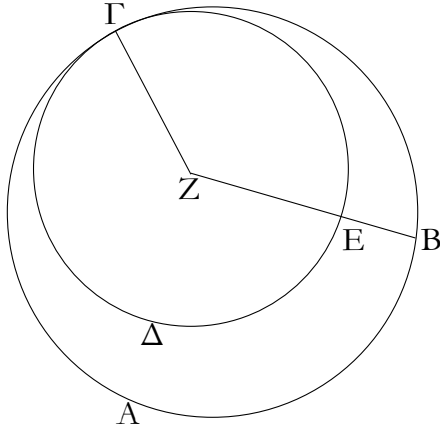
For let the two circles ABC and CDG cut one another at points B and C . I say that they will not have the same center.

For, if possible, let E be (the common center), and let EC have been joined, and let EFG have been drawn through (the two circles), at random. And since point E is the center of the circle ABC , EC is equal to EF . Again, since point E is the center of the circle CDG , EC is equal to EG . But EC was also shown (to be) equal to EF . Thus, EF is also equal to EG , the lesser to the greater. The very thing is impossible. Thus, point E is not

αὐτῶν τὸ αὐτὸ κέντρον· ὅπερ ἔδει δεῖξαι.

ϛ'.

Ἐάν δύο κύκλοι ἐφάπτωνται ἀλλήλων, οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.



Δύο γὰρ κύκλοι οἱ ΑΒΓ, ΓΔΕ ἐφαπτέσθωσαν ἀλλήλων κατὰ τὸ Γ σημεῖον· λέγω, ὅτι οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον.

Εἰ γὰρ δυνατόν, ἔστω τὸ Ζ, καὶ ἐπεζεύχθω ἡ ΖΓ, καὶ διήχθω, ὡς ἔτυχεν, ἡ ΖΕΒ.

Ἐπεὶ οὖν τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου, ἴση ἐστὶν ἡ ΖΓ τῇ ΖΒ. πάλιν, ἐπεὶ τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΓΔΕ κύκλου, ἴση ἐστὶν ἡ ΖΓ τῇ ΖΕ. ἐδείχθη δὲ ἡ ΖΓ τῇ ΖΒ ἴση· καὶ ἡ ΖΕ ἄρα τῇ ΖΒ ἐστὶν ἴση, ἡ ἐλάττω τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ Ζ σημεῖον κέντρον ἐστὶ τῶν ΑΒΓ, ΓΔΕ κύκλων.

Ἐάν ἄρα δύο κύκλοι ἐφάπτωνται ἀλλήλων, οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον· ὅπερ ἔδει δεῖξαι.

ζ'.

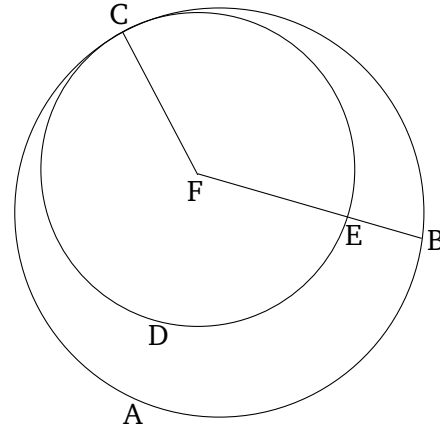
Ἐάν κύκλου ἐπὶ τῆς διαμέτρου ληφθῇ τι σημεῖον, ὃ μὴ ἐστὶ κέντρον τοῦ κύκλου, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσιν εὐθεῖαι τινες, μεγίστη μὲν ἔσται, ἐφ' ἧς τὸ κέντρον, ἐλάχιστη δὲ ἡ λοιπή, τῶν δὲ ἄλλων αἰεὶ ἡ ἑγγιον τῆς διὰ τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ' ἑκάτερα τῆς ἐλαχίστης.

the (common) center of the circles ABC and CDG .

Thus, if two circles cut one another then they will not have the same center. (Which is) the very thing it was required to show.

Proposition 6

If two circles touch one another then they will not have the same center.



For let the two circles ABC and CDE touch one another at point C . I say that they will not have the same center.

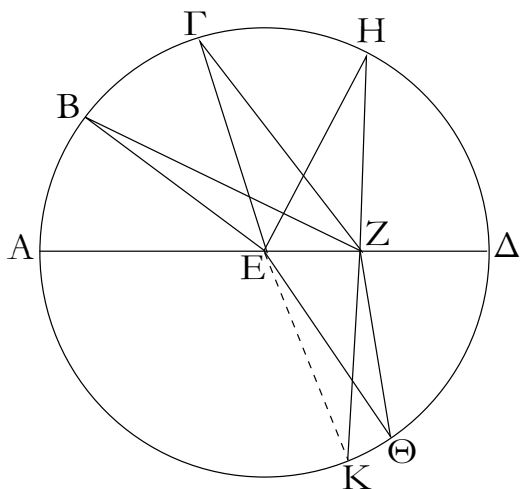
For, if possible, let F be (the common center), and let FC have been joined, and let FEB have been drawn through (the two circles), at random.

Therefore, since point F is the center of the circle ABC , FC is equal to FB . Again, since point F is the center of the circle CDE , FC is equal to FE . But FC was shown (to be) equal to FB . Thus, FE is also equal to FB , the lesser to the greater. The very thing is impossible. Thus, point F is not the (common) center of the circles ABC and CDE .

Thus, if two circles touch one another then they will not have the same center. (Which is) the very thing it was required to show.

Proposition 7

If some point, which is not the center of the circle, is taken on the diameter of a circle, and some straight-lines radiate from the point towards the (circumference of the) circle, then the greatest (straight-line) will be that on which the center (lies), and the least the remainder (of the same diameter). And for the others, a (straight-line) nearer[†] to the (straight-line) through the center is always greater than a (straight-line) further away. And only two equal (straight-lines) will radiate from the point towards the (circumference of the) circle, (one) on each



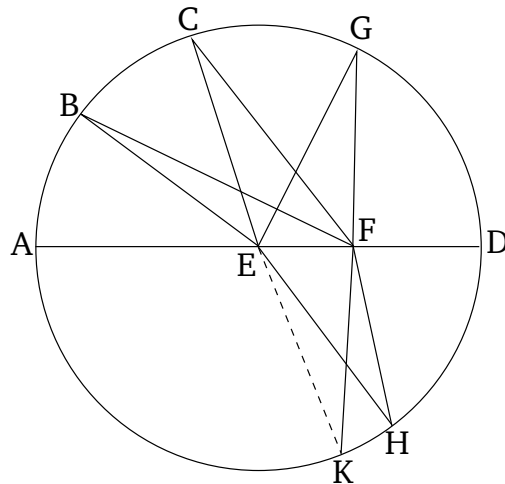
Ἐστω κύκλος ὁ ΑΒΓΔ, διάμετρος δὲ αὐτοῦ ἔστω ἡ ΑΔ, καὶ ἐπὶ τῆς ΑΔ εἰλήφθω τι σημεῖον τὸ Ζ, ὃ μὴ ἔστι κέντρον τοῦ κύκλου, κέντρον δὲ τοῦ κύκλου ἔστω τὸ Ε, καὶ ἀπὸ τοῦ Ζ πρὸς τὸν ΑΒΓΔ κύκλον προσπιπτέωσαν εὐθεῖαι τινες αἱ ΖΒ, ΖΓ, ΖΗ· λέγω, ὅτι μεγίστη μὲν ἔστιν ἡ ΖΑ, ἐλάχιστη δὲ ἡ ΖΔ, τῶν δὲ ἄλλων ἡ μὲν ΖΒ τῆς ΖΓ μείζων, ἡ δὲ ΖΓ τῆς ΖΗ.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΕ, ΓΕ, ΗΕ. καὶ ἐπεὶ παντὸς τριγώνου αἱ δύο πλευραὶ τῆς λοιπῆς μείζονες εἰσιν, αἱ ἄρα ΕΒ, ΕΖ τῆς ΒΖ μείζονες εἰσιν. ἴση δὲ ἡ ΑΕ τῇ ΒΕ [αἱ ἄρα ΒΕ, ΕΖ ἴσαι εἰσὶ τῇ ΑΖ]· μείζων ἄρα ἡ ΑΖ τῆς ΒΖ. πάλιν, ἐπεὶ ἴση ἔστιν ἡ ΒΕ τῇ ΓΕ, κοινὴ δὲ ἡ ΖΕ, δύο δὲ αἱ ΒΕ, ΕΖ δυσὶ ταῖς ΓΕ, ΕΖ ἴσαι εἰσιν. ἀλλὰ καὶ γωνία ἡ ὑπὸ ΒΕΖ γωνίας τῆς ὑπὸ ΓΕΖ μείζων· βάσις ἄρα ἡ ΒΖ βάσεως τῆς ΓΖ μείζων ἔστιν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΖ τῆς ΖΗ μείζων ἔστιν.

Πάλιν, ἐπεὶ αἱ ΗΖ, ΖΕ τῆς ΕΗ μείζονες εἰσιν, ἴση δὲ ἡ ΕΗ τῇ ΕΔ, αἱ ἄρα ΗΖ, ΖΕ τῆς ΕΔ μείζονες εἰσιν. κοινὴ ἄφρησθω ἡ ΕΖ· λοιπὴ ἄρα ἡ ΗΖ λοιπῆς τῆς ΖΔ μείζων ἔστιν. μεγίστη μὲν ἄρα ἡ ΖΑ, ἐλάχιστη δὲ ἡ ΖΔ, μείζων δὲ ἡ μὲν ΖΒ τῆς ΖΓ, ἡ δὲ ΖΓ τῆς ΖΗ.

Λέγω, ὅτι καὶ ἀπὸ τοῦ Ζ σημείου δύο μόνον ἴσαι προσπεσοῦνται πρὸς τὸν ΑΒΓΔ κύκλον ἐφ' ἑκάτερα τῆς ΖΔ ἐλάχιστης. συνεστάτω γὰρ πρὸς τῇ ΕΖ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Ε τῇ ὑπὸ ΗΕΖ γωνίᾳ ἴση ἡ ὑπὸ ΖΕΘ, καὶ ἐπεζεύχθω ἡ ΖΘ. ἐπεὶ οὖν ἴση ἔστιν ἡ ΗΕ τῇ ΕΘ, κοινὴ δὲ ἡ ΕΖ, δύο δὲ αἱ ΗΕ, ΕΖ δυσὶ ταῖς ΘΕ, ΕΖ ἴσαι εἰσιν· καὶ γωνία ἡ ὑπὸ ΗΕΖ γωνία τῇ ὑπὸ ΘΕΖ ἴση· βάσις ἄρα ἡ ΖΗ βάσει τῇ ΖΘ ἴση ἔστιν. λέγω δὴ, ὅτι τῇ ΖΗ ἄλλη ἴση οὐ προσπεσεῖται πρὸς τὸν κύκλον ἀπὸ τοῦ Ζ σημείου. εἰ γὰρ δυνατόν, προσπιπτέτω ἡ ΖΚ. καὶ ἐπεὶ ἡ ΖΚ τῇ ΖΗ ἴση ἔστιν, ἀλλὰ ἡ ΖΘ τῇ ΖΗ [ἴση ἔστιν], καὶ ἡ ΖΚ ἄρα τῇ ΖΘ ἔστιν ἴση, ἡ ἑγγιον τῆς διὰ τοῦ κέντρου τῇ ἀπώτερον ἴση· ὅπερ ἀδύνατον. οὐκ ἄρα ἀπὸ τοῦ Ζ σημείου ἑτέρα τις

(side) of the least (straight-line).



Let $ABCD$ be a circle, and let AD be its diameter, and let some point F , which is not the center of the circle, have been taken on AD . Let E be the center of the circle. And let some straight-lines, FB , FC , and FG , radiate from F towards (the circumference of) circle $ABCD$. I say that FA is the greatest (straight-line), FD the least, and of the others, FB (is) greater than FC , and FC than FG .

For let BE , CE , and GE have been joined. And since for every triangle (any) two sides are greater than the remaining (side) [Prop. 1.20], EB and EF is thus greater than BF . And AE (is) equal to BE [thus, BE and EF is equal to AF]. Thus, AF (is) greater than BF . Again, since BE is equal to CE , and FE (is) common, the two (straight-lines) BE , EF are equal to the two (straight-lines) CE , EF (respectively). But, angle BEF (is) also greater than angle CEF .[‡] Thus, the base BF is greater than the base CF . Thus, the base BF is greater than the base CF [Prop. 1.24]. So, for the same (reasons), CF is also greater than FG .

Again, since GF and FE are greater than EG [Prop. 1.20], and EG (is) equal to ED , GF and FE are thus greater than ED . Let EF have been taken from both. Thus, the remainder GF is greater than the remainder FD . Thus, FA (is) the greatest (straight-line), FD the least, and FB (is) greater than FC , and FC than FG .

I also say that from point F only two equal (straight-lines) will radiate towards (the circumference of) circle $ABCD$, (one) on each (side) of the least (straight-line) FD . For let the (angle) FEH , equal to angle GEF , have been constructed on the straight-line EF , at the point E on it [Prop. 1.23], and let FH have been joined. Therefore, since GE is equal to EH , and EF (is) common,

προσπεσεῖται πρὸς τὸν κύκλον ἴση τῇ HZ : μία ἄρα μόνη.

Ἐάν ἄρα κύκλου ἐπὶ τῆς διαμέτρου ληφθῇ τι σημεῖον, ὃ μὴ ἐστὶ κέντρον τοῦ κύκλου, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσιν εὐθεῖαί τινες, μεγίστη μὲν ἔσται, ἐφ' ἧς τὸ κέντρον, ἐλαχίστη δὲ ἡ λοιπή, τῶν δὲ ἄλλων αἰεὶ ἡ ἔγγιον τῆς διὰ τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ αὐτοῦ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ' ἑκάτερα τῆς ἐλαχίστης· ὅπερ ἔδει δεῖξαι.

the two (straight-lines) GE , EF are equal to the two (straight-lines) HE , EF (respectively). And angle GEF (is) equal to angle HEF . Thus, the base FG is equal to the base FH [Prop. 1.4]. So I say that another (straight-line) equal to FG will not radiate towards (the circumference of) the circle from point F . For, if possible, let FK (so) radiate. And since FK is equal to FG , but FH [is equal] to FG , FK is thus also equal to FH , the nearer to the (straight-line) through the center equal to the further away. The very thing (is) impossible. Thus, another (straight-line) equal to GF will not radiate from the point F towards (the circumference of) the circle. Thus, (there is) only one (such straight-line).

Thus, if some point, which is not the center of the circle, is taken on the diameter of a circle, and some straight-lines radiate from the point towards the (circumference of the) circle, then the greatest (straight-line) will be that on which the center (lies), and the least the remainder (of the same diameter). And for the others, a (straight-line) nearer to the (straight-line) through the center is always greater than a (straight-line) further away. And only two equal (straight-lines) will radiate from the same point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line). (Which is) the very thing it was required to show.

† Presumably, in an angular sense.

‡ This is not proved, except by reference to the figure.

η'.

Ἐάν κύκλου ληφθῇ τι σημεῖον ἐκτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον διαχθῶσιν εὐθεῖαί τινες, ὧν μία μὲν διὰ τοῦ κέντρου, αἱ δὲ λοιπαί, ὡς ἔτυχεν, τῶν μὲν πρὸς τὴν κοίλην περιφέρειαν προσπιπτουσῶν εὐθειῶν μεγίστη μὲν ἐστὶν ἡ διὰ τοῦ κέντρου, τῶν δὲ ἄλλων αἰεὶ ἡ ἔγγιον τῆς διὰ τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, τῶν δὲ πρὸς τὴν κυρτὴν περιφέρειαν προσπιπτουσῶν εὐθειῶν ἐλαχίστη μὲν ἐστὶν ἡ μεταξὺ τοῦ τε σημείου καὶ τῆς διαμέτρου, τῶν δὲ ἄλλων αἰεὶ ἡ ἔγγιον τῆς ἐλαχίστης τῆς ἀπώτερόν ἐστιν ἐλάττων, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ' ἑκάτερα τῆς ἐλαχίστης.

Ἦστω κύκλος ὁ $ABΓ$, καὶ τοῦ $ABΓ$ εἰληφθῶ τι σημεῖον ἐκτός τὸ Δ , καὶ ἀπ' αὐτοῦ διήχθωσαν εὐθεῖαί τινες αἱ ΔA , ΔE , ΔZ , $\Delta Γ$, ἔστω δὲ ἡ ΔA διὰ τοῦ κέντρου. λέγω, ὅτι τῶν μὲν πρὸς τὴν $AEZΓ$ κοίλην περιφέρειαν προσπιπτουσῶν εὐθειῶν μεγίστη μὲν ἐστὶν ἡ διὰ τοῦ κέντρου ἡ ΔA , μείζων δὲ ἡ μὲν ΔE τῆς ΔZ ἢ δὲ ΔZ τῆς $\Delta Γ$, τῶν δὲ πρὸς τὴν $\Theta A K H$ κυρτὴν περιφέρειαν προσπιπτουσῶν εὐθειῶν ἐλαχίστη μὲν ἐστὶν ἡ ΔH ἢ μεταξὺ τοῦ σημείου καὶ τῆς διαμέτρου τῆς AH , αἰεὶ δὲ ἡ ἔγγιον τῆς ΔH ἐλαχίστης ἐλάττων ἐστὶ τῆς ἀπώτερον, ἡ μὲν ΔK τῆς $\Delta \Lambda$, ἡ δὲ $\Delta \Lambda$

Proposition 8

If some point is taken outside a circle, and some straight-lines are drawn from the point to the (circumference of the) circle, one of which (passes) through the center, the remainder (being) random, then for the straight-lines radiating towards the concave (part of the) circumference, the greatest is that (passing) through the center. For the others, a (straight-line) nearer[†] to the (straight-line) through the center is always greater than one further away. For the straight-lines radiating towards the convex (part of the) circumference, the least is that between the point and the diameter. For the others, a (straight-line) nearer to the least (straight-line) is always less than one further away. And only two equal (straight-lines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line).

Let ABC be a circle, and let some point D have been taken outside ABC , and from it let some straight-lines, DA , DE , DF , and DC , have been drawn through (the circle), and let DA be through the center. I say that for the straight-lines radiating towards the concave (part of

ἴσαι εἰσὶν ἑκατέρα ἑκατέρῃ· καὶ γωνία ἡ ὑπὸ $KM\Delta$ γωνία τῇ ὑπὸ $BM\Delta$ ἴση· βάσις ἄρα ἡ ΔK βάσει τῇ ΔB ἴση ἐστίν. λέγω $[\delta\eta]$, ὅτι τῇ ΔK εὐθείᾳ ἄλλη ἴση οὐ προσπεσεῖται πρὸς τὸν κύκλον ἀπὸ τοῦ Δ σημείου. εἰ γὰρ δυνατόν, προσπιπτέτω καὶ ἔστω ἡ ΔN . ἐπεὶ οὖν ἡ ΔK τῇ ΔN ἐστὶν ἴση, ἀλλ' ἡ ΔK τῇ ΔB ἐστὶν ἴση, καὶ ἡ ΔB ἄρα τῇ ΔN ἐστὶν ἴση, ἡ ἑγγιον τῆς ΔH ἐλαχίστης τῇ ἀπώτερον $[\epsilon\sigma\tau\iota\nu]$ ἴση· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα πλείους ἢ δύο ἴσαι πρὸς τὸν $AB\Gamma$ κύκλον ἀπὸ τοῦ Δ σημείου ἐφ' ἑκάτερα τῆς ΔH ἐλαχίστης προσπεσοῦνται.

Ἐὰν ἄρα κύκλου ληφθῇ τι σημεῖον ἐκτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον διαχθῶσιν εὐθεῖαί τινες, ὧν μία μὲν διὰ τοῦ κέντρου αἱ δὲ λοιπαί, ὡς ἔτυχεν, τῶν μὲν πρὸς τὴν κοίλην περιφέρειαν προσπιπτουσῶν εὐθειῶν μεγίστη μὲν ἐστὶν ἡ διὰ τοῦ κέντρου, τῶν δὲ ἄλλων αἰεὶ ἡ ἑγγιον τῆς διὰ τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν, τῶν δὲ πρὸς τὴν κυρτὴν περιφέρειαν προσπιπτουσῶν εὐθειῶν ἐλαχίστη μὲν ἐστὶν ἡ μεταξὺ τοῦ τε σημείου καὶ τῆς διαμέτρου, τῶν δὲ ἄλλων αἰεὶ ἡ ἑγγιον τῆς ἐλαχίστης τῆς ἀπώτερον ἐστὶν ἐλάττων, δύο δὲ μόνον ἴσαι ἀπὸ τοῦ σημείου προσπεσοῦνται πρὸς τὸν κύκλον ἐφ' ἑκάτερα τῆς ἐλαχίστης· ὅπερ ἔδει δεῖξαι.

ate from point D towards (the circumference of) the circle, (one) on each (side) on the least (straight-line), DG . Let the angle DMB , equal to angle KMD , have been constructed on the straight-line MD , at the point M on it [Prop. 1.23], and let DB have been joined. And since MK is equal to MB , and MD (is) common, the two (straight-lines) KM , MD are equal to the two (straight-lines) BM , MD , respectively. And angle KMD (is) equal to angle BMD . Thus, the base DK is equal to the base DB [Prop. 1.4]. [So] I say that another (straight-line) equal to DK will not radiate towards the (circumference of the) circle from point D . For, if possible, let (such a straight-line) radiate, and let it be DN . Therefore, since DK is equal to DN , but DK is equal to DB , then DB is thus also equal to DN , (so that) a (straight-line) nearer to the least (straight-line) DG [is] equal to one further away. The very thing was shown (to be) impossible. Thus, not more than two equal (straight-lines) will radiate towards (the circumference of) circle ABC from point D , (one) on each side of the least (straight-line) DG .

Thus, if some point is taken outside a circle, and some straight-lines are drawn from the point to the (circumference of the) circle, one of which (passes) through the center, the remainder (being) random, then for the straight-lines radiating towards the concave (part of the) circumference, the greatest is that (passing) through the center. For the others, a (straight-line) nearer to the (straight-line) through the center is always greater than one further away. For the straight-lines radiating towards the convex (part of the) circumference, the least is that between the point and the diameter. For the others, a (straight-line) nearer to the least (straight-line) is always less than one further away. And only two equal (straight-lines) will radiate from the point towards the (circumference of the) circle, (one) on each (side) of the least (straight-line). (Which is) the very thing it was required to show.

† Presumably, in an angular sense.

‡ This is not proved, except by reference to the figure.

θ'.

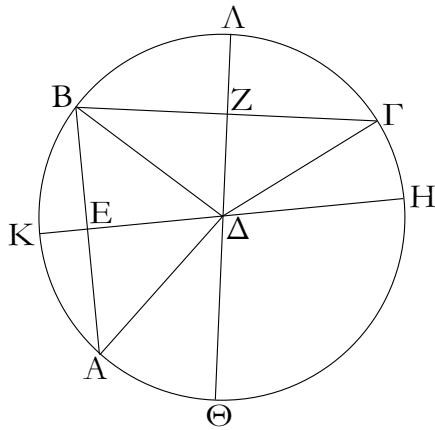
Ἐὰν κύκλου ληφθῇ τι σημεῖον ἐντός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσι πλείους ἢ δύο ἴσαι εὐθεῖαι, τὸ ληφθὲν σημεῖον κέντρον ἐστὶ τοῦ κύκλου.

Ἐστω κύκλος ὁ $AB\Gamma$, ἐντός δὲ αὐτοῦ σημεῖον τὸ Δ , καὶ ἀπὸ τοῦ Δ πρὸς τὸν $AB\Gamma$ κύκλον προσπιπτέωσαν πλείους ἢ δύο ἴσαι εὐθεῖαι αἱ ΔA , ΔB , $\Delta \Gamma$. λέγω, ὅτι τὸ Δ σημεῖον κέντρον ἐστὶ τοῦ $AB\Gamma$ κύκλου.

Proposition 9

If some point is taken inside a circle, and more than two equal straight-lines radiate from the point towards the (circumference of the) circle, then the point taken is the center of the circle.

Let ABC be a circle, and D a point inside it, and let more than two equal straight-lines, DA , DB , and DC , radiate from D towards (the circumference of) circle ABC .



Ἐπεξεύχθωσαν γὰρ αἱ AB, BΓ καὶ τετμήσθωσαν δίσχα κατὰ τὰ E, Z σημεία, καὶ ἐπιζευχθεῖσαι αἱ EΔ, ZΔ διήχθωσαν ἐπὶ τὰ H, K, Θ, Λ σημεία.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ AE τῇ EB, κοινὴ δὲ ἡ EΔ, δύο δὴ αἱ AE, EΔ δύο ταῖς BE, EΔ ἴσαι εἰσὶν· καὶ βάσεις ἡ ΔA βάσει τῇ ΔB ἴση· γωνία ἄρα ἡ ὑπὸ AED γωνία τῇ ὑπὸ BED ἴση ἐστὶν· ὀρθὴ ἄρα ἑκατέρα τῶν ὑπὸ AED, BED γωνιῶν· ἡ HK ἄρα τὴν AB τέμνει δίσχα καὶ πρὸς ὀρθάς. καὶ ἐπεὶ, ἐὰν ἐν κύκλῳ εὐθεῖα τις εὐθεϊάν τινα δίσχα τε καὶ πρὸς ὀρθάς τέμνη, ἐπὶ τῆς τεμνουσῆς ἐστὶ τὸ κέντρον τοῦ κύκλου, ἐπὶ τῆς HK ἄρα ἐστὶ τὸ κέντρον τοῦ κύκλου. διὰ τὰ αὐτὰ δὴ καὶ ἐπὶ τῆς ΘΛ ἐστὶ τὸ κέντρον τοῦ ABΓ κύκλου. καὶ οὐδὲν ἕτερον κοινὸν ἔχουσιν αἱ HK, ΘΛ εὐθεῖαι ἢ τὸ Δ σημεῖον· τὸ Δ ἄρα σημεῖον κέντρον ἐστὶ τοῦ ABΓ κύκλου.

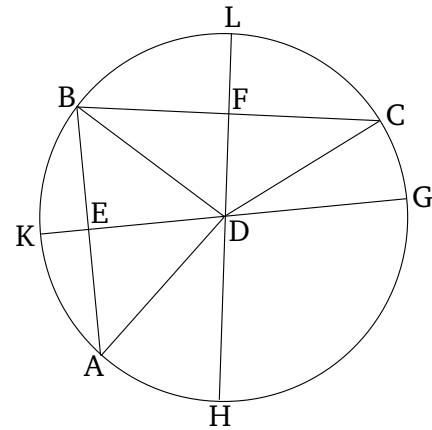
Ἐὰν ἄρα κύκλου ληφθῇ τι σημεῖον ἐντός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσι πλείους ἢ δύο ἴσαι εὐθεῖαι, τὸ ληφθὲν σημεῖον κέντρον ἐστὶ τοῦ κύκλου· ὅπερ εἶδει δεῖξαι.

ι'.

Κύκλος κύκλον οὐ τέμνει κατὰ πλείονα σημεία ἢ δύο.

Εἰ γὰρ δυνατόν, κύκλος ὁ ABΓ κύκλον τὸν ΔEZ τεμνέτω κατὰ πλείονα σημεία ἢ δύο τὰ B, H, Z, Θ, καὶ ἐπιζευχθεῖσαι αἱ BΘ, BH δίσχα τεμνέσθωσαν κατὰ τὰ K, Λ σημεία· καὶ ἀπὸ τῶν K, Λ ταῖς BΘ, BH πρὸς ὀρθάς ἀχθεῖσαι αἱ KΓ, ΛM διήχθωσαν ἐπὶ τὰ A, E σημεία.

I say that point D is the center of circle ABC .



For let AB and BC have been joined, and (then) have been cut in half at points E and F (respectively) [Prop. 1.10]. And ED and FD being joined, let them have been drawn through to points G, K, H , and L .

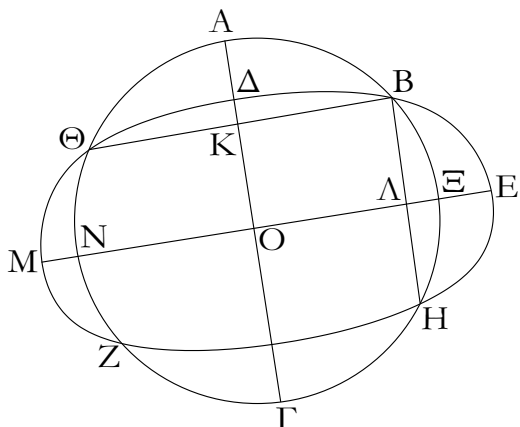
Therefore, since AE is equal to EB , and ED (is) common, the two (straight-lines) AE, ED are equal to the two (straight-lines) BE, ED (respectively). And the base DA (is) equal to the base DB . Thus, angle AED is equal to angle BED [Prop. 1.8]. Thus, angles AED and BED (are) each right-angles [Def. 1.10]. Thus, GK cuts AB in half, and at right-angles. And since, if some straight-line in a circle cuts some (other) straight-line in half, and at right-angles, then the center of the circle is on the former (straight-line) [Prop. 3.1 corr.], the center of the circle is thus on GK . So, for the same (reasons), the center of circle ABC is also on HL . And the straight-lines GK and HL have no common (point) other than point D . Thus, point D is the center of circle ABC .

Thus, if some point is taken inside a circle, and more than two equal straight-lines radiate from the point towards the (circumference of the) circle, then the point taken is the center of the circle. (Which is) the very thing it was required to show.

Proposition 10

A circle does not cut a(nother) circle at more than two points.

For, if possible, let the circle ABC cut the circle DEF at more than two points, B, G, F , and H . And BH and BG being joined, let them (then) have been cut in half at points K and L (respectively). And KC and LM being drawn at right-angles to BH and BG from K and L (respectively) [Prop. 1.11], let them (then) have been drawn through to points A and E (respectively).



Ἐπεὶ οὖν ἐν κύκλῳ τῷ ABΓ εὐθεῖά τις ἡ AG εὐθεῖάν τινα τὴν BH διέχαι καὶ πρὸς ὀρθὰς τέμνει, ἐπὶ τῆς AG ἄρα ἐστὶ τὸ κέντρον τοῦ ABΓ κύκλου. πάλιν, ἐπεὶ ἐν κύκλῳ τῷ ΔΕΖ εὐθεῖά τις ἡ NE εὐθεῖάν τινα τὴν BH διέχαι καὶ πρὸς ὀρθὰς τέμνει, ἐπὶ τῆς NE ἄρα ἐστὶ τὸ κέντρον τοῦ ΔΕΖ κύκλου. ἐδείχθη δὲ καὶ ἐπὶ τῆς AG, καὶ κατ' οὐδὲν συμβάλλουσιν αἱ AG, NE εὐθεῖαι ἢ κατὰ τὸ O· τὸ O ἄρα σημεῖον κέντρον ἐστὶ τοῦ ABΓ κύκλου. ὁμοίως δὲ δείξομεν, ὅτι καὶ τοῦ ΔΕΖ κύκλου κέντρον ἐστὶ τὸ O· δύο ἄρα κύκλων τεμνόντων ἀλλήλους τῶν ABΓ, ΔΕΖ τὸ αὐτὸ ἐστὶ κέντρον τὸ O· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα κύκλος κύκλον τέμνει κατὰ πλείονα σημεία ἢ δύο· ὅπερ ἔδει δεῖξαι.

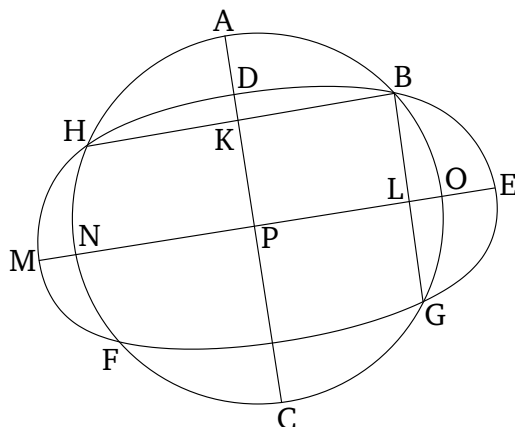
ια'.

Ἐὰν δύο κύκλοι ἐφάπτονται ἀλλήλων ἐντός, καὶ ληφθῇ αὐτῶν τὰ κέντρα, ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη εὐθεῖα καὶ ἐκβαλλομένη ἐπὶ τὴν συναφὴν πεσεῖται τῶν κύκλων.

Δύο γὰρ κύκλοι οἱ ABΓ, ΔΕΖ ἐφαπτόσθωσαν ἀλλήλων ἐντός κατὰ τὸ A σημεῖον, καὶ εἰλήφθω τοῦ μὲν ABΓ κύκλου κέντρον τὸ Z, τοῦ δὲ ΔΕΖ τὸ H· λέγω, ὅτι ἡ ἀπὸ τοῦ H ἐπὶ τὸ Z ἐπιζευγνυμένη εὐθεῖα ἐκβαλλομένη ἐπὶ τὸ A πεσεῖται.

Μὴ γάρ, ἀλλ' εἰ δυνατόν, πιπτέτω ὡς ἡ ZHΘ, καὶ ἐπεζεύχθωσαν αἱ AZ, AH.

Ἐπεὶ οὖν αἱ AH, HZ τῆς ZA, τουτέστι τῆς ZΘ, μείζονες εἰσιν, κοινὴ ἀφηρήσθω ἡ ZH· λοιπὴ ἄρα ἡ AH λοιπῆς τῆς HΘ μείζων ἐστίν. ἴση δὲ ἡ AH τῇ HΔ· καὶ ἡ HΔ ἄρα τῆς HΘ μείζων ἐστίν ἢ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα ἡ ἀπὸ τοῦ Z ἐπὶ τὸ H ἐπιζευγνυμένη εὐθεῖα ἐκτὸς πεσεῖται· κατὰ τὸ A ἄρα ἐπὶ τῆς συναφῆς πεσεῖται.



Therefore, since in circle ABC some straight-line AC cuts some (other) straight-line BH in half, and at right-angles, the center of circle ABC is thus on AC [Prop. 3.1 corr.]. Again, since in the same circle ABC some straight-line NO cuts some (other straight-line) BG in half, and at right-angles, the center of circle ABC is thus on NO [Prop. 3.1 corr.]. And it was also shown (to be) on AC . And the straight-lines AC and NO meet at no other (point) than P . Thus, point P is the center of circle ABC . So, similarly, we can show that P is also the center of circle DEF . Thus, two circles cutting one another, ABC and DEF , have the same center P . The very thing is impossible [Prop. 3.5].

Thus, a circle does not cut a(nother) circle at more than two points. (Which is) the very thing it was required to show.

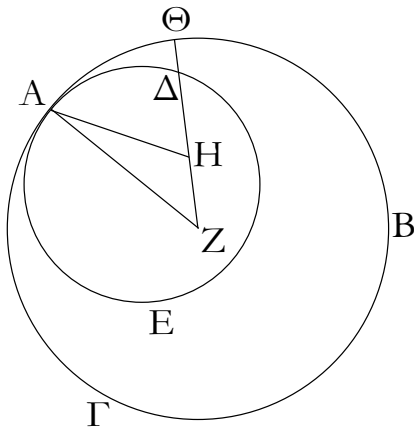
Proposition 11

If two circles touch one another internally, and their centers are found, then the straight-line joining their centers, being produced, will fall upon the point of union of the circles.

For let two circles, ABC and ADE , touch one another internally at point A , and let the center F of circle ABC have been found [Prop. 3.1], and (the center) G of (circle) ADE [Prop. 3.1]. I say that the straight-line joining G to F , being produced, will fall on A .

For (if) not then, if possible, let it fall like FGH (in the figure), and let AF and AG have been joined.

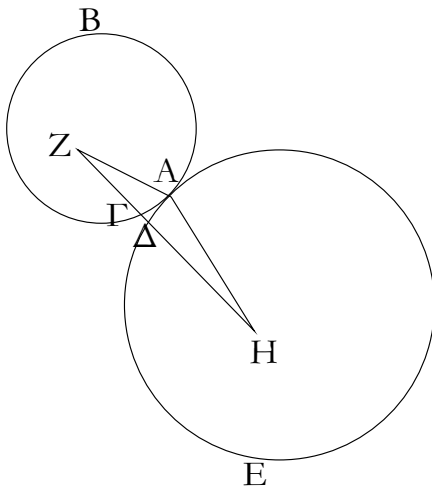
Therefore, since AG and GF is greater than FA , that is to say FH [Prop. 1.20], let FG have been taken from both. Thus, the remainder AG is greater than the remainder GH . And AG (is) equal to GD . Thus, GD is also greater than GH , the lesser than the greater. The very thing is impossible. Thus, the straight-line joining F to G will not fall outside (one circle but inside the other). Thus, it will fall upon the point of union (of the circles)



Ἐάν ἄρα δύο κύκλοι ἐφάπτονται ἀλλήλων ἐντός, [καὶ ληφθῇ αὐτῶν τὰ κέντρα], ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη εὐθεῖα [καὶ ἐκβαλλομένη] ἐπὶ τὴν συναφὴν πεσεῖται τῶν κύκλων· ὅπερ ἔδει δεῖξαι.

ιβ'.

Ἐάν δύο κύκλοι ἐφάπτονται ἀλλήλων ἐκτός, ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπιζευγνυμένη διὰ τῆς ἐπαφῆς ἐλεύσεται.

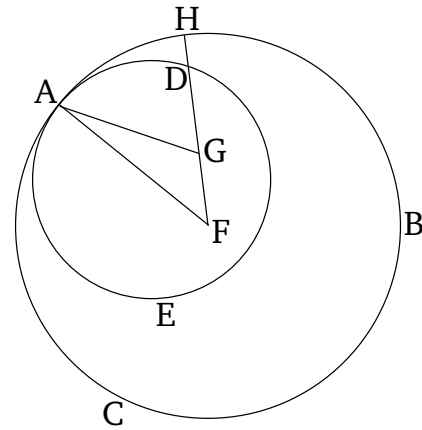


Δύο γὰρ κύκλοι οἱ ABΓ, AΔΕ ἐφαπτέσθωσαν ἀλλήλων ἐκτός κατὰ τὸ A σημεῖον, καὶ εἰλήφθω τοῦ μὲν ABΓ κέντρον τὸ Z, τοῦ δὲ AΔΕ τὸ H· λέγω, ὅτι ἡ ἀπὸ τοῦ Z ἐπὶ τὸ H ἐπιζευγνυμένη εὐθεῖα διὰ τῆς κατὰ τὸ A ἐπαφῆς ἐλεύσεται.

Μὴ γάρ, ἀλλ' εἰ δυνατόν, ἐρχέσθω ὡς ἡ ZΓΔΗ, καὶ ἐπεζεύχθωσαν αἱ AZ, AH.

Ἐπεὶ οὖν τὸ Z σημεῖον κέντρον ἐστὶ τοῦ ABΓ κύκλου, ἴση ἐστὶν ἡ ZA τῇ ZΓ. πάλιν, ἐπεὶ τὸ H σημεῖον κέντρον ἐστὶ τοῦ AΔΕ κύκλου, ἴση ἐστὶν ἡ HA τῇ HΔ. ἐδείχθη

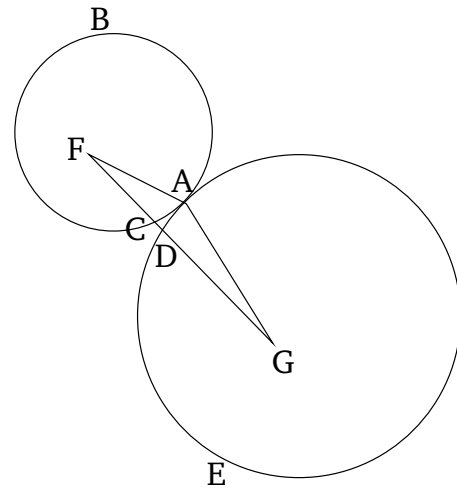
at point A.



Thus, if two circles touch one another internally, [and their centers are found], then the straight-line joining their centers, [being produced], will fall upon the point of union of the circles. (Which is) the very thing it was required to show.

Proposition 12

If two circles touch one another externally then the (straight-line) joining their centers will go through the point of union.



For let two circles, ABC and ADE , touch one another externally at point A , and let the center F of ABC have been found [Prop. 3.1], and (the center) G of ADE [Prop. 3.1]. I say that the straight-line joining F to G will go through the point of union at A .

For (if) not then, if possible, let it go like $FCDG$ (in the figure), and let AF and AG have been joined.

Therefore, since point F is the center of circle ABC , FA is equal to FC . Again, since point G is the center of circle ADE , GA is equal to GD . And FA was also shown

Κύκλος ἄρα κύκλου οὐκ ἐφάπτεται κατὰ πλείονα σημεία ἢ [καθ'] ἓν, ἐάν τε ἐντὸς ἐάν τε ἐκτὸς ἐφάπτηται· ὅπερ ἔδει δεῖξαι.

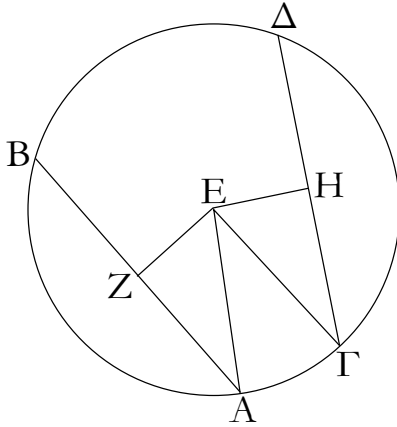
(is) absurd. Thus, a circle does not touch a(nother) circle externally at more than one point. And it was shown that neither (does it) internally.

Thus, a circle does not touch a(nother) circle at more than one point, whether they touch internally or externally. (Which is) the very thing it was required to show.

† The Greek text has “*ABCD*”, which is obviously a mistake.

ιδ'.

Ἐν κύκλῳ αἱ ἴσαι εὐθεῖαι ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου, καὶ αἱ ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου ἴσαι ἀλλήλαις εἰσίν.



Ἐστω κύκλος ὁ ΑΒΓΔ, καὶ ἐν αὐτῷ ἴσαι εὐθεῖαι ἔστωσαν αἱ ΑΒ, ΓΔ· λέγω, ὅτι αἱ ΑΒ, ΓΔ ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου.

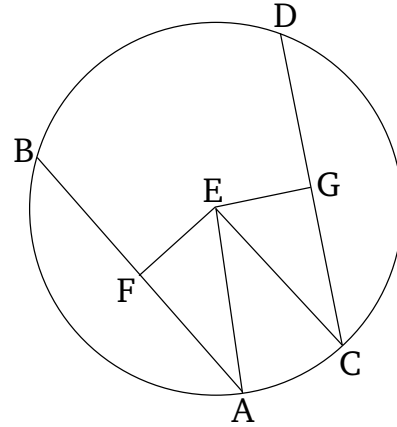
Εἰλήφθω γὰρ τὸ κέντρον τοῦ ΑΒΓΔ κύκλου καὶ ἔστω τὸ Ε, καὶ ἀπὸ τοῦ Ε ἐπὶ τὰς ΑΒ, ΓΔ κάθετοι ἤχθωσαν αἱ ΕΖ, ΕΗ, καὶ ἐπεζεύχθωσαν αἱ ΑΕ, ΕΓ.

Ἐπεὶ οὖν εὐθεῖα τις διὰ τοῦ κέντρου ἡ ΕΖ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΒ πρὸς ὀρθὰς τέμνει, καὶ δίχα αὐτὴν τέμνει. ἴση ἄρα ἡ ΑΖ τῇ ΖΒ· διπλῇ ἄρα ἡ ΑΒ τῆς ΑΖ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓΔ τῆς ΓΗ ἐστὶ διπλῇ· καὶ ἐστὶν ἴση ἡ ΑΒ τῇ ΓΔ· ἴση ἄρα καὶ ἡ ΑΖ τῇ ΓΗ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΕ τῇ ΕΓ, ἴσον καὶ τὸ ἀπὸ τῆς ΑΕ τῷ ἀπὸ τῆς ΕΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΕ ἴσα τὰ ἀπὸ τῶν ΑΖ, ΕΖ· ὀρθὴ γὰρ ἡ πρὸς τῷ Ζ γωνία· τῷ δὲ ἀπὸ τῆς ΕΓ ἴσα τὰ ἀπὸ τῶν ΕΗ, ΗΓ· ὀρθὴ γὰρ ἡ πρὸς τῷ Η γωνία· τὰ ἄρα ἀπὸ τῶν ΑΖ, ΖΕ ἴσα ἐστὶ τοῖς ἀπὸ τῶν ΓΗ, ΗΕ, ὧν τὸ ἀπὸ τῆς ΑΖ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΓΗ· ἴση γάρ ἐστὶν ἡ ΑΖ τῇ ΓΗ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΖΕ τῷ ἀπὸ τῆς ΕΗ ἴσον ἐστίν· ἴση ἄρα ἡ ΕΖ τῇ ΕΗ. ἐν δὲ κύκλῳ ἴσον ἀπέχειν ἀπὸ τοῦ κέντρου εὐθεῖαι λέγονται, ὅταν αἱ ἀπὸ τοῦ κέντρου ἐπ' αὐτὰς κάθετοι ἀγόμεναι ἴσαι ὦσιν· αἱ ἄρα ΑΒ, ΓΔ ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου.

Ἀλλὰ δὴ αἱ ΑΒ, ΓΔ εὐθεῖαι ἴσον ἀπέχονται ἀπὸ τοῦ κέντρου, τουτέστιν ἴση ἔστω ἡ ΕΖ τῇ ΕΗ. λέγω, ὅτι ἴση ἐστὶ καὶ ἡ ΑΒ τῇ ΓΔ.

Proposition 14

In a circle, equal straight-lines are equally far from the center, and (straight-lines) which are equally far from the center are equal to one another.



Let $ABDC^{\dagger}$ be a circle, and let AB and CD be equal straight-lines within it. I say that AB and CD are equally far from the center.

For let the center of circle $ABDC$ have been found [Prop. 3.1], and let it be (at) E . And let EF and EG have been drawn from (point) E , perpendicular to AB and CD (respectively) [Prop. 1.12]. And let AE and EC have been joined.

Therefore, since some straight-line, EF , through the center (of the circle), cuts some (other) straight-line, AB , not through the center, at right-angles, it also cuts it in half [Prop. 3.3]. Thus, AF (is) equal to FB . Thus, AB (is) double AF . So, for the same (reasons), CD is also double CG . And AB is equal to CD . Thus, AF (is) also equal to CG . And since AE is equal to EC , the (square) on AE (is) also equal to the (square) on EC . But, the (sum of the squares) on AF and EF (is) equal to the (square) on AE . For the angle at F (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on EG and GC (is) equal to the (square) on EC . For the angle at G (is) a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on AF and FE is equal to the (sum of the squares) on CG and GE , of which the (square) on AF is equal to the (square) on CG . For AF is equal to CG .

Τῶν γάρ αὐτῶν κατασκευασθέντων ὁμοίως δείζομεν, ὅτι διπλῇ ἐστὶν ἡ μὲν AB τῆς AZ , ἡ δὲ $\Gamma\Delta$ τῆς $\Gamma\Theta$. καὶ ἐπεὶ ἴση ἐστὶν ἡ AE τῇ GE , ἴσον ἐστὶ τὸ ἀπὸ τῆς AE τῷ ἀπὸ τῆς GE . ἀλλὰ τῷ μὲν ἀπὸ τῆς AE ἴσα ἐστὶ τὰ ἀπὸ τῶν EZ , ZA , τῷ δὲ ἀπὸ τῆς GE ἴσα τὰ ἀπὸ τῶν EH , $H\Gamma$. τὰ ἄρα ἀπὸ τῶν EZ , ZA ἴσα ἐστὶ τοῖς ἀπὸ τῶν EH , $H\Gamma$. ὣν τὸ ἀπὸ τῆς EZ τῷ ἀπὸ τῆς EH ἐστὶν ἴσον. ἴση γάρ ἡ EZ τῇ EH . λοιπὸν ἄρα τὸ ἀπὸ τῆς AZ ἴσον ἐστὶ τῷ ἀπὸ τῆς $\Gamma\Theta$. ἴση ἄρα ἡ AZ τῇ $\Gamma\Theta$. καὶ ἐστὶ τῆς μὲν AZ διπλῇ ἡ AB , τῆς δὲ $\Gamma\Theta$ διπλῇ ἡ $\Gamma\Delta$. ἴση ἄρα ἡ AB τῇ $\Gamma\Delta$.

Ἐν κύκλῳ ἄρα αἱ ἴσαι εὐθεῖαι ἴσον ἀπέχουσιν ἀπὸ τοῦ κέντρου, καὶ αἱ ἴσον ἀπέχουσαι ἀπὸ τοῦ κέντρου ἴσαι ἀλλήλαις εἰσὶν. ὅπερ ἔδει δείξαι.

Thus, the remaining (square) on FE is equal to the (remaining square) on EG . Thus, EF (is) equal to EG . And straight-lines in a circle are said to be equally far from the center when perpendicular (straight-lines) which are drawn to them from the center are equal [Def. 3.4]. Thus, AB and CD are equally far from the center.

So, let the straight-lines AB and CD be equally far from the center. That is to say, let EF be equal to EG . I say that AB is also equal to CD .

For, with the same construction, we can, similarly, show that AB is double AF , and CD (double) CG . And since AE is equal to CE , the (square) on AE is equal to the (square) on CE . But, the (sum of the squares) on EF and FA is equal to the (square) on AE [Prop. 1.47]. And the (sum of the squares) on EG and GC (is) equal to the (square) on CE [Prop. 1.47]. Thus, the (sum of the squares) on EF and FA is equal to the (sum of the squares) on EG and GC , of which the (square) on EF is equal to the (square) on EG . For EF (is) equal to EG . Thus, the remaining (square) on AF is equal to the (remaining square) on CG . Thus, AF (is) equal to CG . And AB is double AF , and CD double CG . Thus, AB (is) equal to CD .

Thus, in a circle, equal straight-lines are equally far from the center, and (straight-lines) which are equally far from the center are equal to one another. (Which is) the very thing it was required to show.

† The Greek text has “ $ABCD$ ”, which is obviously a mistake.

ιε'.

Proposition 15

Ἐν κύκλῳ μεγίστη μὲν ἡ διάμετρος, τῶν δὲ ἄλλων αἰεὶ ἡ ἑγγιον τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν.

Ἐστω κύκλος ὁ $AB\Gamma\Delta$, διάμετρος δὲ αὐτοῦ ἔστω ἡ $A\Delta$, κέντρον δὲ τὸ E , καὶ ἑγγιον μὲν τῆς $A\Delta$ διαμέτρου ἔστω ἡ $B\Gamma$, ἀπώτερον δὲ ἡ $Z\Theta$. λέγω, ὅτι μεγίστη μὲν ἐστὶν ἡ $A\Delta$, μείζων δὲ ἡ $B\Gamma$ τῆς $Z\Theta$.

Ἦχθωσαν γάρ ἀπὸ τοῦ E κέντρου ἐπὶ τὰς $B\Gamma$, $Z\Theta$ κάθετοι αἱ $E\Theta$, $E\Lambda$. καὶ ἐπεὶ ἑγγιον μὲν τοῦ κέντρου ἐστὶν ἡ $B\Gamma$, ἀπώτερον δὲ ἡ $Z\Theta$, μείζων ἄρα ἡ $E\Lambda$ τῆς $E\Theta$. κείσθω τῇ $E\Theta$ ἴση ἡ EL , καὶ διὰ τοῦ L τῇ $E\Lambda$ πρὸς ὀρθὰς ἀχθεῖσα ἡ LM διήχθω ἐπὶ τὸ N , καὶ ἐπεξεύχθωσαν αἱ ME , EN , ZE , EH .

Καὶ ἐπεὶ ἴση ἐστὶν ἡ $E\Theta$ τῇ EL , ἴση ἐστὶ καὶ ἡ $B\Gamma$ τῇ MN . πάλιν, ἐπεὶ ἴση ἐστὶν ἡ μὲν AE τῇ EM , ἡ δὲ $E\Delta$ τῇ EN , ἡ ἄρα $A\Delta$ ταῖς ME , EN ἴση ἐστίν. ἀλλ' αἱ μὲν ME , EN τῆς MN μείζονες εἰσιν [καὶ ἡ $A\Delta$ τῆς MN μείζων ἐστίν], ἴση δὲ ἡ MN τῇ $B\Gamma$. ἡ $A\Delta$ ἄρα τῆς $B\Gamma$ μείζων ἐστίν. καὶ ἐπεὶ δύο αἱ ME , EN δύο ταῖς ZE , EH ἴσαι εἰσὶν, καὶ γωνία ἡ ὑπὸ MEN γωνίας τῆς ὑπὸ ZEH μείζων [ἐστίν], βάσις ἄρα

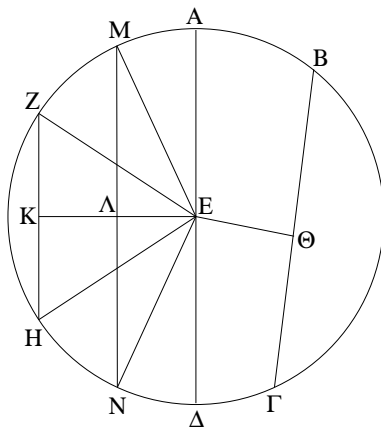
In a circle, a diameter (is) the greatest (straight-line), and for the others, a (straight-line) nearer to the center is always greater than one further away.

Let $ABCD$ be a circle, and let AD be its diameter, and E (its) center. And let BC be nearer to the diameter AD ,† and FG further away. I say that AD is the greatest (straight-line), and BC (is) greater than FG .

For let EH and $E\Lambda$ have been drawn from the center E , at right-angles to BC and FG (respectively) [Prop. 1.12]. And since BC is nearer to the center, and FG further away, $E\Lambda$ (is) thus greater than EH [Def. 3.5]. Let EL be made equal to EH [Prop. 1.3]. And LM being drawn through L , at right-angles to $E\Lambda$ [Prop. 1.11], let it have been drawn through to N . And let ME , EN , FE , and EG have been joined.

And since EH is equal to EL , BC is also equal to MN [Prop. 3.14]. Again, since AE is equal to EM , and ED to EN , AD is thus equal to ME and EN . But, ME and EN is greater than MN [Prop. 1.20] [also AD is

ἡ MN βάσεως τῆς ZH μείζων ἐστίν. ἀλλὰ ἡ MN τῇ $BΓ$ ἐδείχθη ἴση [καὶ ἡ $BΓ$ τῆς ZH μείζων ἐστίν]. μεγίστη μὲν ἄρα ἡ $ΑΔ$ διάμετρος, μείζων δὲ ἡ $BΓ$ τῆς ZH .



Ἐν κύκλῳ ἄρα μεγίστη μὲν ἐστὶν ἡ διάμετρος, τῶν δὲ ἄλλων αἰεὶ ἡ ἔγγιον τοῦ κέντρου τῆς ἀπώτερον μείζων ἐστίν· ὅπερ ἔδει δεῖξαι.

† Euclid should have said “to the center”, rather than “to the diameter AD ”, since BC , AD and FG are not necessarily parallel.

‡ This is not proved, except by reference to the figure.

ις'.

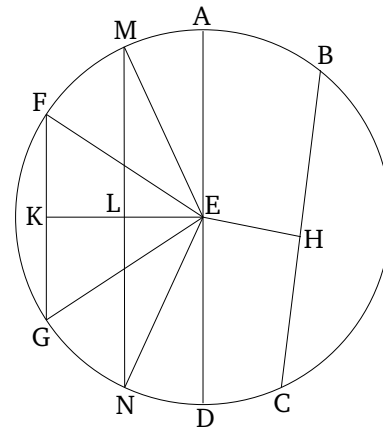
Ἡ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ' ἄκρας ἀγομένη ἐκτὸς πεσεῖται τοῦ κύκλου, καὶ εἰς τὸν μεταξὺ τόπον τῆς τε εὐθείας καὶ τῆς περιφερείας ἑτέρα εὐθεῖα οὐ παρεμπεσεῖται, καὶ ἡ μὲν τοῦ ἡμικυκλίου γωνία ἀπάσης γωνίας ὀξείας εὐθυγράμμου μείζων ἐστίν, ἡ δὲ λοιπὴ ἐλάττω.

Ἐστω κύκλος ὁ $ΑΒΓ$ περὶ κέντρον τὸ $Δ$ καὶ διάμετρον τὴν $ΑΒ$ · λέγω, ὅτι ἡ ἀπὸ τοῦ $Α$ τῇ $ΑΒ$ πρὸς ὀρθὰς ἀπ' ἄκρας ἀγομένη ἐκτὸς πεσεῖται τοῦ κύκλου.

Μὴ γάρ, ἀλλ' εἰ δυνατόν, πιπτέτω ἐντὸς ὡς ἡ $ΓΑ$, καὶ ἐπεζεύχθω ἡ $ΔΓ$.

Ἐπεὶ ἴση ἐστὶν ἡ $ΔΑ$ τῇ $ΔΓ$, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ $ΔΑΓ$ γωνία τῇ ὑπὸ $ΑΓΔ$. ὀρθὴ δὲ ἡ ὑπὸ $ΔΑΓ$ · ὀρθὴ ἄρα καὶ ἡ ὑπὸ $ΑΓΔ$ · τριγώνου δὴ τοῦ $ΑΓΔ$ αἱ δύο γωνίαι αἱ ὑπὸ $ΔΑΓ$, $ΑΓΔ$ δύο ὀρθαῖς ἴσαι εἰσίν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ $Α$ σημείου τῇ $ΒΑ$ πρὸς ὀρθὰς ἀγομένη ἐντὸς πεσεῖται τοῦ κύκλου. ὁμοίως δὲ δεῖξομεν, ὅτι οὐδ' ἐπὶ τῆς περιφερείας· ἐκτὸς ἄρα.

greater than MN], and MN (is) equal to BC . Thus, AD is greater than BC . And since the two (straight-lines) ME , EN are equal to the two (straight-lines) FE , EG (respectively), and angle MEN [is] greater than angle FEG ,[†] the base MN is thus greater than the base FG [Prop. 1.24]. But, MN was shown (to be) equal to BC [(so) BC is also greater than FG]. Thus, the diameter AD (is) the greatest (straight-line), and BC (is) greater than FG .



Thus, in a circle, a diameter (is) the greatest (straight-line), and for the others, a (straight-line) nearer to the center is always greater than one further away. (Which is) the very thing it was required to show.

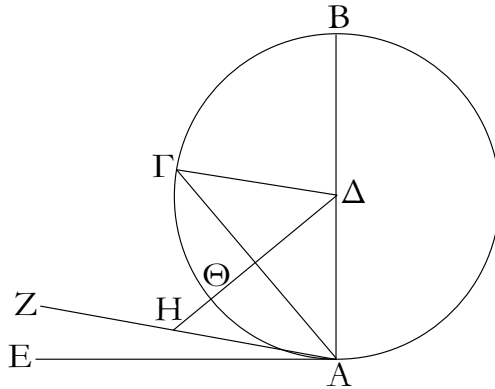
Proposition 16

A (straight-line) drawn at right-angles to the diameter of a circle, from its end, will fall outside the circle. And another straight-line cannot be inserted into the space between the (aforementioned) straight-line and the circumference. And the angle of the semi-circle is greater than any acute rectilinear angle whatsoever, and the remaining (angle is) less (than any acute rectilinear angle).

Let ABC be a circle around the center D and the diameter AB . I say that the (straight-line) drawn from A , at right-angles to AB [Prop 1.11], from its end, will fall outside the circle.

For (if) not then, if possible, let it fall inside, like CA (in the figure), and let DC have been joined.

Since DA is equal to DC , angle DAC is also equal to angle ACD [Prop. 1.5]. And DAC (is) a right-angle. Thus, ACD (is) also a right-angle. So, in triangle ACD , the two angles DAC and ACD are equal to two right-angles. The very thing is impossible [Prop. 1.17]. Thus, the (straight-line) drawn from point A , at right-angles



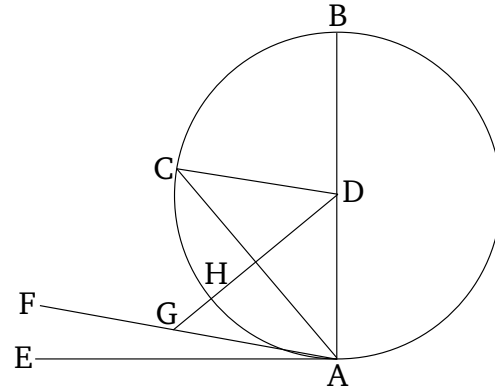
Πιπτέτω ὡς ἡ AE : λέγω δὴ, ὅτι εἰς τὸν μεταξὺ τόπον τῆς τε AE εὐθείας καὶ τῆς $ΓΘΑ$ περιφερείας ἑτέρα εὐθεῖα οὐ παρεμπεσεῖται.

Εἰ γὰρ δυνατόν, παρεμπίπτει ὡς ἡ ZA , καὶ ἄλλω ἀπὸ τοῦ Δ σημείου ἐπὶ τὴν ZA κάθετος ἡ ΔH . καὶ ἐπεὶ ὀρθή ἐστὶν ἡ ὑπὸ $AH\Delta$, ἐλάττων δὲ ὀρθῆς ἡ ὑπὸ ΔAH , μείζων ἄρα ἡ $A\Delta$ τῆς ΔH . ἴση δὲ ἡ ΔA τῇ $\Delta\Theta$: μείζων ἄρα ἡ $\Delta\Theta$ τῆς ΔH , ἢ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα εἰς τὸν μεταξὺ τόπον τῆς τε εὐθείας καὶ τῆς περιφερείας ἑτέρα εὐθεῖα παρεμπεσεῖται.

Λέγω, ὅτι καὶ ἡ μὲν τοῦ ἡμικυκλίου γωνία ἢ περιεχομένη ὑπὸ τε τῆς BA εὐθείας καὶ τῆς $ΓΘΑ$ περιφερείας ἀπάσης γωνίας ὀξείας εὐθυγράμμου μείζων ἐστίν, ἢ δὲ λοιπὴ ἢ περιεχομένη ὑπὸ τε τῆς $ΓΘΑ$ περιφερείας καὶ τῆς AE εὐθείας ἀπάσης γωνίας ὀξείας εὐθυγράμμου ἐλάττων ἐστίν.

Εἰ γὰρ ἐστὶ τις γωνία εὐθύγραμμος μείζων μὲν τῆς περιεχομένης ὑπὸ τε τῆς BA εὐθείας καὶ τῆς $ΓΘΑ$ περιφερείας, ἐλάττων δὲ τῆς περιεχομένης ὑπὸ τε τῆς $ΓΘΑ$ περιφερείας καὶ τῆς AE εὐθείας, εἰς τὸν μεταξὺ τόπον τῆς τε $ΓΘΑ$ περιφερείας καὶ τῆς AE εὐθείας εὐθεῖα παρεμπεσεῖται, ἥτις ποιήσει μείζονα μὲν τῆς περιεχομένης ὑπὸ τε τῆς BA εὐθείας καὶ τῆς $ΓΘΑ$ περιφερείας ὑπὸ εὐθειῶν περιεχομένην, ἐλάττονα δὲ τῆς περιεχομένης ὑπὸ τε τῆς $ΓΘΑ$ περιφερείας καὶ τῆς AE εὐθείας. οὐ παρεμπίπτει δέ· οὐκ ἄρα τῆς περιεχομένης γωνίας ὑπὸ τε τῆς BA εὐθείας καὶ τῆς $ΓΘΑ$ περιφερείας ἔσται μείζων ὀξεῖα ὑπὸ εὐθειῶν περιεχομένη, οὐδὲ μὴν ἐλάττων τῆς περιεχομένης ὑπὸ τε τῆς $ΓΘΑ$ περιφερείας καὶ τῆς AE εὐθείας.

to BA , will not fall inside the circle. So, similarly, we can show that neither (will it fall) on the circumference. Thus, (it will fall) outside (the circle).



Let it fall like AE (in the figure). So, I say that another straight-line cannot be inserted into the space between the straight-line AE and the circumference CHA .

For, if possible, let it be inserted like FA (in the figure), and let DG have been drawn from point D , perpendicular to FA [Prop. 1.12]. And since AGD is a right-angle, and DAG (is) less than a right-angle, AD (is) thus greater than DG [Prop. 1.19]. And DA (is) equal to DH . Thus, DH (is) greater than DG , the lesser than the greater. The very thing is impossible. Thus, another straight-line cannot be inserted into the space between the straight-line (AE) and the circumference.

And I also say that the semi-circular angle contained by the straight-line BA and the circumference CHA is greater than any acute rectilinear angle whatsoever, and the remaining (angle) contained by the circumference CHA and the straight-line AE is less than any acute rectilinear angle whatsoever.

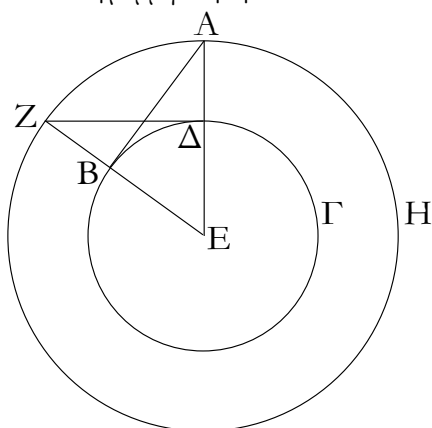
For if any rectilinear angle is greater than the (angle) contained by the straight-line BA and the circumference CHA , or less than the (angle) contained by the circumference CHA and the straight-line AE , then a straight-line can be inserted into the space between the circumference CHA and the straight-line AE —anything which will make (an angle) contained by straight-lines greater than the angle contained by the straight-line BA and the circumference CHA , or less than the (angle) contained by the circumference CHA and the straight-line AE . But (such a straight-line) cannot be inserted. Thus, an acute (angle) contained by straight-lines cannot be greater than the angle contained by the straight-line BA and the circumference CHA , neither (can it be) less than the (angle) contained by the circumference CHA and the straight-line AE .

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἡ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθᾶς ἀπ' ἄκρας ἀγομένη ἐφάπτεται τοῦ κύκλου [καὶ ὅτι εὐθεῖα κύκλου καθ' ἓν μόνον ἐφάπτεται σημεῖον, ἐπειδήπερ καὶ ἡ κατὰ δύο αὐτῶ συμβάλλουσα ἐντὸς αὐτοῦ πίπτουσα ἐδείχθη]. ὅπερ ἔδει δεῖξαι.

15.

Ἀπὸ τοῦ δοθέντος σημείου τοῦ δοθέντος κύκλου ἐφαπτομένην εὐθεΐαν γραμμὴν ἀγαγεῖν.



Ἐστω τὸ μὲν δοθὲν σημεῖον τὸ Α, ὁ δὲ δοθείς κύκλος ὁ ΒΓΔ· δεῖ δὴ ἀπὸ τοῦ Α σημείου τοῦ ΒΓΔ κύκλου ἐφαπτομένην εὐθεῖαν γραμμὴν ἀγαγεῖν.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Ε, καὶ ἐπεζεύχθω ἡ ΑΕ, καὶ κέντρῳ μὲν τῷ Ε διαστήματι δὲ τῷ ΕΑ κύκλος γεγράφθω ὁ ΑΖΗ, καὶ ἀπὸ τοῦ Δ τῇ ΕΑ πρὸς ὀρθὰς ἦχθω ἡ ΔΖ, καὶ ἐπεζεύχθωσαν αἱ ΕΖ, ΑΒ· λέγω, ὅτι ἀπὸ τοῦ Α σημείου τοῦ ΒΓΔ κύκλου ἐφαπτομένη ἦται ἡ ΑΒ.

Ἐπεὶ γὰρ τὸ Ε κέντρον ἐστὶ τῶν ΒΓΔ, ΑΖΗ κύκλων, ἴση ἄρα ἐστὶν ἡ μὲν ΕΑ τῇ ΕΖ, ἡ δὲ ΕΔ τῇ ΕΒ· δύο δὴ αἱ ΑΕ, ΕΒ δύο ταῖς ΖΕ, ΕΔ ἴσαι εἰσὶν· καὶ γωνίαν κοινὴν περιέχουσι τὴν πρὸς τῷ Ε· βάσις ἄρα ἡ ΔΖ βάσει τῇ ΑΒ ἴση ἐστίν, καὶ τὸ ΔΕΖ τρίγωνον τῷ ΕΒΑ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις· ἴση ἄρα ἡ ὑπὸ ΕΔΖ τῇ ὑπὸ ΕΒΑ. ὀρθὴ δὲ ἡ ὑπὸ ΕΔΖ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΕΒΑ. καὶ ἐστὶν ἡ ΕΒ ἐκ τοῦ κέντρου· ἡ δὲ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ' ἄκρας ἀγομένη ἐφαπτεται τοῦ κύκλου· ἡ ΑΒ ἄρα ἐφαπτεται τοῦ ΒΓΔ κύκλου.

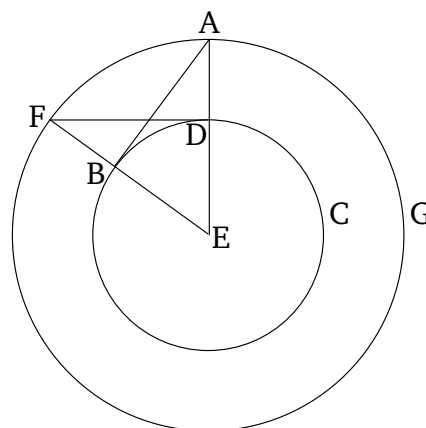
Ἀπὸ τοῦ ἄρα δοθέντος σημείου τοῦ Α τοῦ δοθέντος κύκλου τοῦ ΒΓΔ ἐφαπτομένη εὐθεῖα γραμμὴ ἦται· ἡ ΑΒ· ὅπερ ἔδει ποιῆσαι.

Corollary

So, from this, (it is) manifest that a (straight-line) drawn at right-angles to the diameter of a circle, from its extremity, touches the circle [and that the straight-line touches the circle at a single point, inasmuch as it was also shown that a (straight-line) meeting (the circle) at two (points) falls inside it [Prop. 3.2]]. (Which is) the very thing it was required to show.

Proposition 17

To draw a straight-line touching a given circle from a given point.



Let A be the given point, and BCD the given circle. So it is required to draw a straight-line touching circle BCD from point A .

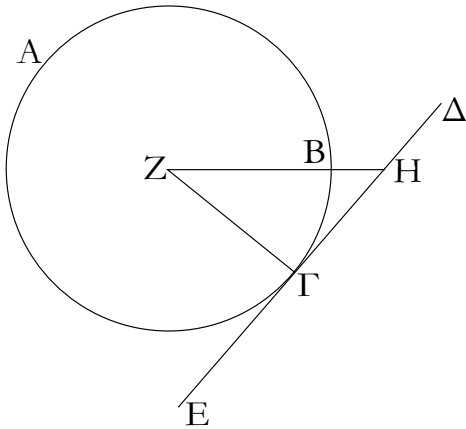
For let the center E of the circle have been found [Prop. 3.1], and let AE have been joined. And let (the circle) AFG have been drawn with center E and radius EA . And let DF have been drawn from (point) D , at right-angles to EA [Prop. 1.11]. And let EF and AB have been joined. I say that the (straight-line) AB has been drawn from point A touching circle BCD .

For since E is the center of circles BCD and AFG , EA is thus equal to EF , and ED to EB . So the two (straight-lines) AE , EB are equal to the two (straight-lines) FE , ED (respectively). And they contain a common angle at E . Thus, the base DF is equal to the base AB , and triangle DEF is equal to triangle EBA , and the remaining angles (are equal) to the (corresponding) remaining angles [Prop. 1.4]. Thus, (angle) EDF (is) equal to EBA . And EDF (is) a right-angle. Thus, EBA (is) also a right-angle. And EB is a radius. And a (straight-line) drawn at right-angles to the diameter of a circle, from its extremity, touches the circle [Prop. 3.16 corr.]. Thus, AB touches circle BCD .

Thus, the straight-line AB has been drawn touching

ιη'.

Ἐάν κύκλου ἐφάπτηται τις εὐθεΐα, ἀπὸ δὲ τοῦ κέντρου ἐπὶ τὴν ἀφὴν ἐπιζευχθῇ τις εὐθεΐα, ἡ ἐπιζευχθεῖσα κάθετος ἔσται ἐπὶ τὴν ἐφαπτομένην.



Κύκλου γὰρ τοῦ ΑΒΓ ἐφαπτέσθω τις εὐθεΐα ἡ ΔΕ κατὰ τὸ Γ σημεῖον, καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓ κύκλου τὸ Ζ, καὶ ἀπὸ τοῦ Ζ ἐπὶ τὸ Γ ἐπεζεύχθω ἡ ΖΓ· λέγω, ὅτι ἡ ΖΓ κάθετός ἐστιν ἐπὶ τὴν ΔΕ.

Εἰ γὰρ μή, ἦχθω ἀπὸ τοῦ Ζ ἐπὶ τὴν ΔΕ κάθετος ἡ ΖΗ.

Ἐπεὶ οὖν ἡ ὑπὸ ΖΗΓ γωνία ὀρθή ἐστιν, ὁξεῖα ἄρα ἐστὶν ἡ ὑπὸ ΖΓΗ· ὑπὸ δὲ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει· μείζων ἄρα ἡ ΖΓ τῆς ΖΗ· ἴση δὲ ἡ ΖΓ τῇ ΖΒ· μείζων ἄρα καὶ ἡ ΖΒ τῆς ΖΗ ἢ ἐλάττων τῆς μείζονος· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ΖΗ κάθετός ἐστιν ἐπὶ τὴν ΔΕ. ὁμοίως δὲ δεῖξομεν, ὅτι οὐδ' ἄλλη τις πλὴν τῆς ΖΓ· ἡ ΖΓ ἄρα κάθετός ἐστιν ἐπὶ τὴν ΔΕ.

Ἐάν ἄρα κύκλου ἐφάπτηται τις εὐθεΐα, ἀπὸ δὲ τοῦ κέντρου ἐπὶ τὴν ἀφὴν ἐπιζευχθῇ τις εὐθεΐα, ἡ ἐπιζευχθεῖσα κάθετος ἔσται ἐπὶ τὴν ἐφαπτομένην· ὅπερ ἔδει δεῖξαι.

ιθ'.

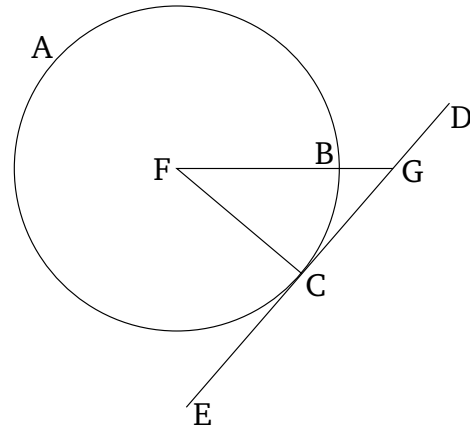
Ἐάν κύκλου ἐφάπτηται τις εὐθεΐα, ἀπὸ δὲ τῆς ἀφῆς τῇ ἐφαπτομένῃ πρὸς ὀρθὰς [γωνίας] εὐθεΐα γραμμὴ ἀχθῇ, ἐπὶ τῆς ἀχθείσης ἔσται τὸ κέντρον τοῦ κύκλου.

Κύκλου γὰρ τοῦ ΑΒΓ ἐφαπτέσθω τις εὐθεΐα ἡ ΔΕ κατὰ τὸ Γ σημεῖον, καὶ ἀπὸ τοῦ Γ τῇ ΔΕ πρὸς ὀρθὰς ἦχθω ἡ ΓΑ· λέγω, ὅτι ἐπὶ τῆς ΑΓ ἐστὶ τὸ κέντρον τοῦ κύκλου.

the given circle BCD from the given point A . (Which is) the very thing it was required to do.

Proposition 18

If some straight-line touches a circle, and some (other) straight-line is joined from the center (of the circle) to the point of contact, then the (straight-line) so joined will be perpendicular to the tangent.



For let some straight-line DE touch the circle ABC at point C , and let the center F of circle ABC have been found [Prop. 3.1], and let FC have been joined from F to C . I say that FC is perpendicular to DE .

For if not, let FG have been drawn from F , perpendicular to DE [Prop. 1.12].

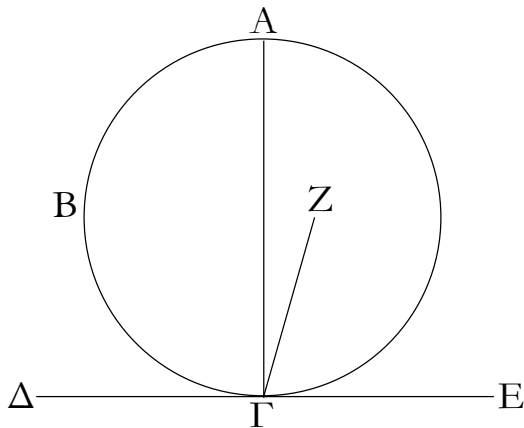
Therefore, since angle FGC is a right-angle, (angle) FCG is thus acute [Prop. 1.17]. And the greater angle is subtended by the greater side [Prop. 1.19]. Thus, FC (is) greater than FG . And FC (is) equal to FB . Thus, FB (is) also greater than FG , the lesser than the greater. The very thing is impossible. Thus, FG is not perpendicular to DE . So, similarly, we can show that neither (is) any other (straight-line) except FC . Thus, FC is perpendicular to DE .

Thus, if some straight-line touches a circle, and some (other) straight-line is joined from the center (of the circle) to the point of contact, then the (straight-line) so joined will be perpendicular to the tangent. (Which is) the very thing it was required to show.

Proposition 19

If some straight-line touches a circle, and a straight-line is drawn from the point of contact, at right-[angles] to the tangent, then the center (of the circle) will be on the (straight-line) so drawn.

For let some straight-line DE touch the circle ABC at point C . And let CA have been drawn from C , at right-



Μή γάρ, ἀλλ' εἰ δυνατόν, ἔστω τὸ Ζ, καὶ ἐπεζεύχθω ἡ ΓΖ.

Ἐπεὶ [οὖν] κύκλου τοῦ ΑΒΓ ἐφάπτεται τις εὐθεΐα ἡ ΔΕ, ἀπὸ δὲ τοῦ κέντρου ἐπὶ τὴν ἀφὴν ἐπέξευκται ἡ ΖΓ, ἡ ΖΓ ἄρα κάθετός ἐστιν ἐπὶ τὴν ΔΕ· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΖΓΕ. ἐστὶ δὲ καὶ ἡ ὑπὸ ΑΓΕ ὀρθή· ἴση ἄρα ἐστὶν ἡ ὑπὸ ΖΓΕ τῇ ὑπὸ ΑΓΕ ἡ ἐλάττων τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὸ Ζ κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου. ὁμοίως δὲ δείξομεν, ὅτι οὐδ' ἄλλο τι πλὴν ἐπὶ τῆς ΑΓ.

Ἐάν ἄρα κύκλου ἐφάπτηται τις εὐθεΐα, ἀπὸ δὲ τῆς ἀφῆς τῇ ἐφαπτομένῃ πρὸς ὀρθὰς εὐθεΐα γραμμὴ ἀχθῇ, ἐπὶ τῆς ἀχθείσης ἔσται τὸ κέντρον τοῦ κύκλου· ὅπερ ἔδει δεῖξαι.

κ'.

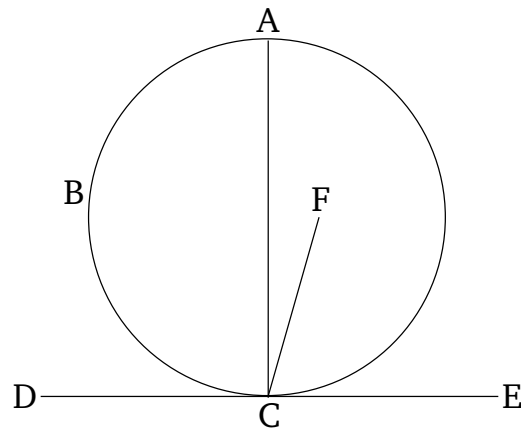
Ἐν κύκλῳ ἡ πρὸς τῷ κέντρῳ γωνία διπλασίων ἐστὶ τῆς πρὸς τῇ περιφερείᾳ, ὅταν τὴν αὐτὴν περιφέρειαν βάσιν ἔχωσιν αἱ γωνίαι.

Ἐστω κύκλος ὁ ΑΒΓ, καὶ πρὸς μὲν τῷ κέντρῳ αὐτοῦ γωνία ἔστω ἡ ὑπὸ ΒΕΓ, πρὸς δὲ τῇ περιφερείᾳ ἡ ὑπὸ ΒΑΓ, ἐχέτωσαν δὲ τὴν αὐτὴν περιφέρειαν βάσιν τὴν ΒΓ· λέγω, ὅτι διπλασίων ἐστὶν ἡ ὑπὸ ΒΕΓ γωνία τῆς ὑπὸ ΒΑΓ.

Ἐπιζευχθεῖσα γὰρ ἡ ΑΕ διήχθω ἐπὶ τὸ Ζ.

Ἐπεὶ οὖν ἴση ἐστὶν ἡ ΕΑ τῇ ΕΒ, ἴση καὶ γωνία ἡ ὑπὸ ΕΑΒ τῇ ὑπὸ ΕΒΑ· αἱ ἄρα ὑπὸ ΕΑΒ, ΕΒΑ γωνίαι τῆς ὑπὸ ΕΑΒ διπλασίους εἰσίν. ἴση δὲ ἡ ὑπὸ ΒΕΖ τᾶς ὑπὸ ΕΑΒ, ΕΒΑ· καὶ ἡ ὑπὸ ΒΕΖ ἄρα τῆς ὑπὸ ΕΑΒ ἐστὶ διπλῇ. διὰ τὰ αὐτὰ δὲ καὶ ἡ ὑπὸ ΖΕΓ τῆς ὑπὸ ΕΑΓ ἐστὶ διπλῇ. ὅλη ἄρα ἡ ὑπὸ ΒΕΓ ὅλης τῆς ὑπὸ ΒΑΓ ἐστὶ διπλῇ.

angles to DE [Prop. 1.11]. I say that the center of the circle is on AC .



For (if) not, if possible, let F be (the center of the circle), and let CF have been joined.

[Therefore], since some straight-line DE touches the circle ABC , and FC has been joined from the center to the point of contact, FC is thus perpendicular to DE [Prop. 3.18]. Thus, FCE is a right-angle. And ACE is also a right-angle. Thus, FCE is equal to ACE , the lesser to the greater. The very thing is impossible. Thus, F is not the center of circle ABC . So, similarly, we can show that neither is any (point) other (than one) on AC .

Thus, if some straight-line touches a circle, and a straight-line is drawn from the point of contact, at right-angles to the tangent, then the center (of the circle) will be on the (straight-line) so drawn. (Which is) the very thing it was required to show.

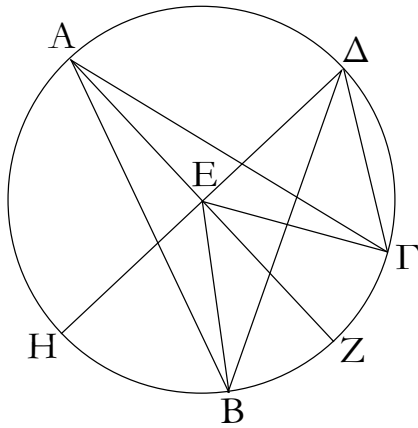
Proposition 20

In a circle, the angle at the center is double that at the circumference, when the angles have the same circumference base.

Let ABC be a circle, and let BEC be an angle at its center, and BAC (one) at (its) circumference. And let them have the same circumference base BC . I say that angle BEC is double (angle) BAC .

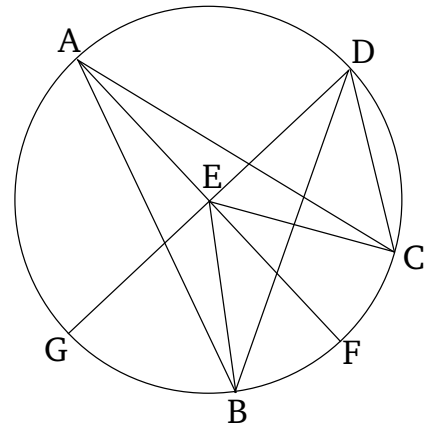
For being joined, let AE have been drawn through to F .

Therefore, since EA is equal to EB , angle EAB (is) also equal to EBA [Prop. 1.5]. Thus, angle EAB and EBA is double (angle) EAB . And BEF (is) equal to EAB and EBA [Prop. 1.32]. Thus, BEF is also double EAB . So, for the same (reasons), FEC is also double EAC . Thus, the whole (angle) BEC is double the whole (angle) BAC .



Κεκλάσθω δὴ πάλιν, καὶ ἔστω ἑτέρα γωνία ἡ ὑπὸ $B\Delta\Gamma$, καὶ ἐπιζευχθεῖσα ἡ ΔE ἐκβεβλήσθω ἐπὶ τὸ H . ὁμοίως δὲ δείξομεν, ὅτι διπλῇ ἐστὶν ἡ ὑπὸ $HE\Gamma$ γωνία τῆς ὑπὸ $E\Delta\Gamma$, ὣν ἡ ὑπὸ HEB διπλῇ ἐστὶ τῆς ὑπὸ $E\Delta B$. λοιπὴ ἄρα ἡ ὑπὸ BEG διπλῇ ἐστὶ τῆς ὑπὸ $B\Delta\Gamma$.

Ἐν κύκλῳ ἄρα ἡ πρὸς τῷ κέντρῳ γωνία διπλασίων ἐστὶ τῆς πρὸς τῇ περιφερείᾳ, ὅταν τὴν αὐτὴν περιφέρειαν βάσιν ἔχωσιν [αἱ γωνίαι]. ὅπερ ἔδει δεῖξαι.

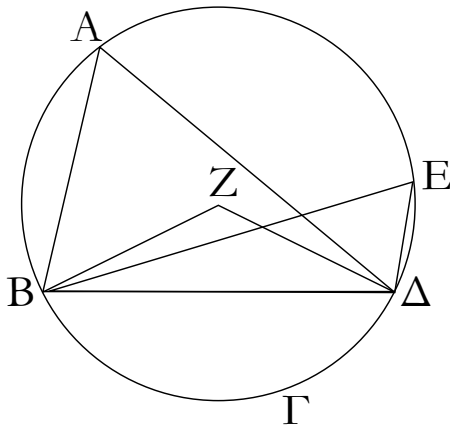


So let another (straight-line) have been inflected, and let there be another angle, BDC . And DE being joined, let it have been produced to G . So, similarly, we can show that angle GEC is double EDC , of which GEB is double EDB . Thus, the remaining (angle) BEC is double the (remaining angle) BDC .

Thus, in a circle, the angle at the center is double that at the circumference, when [the angles] have the same circumference base. (Which is) the very thing it was required to show.

κα'.

Ἐν κύκλῳ αἱ ἐν τῷ αὐτῷ τμήματι γωνίαι ἴσαι ἀλλήλαις εἰσίν.



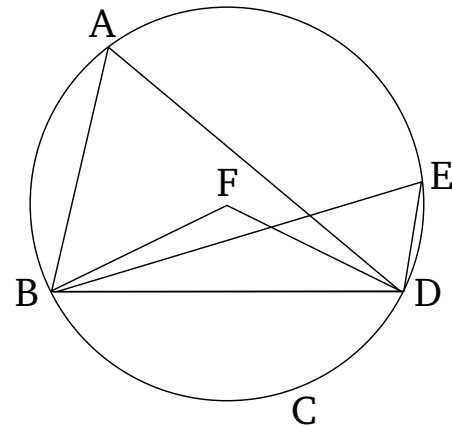
Ἐστω κύκλος ὁ $AB\Gamma\Delta$, καὶ ἐν τῷ αὐτῷ τμήματι τῶν $BAE\Delta$ γωνίαι ἔστωσαν αἱ ὑπὸ $BA\Delta$, $BE\Delta$. λέγω, ὅτι αἱ ὑπὸ $BA\Delta$, $BE\Delta$ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

Εἰλήφθω γάρ τοῦ $AB\Gamma\Delta$ κύκλου τὸ κέντρον, καὶ ἔστω τὸ Z , καὶ ἐπεζεύχθωσαν αἱ BZ , $Z\Delta$.

Καὶ ἐπεὶ ἡ μὲν ὑπὸ $BZ\Delta$ γωνία πρὸς τῷ κέντρῳ ἐστίν, ἡ δὲ ὑπὸ $BA\Delta$ πρὸς τῇ περιφερείᾳ, καὶ ἔχουσι τὴν αὐτὴν περιφέρειαν βάσιν τὴν $B\Gamma\Delta$, ἡ ἄρα ὑπὸ $BZ\Delta$ γωνία διπλασίων ἐστὶ τῆς ὑπὸ $BA\Delta$. διὰ τὰ αὐτὰ δὲ ἡ ὑπὸ $BZ\Delta$ καὶ τῆς ὑπὸ

Proposition 21

In a circle, angles in the same segment are equal to one another.



Let $ABCD$ be a circle, and let BAD and BED be angles in the same segment $BAED$. I say that angles BAD and BED are equal to one another.

For let the center of circle $ABCD$ have been found [Prop. 3.1], and let it be (at point) F . And let BF and FD have been joined.

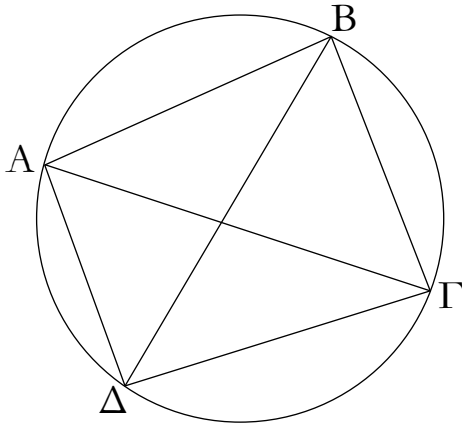
And since angle BFD is at the center, and BAD at the circumference, and they have the same circumference base BCD , angle BFD is thus double BAD [Prop. 3.20].

$BE\Delta$ ἐστὶ διπλῶν· ἴση ἄρα ἡ ὑπὸ $BA\Delta$ τῇ ὑπὸ $BE\Delta$.

Ἐν κύκλῳ ἄρα αἱ ἐν τῷ αὐτῷ τμήματι γωνίαι ἴσαι ἀλλήλαις εἰσὶν· ὅπερ ἔδει δεῖξαι.

κβ'.

Τῶν ἐν τοῖς κύκλοις τετραπλεύρων αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσὶν.



Ἐστω κύκλος ὁ $ABG\Delta$, καὶ ἐν αὐτῷ τετράπλευρον ἔστω τὸ $ABG\Delta$. λέγω, ὅτι αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσὶν.

Ἐπεζεύχθωσαν αἱ AG , BD .

Ἐπεὶ οὖν παντὸς τριγώνου αἱ τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσὶν, τοῦ ABG ἄρα τριγώνου αἱ τρεῖς γωνίαι αἱ ὑπὸ ΓAB , ABG , BGA δυσὶν ὀρθαῖς ἴσαι εἰσὶν. ἴση δὲ ἡ μὲν ὑπὸ ΓAB τῇ ὑπὸ $B\Delta G$. ἐν γὰρ τῷ αὐτῷ τμήματι εἰσι τῷ $BA\Delta G$. ἡ δὲ ὑπὸ ΓGB τῇ ὑπὸ ΔAB . ἐν γὰρ τῷ αὐτῷ τμήματι εἰσι τῷ $\Delta A\Gamma B$. ὅλη ἄρα ἡ ὑπὸ $\Delta\Delta G$ ταῖς ὑπὸ $BA\Gamma$, ΓGB ἴση ἐστίν. κοινὴ προσκείσθω ἡ ὑπὸ ABG . αἱ ἄρα ὑπὸ ABG , $BA\Gamma$, ΓGB ταῖς ὑπὸ ABG , $\Delta\Delta G$ ἴσαι εἰσὶν. ἀλλ' αἱ ὑπὸ ABG , $\Delta\Delta G$ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσὶν. ὁμοίως δὲ δείξομεν, ὅτι καὶ αἱ ὑπὸ $BA\Delta$, ΔGB γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσὶν.

Τῶν ἄρα ἐν τοῖς κύκλοις τετραπλεύρων αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσὶν· ὅπερ ἔδει δεῖξαι.

κγ'.

Ἐπὶ τῆς αὐτῆς εὐθείας δύο τμήματα κύκλων ὅμοια καὶ ἄνισα οὐ συσταθήσεται ἐπὶ τὰ αὐτὰ μέρη.

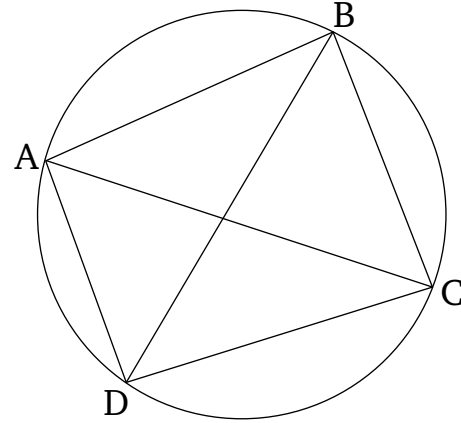
Εἰ γὰρ δυνατόν, ἐπὶ τῆς αὐτῆς εὐθείας τῆς AB δύο τμήματα κύκλων ὅμοια καὶ ἄνισα συνεστάτω ἐπὶ τὰ αὐτὰ μέρη τὰ ΓGB , $\Delta\Delta B$, καὶ διήχθω ἡ $AG\Delta$, καὶ ἐπεζεύχθωσαν

So, for the same (reasons), BFD is also double BED . Thus, BAD (is) equal to BED .

Thus, in a circle, angles in the same segment are equal to one another. (Which is) the very thing it was required to show.

Proposition 22

For quadrilaterals within circles, the (sum of the) opposite angles is equal to two right-angles.



Let $ABCD$ be a circle, and let $ABCD$ be a quadrilateral within it. I say that the (sum of the) opposite angles is equal to two right-angles.

Let AC and BD have been joined.

Therefore, since the three angles of any triangle are equal to two right-angles [Prop. 1.32], the three angles CAB , ABC , and BCA of triangle ABC are thus equal to two right-angles. And CAB (is) equal to BDC . For they are in the same segment $BADC$ [Prop. 3.21]. And ACB (is equal) to ADB . For they are in the same segment $ADCB$ [Prop. 3.21]. Thus, the whole of ADC is equal to BAC and ACB . Let ABC have been added to both. Thus, ABC , BAC , and ACB are equal to ABC and ADC . But, ABC , BAC , and ACB are equal to two right-angles. Thus, ABC and ADC are also equal to two right-angles. Similarly, we can show that angles BAD and DCB are also equal to two right-angles.

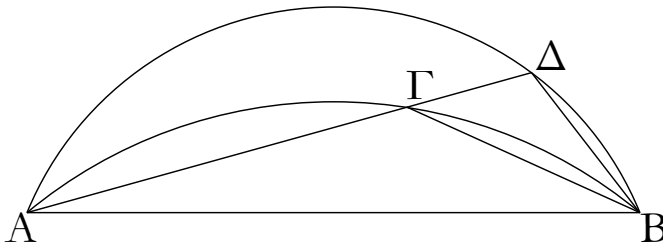
Thus, for quadrilaterals within circles, the (sum of the) opposite angles is equal to two right-angles. (Which is) the very thing it was required to show.

Proposition 23

Two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.

For, if possible, let the two similar and unequal segments of circles, ACB and ADB , have been constructed on the same side of the same straight-line AB . And let

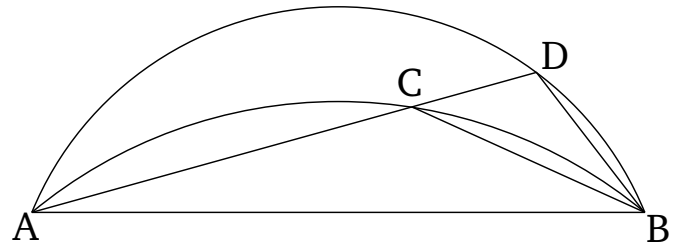
αἱ ΓΒ, ΔΒ.



Ἐπεὶ οὖν ὁμοίον ἐστὶ τὸ ΑΓΒ τμήμα τῷ ΑΔΒ τμήματι, ὁμοία δὲ τμήματα κύκλων ἐστὶ τὰ δεχόμενα γωνίας ἴσας, ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΓΒ γωνία τῇ ὑπὸ ΑΔΒ ἢ ἐκτὸς τῇ ἐντὸς· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα ἐπὶ τῆς αὐτῆς εὐθείας δύο τμήματα κύκλων ὁμοία καὶ ἀνισα συσταθήσεται ἐπὶ τὰ αὐτὰ μέρη· ὅπερ ἔδει δεῖξαι.

ACD have been drawn through (the segments), and let CB and DB have been joined.

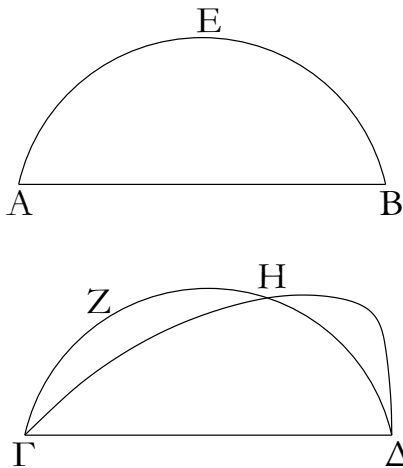


Therefore, since segment ACB is similar to segment ADB , and similar segments of circles are those accepting equal angles [Def. 3.11], angle ACB is thus equal to ADB , the external to the internal. The very thing is impossible [Prop. 1.16].

Thus, two similar and unequal segments of circles cannot be constructed on the same side of the same straight-line.

κδ'.

Τὰ ἐπὶ ἴσων εὐθειῶν ὁμοία τμήματα κύκλων ἴσα ἀλλήλοις ἐστίν.

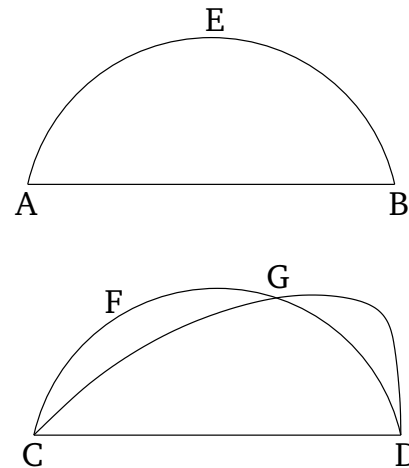


Ἐστωσαν γὰρ ἐπὶ ἴσων εὐθειῶν τῶν ΑΒ, ΓΔ ὁμοία τμήματα κύκλων τὰ ΑΕΒ, ΓΖΔ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΕΒ τμήμα τῷ ΓΖΔ τμήματι.

Ἐφαρμοζομένου γὰρ τοῦ ΑΕΒ τμήματος ἐπὶ τὸ ΓΖΔ καὶ τιθεμένου τοῦ μὲν Α σημείου ἐπὶ τὸ Γ τῆς δὲ ΑΒ εὐθείας ἐπὶ τὴν ΓΔ, ἐφαρμόσει καὶ τὸ Β σημεῖον ἐπὶ τὸ Δ σημείον διὰ τὸ ἴσην εἶναι τὴν ΑΒ τῇ ΓΔ· τῆς δὲ ΑΒ ἐπὶ τὴν ΓΔ ἐφαρμοσάσης ἐφαρμόσει καὶ τὸ ΑΕΒ τμήμα ἐπὶ τὸ ΓΖΔ. εἰ γὰρ ἡ ΑΒ εὐθεῖα ἐπὶ τὴν ΓΔ ἐφαρμόσει, τὸ δὲ ΑΕΒ τμήμα ἐπὶ τὸ ΓΖΔ μὴ ἐφαρμόσει, ἥτοι ἐντὸς αὐτοῦ πεσεῖται ἢ ἐκτὸς ἢ παραλλάξει, ὡς τὸ ΓΗΔ, καὶ κύκλος κύκλον τέμνει κατὰ πλεονα σημεία ἢ δύο· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐφαρμοζομένης τῆς ΑΒ εὐθείας ἐπὶ τὴν ΓΔ οὐκ ἐφαρμόσει καὶ

Proposition 24

Similar segments of circles on equal straight-lines are equal to one another.



For let AEB and CFD be similar segments of circles on the equal straight-lines AB and CD (respectively). I say that segment AEB is equal to segment CFD .

For if the segment AEB is applied to the segment CFD , and point A is placed on (point) C , and the straight-line AB on CD , then point B will also coincide with point D , on account of AB being equal to CD . And if AB coincides with CD then the segment AEB will also coincide with CFD . For if the straight-line AB coincides with CD , and the segment AEB does not coincide with CFD , then it will surely either fall inside it, outside (it),[†] or it will miss like CGD (in the figure), and a circle (will) cut (another) circle at more than two points. The very

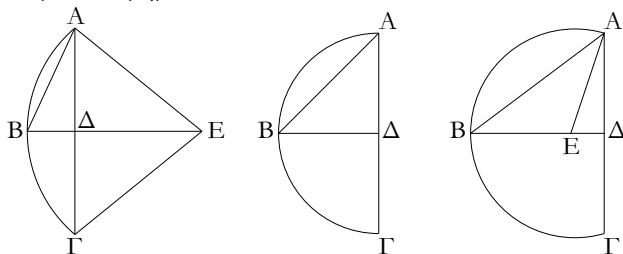
τὸ AEB τμήμα ἐπὶ τὸ $\Gamma Z\Delta$ · ἐφαρμόσει ἄρα, καὶ ἴσον αὐτῷ ἔσται.

Τὰ ἄρα ἐπὶ ἴσων εὐθειῶν ὅμοια τμήματα κύκλων ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

† Both this possibility, and the previous one, are precluded by Prop. 3.23.

κε'.

Κύκλου τμήματος δοθέντος προσαναγράψαι τὸν κύκλον, οὐπὲρ ἐστὶ τμήμα.



Ἐστω τὸ δοθὲν τμήμα κύκλου τὸ $AB\Gamma$ · δεῖ δὴ τοῦ $AB\Gamma$ τμήματος προσαναγράψαι τὸν κύκλον, οὐπὲρ ἐστὶ τμήμα.

Τετμήσθω γὰρ ἡ AG δίχα κατὰ τὸ Δ , καὶ ῥηθῶ ἀπὸ τοῦ Δ σημείου τῇ AG πρὸς ὀρθὰς ἡ ΔB , καὶ ἐπεζεύχθω ἡ AB · ἡ ὑπὸ $AB\Delta$ γωνία ἄρα τῆς ὑπὸ $BA\Delta$ ἥτοι μείζων ἐστὶν ἢ ἴση ἢ ἐλάττω.

Ἐστω πρότερον μείζων, καὶ συνεστάτω πρὸς τῇ BA εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ A τῇ ὑπὸ $AB\Delta$ γωνίᾳ ἴση ἡ ὑπὸ BAE , καὶ διήχθω ἡ ΔB ἐπὶ τὸ E , καὶ ἐπεζεύχθω ἡ EF . ἐπεὶ οὖν ἴση ἐστὶν ἡ ὑπὸ ABE γωνία τῇ ὑπὸ BAE , ἴση ἄρα ἐστὶ καὶ ἡ EB εὐθεῖα τῇ EA . καὶ ἐπεὶ ἴση ἐστὶν ἡ AD τῇ $\Delta\Gamma$, κοινὴ δὲ ἡ ΔE , δύο δὴ αἱ AD , ΔE δύο ταῖς $\Gamma\Delta$, ΔE ἴσαι εἰσὶν ἑκατέρᾳ ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ $AD\Gamma$ γωνία τῇ ὑπὸ $\Gamma\Delta E$ ἐστὶν ἴση· ὀρθὴ γὰρ ἑκατέρᾳ· βάσις ἄρα ἡ AE βάσει τῇ ΓE ἐστὶν ἴση· ἀλλὰ ἡ AE τῇ BE ἐδείχθη ἴση· καὶ ἡ BE ἄρα τῇ ΓE ἐστὶν ἴση· αἱ τρεῖς ἄρα αἱ AE , EB , $E\Gamma$ ἴσαι ἀλλήλαις εἰσὶν· ὁ ἄρα κέντρον τῷ E διαστήματι δὲ ἐνὶ τῶν AE , EB , $E\Gamma$ κύκλος γραφόμενος ῥῆξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἔσται προσαναγεγραμμένος· κύκλου ἄρα τμήματος δοθέντος προσαναγεγραπται ὁ κύκλος· καὶ δῆλον, ὡς τὸ $AB\Gamma$ τμήμα ἐλαττόν ἐστιν ἡμικυκλίου διὰ τὸ τὸ E κέντρον ἐκτὸς αὐτοῦ τυγχάνειν.

Ὁμοίως [δὲ] καὶ ἡ ὑπὸ $AB\Delta$ γωνία ἴση τῇ ὑπὸ $BA\Delta$, τῆς AD ἴσης γενομένης ἑκατέρᾳ τῶν $B\Delta$, $\Delta\Gamma$ αἱ τρεῖς αἱ ΔA , ΔB , $\Delta\Gamma$ ἴσαι ἀλλήλαις ἔσονται, καὶ ἔσται τὸ Δ κέντρον τοῦ προσαναπεπληρωμένου κύκλου, καὶ δηλαδὴ ἔσται τὸ $AB\Gamma$ ἡμικύκλιον.

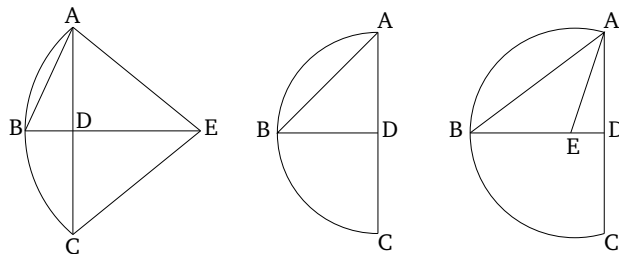
Ἐάν δὲ ἡ ὑπὸ $AB\Delta$ ἐλάττω ἢ τῆς ὑπὸ $BA\Delta$, καὶ συστησώμεθα πρὸς τῇ BA εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ

thing is impossible [Prop. 3.10]. Thus, if the straight-line AB is applied to CD , the segment AEB cannot not also coincide with CFD . Thus, it will coincide, and will be equal to it [C.N. 4].

Thus, similar segments of circles on equal straight-lines are equal to one another. (Which is) the very thing it was required to show.

Proposition 25

For a given segment of a circle, to complete the circle, the very one of which it is a segment.



Let ABC be the given segment of a circle. So it is required to complete the circle for segment ABC , the very one of which it is a segment.

For let AC have been cut in half at (point) D [Prop. 1.10], and let DB have been drawn from point D , at right-angles to AC [Prop. 1.11]. And let AB have been joined. Thus, angle ABD is surely either greater than, equal to, or less than (angle) BAD .

First of all, let it be greater. And let (angle) BAE , equal to angle ABD , have been constructed on the straight-line BA , at the point A on it [Prop. 1.23]. And let DB have been drawn through to E , and let EC have been joined. Therefore, since angle ABE is equal to BAE , the straight-line EB is thus also equal to EA [Prop. 1.6]. And since AD is equal to DC , and DE (is) common, the two (straight-lines) AD , DE are equal to the two (straight-lines) CD , DE , respectively. And angle ADE is equal to angle CDE . For each (is) a right-angle. Thus, the base AE is equal to the base CE [Prop. 1.4]. But, AE was shown (to be) equal to BE . Thus, BE is also equal to CE . Thus, the three (straight-lines) AE , EB , and EC are equal to one another. Thus, if a circle is drawn with center E , and radius one of AE , EB , or EC , it will also go through the remaining points (of the segment), and the (associated circle) will have been completed [Prop. 3.9]. Thus, a circle has been completed from the given segment of a circle. And (it is) clear that the segment ABC is less than a semi-circle, because the center E happens to lie outside it.

τῷ A τῇ ὑπὸ $AB\Delta$ γωνίᾳ ἴσην, ἐντὸς τοῦ $AB\Gamma$ τμήματος πεσεῖται τὸ κέντρον ἐπὶ τῆς ΔB , καὶ ἔσται δηλαδὴ τὸ $AB\Gamma$ τμήμα μείζον ἡμικυκλίου.

Κύκλου ἄρα τμήματος δοθέντος προσαναγέγραπται ὁ κύκλος· ὅπερ ἔδει ποιῆσαι.

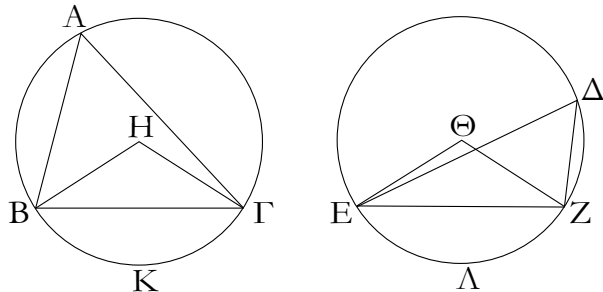
[And], similarly, even if angle ABD is equal to BAD , (since) AD becomes equal to each of BD [Prop. 1.6] and DC , the three (straight-lines) DA , DB , and DC will be equal to one another. And point D will be the center of the completed circle. And ABC will manifestly be a semi-circle.

And if ABD is less than BAD , and we construct (angle BAE), equal to angle ABD , on the straight-line BA , at the point A on it [Prop. 1.23], then the center will fall on DB , inside the segment ABC . And segment ABC will manifestly be greater than a semi-circle.

Thus, a circle has been completed from the given segment of a circle. (Which is) the very thing it was required to do.

κτ'.

Ἐν τοῖς ἴσοις κύκλοις αἱ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ἂν τε πρὸς τοῖς κέντροις ἂν τε πρὸς ταῖς περιφερείαις ὡς βεβηκυῖαι.



Ἐστωσαν ἴσοι κύκλοι οἱ $AB\Gamma$, ΔEZ καὶ ἐν αὐτοῖς ἴσαι γωνίαι ἔστωσαν πρὸς μὲν τοῖς κέντροις αἱ ὑπὸ BHG , $E\Theta Z$, πρὸς δὲ ταῖς περιφερείαις αἱ ὑπὸ BAG , $E\Delta Z$ · λέγω, ὅτι ἴση ἔστιν ἡ $BK\Gamma$ περιφέρεια τῇ ELZ περιφερείᾳ.

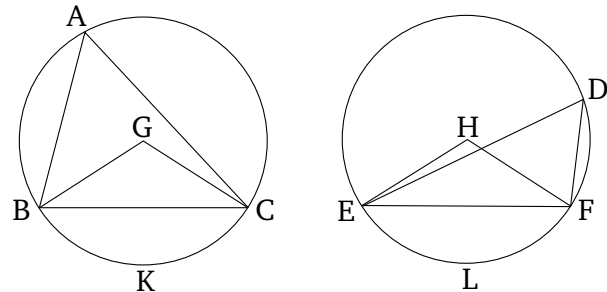
Ἐπεζεύχθωσαν γὰρ αἱ $B\Gamma$, EZ .

Καὶ ἐπεὶ ἴσοι εἰσὶν οἱ $AB\Gamma$, ΔEZ κύκλοι, ἴσαι εἰσὶν αἱ ἐκ τῶν κέντρων· δύο δὲ αἱ BH , $H\Gamma$ δύο ταῖς $E\Theta$, ΘZ ἴσαι· καὶ γωνία ἡ πρὸς τῷ H γωνία τῇ πρὸς τῷ Θ ἴση· βάσεις ἄρα ἡ $B\Gamma$ βάσει τῇ EZ ἔστιν ἴση. καὶ ἐπεὶ ἴση ἔστιν ἡ πρὸς τῷ A γωνία τῇ πρὸς τῷ Δ , ὅμοιον ἄρα ἔστι τὸ BAG τμήμα τῷ $E\Delta Z$ τμήματι· καὶ εἰσὶν ἐπὶ ἴσων εὐθειῶν [τῶν $B\Gamma$, EZ]· τὰ δὲ ἐπὶ ἴσων εὐθειῶν ὅμοια τμήματα κύκλων ἴσα ἀλλήλοις ἔστιν· ἴσον ἄρα τὸ BAG τμήμα τῷ $E\Delta Z$. ἔστι δὲ καὶ ὅλος ὁ $AB\Gamma$ κύκλος ὅλῳ τῷ ΔEZ κύκλῳ ἴσος· λοιπὴ ἄρα ἡ $BK\Gamma$ περιφέρεια τῇ ELZ περιφερείᾳ ἔστιν ἴση.

Ἐν ἄρα τοῖς ἴσοις κύκλοις αἱ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ἂν τε πρὸς τοῖς κέντροις ἂν τε πρὸς ταῖς περιφερείαις ὡς βεβηκυῖαι· ὅπερ ἔδει δεῖξαι.

Proposition 26

In equal circles, equal angles stand upon equal circumferences whether they are standing at the center or at the circumference.



Let ABC and DEF be equal circles, and within them let BGC and EHF be equal angles at the center, and BAC and EDF (equal angles) at the circumference. I say that circumference BKC is equal to circumference ELF .

For let BC and EF have been joined.

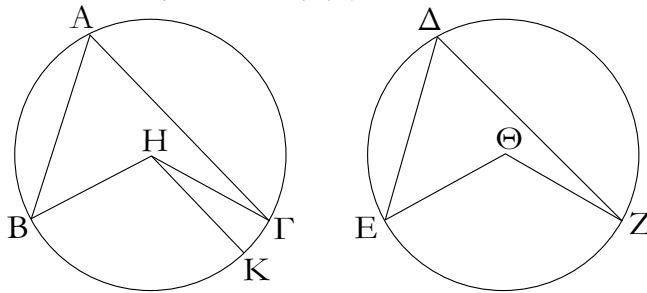
And since circles ABC and DEF are equal, their radii are equal. So the two (straight-lines) BG , GC (are) equal to the two (straight-lines) EH , HF (respectively). And the angle at G (is) equal to the angle at H . Thus, the base BC is equal to the base EF [Prop. 1.4]. And since the angle at A is equal to the (angle) at D , the segment BAC is thus similar to the segment EDF [Def. 3.11]. And they are on equal straight-lines [BC and EF]. And similar segments of circles on equal straight-lines are equal to one another [Prop. 3.24]. Thus, segment BAC is equal to (segment) EDF . And the whole circle ABC is also equal to the whole circle DEF . Thus, the remaining circumference BKC is equal to the (remaining) circumference ELF .

Thus, in equal circles, equal angles stand upon equal circumferences, whether they are standing at the center

or at the circumference. (Which is) the very thing which it was required to show.

κζ'.

Ἐν τοῖς ἴσοις κύκλοις αἱ ἐπὶ ἴσων περιφερειῶν βεβηκυῖαι γωνίαι ἴσαι ἀλλήλαις εἰσίν, ἂν τε πρὸς τοῖς κέντροις ἂν τε πρὸς ταῖς περιφερείαις ὥσι βεβηκυῖαι.



Ἐν γὰρ ἴσοις κύκλοις τοῖς $ABΓ$, $ΔΕΖ$ ἐπὶ ἴσων περιφερειῶν τῶν $ΒΓ$, $ΕΖ$ πρὸς μὲν τοῖς H , $Θ$ κέντροις γωνίαι βεβηκέτωσαν αἱ ὑπὸ BHG , $ΕΘΖ$, πρὸς δὲ ταῖς περιφερείαις αἱ ὑπὸ BAG , $ΕΔΖ$ · λέγω, ὅτι ἡ μὲν ὑπὸ BHG γωνία τῇ ὑπὸ $ΕΘΖ$ ἐστὶν ἴση, ἡ δὲ ὑπὸ BAG τῇ ὑπὸ $ΕΔΖ$ ἐστὶν ἴση.

Εἰ γὰρ ἄνισός ἐστιν ἡ ὑπὸ BHG τῇ ὑπὸ $ΕΘΖ$, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ὑπὸ BHG , καὶ συνεστάτω πρὸς τῇ BH εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ H τῇ ὑπὸ $ΕΘΖ$ γωνίᾳ ἴση ἡ ὑπὸ BHK · αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ὅταν πρὸς τοῖς κέντροις ὦσιν· ἴση ἄρα ἡ BK περιφέρεια τῇ $ΕΖ$ περιφέρειᾳ. ἀλλὰ ἡ $ΕΖ$ τῇ $ΒΓ$ ἐστὶν ἴση· καὶ ἡ BK ἄρα τῇ $ΒΓ$ ἐστὶν ἴση ἢ ἐλάττω τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἄνισός ἐστιν ἡ ὑπὸ BHG γωνία τῇ ὑπὸ $ΕΘΖ$ · ἴση ἄρα. καὶ ἐστὶ τῆς μὲν ὑπὸ BHG ἡμίσεια ἡ πρὸς τῷ A , τῆς δὲ ὑπὸ $ΕΘΖ$ ἡμίσεια ἡ πρὸς τῷ $Δ$ · ἴση ἄρα καὶ ἡ πρὸς τῷ A γωνία τῇ πρὸς τῷ $Δ$.

Ἐν ἄρα τοῖς ἴσοις κύκλοις αἱ ἐπὶ ἴσων περιφερειῶν βεβηκυῖαι γωνίαι ἴσαι ἀλλήλαις εἰσίν, ἂν τε πρὸς τοῖς κέντροις ἂν τε πρὸς ταῖς περιφερείαις ὥσι βεβηκυῖαι· ὅπερ ἔδει δεῖξαι.

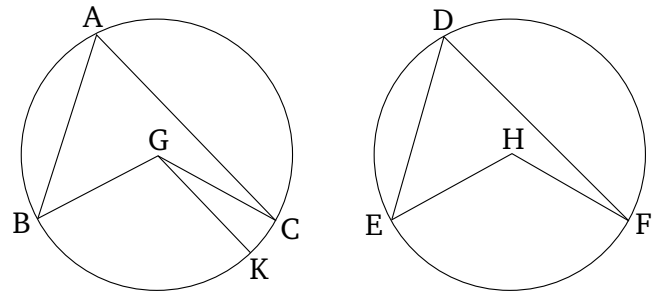
κη'.

Ἐν τοῖς ἴσοις κύκλοις αἱ ἴσαι εὐθεῖαι ἴσας περιφερείας ἀφαιροῦσι τὴν μὲν μείζονα τῇ μείζονι τὴν δὲ ἐλάττωνα τῇ ἐλάττω.

Ἐστῶσαν ἴσοι κύκλοι οἱ $ABΓ$, $ΔΕΖ$, καὶ ἐν τοῖς κύκλοις ἴσαι εὐθεῖαι ἔστωσαν αἱ AB , $ΔΕ$ τὰς μὲν $ΑΓΒ$, $ΑΖΕ$ περιφερείας μείζονας ἀφαιροῦσαι τὰς δὲ AHB , $ΔΘΕ$ ἐλάττωνας· λέγω, ὅτι ἡ μὲν $ΑΓΒ$ μείζων περιφέρεια ἴση ἐστὶ τῇ $ΔΖΕ$ μείζονι περιφέρειᾳ ἡ δὲ AHB ἐλάττων περιφέρεια τῇ $ΔΘΕ$.

Proposition 27

In equal circles, angles standing upon equal circumferences are equal to one another, whether they are standing at the center or at the circumference.



For let the angles BGC and EHF at the centers G and H , and the (angles) BAC and EDF at the circumferences, stand upon the equal circumferences BC and EF , in the equal circles ABC and DEF (respectively). I say that angle BGC is equal to (angle) EHF , and BAC is equal to EDF .

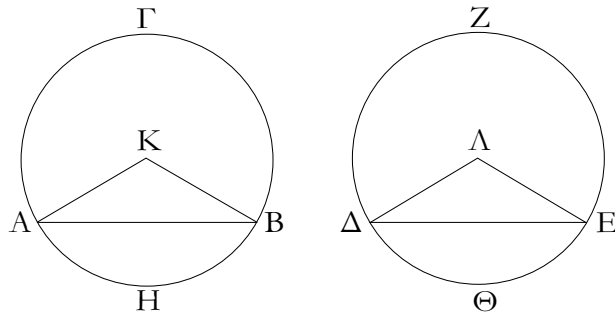
For if BGC is unequal to EHF , one of them is greater. Let BGC be greater, and let the (angle) BGK , equal to angle EHF , have been constructed on the straight-line BG , at the point G on it [Prop. 1.23]. But equal angles (in equal circles) stand upon equal circumferences, when they are at the centers [Prop. 3.26]. Thus, circumference BK (is) equal to circumference EF . But, EF is equal to BC . Thus, BK is also equal to BC , the lesser to the greater. The very thing is impossible. Thus, angle BGC is not unequal to EHF . Thus, (it is) equal. And the (angle) at A is half BGC , and the (angle) at D half EHF [Prop. 3.20]. Thus, the angle at A (is) also equal to the (angle) at D .

Thus, in equal circles, angles standing upon equal circumferences are equal to one another, whether they are standing at the center or at the circumference. (Which is) the very thing it was required to show.

Proposition 28

In equal circles, equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser.

Let ABC and DEF be equal circles, and let AB and DE be equal straight-lines in these circles, cutting off the greater circumferences ACB and DFE , and the lesser (circumferences) AGB and DHE (respectively). I say that the greater circumference ACB is equal to the greater circumference DFE , and the lesser circumfer-

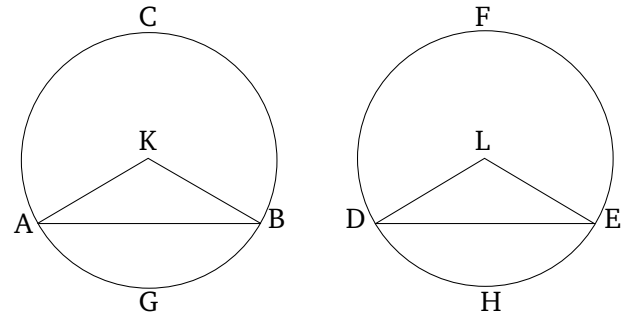


Εἰλήφθω γὰρ τὰ κέντρα τῶν κύκλων τὰ K , Λ , καὶ ἐπεζεύχθωσαν αἱ AK , KB , $\Delta\Lambda$, ΛE .

Καὶ ἐπεὶ ἴσοι κύκλοι εἰσὶν, ἴσαι εἰσὶ καὶ αἱ ἐκ τῶν κέντρων· δύο δὲ αἱ AK , KB δυσὶ ταῖς $\Delta\Lambda$, ΛE ἴσαι εἰσὶν· καὶ βάσις ἡ AB βάσει τῇ ΔE ἴση· γωνία ἄρα ἡ ὑπὸ AKB γωνία τῇ ὑπὸ $\Delta\Lambda E$ ἴση ἐστίν. αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν, ὅταν πρὸς τοῖς κέντροις ὦσιν· ἴση ἄρα ἡ AHB περιφέρεια τῇ $\Delta\Theta E$. ἐστὶ δὲ καὶ ὅλος ὁ $AB\Gamma$ κύκλος ὅλῳ τῷ $\Delta E Z$ κύκλῳ ἴσος· καὶ λοιπὴ ἄρα ἡ $ΑΓΒ$ περιφέρεια λοιπῇ τῇ $\Delta Z E$ περιφερείᾳ ἴση ἐστίν.

Ἐν ἄρα τοῖς ἴσοις κύκλοις αἱ ἴσαι εὐθεῖαι ἴσας περιφερείας ἀφαιροῦσι τὴν μὲν μείζονα τῇ μείζονι τὴν δὲ ἐλάττονα τῇ ἐλάττονι· ὅπερ εἶδει δεῖξαι.

ence AGB to (the lesser) DHE .



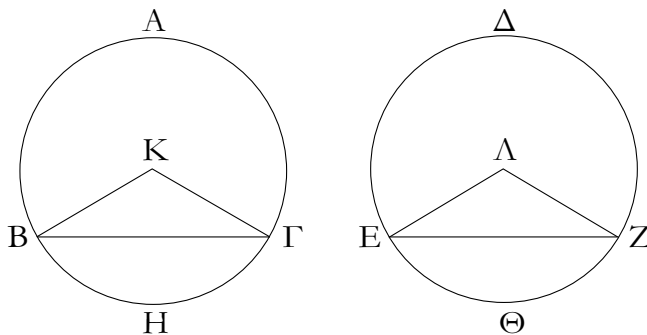
For let the centers of the circles, K and L , have been found [Prop. 3.1], and let AK , KB , DL , and LE have been joined.

And since $(ABC$ and $DEF)$ are equal circles, their radii are also equal [Def. 3.1]. So the two (straight-lines) AK , KB are equal to the two (straight-lines) DL , LE (respectively). And the base AB (is) equal to the base DE . Thus, angle AKB is equal to angle DLE [Prop. 1.8]. And equal angles stand upon equal circumferences, when they are at the centers [Prop. 3.26]. Thus, circumference AGB (is) equal to DHE . And the whole circle ABC is also equal to the whole circle DEF . Thus, the remaining circumference ACB is also equal to the remaining circumference DFE .

Thus, in equal circles, equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser. (Which is) the very thing it was required to show.

κθ'.

Ἐν τοῖς ἴσοις κύκλοις τὰς ἴσας περιφερείας ἴσαι εὐθεῖαι ὑποτείνουσιν.



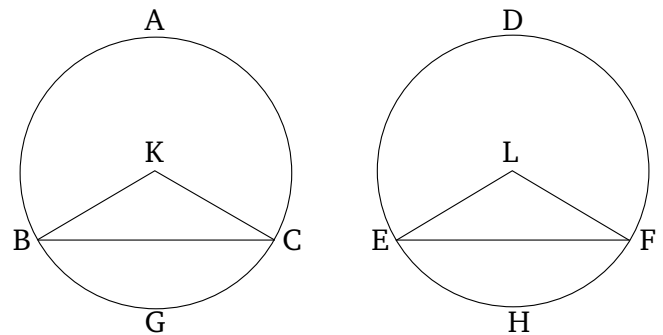
Ἐστωσαν ἴσοι κύκλοι οἱ $AB\Gamma$, $\Delta E Z$, καὶ ἐν αὐτοῖς ἴσαι περιφέρειαι ἀπειλήφθωσαν αἱ $BH\Gamma$, $E\Theta Z$, καὶ ἐπεζεύχθωσαν αἱ $B\Gamma$, $E Z$ εὐθεῖαι· λέγω, ὅτι ἴση ἐστὶν ἡ $B\Gamma$ τῇ $E Z$.

Εἰλήφθω γὰρ τὰ κέντρα τῶν κύκλων, καὶ ἔστω τὰ K , Λ , καὶ ἐπεζεύχθωσαν αἱ BK , $K\Gamma$, $E\Lambda$, ΛZ .

Καὶ ἐπεὶ ἴση ἐστὶν ἡ $BH\Gamma$ περιφέρεια τῇ $E\Theta Z$ περιφερείᾳ,

Proposition 29

In equal circles, equal straight-lines subtend equal circumferences.



Let ABC and DEF be equal circles, and within them let the equal circumferences BGC and EHF have been cut off. And let the straight-lines BC and EF have been joined. I say that BC is equal to EF .

For let the centers of the circles have been found [Prop. 3.1], and let them be (at) K and L . And let BK ,

ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ $B\Gamma\Delta$ τῇ ὑπὸ $\epsilon\lambda\zeta$. καὶ ἐπεὶ ἴσοι εἰσὶν οἱ $AB\Gamma$, $\Delta\epsilon\zeta$ κύκλοι, ἴσαι εἰσὶ καὶ αἱ ἐκ τῶν κέντρων· δύο δὲ αἱ BK , $K\Gamma$ δυσὶ ταῖς $\epsilon\lambda$, $\lambda\zeta$ ἴσαι εἰσὶν· καὶ γωνίας ἴσας περιέχουσιν· βάσις ἄρα ἡ $B\Gamma$ βάσει τῇ $\epsilon\zeta$ ἴση ἐστίν·

Ἐν ἄρα τοῖς ἴσοις κύκλοις τὰς ἴσας περιφερείας ἴσαι εὐθεΐαι ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

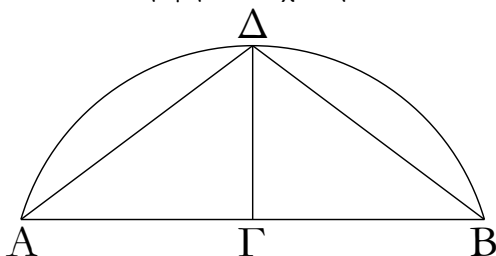
KC , EL , and LF have been joined.

And since the circumference BGC is equal to the circumference EHF , the angle BKC is also equal to (angle) ELF [Prop. 3.27]. And since the circles ABC and DEF are equal, their radii are also equal [Def. 3.1]. So the two (straight-lines) BK , KC are equal to the two (straight-lines) EL , LF (respectively). And they contain equal angles. Thus, the base BC is equal to the base EF [Prop. 1.4].

Thus, in equal circles, equal straight-lines subtend equal circumferences. (Which is) the very thing it was required to show.

λ'.

Τὴν δοθεῖσαν περιφέρειαν δίχα τεμεῖν.



Ἐστω ἡ δοθεῖσα περιφέρεια ἡ $A\Delta B$ · δεῖ δὲ τὴν $A\Delta B$ περιφέρειαν δίχα τεμεῖν.

Ἐπεζεύχθω ἡ AB , καὶ τετμήσθω δίχα κατὰ τὸ Γ , καὶ ἀπὸ τοῦ Γ σημείου τῇ AB εὐθείᾳ πρὸς ὀρθὰς ἦχθω ἡ $\Gamma\Delta$, καὶ ἐπεζεύχθωσαν αἱ $A\Delta$, ΔB .

Καὶ ἐπεὶ ἴση ἐστὶν ἡ $A\Gamma$ τῇ ΓB , κοινὴ δὲ ἡ $\Gamma\Delta$, δύο δὲ αἱ $A\Gamma$, $\Gamma\Delta$ δυσὶ ταῖς $B\Gamma$, $\Gamma\Delta$ ἴσαι εἰσὶν· καὶ γωνία ἡ ὑπὸ $A\Gamma\Delta$ γωνία τῇ ὑπὸ $B\Gamma\Delta$ ἴση· ὀρθὴ γὰρ ἑκατέρω· βάσις ἄρα ἡ $A\Delta$ βάσει τῇ ΔB ἴση ἐστίν. αἱ δὲ ἴσαι εὐθεΐαι ἴσας περιφερείας ἀφαιροῦσι τὴν μὲν μείζονα τῇ μείζονι τὴν δὲ ἐλάττωνα τῇ ἐλάττω· καὶ ἐστὶν ἑκατέρα τῶν $A\Delta$, ΔB περιφερειῶν ἐλάττων ἡμικυκλίου· ἴση ἄρα ἡ $A\Delta$ περιφέρεια τῇ ΔB περιφέρειᾳ.

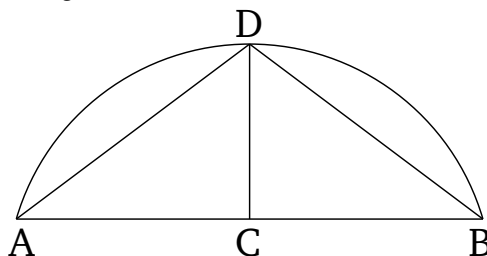
Ἡ ἄρα δοθεῖσα περιφέρεια δίχα τέτμηται κατὰ τὸ Δ σημεῖον· ὅπερ ἔδει ποιῆσαι.

λα'.

Ἐν κύκλῳ ἡ μὲν ἐν τῷ ἡμικυκλίῳ γωνία ὀρθὴ ἐστίν, ἡ δὲ ἐν τῷ μείζονι τμήματι ἐλάττων ὀρθῆς, ἡ δὲ ἐν τῷ ἐλάττωι τμήματι μείζων ὀρθῆς· καὶ ἔπι ἡ μὲν τοῦ μείζονος τμήματος γωνία μείζων ἐστὶν ὀρθῆς, ἡ δὲ τοῦ ἐλάττονος τμήματος γωνία ἐλάττων ὀρθῆς.

Proposition 30

To cut a given circumference in half.



Let ADB be the given circumference. So it is required to cut circumference ADB in half.

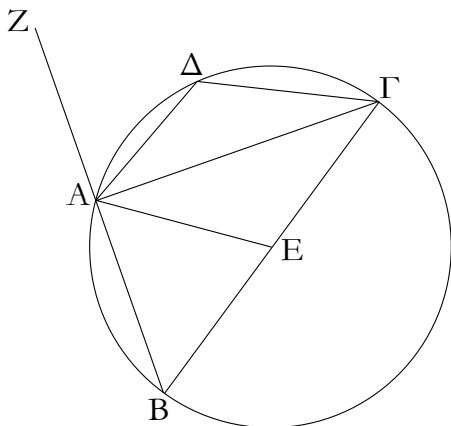
Let AB have been joined, and let it have been cut in half at (point) C [Prop. 1.10]. And let CD have been drawn from point C , at right-angles to AB [Prop. 1.11]. And let AD , and DB have been joined.

And since AC is equal to CB , and CD (is) common, the two (straight-lines) AC , CD are equal to the two (straight-lines) BC , CD (respectively). And angle ACD (is) equal to angle BCD . For (they are) each right-angles. Thus, the base AD is equal to the base DB [Prop. 1.4]. And equal straight-lines cut off equal circumferences, the greater (circumference being equal) to the greater, and the lesser to the lesser [Prop. 1.28]. And the circumferences AD and DB are each less than a semi-circle. Thus, circumference AD (is) equal to circumference DB .

Thus, the given circumference has been cut in half at point D . (Which is) the very thing it was required to do.

Proposition 31

In a circle, the angle in a semi-circle is a right-angle, and that in a greater segment (is) less than a right-angle, and that in a lesser segment (is) greater than a right-angle. And, further, the angle of a segment greater (than a semi-circle) is greater than a right-angle, and the an-



Ἐστω κύκλος ὁ $ABΓΔ$, διάμετρος δὲ αὐτοῦ ἔστω ἡ $ΒΓ$, κέντρον δὲ τὸ $Ε$, καὶ ἐπεζεύχθωσαν αἱ $ΒΑ$, $ΑΓ$, $ΑΔ$, $ΔΓ$. λέγω, ὅτι ἡ μὲν ἐν τῷ $ΒΑΓ$ ἡμικυκλίῳ γωνία ἡ ὑπὸ $ΒΑΓ$ ὀρθή ἐστίν, ἡ δὲ ἐν τῷ $ΑΒΓ$ μείζονι τοῦ ἡμικυκλίου τμήματι γωνία ἡ ὑπὸ $ΑΒΓ$ ἐλάττω ἐστὶν ὀρθῆς, ἡ δὲ ἐν τῷ $ΑΔΓ$ ἐλάττω τοῦ ἡμικυκλίου τμήματι γωνία ἡ ὑπὸ $ΑΔΓ$ μείζων ἐστὶν ὀρθῆς.

Ἐπεζεύχθω ἡ $ΑΕ$, καὶ διήχθω ἡ $ΒΑ$ ἐπὶ τὸ $Ζ$.

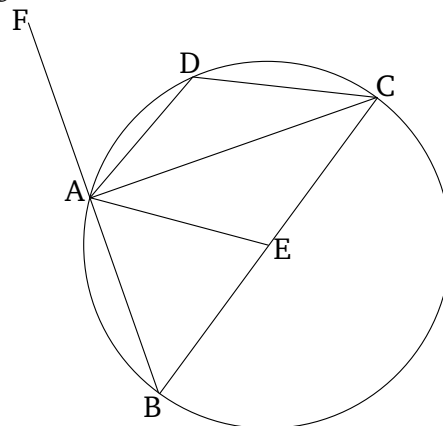
Καὶ ἐπεὶ ἴση ἐστὶν ἡ $ΒΕ$ τῇ $ΕΑ$, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ $ΑΒΕ$ τῇ ὑπὸ $ΒΑΕ$. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ $ΓΕ$ τῇ $ΕΑ$, ἴση ἐστὶ καὶ ἡ ὑπὸ $ΑΓΕ$ τῇ ὑπὸ $ΓΑΕ$. ὅλη ἄρα ἡ ὑπὸ $ΒΑΓ$ δυσὶ ταῖς ὑπὸ $ΑΒΓ$, $ΑΓΒ$ ἴση ἐστίν. ἐστὶ δὲ καὶ ἡ ὑπὸ $ΖΑΓ$ ἐκτὸς τοῦ $ΑΒΓ$ τριγώνου δυσὶ ταῖς ὑπὸ $ΑΒΓ$, $ΑΓΒ$ γωνίαις ἴση· ἴση ἄρα καὶ ἡ ὑπὸ $ΒΑΓ$ γωνία τῇ ὑπὸ $ΖΑΓ$. ὀρθὴ ἄρα ἐκατέρω· ἡ ἄρα ἐν τῷ $ΒΑΓ$ ἡμικυκλίῳ γωνία ἡ ὑπὸ $ΒΑΓ$ ὀρθή ἐστίν.

Καὶ ἐπεὶ τοῦ $ΑΒΓ$ τριγώνου δύο γωνίαι αἱ ὑπὸ $ΑΒΓ$, $ΒΑΓ$ δύο ὀρθῶν ἐλάττων ἐσίν, ὀρθὴ δὲ ἡ ὑπὸ $ΒΑΓ$, ἐλάττων ἄρα ὀρθῆς ἐστὶν ἡ ὑπὸ $ΑΒΓ$ γωνία· καὶ ἐστὶν ἐν τῷ $ΑΒΓ$ μείζονι τοῦ ἡμικυκλίου τμήματι.

Καὶ ἐπεὶ ἐν κύκλῳ τετράπλευρόν ἐστι τὸ $ΑΒΓΔ$, τῶν δὲ ἐν τοῖς κύκλοις τετραπλεύρων αἱ ἀπεναντίον γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσὶν [αἱ ἄρα ὑπὸ $ΑΒΓ$, $ΑΔΓ$ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσὶν], καὶ ἐστὶν ἡ ὑπὸ $ΑΒΓ$ ἐλάττων ὀρθῆς· λοιπὴ ἄρα ἡ ὑπὸ $ΑΔΓ$ γωνία μείζων ὀρθῆς ἐστίν· καὶ ἐστὶν ἐν τῷ $ΑΔΓ$ ἐλάττω τοῦ ἡμικυκλίου τμήματι.

Λέγω, ὅτι καὶ ἡ μὲν τοῦ μείζονος τμήματος γωνία ἡ περιεχομένη ὑπὸ [τε] τῆς $ΑΒΓ$ περιφερείας καὶ τῆς $ΑΓ$ εὐθείας μείζων ἐστὶν ὀρθῆς, ἡ δὲ τοῦ ἐλάττονος τμήματος γωνία ἡ περιεχομένη ὑπὸ [τε] τῆς $ΑΔΓ$ περιφερείας καὶ τῆς $ΑΓ$ εὐθείας ἐλάττων ἐστὶν ὀρθῆς. καὶ ἐστὶν αὐτόθεν φανερόν. ἐπεὶ γὰρ ἡ ὑπὸ τῶν $ΒΑ$, $ΑΓ$ εὐθειῶν ὀρθὴ ἐστίν, ἡ ἄρα ὑπὸ τῆς $ΑΒΓ$ περιφερείας καὶ τῆς $ΑΓ$ εὐθείας περιεχομένη μείζων ἐστὶν ὀρθῆς. πάλιν, ἐπεὶ ἡ ὑπὸ τῶν $ΑΓ$, $ΑΖ$ εὐθειῶν ὀρθὴ ἐστίν, ἡ ἄρα ὑπὸ τῆς $ΓΑ$ εὐθείας καὶ τῆς $ΑΔΓ$ περι-

gle of a segment less (than a semi-circle) is less than a right-angle.



Let $ABCD$ be a circle, and let BC be its diameter, and E its center. And let BA , AC , AD , and DC have been joined. I say that the angle BAC in the semi-circle BAC is a right-angle, and the angle ABC in the segment ABC , (which is) greater than a semi-circle, is less than a right-angle, and the angle ADC in the segment ADC , (which is) less than a semi-circle, is greater than a right-angle.

Let AE have been joined, and let BA have been drawn through to F .

And since BE is equal to EA , angle ABE is also equal to BAE [Prop. 1.5]. Again, since CE is equal to EA , ACE is also equal to CAE [Prop. 1.5]. Thus, the whole (angle) BAC is equal to the two (angles) ABC and ACB . And FAC , (which is) external to triangle ABC , is also equal to the two angles ABC and ACB [Prop. 1.32]. Thus, angle BAC (is) also equal to FAC . Thus, (they are) each right-angles. [Def. 1.10]. Thus, the angle BAC in the semi-circle BAC is a right-angle.

And since the two angles ABC and BAC of triangle ABC are less than two right-angles [Prop. 1.17], and BAC is a right-angle, angle ABC is thus less than a right-angle. And it is in segment ABC , (which is) greater than a semi-circle.

And since $ABCD$ is a quadrilateral within a circle, and for quadrilaterals within circles the (sum of the) opposite angles is equal to two right-angles [Prop. 3.22] [angles ABC and ADC are thus equal to two right-angles], and (angle) ABC is less than a right-angle. The remaining angle ADC is thus greater than a right-angle. And it is in segment ADC , (which is) less than a semi-circle.

I also say that the angle of the greater segment, (namely) that contained by the circumference ABC and the straight-line AC , is greater than a right-angle. And the angle of the lesser segment, (namely) that contained

φερείας περιεχομένη ἐλάττων ἐστὶν ὀρθῆς.

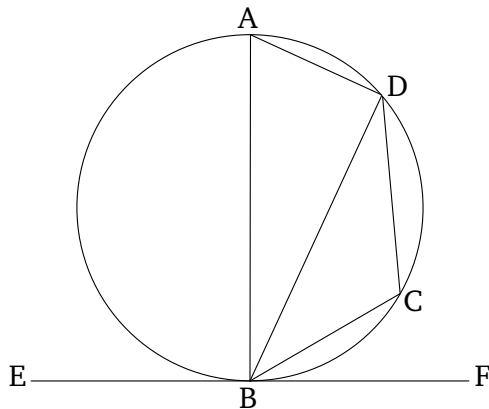
Ἐν κύκλῳ ἄρα ἡ μὲν ἐν τῷ ἡμικυκλίῳ γωνία ὀρθή ἐστίν, ἡ δὲ ἐν τῷ μείζονι τμήματι ἐλάττων ὀρθῆς, ἡ δὲ ἐν τῷ ἐλάττονι [τμήματι] μείζων ὀρθῆς· καὶ ἔπι ἡ μὲν τοῦ μείζονος τμήματος [γωνία] μείζων [ἐστὶν] ὀρθῆς, ἡ δὲ τοῦ ἐλάττονος τμήματος [γωνία] ἐλάττων ὀρθῆς· ὅπερ ἔδει δεῖξαι.

by the circumference $AD[C]$ and the straight-line AC , is less than a right-angle. And this is immediately apparent. For since the (angle contained by) the two straight-lines BA and AC is a right-angle, the (angle) contained by the circumference ABC and the straight-line AC is thus greater than a right-angle. Again, since the (angle contained by) the straight-lines AC and AF is a right-angle, the (angle) contained by the circumference $AD[C]$ and the straight-line CA is thus less than a right-angle.

Thus, in a circle, the angle in a semi-circle is a right-angle, and that in a greater segment (is) less than a right-angle, and that in a lesser [segment] (is) greater than a right-angle. And, further, the [angle] of a segment greater (than a semi-circle) [is] greater than a right-angle, and the [angle] of a segment less (than a semi-circle) is less than a right-angle. (Which is) the very thing it was required to show.

λβ'.

Ἐὰν κύκλου ἐφάπτηται τις εὐθεΐα, ἀπὸ δὲ τῆς ἀφῆς εἰς τὸν κύκλον διαχθῇ τις εὐθεΐα τέμνουσα τὸν κύκλον, ἃς ποιῇ γωνίας πρὸς τῇ ἐφαπτομένῃ, ἴσαι ἔσονται ταῖς ἐν τοῖς ἐναλλάξ τοῦ κύκλου τμήμασι γωνίαις.



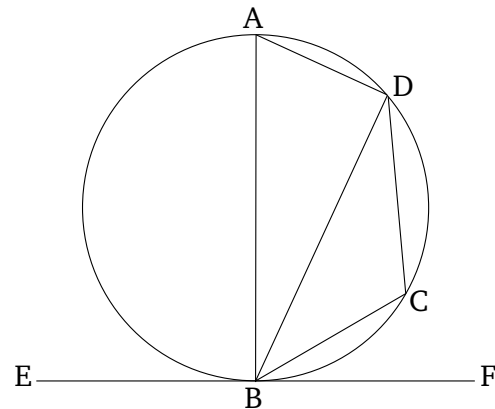
Κύκλου γὰρ τοῦ $AB\Gamma\Delta$ ἐφαπτέσθω τις εὐθεΐα ἡ EZ κατὰ τὸ B σημεῖον, καὶ ἀπὸ τοῦ B σημείου διήχθω τις εὐθεΐα εἰς τὸν $AB\Gamma\Delta$ κύκλον τέμνουσα αὐτὸν ἡ BD . λέγω, ὅτι ἃς ποιῇ γωνίας ἡ BD μετὰ τῆς EZ ἐφαπτομένης, ἴσας ἔσονται ταῖς ἐν τοῖς ἐναλλάξ τμήμασι τοῦ κύκλου γωνίαις, τουτέστιν, ὅτι ἡ μὲν ὑπὸ ZBD γωνία ἴση ἐστὶ τῇ ἐν τῷ $BA\Delta$ τμήματι συνισταμένῃ γωνίᾳ, ἡ δὲ ὑπὸ EBD γωνία ἴση ἐστὶ τῇ ἐν τῷ $\Delta\Gamma B$ τμήματι συνισταμένῃ γωνίᾳ.

Ἦχθω γὰρ ἀπὸ τοῦ B τῇ EZ πρὸς ὀρθὰς ἡ BA , καὶ εἰλήφθω ἐπὶ τῆς BD περιφερείας τυχὸν σημεῖον τὸ Γ , καὶ ἐπεζεύχθωσαν αἱ $A\Delta$, $\Delta\Gamma$, ΓB .

Καὶ ἐπεὶ κύκλου τοῦ $AB\Gamma\Delta$ ἐφάπτεται τις εὐθεΐα ἡ EZ

Proposition 32

If some straight-line touches a circle, and some (other) straight-line is drawn across, from the point of contact into the circle, cutting the circle (in two), then those angles the (straight-line) makes with the tangent will be equal to the angles in the alternate segments of the circle.



For let some straight-line EF touch the circle $ABCD$ at the point B , and let some (other) straight-line BD have been drawn from point B into the circle $ABCD$, cutting it (in two). I say that the angles BD makes with the tangent EF will be equal to the angles in the alternate segments of the circle. That is to say, that angle FBD is equal to the angle constructed in segment BAD , and angle EBD is equal to the angle constructed in segment DCB .

For let BA have been drawn from B , at right-angles to EF [Prop. 1.11]. And let the point C have been taken at random on the circumference BD . And let AD , DC ,

πρὸς ὀρθὰς ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ ΗΒ.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΖ τῇ ΖΒ, κοινὴ δὲ ἡ ΖΗ, δύο δὲ αἱ ΑΖ, ΖΗ δύο ταῖς ΒΖ, ΖΗ ἴσαι εἰσὶν· καὶ γωνία ἡ ὑπὸ ΑΖΗ [γωνία] τῇ ὑπὸ ΒΖΗ ἴση· βάσις ἄρα ἡ ΑΗ βάσει τῇ ΒΗ ἴση ἐστίν. ὁ ἄρα κέντρω μὲν τῷ Η διαστήματι δὲ τῷ ΗΑ κύκλος γραφόμενος ἥξει καὶ διὰ τοῦ Β. γεγράφθω καὶ ἔστω ὁ ΑΒΕ, καὶ ἐπεζεύχθω ἡ ΕΒ. ἐπεὶ οὖν ἀπ' ἄκρας τῆς ΑΕ διαμέτρου ἀπὸ τοῦ Α τῇ ΑΕ πρὸς ὀρθὰς ἐστὶν ἡ ΑΔ, ἡ ΑΔ ἄρα ἐφάπτεται τοῦ ΑΒΕ κύκλου· ἐπεὶ οὖν κύκλου τοῦ ΑΒΕ ἐφάπτεται τις εὐθεῖα ἡ ΑΔ, καὶ ἀπὸ τῆς κατὰ τὸ Α ἀφῆς εἰς τὸν ΑΒΕ κύκλον διήκται τις εὐθεῖα ἡ ΑΒ, ἡ ἄρα ὑπὸ ΔΑΒ γωνία ἴση ἐστὶ τῇ ἐν τῷ ἐναλλάξ τοῦ κύκλου τμήματι γωνία τῇ ὑπὸ ΑΕΒ. ἀλλ' ἡ ὑπὸ ΔΑΒ τῇ πρὸς τῷ Γ ἐστὶν ἴση· καὶ ἡ πρὸς τῷ Γ ἄρα γωνία ἴση ἐστὶ τῇ ὑπὸ ΑΕΒ.

Ἐπὶ τῆς δοθείσης ἄρα εὐθείας τῆς ΑΒ τμήμα κύκλου γέγραπται τὸ ΑΕΒ δεχόμενον γωνίαν τὴν ὑπὸ ΑΕΒ ἴσην τῇ δοθείσῃ τῇ πρὸς τῷ Γ.

Ἀλλὰ δὴ ὀρθὴ ἔστω ἡ πρὸς τῷ Γ· καὶ δεόν πάλιν ἔστω ἐπὶ τῆς ΑΒ γράψαι τμήμα κύκλου δεχόμενον γωνίαν ἴσην τῇ πρὸς τῷ Γ ὀρθῇ [γωνία]. συνεστάτω [πάλιν] τῇ πρὸς τῷ Γ ὀρθῇ γωνία ἴση ἡ ὑπὸ ΒΑΔ, ὡς ἔχει ἐπὶ τῆς δευτέρας καταγραφῆς, καὶ τετμήσθω ἡ ΑΒ δίχα κατὰ τὸ Ζ, καὶ κέντρω τῷ Ζ, διαστήματι δὲ ὁποτέρω τῶν ΖΑ, ΖΒ, κύκλος γεγράφθω ὁ ΑΕΒ.

Ἐφάπτεται ἄρα ἡ ΑΔ εὐθεῖα τοῦ ΑΒΕ κύκλου διὰ τὸ ὀρθὴν εἶναι τὴν πρὸς τῷ Α γωνίαν. καὶ ἴση ἐστὶν ἡ ὑπὸ ΒΑΔ γωνία τῇ ἐν τῷ ΑΕΒ τμήματι· ὀρθὴ γὰρ καὶ αὕτῃ ἐν ἡμικυκλίῳ οὔσα. ἀλλὰ καὶ ἡ ὑπὸ ΒΑΔ τῇ πρὸς τῷ Γ ἴση ἐστίν. καὶ ἡ ἐν τῷ ΑΕΒ ἄρα ἴση ἐστὶ τῇ πρὸς τῷ Γ.

Γέγραπται ἄρα πάλιν ἐπὶ τῆς ΑΒ τμήμα κύκλου τὸ ΑΕΒ δεχόμενον γωνίαν ἴσην τῇ πρὸς τῷ Γ.

Ἀλλὰ δὴ ἡ πρὸς τῷ Γ ἀμβλεία ἔστω· καὶ συνεστάτω αὕτῃ ἴση πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ Α σημείῳ ἡ ὑπὸ ΒΑΔ, ὡς ἔχει ἐπὶ τῆς τρίτης καταγραφῆς, καὶ τῇ ΑΔ πρὸς ὀρθὰς ἥχθω ἡ ΑΕ, καὶ τετμήσθω πάλιν ἡ ΑΒ δίχα κατὰ τὸ Ζ, καὶ τῇ ΑΒ πρὸς ὀρθὰς ἥχθω ἡ ΖΗ, καὶ ἐπεζεύχθω ἡ ΗΒ.

Καὶ ἐπεὶ πάλιν ἴση ἐστὶν ἡ ΑΖ τῇ ΖΒ, καὶ κοινὴ ἡ ΖΗ, δύο δὲ αἱ ΑΖ, ΖΗ δύο ταῖς ΒΖ, ΖΗ ἴσαι εἰσὶν· καὶ γωνία ἡ ὑπὸ ΑΖΗ γωνία τῇ ὑπὸ ΒΖΗ ἴση· βάσις ἄρα ἡ ΑΗ βάσει τῇ ΒΗ ἴση ἐστίν· ὁ ἄρα κέντρω μὲν τῷ Η διαστήματι δὲ τῷ ΗΑ κύκλος γραφόμενος ἥξει καὶ διὰ τοῦ Β. ἐρχέσθω ὡς ὁ ΑΕΒ. καὶ ἐπεὶ τῇ ΑΕ διαμέτρῳ ἀπ' ἄκρας πρὸς ὀρθὰς ἐστὶν ἡ ΑΔ, ἡ ΑΔ ἄρα ἐφάπτεται τοῦ ΑΕΒ κύκλου. καὶ ἀπὸ τῆς κατὰ τὸ Α ἐπαφῆς διήκται ἡ ΑΒ· ἡ ἄρα ὑπὸ ΒΑΔ γωνία ἴση ἐστὶ τῇ ἐν τῷ ἐναλλάξ τοῦ κύκλου τμήματι τῷ ΑΘΒ συνισταμένη γωνία. ἀλλ' ἡ ὑπὸ ΒΑΔ γωνία τῇ πρὸς τῷ Γ ἴση ἐστίν. καὶ ἡ ἐν τῷ ΑΘΒ ἄρα τμήματι γωνία ἴση ἐστὶ τῇ πρὸς τῷ Γ.

Ἐπὶ τῆς ἄρα δοθείσης εὐθείας τῆς ΑΒ γέγραπται τμήμα κύκλου τὸ ΑΘΒ δεχόμενον γωνίαν ἴσην τῇ πρὸς τῷ Γ· ὅπερ ἔδει ποιῆσαι.

And let AB have been cut in half at F [Prop. 1.10]. And let FG have been drawn from point F , at right-angles to AB [Prop. 1.11]. And let GB have been joined.

And since AF is equal to FB , and FG (is) common, the two (straight-lines) AF , FG are equal to the two (straight-lines) BF , FG (respectively). And angle AFG (is) equal to [angle] BFG . Thus, the base AG is equal to the base BG [Prop. 1.4]. Thus, the circle drawn with center G , and radius GA , will also go through B (as well as A). Let it have been drawn, and let it be (denoted) ABE . And let EB have been joined. Therefore, since AD is at the extremity of diameter AE , (namely, point) A , at right-angles to AE , the (straight-line) AD thus touches the circle ABE [Prop. 3.16 corr.]. Therefore, since some straight-line AD touches the circle ABE , and some (other) straight-line AB has been drawn across from the point of contact A into circle ABE , angle DAB is thus equal to the angle AEB in the alternate segment of the circle [Prop. 3.32]. But, DAB is equal to C . Thus, angle C is also equal to AEB .

Thus, a segment ABE of a circle, accepting the angle AEB (which is) equal to the given (angle) C , has been drawn on the given straight-line AB .

And so let C be a right-angle. And let it again be necessary to draw a segment of a circle on AB , accepting an angle equal to the right-[angle] C . Let the (angle) BAD [again] have been constructed, equal to the right-angle C [Prop. 1.23], as in the second diagram (from the left). And let AB have been cut in half at F [Prop. 1.10]. And let the circle ABE have been drawn with center F , and radius either FA or FB .

Thus, the straight-line AD touches the circle ABE , on account of the angle at A being a right-angle [Prop. 3.16 corr.]. And angle BAD is equal to the angle in segment ABE . For (the latter angle), being in a semi-circle, is also a right-angle [Prop. 3.31]. But, BAD is also equal to C . Thus, the (angle) in (segment) ABE is also equal to C .

Thus, a segment ABE of a circle, accepting an angle equal to C , has again been drawn on AB .

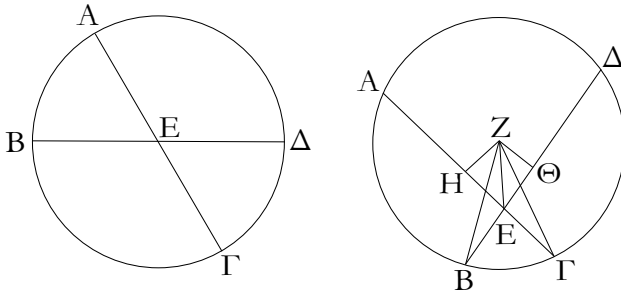
And so let (angle) C be obtuse. And let (angle) BAD , equal to (C), have been constructed on the straight-line AB , at the point A (on it) [Prop. 1.23], as in the third diagram (from the left). And let AE have been drawn, at right-angles to AD [Prop. 1.11]. And let AB have again been cut in half at F [Prop. 1.10]. And let FG have been drawn, at right-angles to AB [Prop. 1.10]. And let GB have been joined.

And again, since AF is equal to FB , and FG (is) common, the two (straight-lines) AF , FG are equal to the two (straight-lines) BF , FG (respectively). And angle AFG (is) equal to angle BFG . Thus, the base AG is

point B , at right-angles to the aforementioned straight-line [Prop. 1.11].

λε'.

Ἐάν ἐν κύκλῳ δύο εὐθεῖαι τέμνωσιν ἀλλήλας, τὸ ὑπὸ τῶν τῆς μιᾶς τμημάτων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν τῆς ἐτέρας τμημάτων περιεχομένῳ ὀρθογωνίῳ.



Ἐν γὰρ κύκλῳ τῷ $AB\Gamma\Delta$ δύο εὐθεῖαι αἱ AG , BD τεμνέτωσαν ἀλλήλας κατὰ τὸ E σημεῖον· λέγω, ὅτι τὸ ὑπὸ τῶν AE , EG περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν DE , EB περιεχομένῳ ὀρθογωνίῳ.

Εἰ μὲν οὖν αἱ AG , BD διὰ τοῦ κέντρου εἰσὶν ὥστε τὸ E κέντρον εἶναι τοῦ $AB\Gamma\Delta$ κύκλου, φανερόν, ὅτι ἴσων οὐσῶν τῶν AE , EG , DE , EB καὶ τὸ ὑπὸ τῶν AE , EG περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν DE , EB περιεχομένῳ ὀρθογωνίῳ.

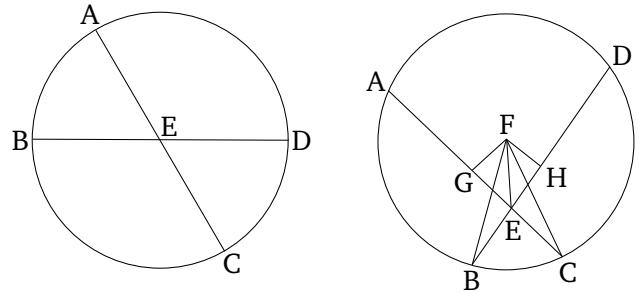
Μὴ ἔστωσαν δὴ αἱ AG , BD διὰ τοῦ κέντρου, καὶ εἰλήφθω τὸ κέντρον τοῦ $AB\Gamma\Delta$, καὶ ἔστω τὸ Z , καὶ ἀπὸ τοῦ Z ἐπὶ τὰς AG , BD εὐθείας κἀνθετοὶ ἤχθωσαν αἱ ZH , $Z\Theta$, καὶ ἐπεζεύχθωσαν αἱ ZB , $Z\Gamma$, ZE .

Καὶ ἐπεὶ εὐθεῖα τις διὰ τοῦ κέντρου ἢ HZ εὐθεῖάν τινα μὴ διὰ τοῦ κέντρου τὴν AG πρὸς ὀρθὰς τέμνει, καὶ δῖχα αὐτὴν τέμνει· ἴση ἄρα ἢ AH τῇ $H\Gamma$. ἐπεὶ οὖν εὐθεῖα ἢ AG τέτμηται εἰς μὲν ἴσα κατὰ τὸ H , εἰς δὲ ἄνισα κατὰ τὸ E , τὸ ἄρα ὑπὸ τῶν AE , EG περιεχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς EH τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς $H\Gamma$ · [κοινόν] προσκείσθω τὸ ἀπὸ τῆς HZ · τὸ ἄρα ὑπὸ τῶν AE , EG μετὰ τῶν ἀπὸ τῶν HE , HZ ἴσον ἐστὶ τοῖς ἀπὸ τῶν GH , HZ . ἀλλὰ τοῖς μὲν ἀπὸ τῶν EH , HZ ἴσον ἐστὶ τὸ ἀπὸ τῆς ZE , τοῖς δὲ ἀπὸ τῶν GH , HZ ἴσον ἐστὶ τὸ ἀπὸ τῆς $Z\Gamma$ · τὸ ἄρα ὑπὸ τῶν AE , EG μετὰ τοῦ ἀπὸ τῆς ZE ἴσον ἐστὶ τῷ ἀπὸ τῆς $Z\Gamma$. ἴση δὲ ἢ $Z\Gamma$ τῇ ZB · τὸ ἄρα ὑπὸ τῶν AE , EG μετὰ τοῦ ἀπὸ τῆς EZ ἴσον ἐστὶ τῷ ἀπὸ τῆς ZB . διὰ τὰ αὐτὰ δὴ καὶ τὸ ὑπὸ τῶν DE , EB μετὰ τοῦ ἀπὸ τῆς ZE ἴσον ἐστὶ τῷ ἀπὸ τῆς ZB . ἐδείχθη δὲ καὶ τὸ ὑπὸ τῶν AE , EG μετὰ τοῦ ἀπὸ τῆς ZE ἴσον τῷ ἀπὸ τῆς ZB · τὸ ἄρα ὑπὸ τῶν AE , EG μετὰ τοῦ ἀπὸ τῆς ZE ἴσον ἐστὶ τῷ ὑπὸ τῶν DE , EB μετὰ τοῦ ἀπὸ τῆς ZE . κοινόν ἀφῆρήσθω τὸ ἀπὸ τῆς ZE · λοιπὸν ἄρα τὸ ὑπὸ τῶν AE , EG περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν DE , EB περιεχομένῳ ὀρθογωνίῳ.

Ἐάν ἄρα ἐν κύκλῳ εὐθεῖαι δύο τέμνωσιν ἀλλήλας, τὸ ὑπὸ τῶν τῆς μιᾶς τμημάτων περιεχόμενον ὀρθογώνιον ἴσον

Proposition 35

If two straight-lines in a circle cut one another then the rectangle contained by the pieces of one is equal to the rectangle contained by the pieces of the other.



For let the two straight-lines AC and BD , in the circle $ABCD$, cut one another at point E . I say that the rectangle contained by AE and EC is equal to the rectangle contained by DE and EB .

In fact, if AC and BD are through the center (as in the first diagram from the left), so that E is the center of circle $ABCD$, then (it is) clear that, AE , EC , DE , and EB being equal, the rectangle contained by AE and EC is also equal to the rectangle contained by DE and EB .

So let AC and DB not be though the center (as in the second diagram from the left), and let the center of $ABCD$ have been found [Prop. 3.1], and let it be (at) F . And let FG and FH have been drawn from F , perpendicular to the straight-lines AC and DB (respectively) [Prop. 1.12]. And let FB , FC , and FE have been joined.

And since some straight-line, GF , through the center, cuts at right-angles some (other) straight-line, AC , not through the center, then it also cuts it in half [Prop. 3.3]. Thus, AG (is) equal to GC . Therefore, since the straight-line AC is cut equally at G , and unequally at E , the rectangle contained by AE and EC plus the square on EG is thus equal to the (square) on GC [Prop. 2.5]. Let the (square) on GF have been added [to both]. Thus, the (rectangle contained) by AE and EC plus the (sum of the squares) on GE and GF is equal to the (sum of the squares) on CG and GF . But, the (square) on FE is equal to the (sum of the squares) on EG and GF [Prop. 1.47], and the (square) on FC is equal to the (sum of the squares) on CG and GF [Prop. 1.47]. Thus, the (rectangle contained) by AE and EC plus the (square) on FE is equal to the (square) on FC . And FC (is) equal to FB . Thus, the (rectangle contained) by AE and EC plus the (square) on FE is equal to the (square) on FB . So, for the same (reasons), the (rectangle contained) by DE and EB plus the (square) on FE is equal

ἐστὶ τῶ ὑπὸ τῶν τῆς ἐτέρας τμημάτων περιεχομένῳ ὀρθογώνῳ· ὅπερ ἔδει δεῖξαι.

to the (square) on FB . And the (rectangle contained) by AE and EC plus the (square) on FE was also shown (to be) equal to the (square) on FB . Thus, the (rectangle contained) by AE and EC plus the (square) on FE is equal to the (rectangle contained) by DE and EB plus the (square) on FE . Let the (square) on FE have been taken from both. Thus, the remaining rectangle contained by AE and EC is equal to the rectangle contained by DE and EB .

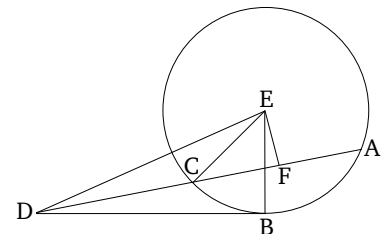
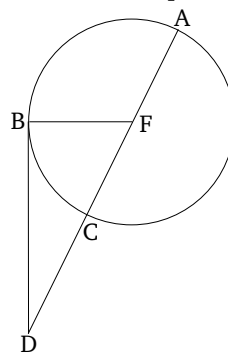
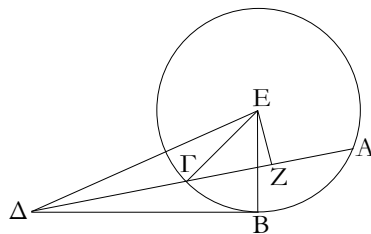
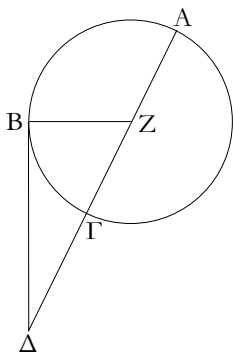
Thus, if two straight-lines in a circle cut one another then the rectangle contained by the pieces of one is equal to the rectangle contained by the pieces of the other. (Which is) the very thing it was required to show.

λζ'.

Proposition 36

Ἐὰν κύκλου ληφθῇ τι σημεῖον ἐκτός, καὶ ἀπ' αὐτοῦ πρὸς τὸν κύκλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν κύκλον, ἡ δὲ ἐφάπτηται, ἔσται τὸ ὑπὸ ὅλης τῆς τεμνούσης καὶ τῆς ἐκτὸς ἀπολαμβανομένης μεταξὺ τοῦ σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῶ ἀπὸ τῆς ἐφαπτομένης τετραγώνῳ.

If some point is taken outside a circle, and two straight-lines radiate from it towards the circle, and (one) of them cuts the circle, and the (other) touches (it), then the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, will be equal to the square on the tangent (line).



Κύκλου γάρ τοῦ $AB\Gamma$ εἰλήφθω τι σημεῖον ἐκτὸς τὸ Δ , καὶ ἀπὸ τοῦ Δ πρὸς τὸν $AB\Gamma$ κύκλον προσπιπτέωσαν δύο εὐθεῖαι αἱ $\Delta\Gamma[A]$, ΔB · καὶ ἡ μὲν $\Delta\Gamma A$ τεμνέτω τὸν $AB\Gamma$ κύκλον, ἡ δὲ $B\Delta$ ἐφαπτέσθω· λέγω, ὅτι τὸ ὑπὸ τῶν $A\Delta$, $\Delta\Gamma$ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῶ ἀπὸ τῆς ΔB τετραγώνῳ.

Ἡ ἄρα $[\Delta]\Gamma A$ ἤτοι διὰ τοῦ κέντρου ἐστὶν ἢ οὐ. ἔστω πρότερον διὰ τοῦ κέντρου, καὶ ἔστω τὸ Z κέντρον τοῦ $AB\Gamma$ κύκλου, καὶ ἐπεζεύχθω ἡ ZB · ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ $ZB\Delta$. καὶ ἐπεὶ εὐθεῖα ἡ $A\Gamma$ δίχα τέμνεται κατὰ τὸ Z , πρόσκειται δὲ αὐτῇ ἡ $\Gamma\Delta$, τὸ ἄρα ὑπὸ τῶν $A\Delta$, $\Delta\Gamma$ μετὰ τοῦ ἀπὸ τῆς $Z\Gamma$ ἴσον ἐστὶ τῶ ἀπὸ τῆς $Z\Delta$. ἴση δὲ ἡ $Z\Gamma$ τῇ ZB · τὸ ἄρα ὑπὸ τῶν $A\Delta$, $\Delta\Gamma$ μετὰ τοῦ ἀπὸ τῆς ZB ἴσον ἐστὶ τῶ ἀπὸ τῆς $Z\Delta$. τῶ δὲ ἀπὸ τῆς $Z\Delta$ ἴσα ἐστὶ τὰ ἀπὸ τῶν ZB , $B\Delta$ · τὸ ἄρα ὑπὸ τῶν $A\Delta$, $\Delta\Gamma$ μετὰ τοῦ ἀπὸ τῆς ZB ἴσον ἐστὶ τοῖς ἀπὸ τῶν ZB , $B\Delta$. κοινὸν ἀφῆρήσθω τὸ ἀπὸ τῆς ZB · λοιπὸν ἄρα τὸ ὑπὸ τῶν $A\Delta$, $\Delta\Gamma$ ἴσον ἐστὶ τῶ ἀπὸ τῆς ΔB

For let some point D have been taken outside circle ABC , and let two straight-lines, $DC[A]$ and DB , radiate from D towards circle ABC . And let DCA cut circle ABC , and let BD touch (it). I say that the rectangle contained by AD and DC is equal to the square on DB .

$[D]CA$ is surely either through the center, or not. Let it first of all be through the center, and let F be the center of circle ABC , and let FB have been joined. Thus, (angle) FBD is a right-angle [Prop. 3.18]. And since straight-line AC is cut in half at F , let CD have been added to it. Thus, the (rectangle contained) by AD and DC plus the (square) on FC is equal to the (square) on FD [Prop. 2.6]. And FC (is) equal to FB . Thus, the (rectangle contained) by AD and DC plus the (square) on FB is equal to the (square) on FD . And the (square) on FD is equal to the (sum of the squares) on FB and BD [Prop. 1.47]. Thus, the (rectangle contained) by AD

ἐφαπτομένης.

Ἀλλὰ δὴ ἡ ΔΓΑ μὴ ἔστω διὰ τοῦ κέντρου τοῦ ΑΒΓ κύκλου, καὶ εἰλήφθω τὸ κέντρον τὸ Ε, καὶ ἀπὸ τοῦ Ε ἐπὶ τὴν ΑΓ κάθετος ἦχθω ἡ ΕΖ, καὶ ἐπεξέχθωσαν αἱ ΕΒ, ΕΓ, ΕΔ· ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΕΒΔ. καὶ ἐπεὶ εὐθεία τις διὰ τοῦ κέντρου ἡ ΕΖ εὐθεϊάν τινα μὴ διὰ τοῦ κέντρου τὴν ΑΓ πρὸς ὀρθὰς τέμνει, καὶ δίχα αὐτὴν τέμνει· ἡ ΑΖ ἄρα τῇ ΖΓ ἐστὶν ἴση. καὶ ἐπεὶ εὐθεία ἡ ΑΓ τέμνεται δίχα κατὰ τὸ Ζ σημεῖον, πρόσκειται δὲ αὐτῇ ἡ ΓΔ, τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΖΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΖΔ. κοινὸν προσκείσθω τὸ ἀπὸ τῆς ΖΕ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τῶν ἀπὸ τῶν ΓΖ, ΖΕ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΖΔ, ΖΕ. τοῖς δὲ ἀπὸ τῶν ΓΖ, ΖΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΓ· ὀρθὴ γὰρ [ἐστὶν] ἡ ὑπὸ ΕΖΓ [γωνία]· τοῖς δὲ ἀπὸ τῶν ΔΖ, ΖΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΕΔ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΕΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΔ. ἴση δὲ ἡ ΕΓ τῇ ΕΒ· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΕΒ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΔ. τῷ δὲ ἀπὸ τῆς ΕΔ ἴσα ἐστὶ τὰ ἀπὸ τῶν ΕΒ, ΒΔ· ὀρθὴ γὰρ ἡ ὑπὸ ΕΒΔ γωνία· τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ μετὰ τοῦ ἀπὸ τῆς ΕΒ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΕΒ, ΒΔ. κοινὸν ἀφηρήσθω τὸ ἀπὸ τῆς ΕΒ· λοιπὸν ἄρα τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ.

Ἐὰν ἄρα κύκλου ληφθῇ τι σημεῖον ἐκτός, καὶ ἀπ' αὐτοῦ πρὸς τὸν κύκλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν κύκλον, ἡ δὲ ἐφάπτεται, ἔσται τὸ ὑπὸ ὅλης τῆς τεμνούσης καὶ τῆς ἐκτός ἀπολαμβανομένης μεταξὺ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς ἐφαπτομένης τετραγώνῳ· ὅπερ ἔδει δεῖξαι.

λζ'.

Ἐὰν κύκλου ληφθῇ τι σημεῖον ἐκτός, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν κύκλον, ἡ δὲ προσπίπτῃ, ἢ δὲ τὸ ὑπὸ [τῆς] ὅλης τῆς τεμνούσης καὶ τῆς ἐκτός ἀπολαμβα-

and DC plus the (square) on FB is equal to the (sum of the squares) on FB and BD . Let the (square) on FB have been subtracted from both. Thus, the remaining (rectangle contained) by AD and DC is equal to the (square) on the tangent DB .

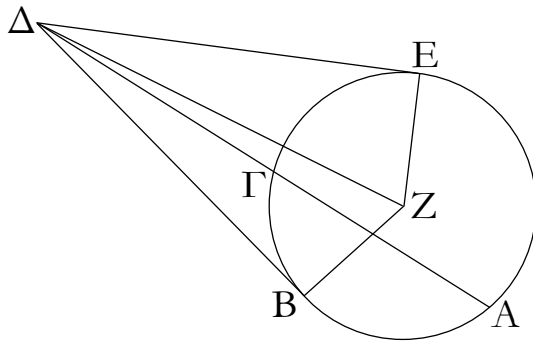
And so let DCA not be through the center of circle ABC , and let the center E have been found, and let EF have been drawn from E , perpendicular to AC [Prop. 1.12]. And let EB , EC , and ED have been joined. (Angle) EBD (is) thus a right-angle [Prop. 3.18]. And since some straight-line, EF , through the center, cuts some (other) straight-line, AC , not through the center, at right-angles, it also cuts it in half [Prop. 3.3]. Thus, AF is equal to FC . And since the straight-line AC is cut in half at point F , let CD have been added to it. Thus, the (rectangle contained) by AD and DC plus the (square) on FC is equal to the (square) on FD [Prop. 2.6]. Let the (square) on FE have been added to both. Thus, the (rectangle contained) by AD and DC plus the (sum of the squares) on CF and FE is equal to the (sum of the squares) on FD and FE . But the (square) on EC is equal to the (sum of the squares) on CF and FE . For [angle] EFC [is] a right-angle [Prop. 1.47]. And the (square) on ED is equal to the (sum of the squares) on DF and FE [Prop. 1.47]. Thus, the (rectangle contained) by AD and DC plus the (square) on EC is equal to the (square) on ED . And EC (is) equal to EB . Thus, the (rectangle contained) by AD and DC plus the (square) on EB is equal to the (square) on ED . And the (sum of the squares) on EB and BD is equal to the (square) on ED . For EBD (is) a right-angle [Prop. 1.47]. Thus, the (rectangle contained) by AD and DC plus the (square) on EB is equal to the (sum of the squares) on EB and BD . Let the (square) on EB have been subtracted from both. Thus, the remaining (rectangle contained) by AD and DC is equal to the (square) on BD .

Thus, if some point is taken outside a circle, and two straight-lines radiate from it towards the circle, and (one) of them cuts the circle, and (the other) touches (it), then the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, will be equal to the square on the tangent (line). (Which is) the very thing it was required to show.

Proposition 37

If some point is taken outside a circle, and two straight-lines radiate from the point towards the circle, and one of them cuts the circle, and the (other) meets (it), and the (rectangle contained) by the whole (straight-

νομένης μεταξύ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς προσπιπτούσης, ἢ προσπίπτουσα ἐφάπεται τοῦ κύκλου.

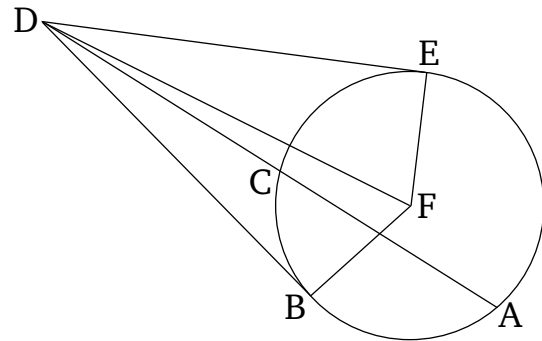


Κύκλου γάρ τοῦ ΑΒΓ εἰλήφθω τι σημεῖον ἐκτὸς τὸ Δ, καὶ ἀπὸ τοῦ Δ πρὸς τὸν ΑΒΓ κύκλον προσπιπτέωσαν δύο εὐθεῖαι αἱ ΔΓΑ, ΔΒ, καὶ ἡ μὲν ΔΓΑ τεμνέτω τὸν κύκλον, ἡ δὲ ΔΒ προσπιπτέτω, ἔστω δὲ τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον τῷ ἀπὸ τῆς ΔΒ. λέγω, ὅτι ἡ ΔΒ ἐφάπτεται τοῦ ΑΒΓ κύκλου.

Ἦχθω γὰρ τοῦ ΑΒΓ ἐφαπτομένη ἡ ΔΕ, καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓ κύκλου, καὶ ἔστω τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΖΕ, ΖΒ, ΖΑ. ἡ ἄρα ὑπὸ ΖΕΔ ὀρθὴ ἐστίν. καὶ ἐπεὶ ἡ ΔΕ ἐφάπτεται τοῦ ΑΒΓ κύκλου, τέμνει δὲ ἡ ΔΓΑ, τὸ ἄρα ὑπὸ τῶν ΑΔ, ΔΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΕ. ἦν δὲ καὶ τὸ ὑπὸ τῶν ΑΔ, ΔΓ ἴσον τῷ ἀπὸ τῆς ΔΒ· τὸ ἄρα ἀπὸ τῆς ΔΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΔΒ· ἴση ἄρα ἡ ΔΕ τῇ ΔΒ. ἐστὶ δὲ καὶ ἡ ΖΕ τῇ ΖΒ ἴση· δύο δὲ αἱ ΔΕ, ΕΖ δύο ταῖς ΔΒ, ΒΖ ἴσαι εἰσίν· καὶ βάσις αὐτῶν κοινὴ ἡ ΖΔ· γωνία ἄρα ἡ ὑπὸ ΔΕΖ γωνία τῇ ὑπὸ ΔΒΖ ἐστίν ἴση. ὀρθὴ δὲ ἡ ὑπὸ ΔΕΖ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ ΔΒΖ. καὶ ἐστίν ἡ ΖΒ ἐκβαλλομένη διάμετρος· ἡ δὲ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθῶς ἀπ' ἄκρας ἀγομένη ἐφάπτεται τοῦ κύκλου· ἡ ΔΒ ἄρα ἐφάπτεται τοῦ ΑΒΓ κύκλου. ὁμοίως δὲ δειχθήσεται, ἂν τὸ κέντρον ἐπὶ τῆς ΑΓ τυγχάνῃ.

Ἐὰν ἄρα κύκλου ληφθῇ τι σημεῖον ἐκτὸς, ἀπὸ δὲ τοῦ σημείου πρὸς τὸν κύκλον προσπίπτωσι δύο εὐθεῖαι, καὶ ἡ μὲν αὐτῶν τέμνῃ τὸν κύκλον, ἡ δὲ προσπίπτῃ, ἢ δὲ τὸ ὑπὸ ὅλης τῆς τεμνούσης καὶ τῆς ἐκτὸς ἀπολαμβανομένης μεταξύ τοῦ τε σημείου καὶ τῆς κυρτῆς περιφερείας ἴσον τῷ ἀπὸ τῆς προσπιπτούσης, ἢ προσπίπτουσα ἐφάπεται τοῦ κύκλου· ὅπερ ἔδει δεῖξαι.

line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, is equal to the (square) on the (straight-line) meeting (the circle), then the (straight-line) meeting (the circle) will touch the circle.



For let some point D have been taken outside circle ABC , and let two straight-lines, DCA and DB , radiate from D towards circle ABC , and let DCA cut the circle, and let DB meet (the circle). And let the (rectangle contained) by AD and DC be equal to the (square) on DB . I say that DB touches circle ABC .

For let DE have been drawn touching ABC [Prop. 3.17], and let the center of the circle ABC have been found, and let it be (at) F . And let FE , FB , and FD have been joined. (Angle) FED is thus a right-angle [Prop. 3.18]. And since DE touches circle ABC , and DCA cuts (it), the (rectangle contained) by AD and DC is thus equal to the (square) on DE [Prop. 3.36]. And the (rectangle contained) by AD and DC was also equal to the (square) on DB . Thus, the (square) on DE is equal to the (square) on DB . Thus, DE (is) equal to DB . And FE is also equal to FB . So the two (straight-lines) DE , EF are equal to the two (straight-lines) DB , BF (respectively). And their base, FD , is common. Thus, angle DEF is equal to angle DBF [Prop. 1.8]. And DEF (is) a right-angle. Thus, DBF (is) also a right-angle. And FB produced is a diameter, And a (straight-line) drawn at right-angles to a diameter of a circle, at its extremity, touches the circle [Prop. 3.16 corr.]. Thus, DB touches circle ABC . Similarly, (the same thing) can be shown, even if the center happens to be on AC .

Thus, if some point is taken outside a circle, and two straight-lines radiate from the point towards the circle, and one of them cuts the circle, and the (other) meets (it), and the (rectangle contained) by the whole (straight-line) cutting (the circle), and the (part of it) cut off outside (the circle), between the point and the convex circumference, is equal to the (square) on the (straight-line) meeting (the circle), then the (straight-line) meeting (the circle) will touch the circle. (Which is) the very thing it

was required to show.

ELEMENTS BOOK 4

*Construction of Rectilinear Figures In and
Around Circles*

Ὅροι.

α'. Σχῆμα εὐθύγραμμον εἰς σχῆμα εὐθύγραμμον ἐγγράφ-
εσθαι λέγεται, ὅταν ἐκάστη τῶν τοῦ ἐγγραφομένου σχήμα-
τος γωνιῶν ἐκάστης πλευρᾶς τοῦ, εἰς ὃ ἐγγράφεται, ᾗπτηται.

β'. Σχῆμα δὲ ὁμοίως περὶ σχῆμα περιγράφεσθαι λέγεται,
ὅταν ἐκάστη πλευρὰ τοῦ περιγραφομένου ἐκάστης γωνίας
τοῦ, περὶ ὃ περιγράφεται, ᾗπτηται.

γ'. Σχῆμα εὐθύγραμμον εἰς κύκλον ἐγγράφεσθαι λέγεται,
ὅταν ἐκάστη γωνία τοῦ ἐγγραφομένου ᾗπτηται τῆς τοῦ
κύκλου περιφερείας.

δ'. Σχῆμα δὲ εὐθύγραμμον περὶ κύκλον περιγράφ-
εσθαι λέγεται, ὅταν ἐκάστη πλευρὰ τοῦ περιγραφομένου
ἐφάπτηται τῆς τοῦ κύκλου περιφερείας.

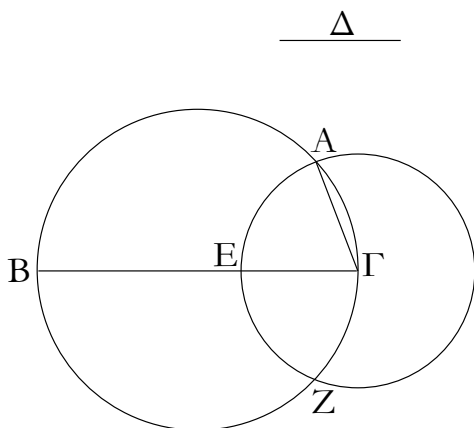
ε'. Κύκλος δὲ εἰς σχῆμα ὁμοίως ἐγγράφεσθαι λέγεται,
ὅταν ἢ τοῦ κύκλου περιφέρεια ἐκάστης πλευρᾶς τοῦ, εἰς ὃ
ἐγγράφεται, ᾗπτηται.

ς'. Κύκλος δὲ περὶ σχῆμα περιγράφεσθαι λέγεται, ὅταν
ἢ τοῦ κύκλου περιφέρεια ἐκάστης γωνίας τοῦ, περὶ ὃ πε-
ριγράφεται, ᾗπτηται.

ζ'. Εὐθεῖα εἰς κύκλον ἐναρμόζεσθαι λέγεται, ὅταν τὰ
πέρατα αὐτῆς ἐπὶ τῆς περιφερείας ᾗ τοῦ κύκλου.

α'.

Εἰς τὸν δοθέντα κύκλον τῇ δοθείσῃ εὐθείᾳ μὴ μείζονι
οὕσῃ τῆς τοῦ κύκλου διαμέτρου ἴσην εὐθεῖαν ἐναρμόσαι.



Ἐστω ὁ δοθείς κύκλος ὁ $ABΓ$, ἡ δὲ δοθεῖσα εὐθεῖα μὴ
μείζων τῆς τοῦ κύκλου διαμέτρου ἡ $Δ$. δεῖ δὲ εἰς τὸν $ABΓ$
κύκλον τῇ $Δ$ εὐθείᾳ ἴσην εὐθεῖαν ἐναρμόσαι.

Ἦχθω τοῦ $ABΓ$ κύκλου διάμετρος ἡ $BΓ$. εἰ μὲν οὖν ἴση
ἐστὶν ἡ $BΓ$ τῇ $Δ$, γεγονὸς ἂν εἴη τὸ ἐπιταχθέν· ἐνήρμοσται

Definitions

1. A rectilinear figure is said to be inscribed in a(nother) rectilinear figure when the respective angles of the inscribed figure touch the respective sides of the (figure) in which it is inscribed.

2. And, similarly, a (rectilinear) figure is said to be circumscribed about a(nother rectilinear) figure when the respective sides of the circumscribed (figure) touch the respective angles of the (figure) about which it is circumscribed.

3. A rectilinear figure is said to be inscribed in a circle when each angle of the inscribed (figure) touches the circumference of the circle.

4. And a rectilinear figure is said to be circumscribed about a circle when each side of the circumscribed (figure) touches the circumference of the circle.

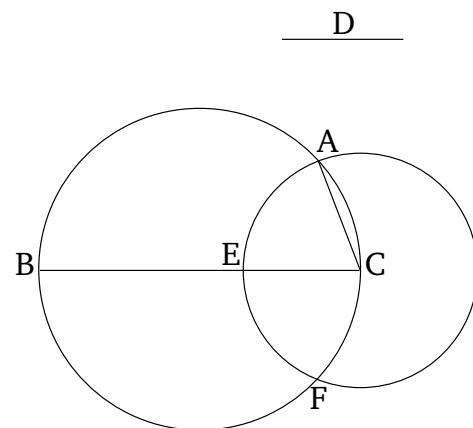
5. And, similarly, a circle is said to be inscribed in a (rectilinear) figure when the circumference of the circle touches each side of the (figure) in which it is inscribed.

6. And a circle is said to be circumscribed about a rectilinear (figure) when the circumference of the circle touches each angle of the (figure) about which it is circumscribed.

7. A straight-line is said to be inserted into a circle when its extremities are on the circumference of the circle.

Proposition 1

To insert a straight-line equal to a given straight-line into a circle, (the latter straight-line) not being greater than the diameter of the circle.



Let ABC be the given circle, and D the given straight-line (which is) not greater than the diameter of the circle. So it is required to insert a straight-line, equal to the straight-line D , into the circle ABC .

Let a diameter BC of circle ABC have been drawn.†

γὰρ εἰς τὸν $AB\Gamma$ κύκλον τῇ Δ εὐθείᾳ ἴση ἢ $B\Gamma$. εἰ δὲ μείζων ἔστιν ἢ $B\Gamma$ τῆς Δ , κείσθω τῇ Δ ἴση ἢ ΓE , καὶ κέντρῳ τῷ Γ διαστήματι δὲ τῷ ΓE κύκλος γεγράφθω ὁ EAZ , καὶ ἐπεξεύχθω ἡ ΓA .

Ἐπεὶ οὖν το Γ σημεῖον κέντρον ἐστὶ τοῦ EAZ κύκλου, ἴση ἔστιν ἡ ΓA τῇ ΓE . ἀλλὰ τῇ Δ ἢ ΓE ἐστὶν ἴση· καὶ ἡ Δ ἄρα τῇ ΓA ἐστὶν ἴση.

Εἰς ἄρα τὸν δοθέντα κύκλον τὸν $AB\Gamma$ τῇ δοθείσῃ εὐθείᾳ τῇ Δ ἴση ἐνήρμοσται ἡ ΓA . ὅπερ ἔδει ποιῆσαι.

Therefore, if BC is equal to D then that (which) was prescribed has taken place. For the (straight-line) BC , equal to the straight-line D , has been inserted into the circle ABC . And if BC is greater than D then let CE be made equal to D [Prop. 1.3], and let the circle EAF have been drawn with center C and radius CE . And let CA have been joined.

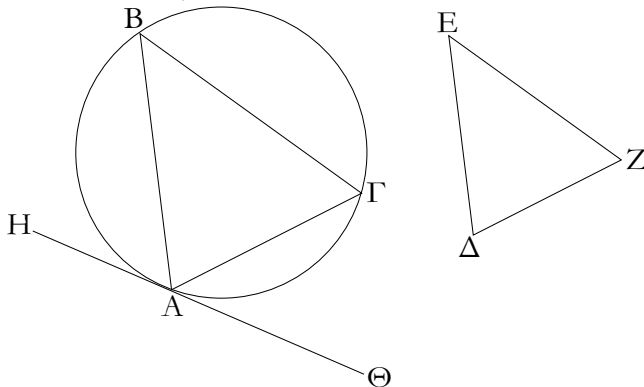
Therefore, since the point C is the center of circle EAF , CA is equal to CE . But, CE is equal to D . Thus, D is also equal to CA .

Thus, CA , equal to the given straight-line D , has been inserted into the given circle ABC . (Which is) the very thing it was required to do.

† Presumably, by finding the center of the circle [Prop. 3.1], and then drawing a line through it.

β'.

Εἰς τὸν δοθέντα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον ἐγγράψαι.



Ἐστω ὁ δοθείς κύκλος ὁ $AB\Gamma$, τὸ δὲ δοθὲν τρίγωνον τὸ $\Delta E Z$: δεῖ δὴ εἰς τὸν $AB\Gamma$ κύκλον τῷ $\Delta E Z$ τριγώνῳ ἰσογώνιον τρίγωνον ἐγγράψαι.

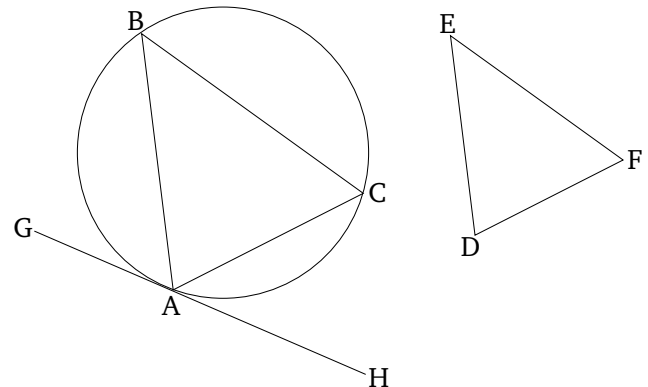
Ἦχθω τοῦ $AB\Gamma$ κύκλου ἐφαπτομένη ἡ $H\Theta$ κατὰ τὸ A , καὶ συνεστάτω πρὸς τῇ $A\Theta$ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ A τῇ ὑπὸ $\Delta E Z$ γωνίᾳ ἴση ἢ ὑπὸ $\Theta A\Gamma$, πρὸς δὲ τῇ AH εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ A τῇ ὑπὸ $\Delta Z E$ [γωνίᾳ] ἴση ἢ ὑπὸ $H A B$, καὶ ἐπεξεύχθω ἡ $B\Gamma$.

Ἐπεὶ οὖν κύκλου τοῦ $AB\Gamma$ ἐφάπτεται τις εὐθεῖα ἡ $A\Theta$, καὶ ἀπὸ τῆς κατὰ τὸ A ἐπαφῆς εἰς τὸν κύκλον διῆκται εὐθεῖα ἡ $A\Gamma$, ἡ ἄρα ὑπὸ $\Theta A\Gamma$ ἴση ἐστὶ τῇ ἐν τῷ ἐναλλάξ τοῦ κύκλου τμήματι γωνίᾳ τῇ ὑπὸ $AB\Gamma$. ἀλλ' ἡ ὑπὸ $\Theta A\Gamma$ τῇ ὑπὸ $\Delta E Z$ ἐστὶν ἴση· καὶ ἡ ὑπὸ $AB\Gamma$ ἄρα γωνία τῇ ὑπὸ $\Delta E Z$ ἐστὶν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ $A\Gamma B$ τῇ ὑπὸ $\Delta Z E$ ἐστὶν ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ $B A\Gamma$ λοιπὴ τῇ ὑπὸ $E \Delta Z$ ἐστὶν ἴση [ἰσογώνιον ἄρα ἐστὶ τὸ $AB\Gamma$ τρίγωνον τῷ $\Delta E Z$ τριγώνῳ, καὶ ἐγγέγραπται εἰς τὸν $AB\Gamma$ κύκλον].

Εἰς τὸν δοθέντα ἄρα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

Proposition 2

To inscribe a triangle, equiangular with a given triangle, in a given circle.



Let ABC be the given circle, and DEF the given triangle. So it is required to inscribe a triangle, equiangular with triangle DEF , in circle ABC .

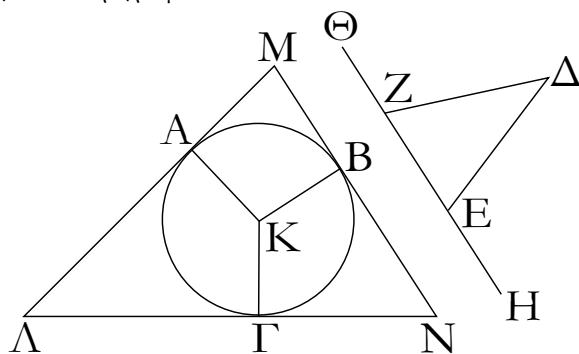
Let GH have been drawn touching circle ABC at A .† And let (angle) HAC , equal to angle DEF , have been constructed on the straight-line AH at the point A on it, and (angle) GAB , equal to [angle] DFE , on the straight-line AG at the point A on it [Prop. 1.23]. And let BC have been joined.

Therefore, since some straight-line AH touches the circle ABC , and the straight-line AC has been drawn across (the circle) from the point of contact A , (angle) HAC is thus equal to the angle ABC in the alternate segment of the circle [Prop. 3.32]. But, HAC is equal to DEF . Thus, angle ABC is also equal to DEF . So, for the same (reasons), ACB is also equal to DFE . Thus, the remaining (angle) BAC is equal to the remaining (angle) EDF [Prop. 1.32]. [Thus, triangle ABC is equiangular with triangle DEF , and has been inscribed in circle

† See the footnote to Prop. 3.34.

Υ'.

Περὶ τὸν δοθέντα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τριγώνον περιγράφαι.



Ἐστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ, τὸ δὲ δοθὲν τρίγωνον τὸ ΔΕΖ· δεῖ δὴ περὶ τὸν ΑΒΓ κύκλον τῷ ΔΕΖ τριγώνῳ ἰσογώνιον τριγώνον περιγράφαι.

Ἐκβεβλήσθω ἡ ΕΖ ἐφ' ἐκάτερα τὰ μέρη κατὰ τὰ Η, Θ σημεία, καὶ εἰλήφθω τοῦ ΑΒΓ κύκλου κέντρον τὸ Κ, καὶ διήχθω, ὡς ἔτυχεν, εὐθεῖα ἡ ΚΒ, καὶ συνεστάτω πρὸς τῇ ΚΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Κ τῇ μὲν ὑπὸ ΔΕΗ γωνίᾳ ἴση ἡ ὑπὸ ΒΚΑ, τῇ δὲ ὑπὸ ΔΖΘ ἴση ἡ ὑπὸ ΒΚΓ, καὶ διὰ τῶν Α, Β, Γ σημείων ἤχθωσαν ἐφαπτόμεναι τοῦ ΑΒΓ κύκλου αἱ ΛΑΜ, ΜΒΝ, ΝΓΛ.

Καὶ ἐπεὶ ἐφαπτόνται τοῦ ΑΒΓ κύκλου αἱ ΛΜ, ΜΝ, ΝΛ κατὰ τὰ Α, Β, Γ σημεία, ἀπὸ δὲ τοῦ Κ κέντρου ἐπὶ τὰ Α, Β, Γ σημεία ἐπεζευγμένα εἰσὶν αἱ ΚΑ, ΚΒ, ΚΓ, ὀρθαὶ ἄρα εἰσὶν αἱ πρὸς τοῖς Α, Β, Γ σημείοις γωνίαι. καὶ ἐπεὶ τοῦ ΑΜΒΚ τετραπλεύρου αἱ τέσσαρες γωνίαι τέτρασιν ὀρθαῖς ἴσαι εἰσὶν, ἐπειδὴ περ καὶ εἰς δύο τρίγωνα διαιρεῖται τὸ ΑΜΒΚ, καὶ εἰσὶν ὀρθαὶ αἱ ὑπὸ ΚΑΜ, ΚΒΜ γωνίαι, λοιπαὶ ἄρα αἱ ὑπὸ ΑΚΒ, ΑΜΒ δυσὶν ὀρθαῖς ἴσαι εἰσὶν. εἰσὶ δὲ καὶ αἱ ὑπὸ ΔΕΗ, ΔΕΖ δυσὶν ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΑΚΒ, ΑΜΒ ταῖς ὑπὸ ΔΕΗ, ΔΕΖ ἴσαι εἰσὶν, ὧν ἡ ὑπὸ ΑΚΒ τῇ ὑπὸ ΔΕΗ ἐστὶν ἴση· λοιπὴ ἄρα ἡ ὑπὸ ΑΜΒ λοιπῇ τῇ ὑπὸ ΔΕΖ ἐστὶν ἴση. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἡ ὑπὸ ΑΝΒ τῇ ὑπὸ ΔΖΕ ἐστὶν ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΜΑΝ [λοιπῇ] τῇ ὑπὸ ΕΔΖ ἐστὶν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΜΝ τρίγωνον τῷ ΔΕΖ τριγώνῳ· καὶ περιέγραπται περὶ τὸν ΑΒΓ κύκλον.

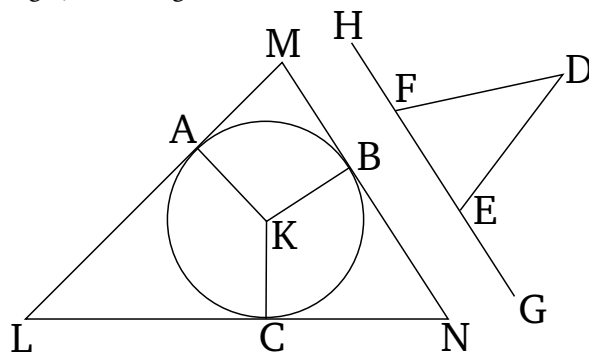
Περὶ τὸν δοθέντα ἄρα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τριγώνον περιέγραπται· ὅπερ ἔδει ποιῆσαι.

ABC].

Thus, a triangle, equiangular with the given triangle, has been inscribed in the given circle. (Which is) the very thing it was required to do.

Proposition 3

To circumscribe a triangle, equiangular with a given triangle, about a given circle.



Let *ABC* be the given circle, and *DEF* the given triangle. So it is required to circumscribe a triangle, equiangular with triangle *DEF*, about circle *ABC*.

Let *EF* have been produced in each direction to points *G* and *H*. And let the center *K* of circle *ABC* have been found [Prop. 3.1]. And let the straight-line *KB* have been drawn, at random, across (*ABC*). And let (angle) *BKA*, equal to angle *DEG*, have been constructed on the straight-line *KB* at the point *K* on it, and (angle) *BKC*, equal to *DFH* [Prop. 1.23]. And let the (straight-lines) *LAM*, *MBN*, and *NCL* have been drawn through the points *A*, *B*, and *C* (respectively), touching the circle *ABC*.†

And since *LM*, *MN*, and *NL* touch circle *ABC* at points *A*, *B*, and *C* (respectively), and *KA*, *KB*, and *KC* are joined from the center *K* to points *A*, *B*, and *C* (respectively), the angles at points *A*, *B*, and *C* are thus right-angles [Prop. 3.18]. And since the (sum of the) four angles of quadrilateral *AMBK* is equal to four right-angles, inasmuch as *AMBK* (can) also (be) divided into two triangles [Prop. 1.32], and angles *KAM* and *KBM* are (both) right-angles, the (sum of the) remaining (angles), *AKB* and *AMB*, is thus equal to two right-angles. And *DEG* and *DEF* is also equal to two right-angles [Prop. 1.13]. Thus, *AKB* and *AMB* is equal to *DEG* and *DEF*, of which *AKB* is equal to *DEG*. Thus, the remainder *AMB* is equal to the remainder *DEF*. So, similarly, it can be shown that *LNB* is also equal to *DFE*. Thus, the remaining (angle) *MLN* is also equal to the

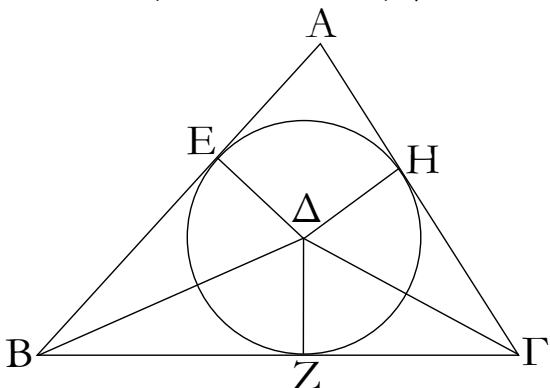
[remaining] (angle) EDF [Prop. 1.32]. Thus, triangle LMN is equiangular with triangle DEF . And it has been drawn around circle ABC .

Thus, a triangle, equiangular with the given triangle, has been circumscribed about the given circle. (Which is) the very thing it was required to do.

† See the footnote to Prop. 3.34.

δ'.

Εἰς τὸ δοθὲν τρίγωνον κύκλον ἐγγράψαι.



Ἐστω τὸ δοθὲν τρίγωνον τὸ $AB\Gamma$. δεῖ δὴ εἰς τὸ $AB\Gamma$ τρίγωνον κύκλον ἐγγράψαι.

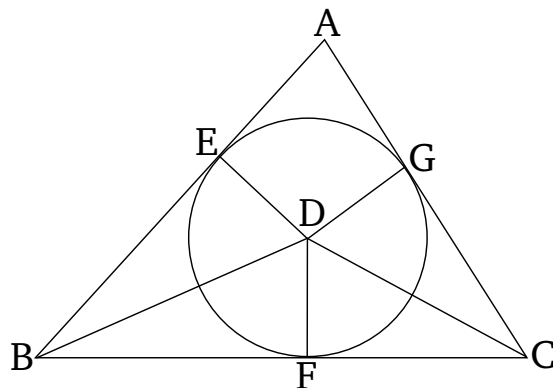
Τετμήσθωσαν αἱ ὑπὸ $AB\Gamma$, AGB γωνίαι διχα ταῖς BD , GD εὐθείαις, καὶ συμβαλλέτωσαν ἀλλήλαις κατὰ τὸ Δ σημεῖον, καὶ ῥηθῶσαν ἀπὸ τοῦ Δ ἐπὶ τὰς AB , $B\Gamma$, GA εὐθείας κάθετοι αἱ DE , ΔZ , ΔH .

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ $AB\Delta$ γωνία τῇ ὑπὸ $GB\Delta$, ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ $BE\Delta$ ὀρθὴ τῇ ὑπὸ $BZ\Delta$ ἴση, δύο δὴ τρίγωνά ἐστι τὰ $EB\Delta$, $ZB\Delta$ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν ὑποτείνουσιν ὑπὸ μίαν τῶν ἴσων γωνιῶν κοινὴν αὐτῶν τὴν $B\Delta$. καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξουσιν· ἴση ἄρα ἡ ΔE τῇ ΔZ . διὰ τὰ αὐτὰ δὴ καὶ ἡ ΔH τῇ ΔZ ἐστὶν ἴση. αἱ τρεῖς ἄρα εὐθεῖαι αἱ ΔE , ΔZ , ΔH ἴσαι ἀλλήλαις εἰσὶν· ὁ ἄρα κέντρῳ τῷ Δ καὶ διαστήματι ἐνὶ τῶν E , Z , H κύκλος γραφόμενος ῥῆξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἐφάπεται τῶν AB , $B\Gamma$, GA εὐθειῶν διὰ τὸ ὀρθὰς εἶναι τὰς πρὸς ταῖς E , Z , H σημείοις γωνίας. εἰ γὰρ τεμεῖ αὐτάς, ἔσται ἡ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ' ἄκρας ἀγομένη ἐντὸς πίπτουσα τοῦ κύκλου· ὅπερ ἄτοπον ἐδείχθη· οὐκ ἄρα ὁ κέντρῳ τῷ Δ διαστήματι δὲ ἐνὶ τῶν E , Z , H γραφόμενος κύκλος τεμεῖ τὰς AB , $B\Gamma$, GA εὐθείας· ἐφάπεται ἄρα αὐτῶν, καὶ ἔσται ὁ κύκλος ἐγγεγραμμένος εἰς τὸ $AB\Gamma$ τρίγωνον. ἐγγεγράφθω ὡς ὁ ZHE .

Εἰς ἄρα τὸ δοθὲν τρίγωνον τὸ $AB\Gamma$ κύκλος ἐγγέγραπται ὁ EZH . ὅπερ ἔδει ποιῆσαι.

Proposition 4

To inscribe a circle in a given triangle.



Let ABC be the given triangle. So it is required to inscribe a circle in triangle ABC .

Let the angles ABC and ACB have been cut in half by the straight-lines BD and CD (respectively) [Prop. 1.9], and let them meet one another at point D , and let DE , DF , and DG have been drawn from point D , perpendicular to the straight-lines AB , BC , and CA (respectively) [Prop. 1.12].

And since angle ABD is equal to CBD , and the right-angle BED is also equal to the right-angle BFD , EBD and FBD are thus two triangles having two angles equal to two angles, and one side equal to one side—the (one) subtending one of the equal angles (which is) common to the (triangles)—(namely), BD . Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, DE (is) equal to DF . So, for the same (reasons), DG is also equal to DF . Thus, the three straight-lines DE , DF , and DG are equal to one another. Thus, the circle drawn with center D , and radius one of E , F , or G ,† will also go through the remaining points, and will touch the straight-lines AB , BC , and CA , on account of the angles at E , F , and G being right-angles. For if it cuts (one of) them then it will be a (straight-line) drawn at right-angles to a diameter of the circle, from its extremity, falling inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center D , and radius one of E , F ,

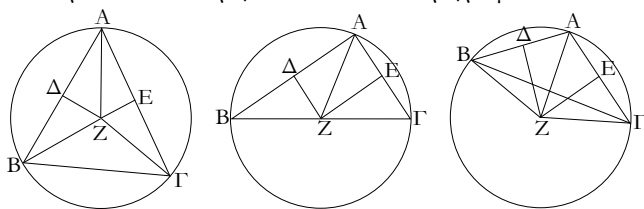
or G , does not cut the straight-lines AB , BC , and CA . Thus, it will touch them and will be the circle inscribed in triangle ABC . Let it have been (so) inscribed, like FGE (in the figure).

Thus, the circle EFG has been inscribed in the given triangle ABC . (Which is) the very thing it was required to do.

† Here, and in the following propositions, it is understood that the radius is actually one of DE , DF , or DG .

ε'.

Περί τὸ δοθέν τρίγωνον κύκλον περιγράψαι.



Ἐστω τὸ δοθέν τρίγωνον τὸ $ABΓ$. δεῖ δὲ περὶ τὸ δοθέν τρίγωνον τὸ $ABΓ$ κύκλον περιγράψαι.

Τετμήσθωσαν αἱ AB , AC εὐθεῖαι δίχα κατὰ τὰ $Δ$, $Ε$ σημεία, καὶ ἀπὸ τῶν $Δ$, $Ε$ σημείων ταῖς AB , AC πρὸς ὀρθὰς ἤχθωσαν αἱ $ΔΖ$, $ΕΖ$: συμπεσοῦνται δὴ ἤτοι ἐντὸς τοῦ $ABΓ$ τριγώνου ἢ ἐπὶ τῆς $ΒΓ$ εὐθείας ἢ ἐκτὸς τῆς $ΒΓ$.

Συμπιπτόμεναι πρότερον ἐντὸς κατὰ τὸ $Ζ$, καὶ ἐπεξεύχθωσαν αἱ ZB , $ZΓ$, ZA . καὶ ἐπεὶ ἴση ἐστὶν ἡ AD τῇ DB , κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ $ΔΖ$, βάσις ἄρα ἡ AZ βάσει τῇ ZB ἐστὶν ἴση. ὁμοίως δὲ δείξομεν, ὅτι καὶ ἡ $ΓΖ$ τῇ AZ ἐστὶν ἴση· ὥστε καὶ ἡ ZB τῇ $ZΓ$ ἐστὶν ἴση· αἱ τρεῖς ἄρα αἱ ZA , ZB , $ZΓ$ ἴσαι ἀλλήλαις εἰσὶν. ὁ ἄρα κέντρον τῷ Z διαστήματι δὲ ἐνὶ τῶν A , B , $Γ$ κύκλος γραφόμενος ἔξει καὶ διὰ τῶν λοιπῶν σημείων, καὶ ἔσται περιγεγραμμένος ὁ κύκλος περὶ τὸ $ABΓ$ τρίγωνον. περιγεγράφθω ὡς ὁ $ABΓ$.

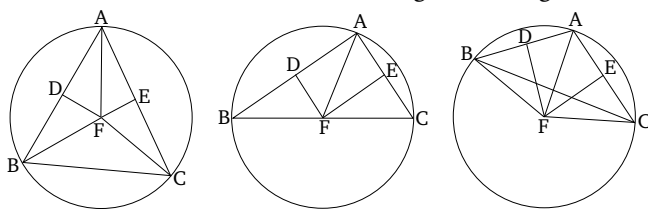
Ἀλλὰ δὴ αἱ $ΔΖ$, $ΕΖ$ συμπιπτόμεναι ἐπὶ τῆς $ΒΓ$ εὐθείας κατὰ τὸ $Ζ$, ὡς ἔχει ἐπὶ τῆς δευτέρας καταγραφῆς, καὶ ἐπεξεύχθω ἡ AZ . ὁμοίως δὲ δείξομεν, ὅτι τὸ $Ζ$ σημεῖον κέντρον ἐστὶ τοῦ περὶ τὸ $ABΓ$ τρίγωνον περιγεγραμμένου κύκλου.

Ἀλλὰ δὴ αἱ $ΔΖ$, $ΕΖ$ συμπιπτόμεναι ἐκτὸς τοῦ $ABΓ$ τριγώνου κατὰ τὸ $Ζ$ πάλιν, ὡς ἔχει ἐπὶ τῆς τρίτης καταγραφῆς, καὶ ἐπεξεύχθωσαν αἱ AZ , BZ , $ΓΖ$. καὶ ἐπεὶ πάλιν ἴση ἐστὶν ἡ AD τῇ DB , κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ $ΔΖ$, βάσις ἄρα ἡ AZ βάσει τῇ BZ ἐστὶν ἴση. ὁμοίως δὲ δείξομεν, ὅτι καὶ ἡ $ΓΖ$ τῇ AZ ἐστὶν ἴση· ὥστε καὶ ἡ BZ τῇ $ΓΖ$ ἐστὶν ἴση· ὁ ἄρα [πάλιν] κέντρον τῷ $Ζ$ διαστήματι δὲ ἐνὶ τῶν ZA , ZB , $ZΓ$ κύκλος γραφόμενος ἔξει καὶ διὰ τῶν λοιπῶν σημείων, καὶ ἔσται περιγεγραμμένος περὶ τὸ $ABΓ$ τρίγωνον.

Περί τὸ δοθέν ἄρα τρίγωνον κύκλος περιγράφεται· ὅπερ ἔδει ποιῆσαι.

Proposition 5

To circumscribe a circle about a given triangle.



Let ABC be the given triangle. So it is required to circumscribe a circle about the given triangle ABC .

Let the straight-lines AB and AC have been cut in half at points D and E (respectively) [Prop. 1.10]. And let DF and EF have been drawn from points D and E , at right-angles to AB and AC (respectively) [Prop. 1.11]. So (DF and EF) will surely either meet inside triangle ABC , on the straight-line BC , or beyond BC .

Let them, first of all, meet inside (triangle ABC) at (point) F , and let FB , FC , and FA have been joined. And since AD is equal to DB , and DF is common and at right-angles, the base AF is thus equal to the base FB [Prop. 1.4]. So, similarly, we can show that CF is also equal to AF . So that FB is also equal to FC . Thus, the three (straight-lines) FA , FB , and FC are equal to one another. Thus, the circle drawn with center F , and radius one of A , B , or C , will also go through the remaining points. And the circle will have been circumscribed about triangle ABC . Let it have been (so) circumscribed, like ABC (in the first diagram from the left).

And so, let DF and EF meet on the straight-line BC at (point) F , like in the second diagram (from the left). And let AF have been joined. So, similarly, we can show that point F is the center of the circle circumscribed about triangle ABC .

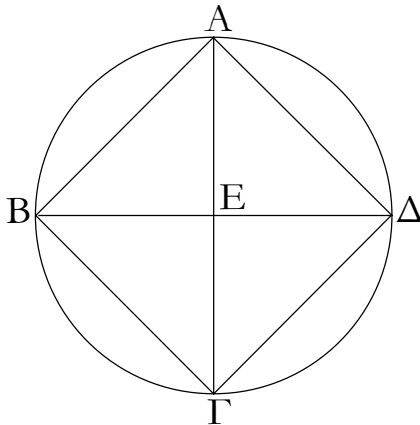
And so, let DF and EF meet outside triangle ABC , again at (point) F , like in the third diagram (from the left). And let AF , BF , and CF have been joined. And, again, since AD is equal to DB , and DF is common and at right-angles, the base AF is thus equal to the base BF [Prop. 1.4]. So, similarly, we can show that CF is also equal to AF . So that BF is also equal to FC . Thus,

[again] the circle drawn with center F , and radius one of FA , FB , and FC , will also go through the remaining points. And it will have been circumscribed about triangle ABC .

Thus, a circle has been circumscribed about the given triangle. (Which is) the very thing it was required to do.

ε'.

Εἰς τὸν δοθέντα κύκλον τετράγωνον ἐγγράψαι.



Ἐστω ἡ δοθεὶς κύκλος ὁ $ABΓΔ$. δεῖ δὲ εἰς τὸν $ABΓΔ$ κύκλον τετράγωνον ἐγγράψαι.

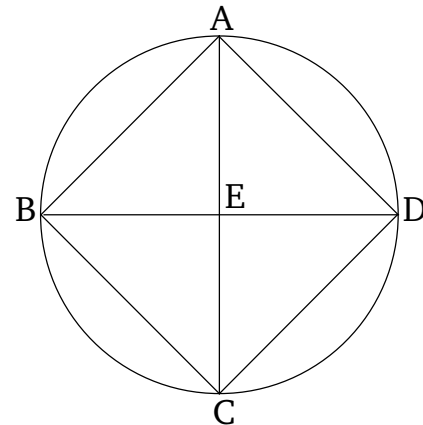
Ἦχθωσαν τοῦ $ABΓΔ$ κύκλου δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ $ΑΓ$, $ΒΔ$, καὶ ἐπεζεύχθωσαν αἱ $ΑΒ$, $ΒΓ$, $ΓΔ$, $ΔΑ$.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ $ΒΕ$ τῇ $ΕΔ$. κέντρον γὰρ τὸ $Ε$. κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ $ΕΑ$, βάσεις ἄρα ἡ $ΑΒ$ βάσει τῇ $ΑΔ$ ἴση ἐστίν. διὰ τὰ αὐτὰ δὲ καὶ ἑκατέρω τῶν $ΒΓ$, $ΓΔ$ ἑκατέρω τῶν $ΑΒ$, $ΑΔ$ ἴση ἐστίν· ἰσόπλευρον ἄρα ἐστὶ τὸ $ΑΒΓΔ$ τετράπλευρον. λέγω δὲ, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ ἡ $ΒΔ$ εὐθεῖα διάμετρος ἐστὶ τοῦ $ΑΒΓΔ$ κύκλου, ἡμικύκλιον ἄρα ἐστὶ τὸ $ΒΑΔ$. ὀρθὴ ἄρα ἡ ὑπὸ $ΒΑΔ$ γωνία. διὰ τὰ αὐτὰ δὲ καὶ ἑκάστη τῶν ὑπὸ $ΑΒΓ$, $ΒΓΔ$, $ΓΔΑ$ ὀρθὴ ἐστίν· ὀρθογώνιον ἄρα ἐστὶ τὸ $ΑΒΓΔ$ τετράπλευρον. ἐδείχθη δὲ καὶ ἰσόπλευρον· τετράγωνον ἄρα ἐστίν. καὶ ἐγγέγραπται εἰς τὸν $ΑΒΓΔ$ κύκλον.

Εἰς ἄρα τὸν δοθέντα κύκλον τετράγωνον ἐγγέγραπται τὸ $ΑΒΓΔ$. ὅπερ ἔδει ποιῆσαι.

Proposition 6

To inscribe a square in a given circle.



Let $ABCD$ be the given circle. So it is required to inscribe a square in circle $ABCD$.

Let two diameters of circle $ABCD$, AC and BD , have been drawn at right-angles to one another.[†] And let AB , BC , CD , and DA have been joined.

And since BE is equal to ED , for E (is) the center (of the circle), and EA is common and at right-angles, the base AB is thus equal to the base AD [Prop. 1.4]. So, for the same (reasons), each of BC and CD is equal to each of AB and AD . Thus, the quadrilateral $ABCD$ is equilateral. So I say that (it is) also right-angled. For since the straight-line BD is a diameter of circle $ABCD$, BAD is thus a semi-circle. Thus, angle BAD (is) a right-angle [Prop. 3.31]. So, for the same (reasons), (angles) ABC , BCD , and CDA are also each right-angles. Thus, the quadrilateral $ABCD$ is right-angled. And it was also shown (to be) equilateral. Thus, it is a square [Def. 1.22]. And it has been inscribed in circle $ABCD$.

Thus, the square $ABCD$ has been inscribed in the given circle. (Which is) the very thing it was required to do.

ζ'.

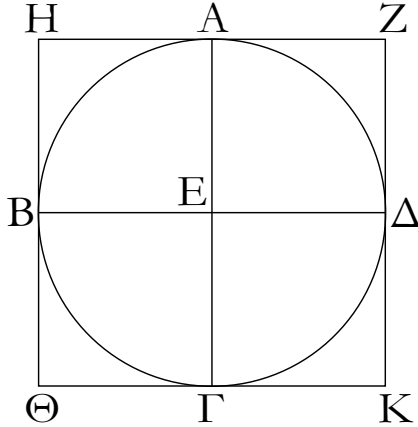
Περὶ τὸν δοθέντα κύκλον τετράγωνον περιγράψαι.

Proposition 7

To circumscribe a square about a given circle.

[†] Presumably, by finding the center of the circle [Prop. 3.1], drawing a line through it, and then drawing a second line through it, at right-angles to the first [Prop. 1.11].

Ἐστω ὁ δοθεὶς κύκλος ὁ $AB\Gamma\Delta$. δεῖ δὴ περὶ τὸν $AB\Gamma\Delta$ κύκλον τετράγωνον περιγράψαι.

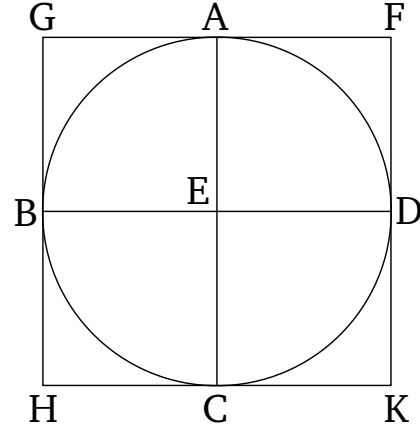


Ἦχθωσαν τοῦ $AB\Gamma\Delta$ κύκλου δύο διάμετροι πρὸς ὀρθὰς ἀλλήλαις αἱ $ΑΓ$, $ΒΔ$, καὶ διὰ τῶν A , B , Γ , Δ σημείων ἤχθωσαν ἐφαπτόμεναι τοῦ $AB\Gamma\Delta$ κύκλου αἱ ZH , $H\Theta$, ΘK , KZ .

Ἐπεὶ οὖν ἐφάπτεται ἡ ZH τοῦ $AB\Gamma\Delta$ κύκλου, ἀπὸ δὲ τοῦ E κέντρου ἐπὶ τὴν κατὰ τὸ A ἐπαφὴν ἐπέzeugται ἡ EA , αἱ ἄρα πρὸς τῷ A γωνίαι ὀρθαὶ εἰσιν. διὰ τὰ αὐτὰ δὴ καὶ αἱ πρὸς τοῖς B , Γ , Δ σημείοις γωνίαι ὀρθαὶ εἰσιν. καὶ ἐπεὶ ὀρθὴ ἐστὶν ἡ ὑπὸ AEB γωνία, ἐστὶ δὲ ὀρθὴ καὶ ἡ ὑπὸ EBH , παράλληλος ἄρα ἐστὶν ἡ $H\Theta$ τῇ $ΑΓ$. διὰ τὰ αὐτὰ δὴ καὶ ἡ $ΑΓ$ τῇ ZK ἐστὶ παράλληλος. ὥστε καὶ ἡ $H\Theta$ τῇ ZK ἐστὶ παράλληλος. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἑκατέρα τῶν HZ , ΘK τῇ $BE\Delta$ ἐστὶ παράλληλος. παραλληλόγραμμα ἄρα ἐστὶ τὰ HK , $H\Gamma$, AK , ZB , BK . ἴση ἄρα ἐστὶν ἡ μὲν HZ τῇ ΘK , ἡ δὲ $H\Theta$ τῇ ZK . καὶ ἐπεὶ ἴση ἐστὶν ἡ $ΑΓ$ τῇ $ΒΔ$, ἀλλὰ καὶ ἡ μὲν $ΑΓ$ ἑκατέρᾳ τῶν $H\Theta$, ZK , ἡ δὲ $ΒΔ$ ἑκατέρᾳ τῶν HZ , ΘK ἐστὶν ἴση [καὶ ἑκατέρα ἄρα τῶν $H\Theta$, ZK ἑκατέρᾳ τῶν HZ , ΘK ἐστὶν ἴση], ἰσόπλευρον ἄρα ἐστὶ τὸ $ZH\Theta K$ τετράπλευρον. λέγω δὴ, ὅτι καὶ ὀρθογώνιον. ἐπεὶ γὰρ παραλληλόγραμμον ἐστὶ τὸ $HBEA$, καὶ ἐστὶν ὀρθὴ ἡ ὑπὸ AEB , ὀρθὴ ἄρα καὶ ἡ ὑπὸ AHB . ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ πρὸς τοῖς Θ , K , Z γωνίαι ὀρθαὶ εἰσιν. ὀρθογώνιον ἄρα ἐστὶ τὸ $ZH\Theta K$. ἐδείχθη δὲ καὶ ἰσόπλευρον· τετράγωνον ἄρα ἐστὶν. καὶ περιέγραπται περὶ τὸν $AB\Gamma\Delta$ κύκλον.

Περὶ τὸν δοθέντα ἄρα κύκλον τετράγωνον περιέγραπται· ὅπερ ἔδει ποιῆσαι.

Let $ABCD$ be the given circle. So it is required to circumscribe a square about circle $ABCD$.



Let two diameters of circle $ABCD$, AC and BD , have been drawn at right-angles to one another.[†] And let FG , GH , HK , and KF have been drawn through points A , B , C , and D (respectively), touching circle $ABCD$.[‡]

Therefore, since FG touches circle $ABCD$, and EA has been joined from the center E to the point of contact A , the angles at A are thus right-angles [Prop. 3.18]. So, for the same (reasons), the angles at points B , C , and D are also right-angles. And since angle AEB is a right-angle, and EBG is also a right-angle, GH is thus parallel to AC [Prop. 1.29]. So, for the same (reasons), AC is also parallel to FK . So that GH is also parallel to FK [Prop. 1.30]. So, similarly, we can show that GF and HK are each parallel to BED . Thus, GK , GC , AK , FB , and BK are (all) parallelograms. Thus, GF is equal to HK , and GH to FK [Prop. 1.34]. And since AC is equal to BD , but AC (is) also (equal) to each of GH and FK , and BD is equal to each of GF and HK [Prop. 1.34] [and each of GH and FK is thus equal to each of GF and HK], the quadrilateral $FGHK$ is thus equilateral. So I say that (it is) also right-angled. For since $GBEA$ is a parallelogram, and AEB is a right-angle, AGB is thus also a right-angle [Prop. 1.34]. So, similarly, we can show that the angles at H , K , and F are also right-angles. Thus, $FGHK$ is right-angled. And it was also shown (to be) equilateral. Thus, it is a square [Def. 1.22]. And it has been circumscribed about circle $ABCD$.

Thus, a square has been circumscribed about the given circle. (Which is) the very thing it was required to do.

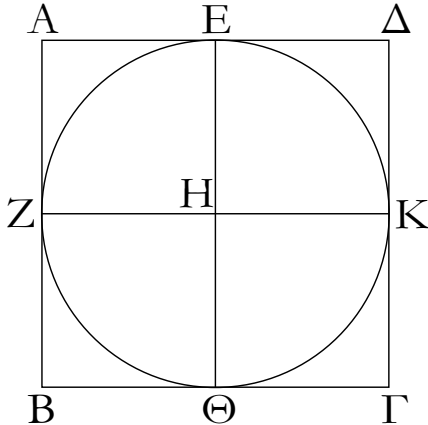
[†] See the footnote to the previous proposition.

[‡] See the footnote to Prop. 3.34.

η'.

Εἰς τὸ δοθὲν τετράγωνον κύκλον ἐγγράψαι.

Ἐστω τὸ δοθὲν τετράγωνον τὸ ΑΒΓΔ. δεῖ δὴ εἰς τὸ ΑΒΓΔ τετράγωνον κύκλον ἐγγράψαι.



Τετμήσθω ἑκατέρα τῶν ΑΔ, ΑΒ δίχα κατὰ τὰ Ε, Ζ σημεία, καὶ διὰ μὲν τοῦ Ε ὁποτέρᾳ τῶν ΑΒ, ΓΔ παράλληλος ἦχθω ὁ ΕΘ, διὰ δὲ τοῦ Ζ ὁποτέρᾳ τῶν ΑΔ, ΒΓ παράλληλος ἦχθω ἡ ΖΚ· παραλληλόγραμμον ἄρα ἐστὶν ἕκαστον τῶν ΑΚ, ΚΒ, ΑΘ, ΘΔ, ΑΗ, ΗΓ, ΒΗ, ΗΔ, καὶ αἱ ἀπεναντίον αὐτῶν πλευραὶ δηλονότι ἴσαι [εἰσὶν]. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΑΒ, καὶ ἐστὶ τῆς μὲν ΑΔ ἡμίσεια ἡ ΑΕ, τῆς δὲ ΑΒ ἡμίσεια ἡ ΑΖ, ἴση ἄρα καὶ ἡ ΑΕ τῇ ΑΖ· ὥστε καὶ αἱ ἀπεναντίον· ἴση ἄρα καὶ ἡ ΖΗ τῇ ΗΕ. ὁμοίως δὴ δεῖξομεν, ὅτι καὶ ἑκατέρα τῶν ΗΘ, ΗΚ ἑκατέρᾳ τῶν ΖΗ, ΗΕ ἐστὶν ἴση· αἱ τέσσαρες ἄρα αἱ ΗΕ, ΗΖ, ΗΘ, ΗΚ ἴσαι ἀλλήλαις [εἰσὶν]. ὁ ἄρα κέντρω μὲν τῷ Η διαστήματι δὲ ἐνὶ τῶν Ε, Ζ, Θ, Κ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων· καὶ ἐφάπτεται τῶν ΑΒ, ΒΓ, ΓΔ, ΔΑ εὐθειῶν διὰ τὸ ὀρθὰς εἶναι τὰς πρὸς τοῖς Ε, Ζ, Θ, Κ γωνίας· εἰ γὰρ τεμεῖ ὁ κύκλος τὰς ΑΒ, ΒΓ, ΓΔ, ΔΑ, ἡ τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ' ἄκρας ἀγομένη ἐντὸς πεσεῖται τοῦ κύκλου· ὅπερ ἄτοπον ἐδείχθη. οὐκ ἄρα ὁ κέντρω τῷ Η διαστήματι δὲ ἐνὶ τῶν Ε, Ζ, Θ, Κ κύκλος γραφόμενος τεμεῖ τὰς ΑΒ, ΒΓ, ΓΔ, ΔΑ εὐθείας. ἐφάπτεται ἄρα αὐτῶν καὶ ἔσται ἐγγεγραμμένος εἰς τὸ ΑΒΓΔ τετράγωνον.

Εἰς ἄρα τὸ δοθὲν τετράγωνον κύκλος ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

θ'.

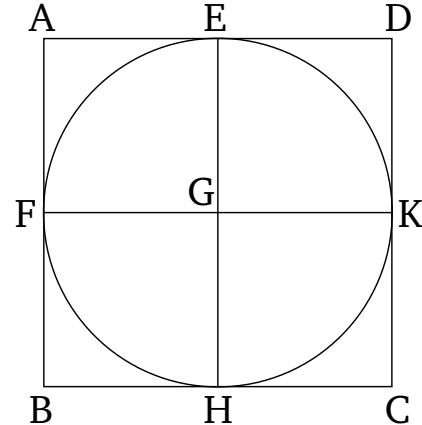
Περὶ τὸ δοθὲν τετράγωνον κύκλον περιγράψαι.

Ἐστω τὸ δοθὲν τετράγωνον τὸ ΑΒΓΔ· δεῖ δὴ περὶ τὸ ΑΒΓΔ τετράγωνον κύκλον περιγράψαι.

Proposition 8

To inscribe a circle in a given square.

Let the given square be $ABCD$. So it is required to inscribe a circle in square $ABCD$.



Let AD and AB each have been cut in half at points E and F (respectively) [Prop. 1.10]. And let EH have been drawn through E , parallel to either of AB or CD , and let FK have been drawn through F , parallel to either of AD or BC [Prop. 1.31]. Thus, AK , KB , AH , HD , AG , GC , BG , and GD are each parallelograms, and their opposite sides [are] manifestly equal [Prop. 1.34]. And since AD is equal to AB , and AE is half of AD , and AF half of AB , AE (is) thus also equal to AF . So that the opposite (sides are) also (equal). Thus, FG (is) also equal to GE . So, similarly, we can also show that each of GH and GK is equal to each of FG and GE . Thus, the four (straight-lines) GE , GF , GH , and GK [are] equal to one another. Thus, the circle drawn with center G , and radius one of E , F , H , or K , will also go through the remaining points. And it will touch the straight-lines AB , BC , CD , and DA , on account of the angles at E , F , H , and K being right-angles. For if the circle cuts AB , BC , CD , or DA , then a (straight-line) drawn at right-angles to a diameter of the circle, from its extremity, will fall inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center G , and radius one of E , F , H , or K , does not cut the straight-lines AB , BC , CD , or DA . Thus, it will touch them, and will have been inscribed in the square $ABCD$.

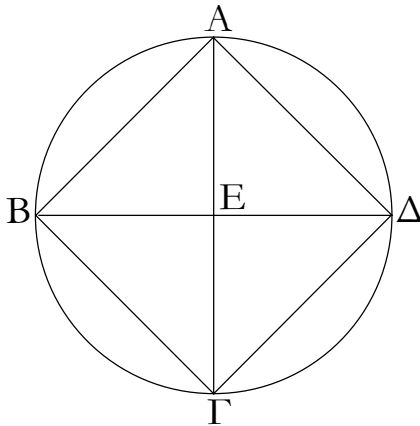
Thus, a circle has been inscribed in the given square. (Which is) the very thing it was required to do.

Proposition 9

To circumscribe a circle about a given square.

Let $ABCD$ be the given square. So it is required to circumscribe a circle about square $ABCD$.

Ἐπιζευχθεῖσαι γὰρ αἱ ΑΓ, ΒΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε.



Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΔΑ τῇ ΑΒ, κοινὴ δὲ ἡ ΑΓ, δύο δὴ αἱ ΔΑ, ΑΓ δυοὶ ταῖς ΒΑ, ΑΓ ἴσαι εἰσὶν· καὶ βάσεις ἡ ΔΓ βάσει τῇ ΒΓ ἴση· γωνία ἄρα ἡ ὑπὸ ΔΑΓ γωνία τῇ ὑπὸ ΒΑΓ ἴση ἐστὶν· ἡ ἄρα ὑπὸ ΔΑΒ γωνία δίχα τέτμηται ὑπὸ τῆς ΑΓ. ὁμοίως δὲ δεῖξομεν, ὅτι καὶ ἐκάστη τῶν ὑπὸ ΑΒΓ, ΒΓΔ, ΓΔΑ δίχα τέτμηται ὑπὸ τῶν ΑΓ, ΔΒ εὐθειῶν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔΑΒ γωνία τῇ ὑπὸ ΑΒΓ, καὶ ἐστὶ τῆς μὲν ὑπὸ ΔΑΒ ἡμίσεια ἡ ὑπὸ ΕΑΒ, τῆς δὲ ὑπὸ ΑΒΓ ἡμίσεια ἡ ὑπὸ ΕΒΑ, καὶ ἡ ὑπὸ ΕΑΒ ἄρα τῇ ὑπὸ ΕΒΑ ἐστὶν ἴση· ὥστε καὶ πλευρὰ ἡ ΕΑ τῇ ΕΒ ἐστὶν ἴση. ὁμοίως δὲ δεῖξομεν, ὅτι καὶ ἐκάτερα τῶν ΕΑ, ΕΒ [εὐθειῶν] ἐκάτερα τῶν ΕΓ, ΕΔ ἴση ἐστὶν. αἱ τέσσαρες ἄρα αἱ ΕΑ, ΕΒ, ΕΓ, ΕΔ ἴσαι ἀλλήλαις εἰσὶν. ὁ ἄρα κέντρον τῷ Ε καὶ διαστήματι ἐνὶ τῶν Α, Β, Γ, Δ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἔσται περιγεγραμμένος περὶ τὸ ΑΒΓΔ τετράγωνον. περιγεγράφθω ὡς ὁ ΑΒΓΔ.

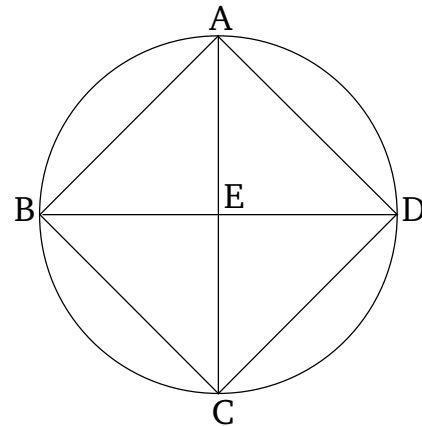
Περὶ τὸ δοθὲν ἄρα τετράγωνον κύκλος περιγράφεται ὅπερ ἔδει ποιῆσαι.

ι'.

Ἴσοσκελὲς τρίγωνον συστήσασθαι ἔχον ἐκάτεραν τῶν πρὸς τῇ βάσει γωνιῶν διπλασίονα τῆς λοιπῆς.

Ἐκκεῖσθω τις εὐθεῖα ἡ ΑΒ, καὶ τετμήσθω κατὰ τὸ Γ σημεῖον, ὥστε τὸ ὑπὸ τῶν ΑΒ, ΒΓ περιεχόμενον ὀρθογώνιον ἴσον εἶναι τῷ ἀπὸ τῆς ΓΑ τετραγώνῳ· καὶ κέντρον τῷ Α καὶ διαστήματι τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΔΕ, καὶ ἐνηρμόσθω εἰς τὸν ΒΔΕ κύκλον τῇ ΑΓ εὐθείᾳ μὴ μείζονι οὐσῇ τῆς τοῦ ΒΔΕ κύκλου διαμέτρου ἴση εὐθεῖα ἡ ΒΔ· καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΓ, καὶ περιγεγράφθω περὶ τὸ ΑΓΔ τρίγωνον κύκλος ὁ ΑΓΔ.

AC and BD being joined, let them cut one another at E.



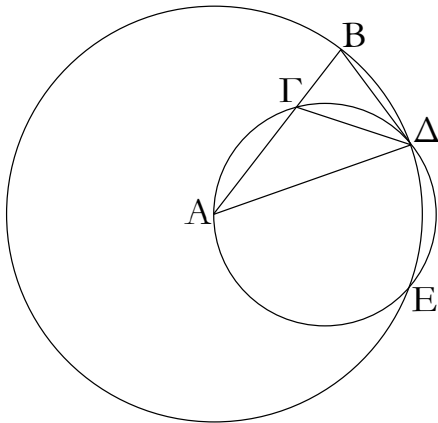
And since DA is equal to AB , and AC (is) common, the two (straight-lines) DA , AC are thus equal to the two (straight-lines) BA , AC . And the base DC (is) equal to the base BC . Thus, angle DAC is equal to angle BAC [Prop. 1.8]. Thus, the angle DAB has been cut in half by AC . So, similarly, we can show that ABC , BCD , and CDA have each been cut in half by the straight-lines AC and DB . And since angle DAB is equal to ABC , and EAB is half of DAB , and EBA half of ABC , EAB is thus also equal to EBA . So that side EA is also equal to EB [Prop. 1.6]. So, similarly, we can show that each of the [straight-lines] EA and EB are also equal to each of EC and ED . Thus, the four (straight-lines) EA , EB , EC , and ED are equal to one another. Thus, the circle drawn with center E , and radius one of A , B , C , or D , will also go through the remaining points, and will have been circumscribed about the square $ABCD$. Let it have been (so) circumscribed, like $ABCD$ (in the figure).

Thus, a circle has been circumscribed about the given square. (Which is) the very thing it was required to do.

Proposition 10

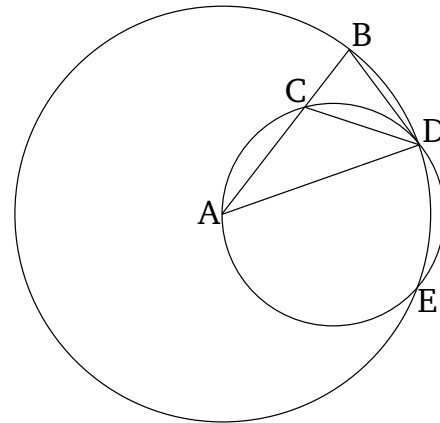
To construct an isosceles triangle having each of the angles at the base double the remaining (angle).

Let some straight-line AB be taken, and let it have been cut at point C so that the rectangle contained by AB and BC is equal to the square on CA [Prop. 2.11]. And let the circle BDE have been drawn with center A , and radius AB . And let the straight-line BD , equal to the straight-line AC , being not greater than the diameter of circle BDE , have been inserted into circle BDE [Prop. 4.1]. And let AD and DC have been joined. And let the circle ACD have been circumscribed about triangle ACD [Prop. 4.5].



Καὶ ἐπεὶ τὸ ὑπὸ τῶν AB, BG ἴσον ἐστὶ τῷ ἀπὸ τῆς AG, ἴση δὲ ἡ AG τῇ BD, τὸ ἄρα ὑπὸ τῶν AB, BG ἴσον ἐστὶ τῷ ἀπὸ τῆς BD. καὶ ἐπεὶ κύκλου τοῦ AGD εἰληπταὶ τι σημεῖον ἐκτὸς τὸ B, καὶ ἀπὸ τοῦ B πρὸς τὸν AGD κύκλον προσπεπτώκασι δύο εὐθεῖαι αἱ BA, BD, καὶ ἡ μὲν αὐτῶν τέμνει, ἡ δὲ προσπίπτει, καὶ ἐστὶ τὸ ὑπὸ τῶν AB, BG ἴσον τῷ ἀπὸ τῆς BD, ἡ BD ἄρα ἐφάπτεται τοῦ AGD κύκλου. ἐπεὶ οὖν ἐφάπτεται μὲν ἡ BD, ἀπὸ δὲ τῆς κατὰ τὸ Δ ἐπαφῆς διῆκται ἡ ΔΓ, ἡ ἄρα ὑπὸ BΔΓ γωνία ἴση ἐστὶ τῇ ἐν τῷ ἐναλλὰξ τοῦ κύκλου τμήματι γωνίᾳ τῇ ὑπὸ ΔAG. ἐπεὶ οὖν ἴση ἐστὶν ἡ ὑπὸ BΔΓ τῇ ὑπὸ ΔAG, κοινὴ προσκείσθω ἡ ὑπὸ ΓΔA· ὅλη ἄρα ἡ ὑπὸ BΔA ἴση ἐστὶ δυσὶ ταῖς ὑπὸ ΓΔA, ΔAG. ἀλλὰ ταῖς ὑπὸ ΓΔA, ΔAG ἴση ἐστὶν ἡ ἐκτὸς ἡ ὑπὸ BΓΔ· καὶ ἡ ὑπὸ BΔA ἄρα ἴση ἐστὶ τῇ ὑπὸ BΓΔ. ἀλλὰ ἡ ὑπὸ BΔA τῇ ὑπὸ ΓBΔ ἐστὶν ἴση, ἐπεὶ καὶ πλευρὰ ἡ AD τῇ AB ἐστὶν ἴση· ὥστε καὶ ἡ ὑπὸ ΔBA τῇ ὑπὸ BΓΔ ἐστὶν ἴση. αἱ τρεῖς ἄρα αἱ ὑπὸ BΔA, ΔBA, BΓA ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΔBG γωνία τῇ ὑπὸ BΓΔ, ἴση ἐστὶ καὶ πλευρὰ ἡ BD πλευρᾷ τῇ ΔΓ. ἀλλὰ ἡ BD τῇ ΓA ὑπόκειται ἴση· καὶ ἡ ΓA ἄρα τῇ ΓΔ ἐστὶν ἴση· ὥστε καὶ γωνία ἡ ὑπὸ ΓΔA γωνία τῇ ὑπὸ ΔAG ἐστὶν ἴση· αἱ ἄρα ὑπὸ ΓΔA, ΔAG τῆς ὑπὸ ΔAG εἰσι διπλασίους. ἴση δὲ ἡ ὑπὸ BΓΔ ταῖς ὑπὸ ΓΔA, ΔAG· καὶ ἡ ὑπὸ BΓΔ ἄρα τῆς ὑπὸ ΓΔA ἐστὶ διπλῇ. ἴση δὲ ἡ ὑπὸ BΓΔ ἑκατέρᾳ τῶν ὑπὸ BΔA, ΔBA· καὶ ἑκατέρᾳ ἄρα τῶν ὑπὸ BΔA, ΔBA τῆς ὑπὸ ΔAB ἐστὶ διπλῇ.

Ἰσοσκελὲς ἄρα τρίγωνον συνέσταται τὸ $AB\Delta$ ἔχον
ἐκατέραν τῶν πρὸς τῇ ΔB βάσει γωνιῶν διπλασίονα τῆς
λοιπῆς· ὅπερ ἔδει ποιῆσαι.



And since the (rectangle contained) by AB and BC is equal to the (square) on AC , and AC (is) equal to BD , the (rectangle contained) by AB and BC is thus equal to the (square) on BD . And since some point B has been taken outside of circle ACD , and two straight-lines BA and BD have radiated from B towards the circle ACD , and (one) of them cuts (the circle), and (the other) meets (the circle), and the (rectangle contained) by AB and BC is equal to the (square) on BD , BD thus touches circle ACD [Prop. 3.37]. Therefore, since BD touches (the circle), and DC has been drawn across (the circle) from the point of contact D , the angle BDC is thus equal to the angle DAC in the alternate segment of the circle [Prop. 3.32]. Therefore, since BDC is equal to DAC , let CDA have been added to both. Thus, the whole of BDA is equal to the two (angles) CDA and DAC . But, the external (angle) BCD is equal to CDA and DAC [Prop. 1.32]. Thus, BDA is also equal to BCD . But, BDA is equal to CBD , since the side AD is also equal to AB [Prop. 1.5]. So that DBA is also equal to BCD . Thus, the three (angles) BDA , DBA , and BCD are equal to one another. And since angle DBC is equal to BCD , side BD is also equal to side DC [Prop. 1.6]. But, BD was assumed (to be) equal to CA . Thus, CA is also equal to CD . So that angle CDA is also equal to angle DAC [Prop. 1.5]. Thus, CDA and DAC is double DAC . But BCD (is) equal to CDA and DAC . Thus, BCD is also double CAD . And BCD (is) equal to to each of BDA and DBA . Thus, BDA and DBA are each double DAB .

Thus, the isosceles triangle ABD has been constructed having each of the angles at the base BD double the remaining (angle). (Which is) the very thing it was required to do.

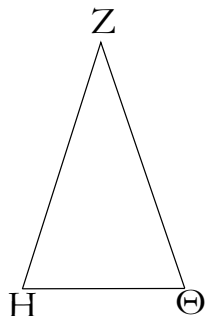
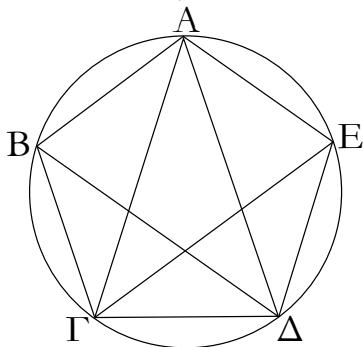
 α' .

Proposition 11

Εἰς τὸν δοθέντα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ

To inscribe an equilateral and equiangular pentagon

ἰσογώνιον ἐγγράψαι.



Ἐστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔΕ· δεῖ δὴ εἰς τὸν ΑΒΓΔΕ κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Ἐκκεῖσθω τρίγωνον ἰσοσκελὲς τὸ ΖΗΘ διπλασίονα ἔχον ἑκατέραν τῶν πρὸς τοῖς Η, Θ γωνιῶν τῆς πρὸς τῷ Ζ, καὶ ἐγγεγράφῃω εἰς τὸν ΑΒΓΔΕ κύκλον τῷ ΖΗΘ τριγώνῳ ἰσογώνιον τρίγωνον τὸ ΑΓΔ, ὥστε τῇ μὲν πρὸς τῷ Ζ γωνίᾳ ἴσην εἶναι τὴν ὑπὸ ΓΑΔ, ἑκατέραν δὲ τῶν πρὸς τοῖς Η, Θ ἴσην ἑκατέρᾳ τῶν ὑπὸ ΑΓΔ, ΓΔΑ· καὶ ἑκατέρα ἄρα τῶν ὑπὸ ΑΓΔ, ΓΔΑ τῆς ὑπὸ ΓΑΔ ἐστὶ διπλῇ. τετμήσθω δὴ ἑκατέρα τῶν ὑπὸ ΑΓΔ, ΓΔΑ δίχα ὑπὸ ἑκατέρας τῶν ΓΕ, ΔΒ εὐθειῶν, καὶ ἐπεξεύχθωσαν αἱ ΑΒ, ΒΓ, ΔΕ, ΕΑ.

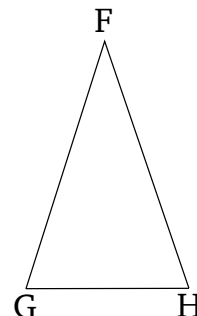
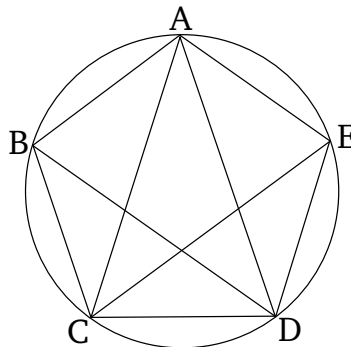
Ἐπεὶ οὖν ἑκατέρα τῶν ὑπὸ ΑΓΔ, ΓΔΑ γωνιῶν διπλασίον ἐστὶ τῆς ὑπὸ ΓΑΔ, καὶ τετμημέναι εἰσὶ δίχα ὑπὸ τῶν ΓΕ, ΔΒ εὐθειῶν, αἱ πέντε ἄρα γωνίαι αἱ ὑπὸ ΔΑΓ, ΑΓΕ, ΕΓΔ, ΓΔΒ, ΒΔΑ ἴσαι ἀλλήλαις εἰσὶν. αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν· αἱ πέντε ἄρα περιφέρειαι αἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ ἴσαι ἀλλήλαις εἰσὶν. ὑπὸ δὲ τὰς ἴσας περιφέρειας ἴσαι εὐθεῖαι ὑποτείνουσιν· αἱ πέντε ἄρα εὐθεῖαι αἱ ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ ἴσαι ἀλλήλαις εἰσὶν· ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον. λέγω δὴ, ὅτι καὶ ἰσογώνιον. ἐπεὶ γὰρ ἡ ΑΒ περιφέρεια τῇ ΔΕ περιφέρειᾳ ἐστὶν ἴση, κοινὴ προσκεῖσθω ἡ ΒΓΔ· ὅλη ἄρα ἡ ΑΒΓΔ περιφέρεια ὅλη τῇ ΕΔΓΒ περιφέρειᾳ ἐστὶν ἴση. καὶ βεβήκεν ἐπὶ μὲν τῆς ΑΒΓΔ περιφερείας γωνία ἡ ὑπὸ ΑΕΔ, ἐπὶ δὲ τῆς ΕΔΓΒ περιφερείας γωνία ἡ ὑπὸ ΒΑΕ· καὶ ἡ ὑπὸ ΒΑΕ ἄρα γωνία τῇ ὑπὸ ΑΕΔ ἐστὶν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἑκάστη τῶν ὑπὸ ΑΒΓ, ΒΓΔ, ΓΔΕ γωνιῶν ἑκατέρᾳ τῶν ὑπὸ ΒΑΕ, ΑΕΔ ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓΔΕ πεντάγωνον. ἐδείχθη δὲ καὶ ἰσόπλευρον.

Εἰς ἄρα τὸν δοθέντα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

ιβ'.

Περὶ τὸν δοθέντα κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον περιγράψαι.

in a given circle.



Let $ABCDE$ be the given circle. So it is required to inscribed an equilateral and equiangular pentagon in circle $ABCDE$.

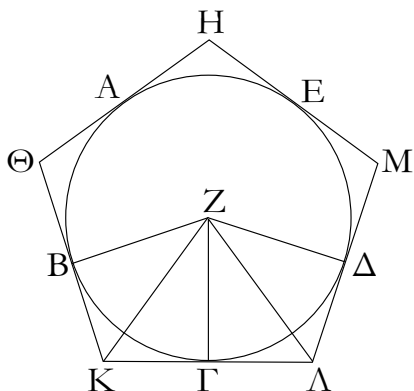
Let the the isosceles triangle FGH be set up having each of the angles at G and H double the (angle) at F [Prop. 4.10]. And let triangle ACD , equiangular to FGH , have been inscribed in circle $ABCDE$, such that CAD is equal to the angle at F , and the (angles) at G and H (are) equal to ACD and CDA , respectively [Prop. 4.2]. Thus, ACD and CDA are each double CAD . So let ACD and CDA have been cut in half by the straight-lines CE and DB , respectively [Prop. 1.9]. And let AB , BC , DE and EA have been joined.

Therefore, since angles ACD and CDA are each double CAD , and are cut in half by the straight-lines CE and DB , the five angles DAC , ACE , ECD , CDB , and BDA are thus equal to one another. And equal angles stand upon equal circumferences [Prop. 3.26]. Thus, the five circumferences AB , BC , CD , DE , and EA are equal to one another [Prop. 3.29]. Thus, the pentagon $ABCDE$ is equilateral. So I say that (it is) also equiangular. For since the circumference AB is equal to the circumference DE , let BCD have been added to both. Thus, the whole circumference $ABCD$ is equal to the whole circumference $EDCB$. And the angle AED stands upon circumference $ABCD$, and angle BAE upon circumference $EDCB$. Thus, angle BAE is also equal to AED [Prop. 3.27]. So, for the same (reasons), each of the angles ABC , BCD , and CDE is also equal to each of BAE and AED . Thus, pentagon $ABCDE$ is equiangular. And it was also shown (to be) equilateral.

Thus, an equilateral and equiangular pentagon has been inscribed in the given circle. (Which is) the very thing it was required to do.

Proposition 12

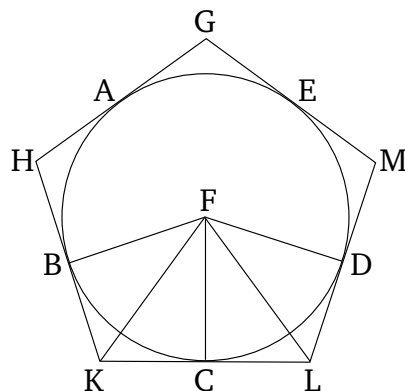
To circumscribe an equilateral and equiangular pentagon about a given circle.



Ἐστω ὁ δοθεὶς κύκλος ὁ ΑΒΓΔΕ· δεῖ δὲ περὶ τὸν ΑΒΓΔΕ κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον περιγράψαι.

Νενοήσθω τοῦ ἐγγεγραμμένου πενταγώνου τῶν γωνιῶν σημεῖα τὰ Α, Β, Γ, Δ, Ε, ὥστε ἴσας εἶναι τὰς ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ περιφερείας· καὶ διὰ τῶν Α, Β, Γ, Δ, Ε ἤχθωσαν τοῦ κύκλου ἐφαπτόμεναι αἱ ΗΘ, ΘΚ, ΚΛ, ΛΜ, ΜΗ, καὶ εἰληφθῶ τοῦ ΑΒΓΔΕ κύκλου κέντρον τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΖΒ, ΖΚ, ΖΓ, ΖΛ, ΖΔ.

Καὶ ἐπεὶ ἡ μὲν ΚΛ εὐθεῖα ἐφάπτεται τοῦ ΑΒΓΔΕ κατὰ τὸ Γ, ἀπὸ δὲ τοῦ Ζ κέντρου ἐπὶ τὴν κατὰ τὸ Γ ἐπαφὴν ἐπέξευκται ἡ ΖΓ, ἡ ΖΓ ἄρα κάθετός ἐστιν ἐπὶ τὴν ΚΛ· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν πρὸς τῷ Γ γωνιών. διὰ τὰ αὐτὰ δὴ καὶ αἱ πρὸς τοῖς Β, Δ σημείοις γωνίαι ὀρθαὶ εἰσιν. καὶ ἐπεὶ ὀρθὴ ἐστὶν ἡ ὑπὸ ΖΓΚ γωνία, τὸ ἄρα ἀπὸ τῆς ΖΚ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΖΓ, ΓΚ. διὰ τὰ αὐτὰ δὴ καὶ τοῖς ἀπὸ τῶν ΖΒ, ΒΚ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΖΚ· ὥστε τὰ ἀπὸ τῶν ΖΓ, ΓΚ τοῖς ἀπὸ τῶν ΖΒ, ΒΚ ἐστὶν ἴσα, ὧν τὸ ἀπὸ τῆς ΖΓ τῷ ἀπὸ τῆς ΖΒ ἐστὶν ἴσον· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΓΚ τῷ ἀπὸ τῆς ΒΚ ἐστὶν ἴσον. ἴση ἄρα ἡ ΒΚ τῇ ΓΚ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΖΒ τῇ ΖΓ, καὶ κοινὴ ἡ ΖΚ, δύο δὴ αἱ ΒΖ, ΖΚ δυοὶ ταῖς ΓΖ, ΖΚ ἴσαι εἰσίν· καὶ βάσεις ἡ ΒΚ βάσει τῇ ΓΚ [ἐστίν] ἴση· γωνία ἄρα ἡ μὲν ὑπὸ ΒΖΚ [γωνία] τῇ ὑπὸ ΚΖΓ ἐστὶν ἴση· ἡ δὲ ὑπὸ ΒΚΖ τῇ ὑπὸ ΖΚΓ· διπλῇ ἄρα ἡ μὲν ὑπὸ ΒΖΓ τῆς ὑπὸ ΚΖΓ, ἡ δὲ ὑπὸ ΒΚΓ τῆς ὑπὸ ΖΚΓ. διὰ τὰ αὐτὰ δὴ καὶ ἡ μὲν ὑπὸ ΓΖΔ τῆς ὑπὸ ΓΖΑ ἐστὶ διπλῇ, ἡ δὲ ὑπὸ ΔΛΓ τῆς ὑπὸ ΖΛΓ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΓ περιφέρεια τῇ ΓΔ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ΒΖΓ τῇ ὑπὸ ΓΖΔ. καὶ ἐστὶν ἡ μὲν ὑπὸ ΒΖΓ τῆς ὑπὸ ΚΖΓ διπλῇ, ἡ δὲ ὑπὸ ΔΖΓ τῆς ὑπὸ ΛΖΓ· ἴση ἄρα καὶ ἡ ὑπὸ ΚΖΓ τῇ ὑπὸ ΛΖΓ· ἐστὶ δὲ καὶ ἡ ὑπὸ ΖΓΚ γωνία τῇ ὑπὸ ΖΓΛ ἴση. δύο δὴ τρίγωνά ἐστι τὰ ΖΚΓ, ΖΛΓ τὰς δύο γωνίας ταῖς δυοὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μὴ πλευρᾷ ἴσην κοινὴν αὐτῶν τὴν ΖΓ· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξει καὶ τὴν λοιπὴν γωνίαν τῇ λοιπῇ γωνίᾳ· ἴση ἄρα ἡ μὲν ΚΓ εὐθεῖα τῇ ΓΛ, ἡ δὲ ὑπὸ ΖΚΓ γωνία τῇ ὑπὸ ΖΛΓ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΚΓ τῇ ΓΛ, διπλῇ ἄρα ἡ ΚΛ τῆς ΚΓ. διὰ τὰ αὐτὰ δὴ δειχθήσεται καὶ ἡ ΘΚ τῆς ΒΚ διπλῇ. καὶ ἐστὶν ἡ ΒΚ τῇ ΚΓ ἴση· καὶ ἡ ΘΚ ἄρα τῇ ΚΛ ἐστὶν ἴση. ὁμοίως δὴ δειχθήσεται



Let $ABCDE$ be the given circle. So it is required to circumscribe an equilateral and equiangular pentagon about circle $ABCDE$.

Let A, B, C, D , and E have been conceived as the angular points of a pentagon having been inscribed (in circle $ABCDE$) [Prop. 3.11], such that the circumferences AB, BC, CD, DE , and EA are equal. And let GH, HK, KL, LM , and MG have been drawn through (points) A, B, C, D , and E (respectively), touching the circle.[†] And let the center F of the circle $ABCDE$ have been found [Prop. 3.1]. And let FB, FK, FC, FL , and FD have been joined.

And since the straight-line KL touches (circle) $ABCDE$ at C , and FC has been joined from the center F to the point of contact C , FC is thus perpendicular to KL [Prop. 3.18]. Thus, each of the angles at C is a right-angle. So, for the same (reasons), the angles at B and D are also right-angles. And since angle FCK is a right-angle, the (square) on FK is thus equal to the (sum of the squares) on FC and CK [Prop. 1.47]. So, for the same (reasons), the (square) on FK is also equal to the (sum of the squares) on FB and BK . So that the (sum of the squares) on FC and CK is equal to the (sum of the squares) on FB and BK , of which the (square) on FC is equal to the (square) on FB . Thus, the remaining (square) on CK is equal to the remaining (square) on BK . Thus, BK (is) equal to CK . And since FB is equal to FC , and FK (is) common, the two (straight-lines) BF, FK are equal to the two (straight-lines) CF, FK . And the base BK [is] equal to the base CK . Thus, angle BFK is equal to [angle] KFC [Prop. 1.8]. And BKF (is equal) to FKC [Prop. 1.8]. Thus, BFC (is) double KFC , and BKC (is double) FKC . So, for the same (reasons), CFD is also double CFL , and DLC (is also double) FLC . And since circumference BC is equal to CD , angle BFC is also equal to CFD [Prop. 3.27]. And BFC is double KFC , and DFC (is double) LFC . Thus, KFC is also equal to LFC . And angle FCK is also equal to FCL . So, FKC and FLC are two triangles hav-

ἡ ΒΓ τῇ ΓΔ, κοινὴ δὲ ἡ ΓΖ, δύο δὲ αἱ ΒΓ, ΓΖ δυσὶ ταῖς ΔΓ, ΓΖ ἴσαι εἰσὶν· καὶ γωνία ἡ ὑπὸ ΒΓΖ γωνία τῇ ὑπὸ ΔΓΖ [ἐστίν] ἴση· βάσις ἄρα ἡ ΒΖ βάσει τῇ ΔΖ ἐστὶν ἴση, καὶ τὸ ΒΓΖ τρίγωνον τῷ ΔΓΖ τριγώνῳ ἐστὶν ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν· ἴση ἄρα ἡ ὑπὸ ΓΒΖ γωνία τῇ ὑπὸ ΓΔΖ. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ ὑπὸ ΓΔΕ τῆς ὑπὸ ΓΔΖ, ἴση δὲ ἡ μὲν ὑπὸ ΓΔΕ τῇ ὑπὸ ΑΒΓ, ἡ δὲ ὑπὸ ΓΔΖ τῇ ὑπὸ ΓΒΖ, καὶ ἡ ὑπὸ ΓΒΑ ἄρα τῆς ὑπὸ ΓΒΖ ἐστὶ διπλῇ· ἴση ἄρα ἡ ὑπὸ ΑΒΖ γωνία τῇ ὑπὸ ΖΒΓ· ἡ ἄρα ὑπὸ ΑΒΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΒΖ εὐθείας. ὁμοίως δὲ δειχθήσεται, ὅτι καὶ ἑκατέρα τῶν ὑπὸ ΒΑΕ, ΑΕΔ δίχα τέτμηται ὑπὸ ἑκατέρας τῶν ΖΑ, ΖΕ εὐθειῶν. ἤχθωσαν δὲ ἀπὸ τοῦ Ζ σημείου ἐπὶ τὰς ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ εὐθείας κάθετοι αἱ ΖΗ, ΖΘ, ΖΚ, ΖΛ, ΖΜ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ ΘΓΖ γωνία τῇ ὑπὸ ΚΓΖ, ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ ΖΘΓ [ὀρθῇ] τῇ ὑπὸ ΖΚΓ ἴση, δύο δὲ τρίγωνά ἐστι τὰ ΖΘΓ, ΖΚΓ τὰς δύο γωνίας δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μὴ πλευρᾷ ἴσην κοινήν αὐτῶν τὴν ΖΓ ὑποτείνουσιν ὑπὸ μίαν τῶν ἴσων γωνιῶν· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξει· ἴση ἄρα ἡ ΖΘ κάθετος τῇ ΖΚ καθετῷ. ὁμοίως δὲ δειχθήσεται, ὅτι καὶ ἑκάστη τῶν ΖΑ, ΖΜ, ΖΗ ἑκατέρᾳ τῶν ΖΘ, ΖΚ ἴση ἐστίν· αἱ πέντε ἄρα εὐθεῖαι αἱ ΖΗ, ΖΘ, ΖΚ, ΖΛ, ΖΜ ἴσαι ἀλλήλαις εἰσὶν. ὁ ἄρα κέντρῳ τῷ Ζ διαστήματι δὲ ἐνὶ τῶν Η, Θ, Κ, Λ, Μ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἐφάπεται τῶν ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ εὐθειῶν διὰ τὸ ὀρθὰς εἶναι τὰς πρὸς τοῖς Η, Θ, Κ, Λ, Μ σημείοις γωνίας. εἰ γὰρ οὐκ ἐφάπεται αὐτῶν, ἀλλὰ τεμεῖ αὐτάς, συμβήσεται τὴν τῇ διαμέτρῳ τοῦ κύκλου πρὸς ὀρθὰς ἀπ' ἄκρας ἀγομένην ἐντὸς πίπτειν τοῦ κύκλου· ὅπερ ἄτοπον ἐδείχθη. οὐκ ἄρα ὁ κέντρῳ τῷ Ζ διαστήματι δὲ ἐνὶ τῶν Η, Θ, Κ, Λ, Μ σημείων γραφόμενος κύκλος τεμεῖ τὰς ΑΒ, ΒΓ, ΓΔ, ΔΕ, ΕΑ εὐθείας· ἐφάπεται ἄρα αὐτῶν. γεγράφθω ὡς ὁ ΗΘΚΛΜ.

Εἰς ἄρα τὸ δοθὲν πεντάγωνον, ὃ ἐστὶν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλος ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

ιδ'.

Περὶ τὸ δοθὲν πεντάγωνον, ὃ ἐστὶν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλον περιγράψαι.

Ἔστω τὸ δοθὲν πεντάγωνον, ὃ ἐστὶν ἰσόπλευρόν τε καὶ

straight-lines FB , FA , and FE have been joined. And since BC is equal to CD , and CF (is) common, the two (straight-lines) BC , CF are equal to the two (straight-lines) DC , CF . And angle BCF [is] equal to angle DCF . Thus, the base BF is equal to the base DF , and triangle BCF is equal to triangle DCF , and the remaining angles will be equal to the (corresponding) remaining angles which the equal sides subtend [Prop. 1.4]. Thus, angle CBF (is) equal to CDF . And since CDE is double CDF , and CDE (is) equal to ABC , and CDF to CBF , CBA is thus also double CBF . Thus, angle ABF is equal to FBC . Thus, angle ABC has been cut in half by the straight-line BF . So, similarly, it can be shown that BAE and AED have been cut in half by the straight-lines FA and FE , respectively. So let FG , FH , FK , FL , and FM have been drawn from point F , perpendicular to the straight-lines AB , BC , CD , DE , and EA (respectively) [Prop. 1.12]. And since angle HCF is equal to KCF , and the right-angle FHC is also equal to the [right-angle] FKC , FHC and FKC are two triangles having two angles equal to two angles, and one side equal to one side, (namely) their common (side) FC , subtending one of the equal angles. Thus, they will also have the remaining sides equal to the (corresponding) remaining sides [Prop. 1.26]. Thus, the perpendicular FH (is) equal to the perpendicular FK . So, similarly, it can be shown that FL , FM , and FG are each equal to each of FH and FK . Thus, the five straight-lines FG , FH , FK , FL , and FM are equal to one another. Thus, the circle drawn with center F , and radius one of G , H , K , L , or M , will also go through the remaining points, and will touch the straight-lines AB , BC , CD , DE , and EA , on account of the angles at points G , H , K , L , and M being right-angles. For if it does not touch them, but cuts them, it follows that a (straight-line) drawn at right-angles to the diameter of the circle, from its extremity, falls inside the circle. The very thing was shown (to be) absurd [Prop. 3.16]. Thus, the circle drawn with center F , and radius one of G , H , K , L , or M , does not cut the straight-lines AB , BC , CD , DE , or EA . Thus, it will touch them. Let it have been drawn, like $GHKLM$ (in the figure).

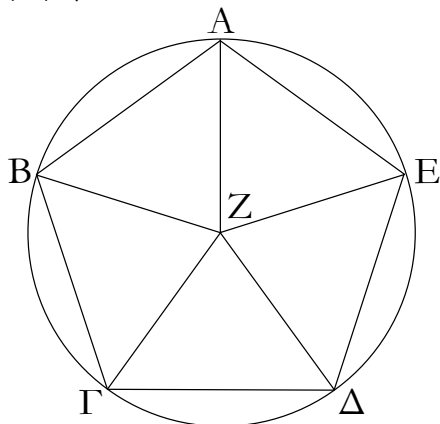
Thus, a circle has been inscribed in the given pentagon which is equilateral and equiangular. (Which is) the very thing it was required to do.

Proposition 14

To circumscribe a circle about a given pentagon which is equilateral and equiangular.

Let $ABCDE$ be the given pentagon which is equilat-

ἰσογώνιον, τὸ $ABΓΔΕ$ · δεῖ δὴ περὶ τὸ $ABΓΔΕ$ πεντάγωνον κύκλον περιγεράψαι.



Τετμήσθω δὴ ἑκάτερα τῶν ὑπὸ $ΒΓΔ$, $ΓΔΕ$ γωνιῶν δίχα ὑπὸ ἑκατέρας τῶν $ΓΖ$, $ΔΖ$, καὶ ἀπὸ τοῦ Z σημείου, καθ' ὃ συμβάλλουσιν αἱ εὐθεῖαι, ἐπὶ τὰ B , A , E σημεῖα ἐπεξεύχθωσαν εὐθεῖαι αἱ ZB , ZA , ZE . ὁμοίως δὴ τῷ πρὸ τούτου δειχθήσεται, ὅτι καὶ ἑκάστη τῶν ὑπὸ $ΓΒΑ$, $ΒΑΕ$, $ΑΕΔ$ γωνιῶν δίχα τέμνεται ὑπὸ ἑκάστης τῶν ZB , ZA , ZE εὐθειῶν. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ $ΒΓΔ$ γωνία τῇ ὑπὸ $ΓΔΕ$, καὶ ἐστὶ τῆς μὲν ὑπὸ $ΒΓΔ$ ἡμίσεια ἡ ὑπὸ $ΖΓΔ$, τῆς δὲ ὑπὸ $ΓΔΕ$ ἡμίσεια ἡ ὑπὸ $ΓΔΖ$, καὶ ἡ ὑπὸ $ΖΓΔ$ ἄρα τῇ ὑπὸ $ΖΔΓ$ ἐστὶν ἴση· ὥστε καὶ πλευρὰ ἡ $ZΓ$ πλευρᾷ τῇ $ZΔ$ ἐστὶν ἴση. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἑκάστη τῶν ZB , ZA , ZE ἑκατέρᾳ τῶν $ZΓ$, $ZΔ$ ἐστὶν ἴση· αἱ πέντε ἄρα εὐθεῖαι αἱ ZA , ZB , $ZΓ$, $ZΔ$, ZE ἴσαι ἀλλήλαις εἰσὶν. ὁ ἄρα κέντρω τῷ Z καὶ διαστήματι ἐνὶ τῶν ZA , ZB , $ZΓ$, $ZΔ$, ZE κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἔσται περιγεγραμμένος. περιγεγράφθω καὶ ἔστω ὁ $ABΓΔΕ$.

Περὶ ἄρα τὸ δοθὲν πεντάγωνον, ὃ ἐστὶν ἰσόπλευρόν τε καὶ ἰσογώνιον, κύκλος περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

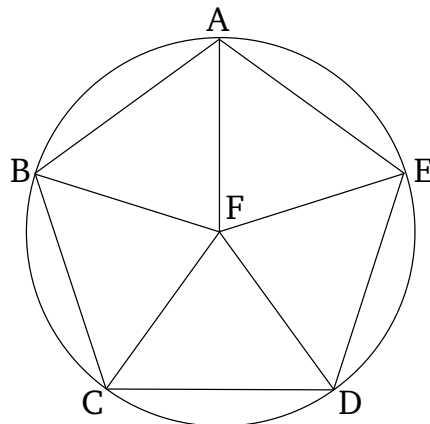
ιε'.

Εἰς τὸν δοθέντα κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Ἐστω ὁ δοθεὶς κύκλος ὁ $ABΓΔΕΖ$ · δεῖ δὴ εἰς τὸν $ABΓΔΕΖ$ κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Ἦχθω τοῦ $ABΓΔΕΖ$ κύκλου διάμετρος ἡ $ΑΔ$, καὶ εἰλήφθω τὸ κέντρον τοῦ κύκλου τὸ H , καὶ κέντρῳ μὲν τῷ $Δ$ διαστήματι δὲ τῷ $ΔH$ κύκλος γεγράφθω ὁ $ΕΗΓΘ$, καὶ ἐπιζευχθεῖσαι αἱ $ΕΗ$, $ΓΗ$ διήχθωσαν ἐπὶ τὰ B , Z σημεῖα, καὶ ἐπεξεύχθωσαν αἱ AB , $ΒΓ$, $ΓΔ$, $ΔΕ$, $ΕΖ$, $ΖΑ$ · λέγω, ὅτι

eral and equiangular. So it is required to circumscribe a circle about the pentagon $ABCDE$.



So let angles BCD and CDE have been cut in half by the (straight-lines) CF and DF , respectively [Prop. 1.9]. And let the straight-lines FB , FA , and FE have been joined from point F , at which the straight-lines meet, to the points B , A , and E (respectively). So, similarly, to the (proposition) before this (one), it can be shown that angles CBA , BAE , and AED have also been cut in half by the straight-lines FB , FA , and FE , respectively. And since angle BCD is equal to CDE , and FCD is half of BCD , and CDF half of CDE , FCD is thus also equal to FDC . So that side FC is also equal to side FD [Prop. 1.6]. So, similarly, it can be shown that FB , FA , and FE are also each equal to each of FC and FD . Thus, the five straight-lines FA , FB , FC , FD , and FE are equal to one another. Thus, the circle drawn with center F , and radius one of FA , FB , FC , FD , or FE , will also go through the remaining points, and will have been circumscribed. Let it have been (so) circumscribed, and let it be $ABCDE$.

Thus, a circle has been circumscribed about the given pentagon, which is equilateral and equiangular. (Which is) the very thing it was required to do.

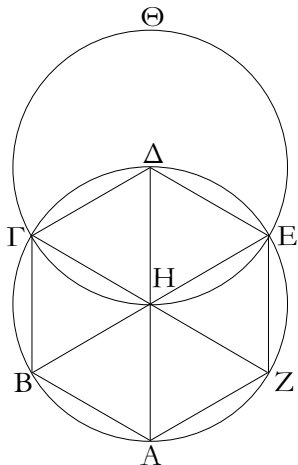
Proposition 15

To inscribe an equilateral and equiangular hexagon in a given circle.

Let $ABCDEF$ be the given circle. So it is required to inscribe an equilateral and equiangular hexagon in circle $ABCDEF$.

Let the diameter AD of circle $ABCDEF$ have been drawn,[†] and let the center G of the circle have been found [Prop. 3.1]. And let the circle $EGCH$ have been drawn, with center D , and radius DG . And EG and CG being joined, let them have been drawn across (the cir-

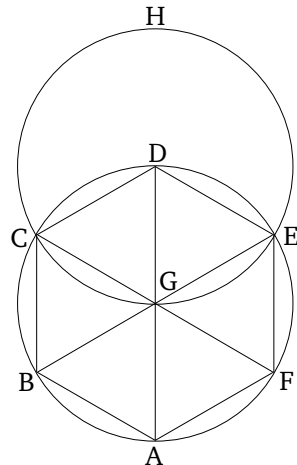
τὸ $ABΓΔEZ$ ἑξάγωνον ἰσόπλευρόν τε ἐστὶ καὶ ἰσογώνιον.



Ἐπεὶ γὰρ τὸ H σημεῖον κέντρον ἐστὶ τοῦ $ABΓΔEZ$ κύκλου, ἴση ἐστὶν ἡ HE τῇ HA . πάλιν, ἐπεὶ τὸ $Δ$ σημεῖον κέντρον ἐστὶ τοῦ $ΗΓΘ$ κύκλου, ἴση ἐστὶν ἡ $ΔE$ τῇ $ΔH$. ἀλλ' ἡ HE τῇ HA ἐδείχθη ἴση· καὶ ἡ HE ἄρα τῇ ED ἴση ἐστὶν· ἰσόπλευρον ἄρα ἐστὶ τὸ $EHΔ$ τρίγωνον· καὶ αἱ τρεῖς ἄρα αὐτοῦ γωνίαι αἱ ὑπὸ $EHΔ$, $HΔE$, $ΔEH$ ἴσαι ἀλλήλαις εἰσὶν, ἐπειδὴ περ τῶν ἰσοσκελῶν τριγώνων αἱ πρὸς τῇ βάσει γωνίαι ἴσαι ἀλλήλαις εἰσὶν· καὶ εἰσὶν αἱ τρεῖς τοῦ τριγώνου γωνίαι δυσὶν ὁρθαῖς ἴσαι· ἡ ἄρα ὑπὸ $EHΔ$ γωνία τρίτον ἐστὶ δύο ὁρθῶν. ὁμοίως δὲ δειχθήσεται καὶ ἡ ὑπὸ $ΔΗΓ$ τρίτον δύο ὁρθῶν. καὶ ἐπεὶ ἡ $ΓH$ εὐθεῖα ἐπὶ τὴν EB σταθεῖσα τὰς ἐφεξῆς γωνίας τὰς ὑπὸ $EHΓ$, $ΓHB$ δυσὶν ὁρθαῖς ἴσας ποιεῖ, καὶ λοιπὴ ἄρα ἡ ὑπὸ $ΓHB$ τρίτον ἐστὶ δύο ὁρθῶν· αἱ ἄρα ὑπὸ $EHΔ$, $ΔΗΓ$, $ΓHB$ γωνίαι ἴσαι ἀλλήλαις εἰσὶν· ὥστε καὶ αἱ κατὰ κορυφὴν αὐταῖς αἱ ὑπὸ BHA , AHZ , ZHE ἴσαι εἰσὶν [ταῖς ὑπὸ $EHΔ$, $ΔΗΓ$, $ΓHB$]. αἱ ἔξ ἄρα γωνίαι αἱ ὑπὸ $EHΔ$, $ΔΗΓ$, $ΓHB$, BHA , AHZ , ZHE ἴσαι ἀλλήλαις εἰσὶν. αἱ δὲ ἴσαι γωνίαι ἐπὶ ἴσων περιφερειῶν βεβήκασιν· αἱ ἔξ ἄρα περιφέρειαι αἱ AB , $ΒΓ$, $ΓΔ$, $ΔE$, EZ , ZA ἴσαι ἀλλήλαις εἰσὶν. ὑπὸ δὲ τὰς ἴσας περιφέρειας αἱ ἴσαι εὐθεῖαι ὑποτείνουσιν· αἱ ἔξ ἄρα εὐθεῖαι ἴσαι ἀλλήλαις εἰσὶν· ἰσόπλευρον ἄρα ἐστὶ τὸ $ABΓΔEZ$ ἑξάγωνον. λέγω δὴ, ὅτι καὶ ἰσογώνιον. ἐπεὶ γὰρ ἴση ἐστὶν ἡ ZA περιφέρεια τῇ ED περιφέρειᾳ, κοινὴ προσκείσθω ἡ $ABΓΔ$ περιφέρεια· ὅλη ἄρα ἡ $ZABΓΔ$ ὅλη τῇ $EDΓBA$ ἐστὶν ἴση· καὶ βέβηκεν ἐπὶ μὲν τῆς $ZABΓΔ$ περιφέρειας ἡ ὑπὸ $ZEΔ$ γωνία, ἐπὶ δὲ τῆς $EDΓBA$ περιφέρειας ἡ ὑπὸ AZE γωνία· ἴση ἄρα ἡ ὑπὸ AZE γωνία τῇ ὑπὸ $ΔEZ$. ὁμοίως δὲ δειχθήσεται, ὅτι καὶ αἱ λοιπαὶ γωνίαι τοῦ $ABΓΔEZ$ ἑξαγώνου κατὰ μίαν ἴσαι εἰσὶν ἑκατέρω τῶν ὑπὸ AZE , $ZEΔ$ γωνιῶν· ἰσογώνιον ἄρα ἐστὶ τὸ $ABΓΔEZ$ ἑξάγωνον. ἐδείχθη δὲ καὶ ἰσόπλευρον· καὶ ἐγγέγραπται εἰς τὸν $ABΓΔEZ$ κύκλον.

Εἰς ἄρα τὸν δοθέντα κύκλον ἑξάγωνον ἰσόπλευρόν τε

cle) to points B and F (respectively). And let AB , BC , CD , DE , EF , and FA have been joined. I say that the hexagon $ABCDEF$ is equilateral and equiangular.



For since point G is the center of circle $ABCDEF$, GE is equal to GD . Again, since point D is the center of circle GCH , DE is equal to DG . But, GE was shown (to be) equal to GD . Thus, GE is also equal to ED . Thus, triangle EGD is equilateral. Thus, its three angles EGD , GDE , and DEG are also equal to one another, inasmuch as the angles at the base of isosceles triangles are equal to one another [Prop. 1.5]. And the three angles of the triangle are equal to two right-angles [Prop. 1.32]. Thus, angle EGD is one third of two right-angles. So, similarly, DGC can also be shown (to be) one third of two right-angles. And since the straight-line CG , standing on EB , makes adjacent angles EGC and CGB equal to two right-angles [Prop. 1.13], the remaining angle CGB is thus also one third of two right-angles. Thus, angles EGD , DGC , and CGB are equal to one another. And hence the (angles) opposite to them BGA , AGF , and FGE are also equal [to EGD , DGC , and CGB (respectively)] [Prop. 1.15]. Thus, the six angles EGD , DGC , CGB , BGA , AGF , and FGE are equal to one another. And equal angles stand on equal circumferences [Prop. 3.26]. Thus, the six circumferences AB , BC , CD , DE , EF , and FA are equal to one another. And equal circumferences are subtended by equal straight-lines [Prop. 3.29]. Thus, the six straight-lines (AB , BC , CD , DE , EF , and FA) are equal to one another. Thus, hexagon $ABCDEF$ is equilateral. So, I say that (it is) also equiangular. For since circumference FA is equal to circumference ED , let circumference $ABCD$ have been added to both. Thus, the whole of $FABCD$ is equal to the whole of $EDCBA$. And angle FED stands on circumference $FABCD$, and angle AFE on circumference $EDCBA$. Thus, angle AFE is equal

καὶ ἰσογώνιον ἐγγέγραπται· ὅπερ ἔδει ποιῆσαι.

Πόρισμα.

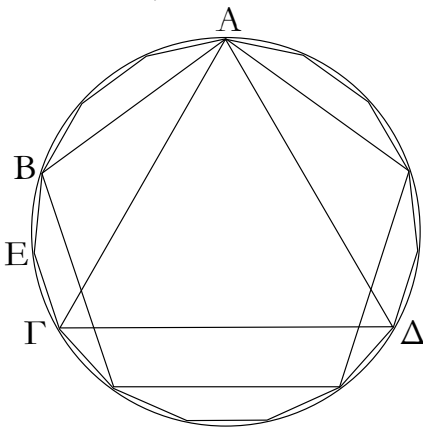
Ἐκ δὴ τούτου φανερόν, ὅτι ἡ τοῦ ἑξαγώνου πλευρὰ ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου τοῦ κύκλου.

Ὅμοιως δὲ τοῖς ἐπὶ τοῦ πενταγώνου ἐὰν διὰ τῶν κατὰ τὸν κύκλον διαιρέσεων ἐφαπτομένας τοῦ κύκλου ἀγάγωμεν, περιγραφήσεται περὶ τὸν κύκλον ἑξάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἀκολούθως τοῖς ἐπὶ τοῦ πενταγώνου εἰρημένοις. καὶ ἔτι διὰ τῶν ὁμοίων τοῖς ἐπὶ τοῦ πενταγώνου εἰρημένοις εἰς τὸ δοθὲν ἑξάγωνον κύκλον ἐγγράψομεν τε καὶ περιγράψομεν· ὅπερ ἔδει ποιῆσαι.

† See the footnote to Prop. 4.6.

ις'.

Εἰς τὸν δοθέντα κύκλον πεντεκαίδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.



Ἐστω ὁ δοθείς κύκλος ὁ ΑΒΓΔ· δεῖ δὴ εἰς τὸν ΑΒΓΔ κύκλον πεντεκαίδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον ἐγγράψαι.

Ἐγγεγράψθω εἰς τὸν ΑΒΓΔ κύκλον τριγώνου μὲν ἰσοπλεύρου τοῦ εἰς αὐτὸν ἐγγεγραμμένου πλευρὰ ἡ ΑΓ, πενταγώνου δὲ ἰσοπλεύρου ἡ ΑΒ· οἷων ἄρα ἐστὶν ὁ ΑΒΓΔ κύκλος ἴσων τμημάτων δεκαπέντε, τοιούτων ἡ μὲν ΑΒΓ περιφέρεια τρίτον οὔσα τοῦ κύκλου ἔσται πέντε, ἡ δὲ ΑΒ περιφέρεια πέμpton οὔσα τοῦ κύκλου ἔσται τριῶν· λοιπὴ ἄρα

to DEF [Prop. 3.27]. Similarly, it can also be shown that the remaining angles of hexagon $ABCDEF$ are individually equal to each of the angles AFE and FED . Thus, hexagon $ABCDEF$ is equiangular. And it was also shown (to be) equilateral. And it has been inscribed in circle $ABCDE$.

Thus, an equilateral and equiangular hexagon has been inscribed in the given circle. (Which is) the very thing it was required to do.

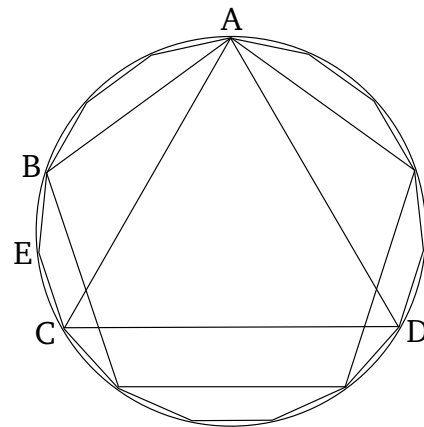
Corollary

So, from this, (it is) manifest that a side of the hexagon is equal to the radius of the circle.

And similarly to a pentagon, if we draw tangents to the circle through the (sixfold) divisions of the (circumference of the) circle, an equilateral and equiangular hexagon can be circumscribed about the circle, analogously to the aforementioned pentagon. And, further, by (means) similar to the aforementioned pentagon, we can inscribe and circumscribe a circle in (and about) a given hexagon. (Which is) the very thing it was required to do.

Proposition 16

To inscribe an equilateral and equiangular fifteen-sided figure in a given circle.



Let $ABCD$ be the given circle. So it is required to inscribe an equilateral and equiangular fifteen-sided figure in circle $ABCD$.

Let the side AC of an equilateral triangle inscribed in (the circle) [Prop. 4.2], and (the side) AB of an (inscribed) equilateral pentagon [Prop. 4.11], have been inscribed in circle $ABCD$. Thus, just as the circle $ABCD$ is (made up) of fifteen equal pieces, the circumference ABC , being a third of the circle, will be (made up) of five

ἡ ΒΓ τῶν ἴσων δύο. τετμήσθω ἡ ΒΓ δίχα κατὰ τὸ Ε· ἑκατέρω ἄρα τῶν ΒΕ, ΕΓ περιφερειῶν πεντεκαιδέκατόν ἐστι τοῦ ΑΒΓΔ κύκλου.

Ἐὰν ἄρα ἐπιζεύξαντες τὰς ΒΕ, ΕΓ ἴσας αὐταῖς κατὰ τὸ συνεχὲς εὐθείας ἐναρμόσωμεν εἰς τὸν ΑΒΓΔ[Ε] κύκλον, ἔσται εἰς αὐτὸν ἐγγεγραμμένον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον· ὅπερ ἔδει ποιῆσαι.

Ὅμοίως δὲ τοῖς ἐπὶ τοῦ πενταγώνου ἐὰν διὰ τῶν κατὰ τὸν κύκλον διαιρέσεων ἐφαπτομένας τοῦ κύκλου ἀγάγωμεν, περιγραφήσεται περὶ τὸν κύκλον πεντεκαιδεκάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον. ἔτι δὲ διὰ τῶν ὁμοίων τοῖς ἐπὶ τοῦ πενταγώνου δείξεων καὶ εἰς τὸ δοθὲν πεντεκαιδεκάγωνον κύκλον ἐγγράψομεν τε καὶ περιγράψομεν· ὅπερ ἔδει ποιῆσαι.

such (pieces), and the circumference AB , being a fifth of the circle, will be (made up) of three. Thus, the remainder BC (will be made up) of two equal (pieces). Let (circumference) BC have been cut in half at E [Prop. 3.30]. Thus, each of the circumferences BE and EC is one fifteenth of the circle $ABCDE$.

Thus, if, joining BE and EC , we continuously insert straight-lines equal to them into circle $ABCD[E]$ [Prop. 4.1], then an equilateral and equiangular fifteen-sided figure will have been inserted into (the circle). (Which is) the very thing it was required to do.

And similarly to the pentagon, if we draw tangents to the circle through the (fifteenfold) divisions of the (circumference of the) circle, we can circumscribe an equilateral and equiangular fifteen-sided figure about the circle. And, further, through similar proofs to the pentagon, we can also inscribe and circumscribe a circle in (and about) a given fifteen-sided figure. (Which is) the very thing it was required to do.

ELEMENTS BOOK 5

Proportion[†]

[†]The theory of proportion set out in this book is generally attributed to Eudoxus of Cnidus. The novel feature of this theory is its ability to deal with irrational magnitudes, which had hitherto been a major stumbling block for Greek mathematicians. Throughout the footnotes in this book, α , β , γ , *etc.*, denote general (possibly irrational) magnitudes, whereas m , n , l , *etc.*, denote positive integers.

Ὅροι.

α'. Μέρος ἐστὶ μέγεθος μεγέθους τὸ ἑλασσον τοῦ μείζονος, ὅταν καταμετρηῇ τὸ μείζον.

β'. Πολλαπλάσιον δὲ τὸ μείζον τοῦ ἐλάττονος, ὅταν καταμετρηῇται ὑπὸ τοῦ ἐλάττονος.

γ'. Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἢ κατὰ πηλικότητά ποια σχέσις.

δ'. Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.

ε'. Ἐν τῷ αὐτῷ λόγῳ μεγέθη λέγεται εἶναι πρῶτον πρὸς δεύτερον καὶ τρίτον πρὸς τέταρτον, ὅταν τὰ τοῦ πρώτου καὶ τρίτου ἰσάκεις πολλαπλάσια τῶν τοῦ δευτέρου καὶ τετάρτου ἰσάκεις πολλαπλασίων καθ' ὅποιον οὖν πολλαπλασιασμὸν ἑκάτερον ἑκατέρου ἢ ἅμα ὑπερέχη ἢ ἅμα ἴσα ἢ ἢ ἅμα ἐλλείπη ληφθέντα κατάλληλα.

ς'. Τὰ δὲ τὸν αὐτὸν ἔχοντα λόγον μεγέθη ἀνάλογον καλεῖσθω.

ζ'. Ὅταν δὲ τῶν ἰσάκεις πολλαπλασίων τὸ μὲν τοῦ πρώτου πολλαπλάσιον ὑπερέχη τοῦ τοῦ δευτέρου πολλαπλασίου, τὸ δὲ τοῦ τρίτου πολλαπλάσιον μὴ ὑπερέχη τοῦ τοῦ τετάρτου πολλαπλασίου, τότε τὸ πρῶτον πρὸς τὸ δεύτερον μείζονα λόγον ἔχειν λέγεται, ἥπερ τὸ τρίτον πρὸς τὸ τέταρτον.

η'. Ἀναλογία δὲ ἐν τρισὶν ὅροις ἐλαχίστη ἐστίν.

θ'. Ὅταν δὲ τρία μεγέθη ἀνάλογον ἦ, τὸ πρῶτον πρὸς τὸ τρίτον διπλασίονα λόγον ἔχειν λέγεται ἥπερ πρὸς τὸ δεύτερον.

ι'. Ὅταν δὲ τέσσαρα μεγέθη ἀνάλογον ἦ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχειν λέγεται ἥπερ πρὸς τὸ δεύτερον, καὶ αἰ ἐξῆς ὁμοίως, ὡς ἂν ἡ ἀναλογία ὑπάρχη.

ια'. Ὁμόλογα μεγέθη λέγεται τὰ μὲν ἡγούμενα τοῖς ἡγουμένοις τὰ δὲ ἐπόμενα τοῖς ἐπομένοις.

ιβ'. Ἐναλλάξ λόγος ἐστὶ λήψις τοῦ ἡγουμένου πρὸς τὸ ἡγούμενον καὶ τοῦ ἐπομένου πρὸς τὸ ἐπόμενον.

ιγ'. Ἀνάπαλιν λόγος ἐστὶ λήψις τοῦ ἐπομένου ὡς ἡγούμενον πρὸς τὸ ἡγούμενον ὡς ἐπόμενον.

ιδ'. Σύνθεσις λόγου ἐστὶ λήψις τοῦ ἡγουμένου μετὰ τοῦ ἐπομένου ὡς ἐνὸς πρὸς αὐτὸ τὸ ἐπόμενον.

ιε'. Διαίρεσις λόγου ἐστὶ λήψις τῆς ὑπεροχῆς, ἢ ὑπερέχει τὸ ἡγούμενον τοῦ ἐπομένου, πρὸς αὐτὸ τὸ ἐπόμενον.

ις'. Ἀναστροφὴ λόγου ἐστὶ λήψις τοῦ ἡγουμένου πρὸς τὴν ὑπεροχὴν, ἢ ὑπερέχει τὸ ἡγούμενον τοῦ ἐπομένου.

ιζ'. Δι' ἴσου λόγος ἐστὶ πλείονων ὄντων μεγεθῶν καὶ ἄλλων αὐτοῖς ἴσων τὸ πλῆθος σύνδυο λαμβανομένων καὶ ἐν τῷ αὐτῷ λόγῳ, ὅταν ἢ ὡς ἐν τοῖς πρώτοις μεγέθεσι τὸ πρῶτον πρὸς τὸ ἔσχατον, οὕτως ἐν τοῖς δευτέροις μεγέθεσι τὸ πρῶτον πρὸς τὸ ἔσχατον ἢ ἄλλως· λήψις τῶν ἄκρων

Definitions

1. A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater.[†]

2. And the greater (magnitude is) a multiple of the lesser when it is measured by the lesser.

3. A ratio is a certain type of condition with respect to size of two magnitudes of the same kind.[‡]

4. (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another.[§]

5. Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and the third either both exceed, are both equal to, or are both less than, equal multiples of the second and the fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.[¶]

6. And let magnitudes having the same ratio be called proportional.*

7. And when for equal multiples (as in Def. 5), the multiple of the first (magnitude) exceeds the multiple of the second, and the multiple of the third (magnitude) does not exceed the multiple of the fourth, then the first (magnitude) is said to have a greater ratio to the second than the third (magnitude has) to the fourth.

8. And a proportion in three terms is the smallest (possible).[§]

9. And when three magnitudes are proportional, the first is said to have to the third the squared^{||} ratio of that (it has) to the second.^{††}

10. And when four magnitudes are (continuously) proportional, the first is said to have to the fourth the cubed^{‡‡} ratio of that (it has) to the second.^{§§} And so on, similarly, in successive order, whatever the (continuous) proportion might be.

11. These magnitudes are said to be corresponding (magnitudes): the leading to the leading (of two ratios), and the following to the following.

12. An alternate ratio is a taking of the (ratio of the) leading (magnitude) to the leading (of two equal ratios), and (setting it equal to) the (ratio of the) following (magnitude) to the following.^{¶¶}

13. An inverse ratio is a taking of the (ratio of the) following (magnitude) as the leading and the leading (magnitude) as the following.^{**}

14. A composition of a ratio is a taking of the (ratio of the) leading plus the following (magnitudes), as one, to the following (magnitude) by itself.^{§§}

καθ' ὑπεξαίρεσιν τῶν μέσων.

ιη'. Τεταραγμένη δὲ ἀναλογία ἐστίν, ὅταν τριῶν ὄντων μεγεθῶν καὶ ἄλλων αὐτοῖς ἴσων τὸ πλῆθος γίνηται ὥς μὲν ἐν τοῖς πρώτοις μεγέθεσιν ἡγούμενον πρὸς ἐπόμενον, οὕτως ἐν τοῖς δευτέροις μεγέθεσιν ἡγούμενον πρὸς ἐπόμενον, ὥς δὲ ἐν τοῖς πρώτοις μεγέθεσιν ἐπόμενον πρὸς ἄλλο τι, οὕτως ἐν τοῖς δευτέροις ἄλλο τι πρὸς ἡγούμενον.

15. A separation of a ratio is a taking of the (ratio of the) excess by which the leading (magnitude) exceeds the following to the following (magnitude) by itself.^{|||}

16. A conversion of a ratio is a taking of the (ratio of the) leading (magnitude) to the excess by which the leading (magnitude) exceeds the following.^{†††}

17. There being several magnitudes, and other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, a ratio via equality (or *ex aequali*) occurs when as the first is to the last in the first (set of) magnitudes, so the first (is) to the last in the second (set of) magnitudes. Or alternately, (it is) a taking of the (ratio of the) outer (magnitudes) by the removal of the inner (magnitudes).^{†††}

18. There being three magnitudes, and other (magnitudes) of equal number to them, a perturbed proportion occurs when as the leading is to the following in the first (set of) magnitudes, so the leading (is) to the following in the second (set of) magnitudes, and as the following (is) to some other (*i.e.*, the remaining magnitude) in the first (set of) magnitudes, so some other (is) to the leading in the second (set of) magnitudes.^{§§§}

† In other words, α is said to be a part of β if $\beta = m\alpha$.

‡ In modern notation, the ratio of two magnitudes, α and β , is denoted $\alpha : \beta$.

§ In other words, α has a ratio with respect to β if $m\alpha > \beta$ and $n\beta > \alpha$, for some m and n .

¶ In other words, $\alpha : \beta :: \gamma : \delta$ if and only if $m\alpha > n\beta$ whenever $m\gamma > n\delta$, and $m\alpha = n\beta$ whenever $m\gamma = n\delta$, and $m\alpha < n\beta$ whenever $m\gamma < n\delta$, for all m and n . This definition is the kernel of Eudoxus' theory of proportion, and is valid even if α , β , etc., are irrational.

* Thus if α and β have the same ratio as γ and δ then they are proportional. In modern notation, $\alpha : \beta :: \gamma : \delta$.

§ In modern notation, a proportion in three terms— α , β , and γ —is written: $\alpha : \beta :: \beta : \gamma$.

|| Literally, "double".

†† In other words, if $\alpha : \beta :: \beta : \gamma$ then $\alpha : \gamma :: \alpha^2 : \beta^2$.

‡‡ Literally, "triple".

§§ In other words, if $\alpha : \beta :: \beta : \gamma :: \gamma : \delta$ then $\alpha : \delta :: \alpha^3 : \beta^3$.

¶¶ In other words, if $\alpha : \beta :: \gamma : \delta$ then the alternate ratio corresponds to $\alpha : \gamma :: \beta : \delta$.

** In other words, if $\alpha : \beta$ then the inverse ratio corresponds to $\beta : \alpha$.

§§ In other words, if $\alpha : \beta$ then the composed ratio corresponds to $\alpha + \beta : \beta$.

||| In other words, if $\alpha : \beta$ then the separated ratio corresponds to $\alpha - \beta : \beta$.

††† In other words, if $\alpha : \beta$ then the converted ratio corresponds to $\alpha : \alpha - \beta$.

‡‡‡ In other words, if α, β, γ are the first set of magnitudes, and δ, ϵ, ζ the second set, and $\alpha : \beta :: \gamma : \delta :: \delta : \epsilon :: \epsilon : \zeta$, then the ratio via equality (or *ex aequali*) corresponds to $\alpha : \gamma :: \delta : \zeta$.

§§§ In other words, if α, β, γ are the first set of magnitudes, and δ, ϵ, ζ the second set, and $\alpha : \beta :: \delta : \epsilon$ as well as $\beta : \gamma :: \zeta : \delta$, then the proportion is said to be perturbed.

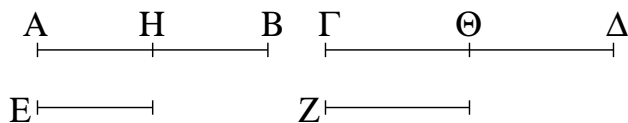
α'.

Proposition 1[†]

Ἐάν ἡ ὅποσαοῦν μεγέθη ὅποσωνοῦν μεγεθῶν ἴσων τὸ πλῆθος ἑκάστου ἑκάστου ἰσάκεις πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ἐν τῶν μεγεθῶν ἐνός, τοσαυταπλάσια ἔσται καὶ τὰ

If there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), then as many

πάντα τῶν πάντων.

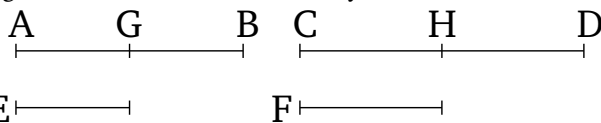


Ἐστω ὅποσαοῦν μεγέθη τὰ $AB, \Gamma\Delta$ ὅποσωνοῦν μεγεθῶν τῶν E, Z ἴσων τὸ πλῆθος ἕκαστον ἑκάστου ἰσάκεις πολλαπλάσιον· λέγω, ὅτι ὁσαπλάσιόν ἐστι τὸ AB τοῦ E , τοσαυταπλάσια ἔσται καὶ τὰ $AB, \Gamma\Delta$ τῶν E, Z .

Ἐπεὶ γὰρ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ AB τοῦ E καὶ τὸ $\Gamma\Delta$ τοῦ Z , ὅσα ἄρα ἐστὶν ἐν τῷ AB μεγέθη ἴσα τῷ E , τοσαῦτα καὶ ἐν τῷ $\Gamma\Delta$ ἴσα τῷ Z . διηγήσθω τὸ μὲν AB εἰς τὰ τῷ E μεγέθη ἴσα τὰ AH, HB , τὸ δὲ $\Gamma\Delta$ εἰς τὰ τῷ Z ἴσα τὰ $\Gamma\Theta, \Theta\Delta$ · ἔσται δὴ ἴσον τὸ πλῆθος τῶν AH, HB τῷ πλῆθει τῶν $\Gamma\Theta, \Theta\Delta$. καὶ ἐπεὶ ἴσον ἐστὶ τὸ μὲν AH τῷ E , τὸ δὲ $\Gamma\Theta$ τῷ Z , ἴσον ἄρα τὸ AH τῷ E , καὶ τὰ $AH, \Gamma\Theta$ τοῖς E, Z . διὰ τὰ αὐτὰ δὴ ἴσον ἐστὶ τὸ HB τῷ E , καὶ τὰ $HB, \Theta\Delta$ τοῖς E, Z · ὅσα ἄρα ἐστὶν ἐν τῷ AB ἴσα τῷ E , τοσαῦτα καὶ ἐν τοῖς $AB, \Gamma\Delta$ ἴσα τοῖς E, Z · ὁσαπλάσιον ἄρα ἐστὶ τὸ AB τοῦ E , τοσαυταπλάσια ἔσται καὶ τὰ $AB, \Gamma\Delta$ τῶν E, Z .

Ἐὰν ἄρα ἡ ὅποσαοῦν μεγέθη ὅποσωνοῦν μεγεθῶν ἴσων τὸ πλῆθος ἕκαστον ἑκάστου ἰσάκεις πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ἐν τῶν μεγεθῶν ἐνός, τοσαυταπλάσια ἔσται καὶ τὰ πάντα τῶν πάντων· ὅπερ εἶδει δεῖξαι.

times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second).



Let there be any number of magnitudes whatsoever, AB, CD , (which are) equal multiples, respectively, of some (other) magnitudes, E, F , of equal number (to them). I say that as many times as AB is (divisible) by E , so many times will AB, CD also be (divisible) by E, F .

For since AB, CD are equal multiples of E, F , thus as many magnitudes as (there) are in AB equal to E , so many (are there) also in CD equal to F . Let AB have been divided into magnitudes AG, GB , equal to E , and CD into (magnitudes) CH, HD , equal to F . So, the number of (divisions) AG, GB will be equal to the number of (divisions) CH, HD . And since AG is equal to E , and CH to F , AG (is) thus equal to E , and AG, CH to E, F . So, for the same (reasons), GB is equal to E , and GB, HD to E, F . Thus, as many (magnitudes) as (there) are in AB equal to E , so many (are there) also in AB, CD equal to E, F . Thus, as many times as AB is (divisible) by E , so many times will AB, CD also be (divisible) by E, F .

Thus, if there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), then as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads $m\alpha + m\beta + \dots = m(\alpha + \beta + \dots)$.

β'.

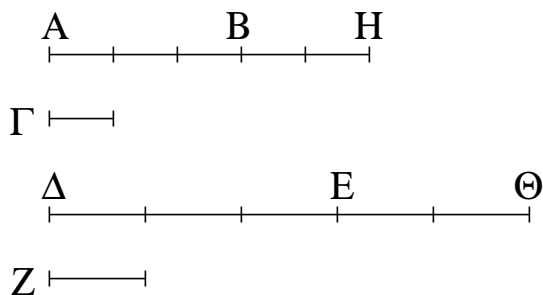
Ἐὰν πρῶτον δευτέρου ἰσάκεις ἡ πολλαπλάσιον καὶ τρίτον τετάρτου, ἡ δὲ καὶ πέμπτου δευτέρου ἰσάκεις πολλαπλάσιον καὶ ἕκτον τετάρτου, καὶ συντεθὲν πρῶτον καὶ πέμπτου δευτέρου ἰσάκεις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τετάρτου.

Πρῶτον γὰρ τὸ AB δευτέρου τοῦ Γ ἰσάκεις ἔστω πολλαπλάσιον καὶ τρίτον τὸ ΔE τετάρτου τοῦ Z , ἔστω δὲ καὶ πέμπτου τὸ BH δευτέρου τοῦ Γ ἰσάκεις πολλαπλάσιον καὶ ἕκτον τὸ $E\Theta$ τετάρτου τοῦ Z · λέγω, ὅτι καὶ συντεθὲν πρῶτον καὶ πέμπτου τὸ AH δευτέρου τοῦ Γ ἰσάκεις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τὸ $\Delta\Theta$ τετάρτου τοῦ Z .

Proposition 2†

If a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and a fifth (magnitude) and a sixth (are) also equal multiples of the second and fourth (respectively), then the first (magnitude) and the fifth, being added together, and the third and the sixth, (being added together), will also be equal multiples of the second (magnitude) and the fourth (respectively).

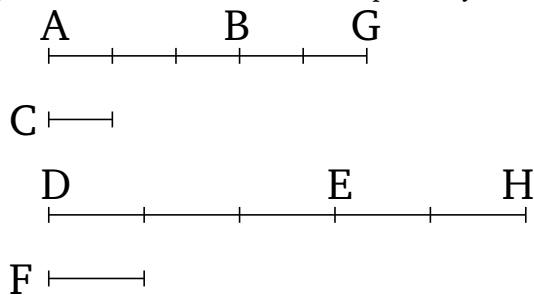
For let a first (magnitude) AB and a third DE be equal multiples of a second C and a fourth F (respectively). And let a fifth (magnitude) BG and a sixth EH also be (other) equal multiples of the second C and the fourth F (respectively). I say that the first (magnitude) and the fifth, being added together, (to give) AG , and the third (magnitude) and the sixth, (being added together,



Ἐπεὶ γὰρ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ AB τοῦ Γ καὶ τὸ ΔΕ τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ AB ἴσα τῷ Γ, τοσαῦτα καὶ ἐν τῷ ΔΕ ἴσα τῷ Ζ. διὰ τὰ αὐτὰ δὲ καὶ ὅσα ἐστὶν ἐν τῷ BH ἴσα τῷ Γ, τοσαῦτα καὶ ἐν τῷ ΕΘ ἴσα τῷ Ζ· ὅσα ἄρα ἐστὶν ἐν ὅλῳ τῷ AH ἴσα τῷ Γ, τοσαῦτα καὶ ἐν ὅλῳ τῷ ΔΘ ἴσα τῷ Ζ· ὁσαπλάσιον ἄρα ἐστὶ τὸ AH τοῦ Γ, τοσαυταπλάσιον ἔσται καὶ τὸ ΔΘ τοῦ Ζ. καὶ συντεθὲν ἄρα πρῶτον καὶ πέμπτον τὸ AH δευτέρου τοῦ Γ ἰσάκεις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τὸ ΔΘ τετάρτου τοῦ Ζ.

Ἐὰν ἄρα πρῶτον δευτέρου ἰσάκεις ἢ πολλαπλάσιον καὶ τρίτον τετάρτου, ἢ δὲ καὶ πέμπτον δευτέρου ἰσάκεις πολλαπλάσιον καὶ ἕκτον τετάρτου, καὶ συντεθὲν πρῶτον καὶ πέμπτον δευτέρου ἰσάκεις ἔσται πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τετάρτου· ὅπερ ἔδει δεῖξαι.

to give) DH , will also be equal multiples of the second (magnitude) C and the fourth F (respectively).



For since AB and DE are equal multiples of C and F (respectively), thus as many (magnitudes) as (there) are in AB equal to C , so many (are there) also in DE equal to F . And so, for the same (reasons), as many (magnitudes) as (there) are in BG equal to C , so many (are there) also in EH equal to F . Thus, as many (magnitudes) as (there) are in the whole of AG equal to C , so many (are there) also in the whole of DH equal to F . Thus, as many times as AG is (divisible) by C , so many times will DH also be divisible by F . Thus, the first (magnitude) and the fifth, being added together, (to give) AG , and the third (magnitude) and the sixth, (being added together, to give) DH , will also be equal multiples of the second (magnitude) C and the fourth F (respectively).

Thus, if a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and a fifth (magnitude) and a sixth (are) also equal multiples of the second and fourth (respectively), then the first (magnitude) and the fifth, being added together, and the third and sixth, (being added together), will also be equal multiples of the second (magnitude) and the fourth (respectively). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads $m\alpha + n\alpha = (m+n)\alpha$.

γ'.

Ἐὰν πρῶτον δευτέρου ἰσάκεις ἢ πολλαπλάσιον καὶ τρίτον τετάρτου, ληφθῇ δὲ ἰσάκεις πολλαπλάσια τοῦ τε πρώτου καὶ τρίτου, καὶ δι' ἴσου τῶν ληφθέντων ἑκάτερον ἑκατέρου ἰσάκεις ἔσται πολλαπλάσιον τὸ μὲν τοῦ δευτέρου τὸ δὲ τοῦ τετάρτου.

Πρῶτον γὰρ τὸ A δευτέρου τοῦ B ἰσάκεις ἔστω πολλαπλάσιον καὶ τρίτον τὸ Γ τετάρτου τοῦ Δ, καὶ εἰλήφθω τῶν A, Γ ἰσάκεις πολλαπλάσια τὰ EZ, ΗΘ· λέγω, ὅτι ἰσάκεις ἐστὶ πολλαπλάσιον τὸ EZ τοῦ B καὶ τὸ ΗΘ τοῦ Δ.

Ἐπεὶ γὰρ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ EZ τοῦ A καὶ τὸ ΗΘ τοῦ Γ, ὅσα ἄρα ἐστὶν ἐν τῷ EZ ἴσα τῷ A, τοσαῦτα καὶ ἐν τῷ ΗΘ ἴσα τῷ Γ. διηρῆσθω τὸ μὲν EZ εἰς τὰ τῷ A μεγέθη ἴσα τὰ EK, KZ, τὸ δὲ ΗΘ εἰς τὰ τῷ Γ ἴσα τὰ ΗΛ,

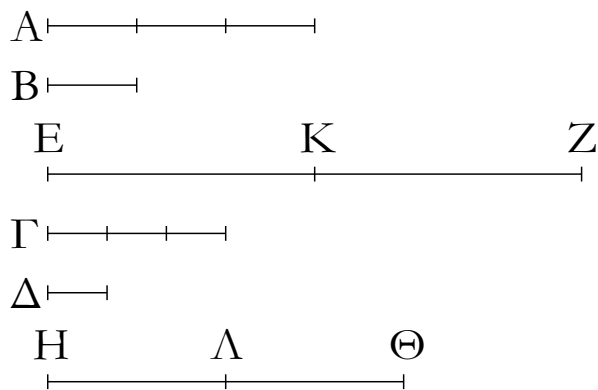
Proposition 3[†]

If a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and equal multiples are taken of the first and the third, then, via equality, the (magnitudes) taken will also be equal multiples of the second (magnitude) and the fourth, respectively.

For let a first (magnitude) A and a third C be equal multiples of a second B and a fourth D (respectively), and let the equal multiples EF and GH have been taken of A and C (respectively). I say that EF and GH are equal multiples of B and D (respectively).

For since EF and GH are equal multiples of A and C (respectively), thus as many (magnitudes) as (there) are in EF equal to A , so many (are there) also in GH

ΛΘ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΕΚ, ΚΖ τῷ πλῆθει τῶν ΗΛ, ΛΘ. καὶ ἐπεὶ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ Α τοῦ Β καὶ τὸ Γ τοῦ Δ, ἴσον δὲ τὸ μὲν ΕΚ τῷ Α, τὸ δὲ ΗΛ τῷ Γ, ἰσάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ ΕΚ τοῦ Β καὶ τὸ ΗΛ τοῦ Δ. διὰ τὰ αὐτὰ δὴ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ ΚΖ τοῦ Β καὶ τὸ ΛΘ τοῦ Δ. ἐπεὶ οὖν πρῶτον τὸ ΕΚ δευτέρου τοῦ Β ἰσάκεις ἐστὶ πολλαπλάσιον καὶ τρίτον τὸ ΗΛ τετάρτου τοῦ Δ, ἔστι δὲ καὶ πέμπτον τὸ ΚΖ δευτέρου τοῦ Β ἰσάκεις πολλαπλάσιον καὶ ἕκτον τὸ ΛΘ τετάρτου τοῦ Δ, καὶ συντεθὲν ἄρα πρῶτον καὶ πέμπτον τὸ ΕΖ δευτέρου τοῦ Β ἰσάκεις ἐστὶ πολλαπλάσιον καὶ τρίτον καὶ ἕκτον τὸ ΗΘ τετάρτου τοῦ Δ.



Ἐὰν ἄρα πρῶτον δευτέρου ἰσάκεις ἢ πολλαπλάσιον καὶ τρίτον τετάρτου, ληφθῇ δὲ τοῦ πρώτου καὶ τρίτου ἰσάκεις πολλαπλάσια, καὶ δι' ἴσου τῶν ληφθέντων ἑκάτερον ἑκατέρου ἰσάκεις ἔσται πολλαπλάσιον τὸ μὲν τοῦ δευτέρου τὸ δὲ τοῦ τετάρτου· ὅπερ ἔδει δεῖξαι.

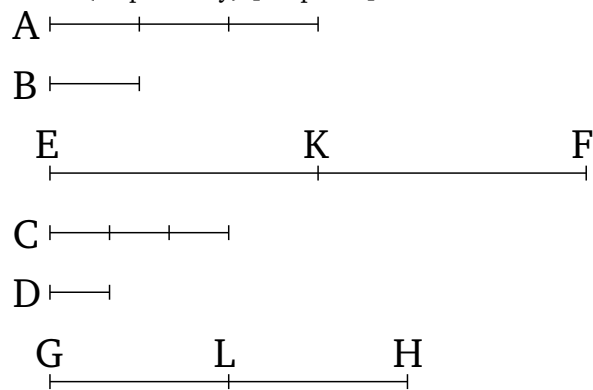
† In modern notation, this proposition reads $m(n\alpha) = (m\,n)\alpha$.

δ'.

Ἐὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχη λόγον καὶ τρίτον πρὸς τέταρτον, καὶ τὰ ἰσάκεις πολλαπλάσια τοῦ τε πρώτου καὶ τρίτου πρὸς τὰ ἰσάκεις πολλαπλάσια τοῦ δευτέρου καὶ τετάρτου καθ' ὅποιον οὖν πολλαπλασιασμὸν τὸν αὐτὸν ἔξει λόγον ληφθέντα κατάλληλα.

Πρῶτον γάρ τὸ Α πρὸς δεύτερον τὸ Β τὸν αὐτὸν ἔχεται λόγον καὶ τρίτον τὸ Γ πρὸς τέταρτον τὸ Δ, καὶ εἰλήφθω τῶν μὲν Α, Γ ἰσάκεις πολλαπλάσια τὰ Ε, Ζ, τῶν δὲ Β, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ Η, Θ· λέγω, ὅτι ἐστὶν ὡς τὸ Ε πρὸς τὸ Η, οὕτως τὸ Ζ πρὸς τὸ Θ.

equal to C . Let EF have been divided into magnitudes EK, KF equal to A , and GH into (magnitudes) GL, LH equal to C . So, the number of (magnitudes) EK, KF will be equal to the number of (magnitudes) GL, LH . And since A and C are equal multiples of B and D (respectively), and EK (is) equal to A , and GL to C , EK and GL are thus equal multiples of B and D (respectively). So, for the same (reasons), KF and LH are equal multiples of B and D (respectively). Therefore, since the first (magnitude) EK and the third GL are equal multiples of the second B and the fourth D (respectively), and the fifth (magnitude) KF and the sixth LH are also equal multiples of the second B and the fourth D (respectively), then the first (magnitude) and fifth, being added together, (to give) EF , and the third (magnitude) and sixth, (being added together, to give) GH , are thus also equal multiples of the second (magnitude) B and the fourth D (respectively) [Prop. 5.2].

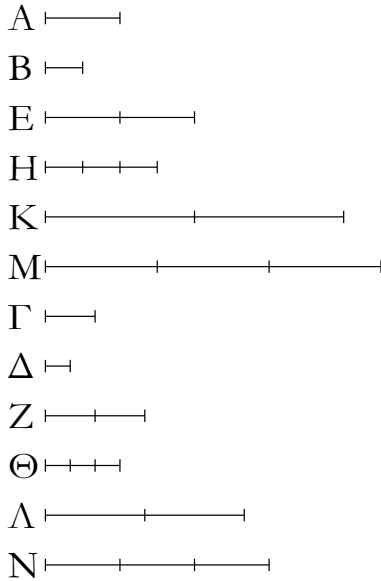


Thus, if a first (magnitude) and a third are equal multiples of a second and a fourth (respectively), and equal multiples are taken of the first and the third, then, via equality, the (magnitudes) taken will also be equal multiples of the second (magnitude) and the fourth, respectively. (Which is) the very thing it was required to show.

Proposition 4†

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth then equal multiples of the first (magnitude) and the third will also have the same ratio to equal multiples of the second and the fourth, being taken in corresponding order, according to any kind of multiplication whatsoever.

For let a first (magnitude) A have the same ratio to a second B that a third C (has) to a fourth D . And let equal multiples E and F have been taken of A and C (respectively), and other random equal multiples G and

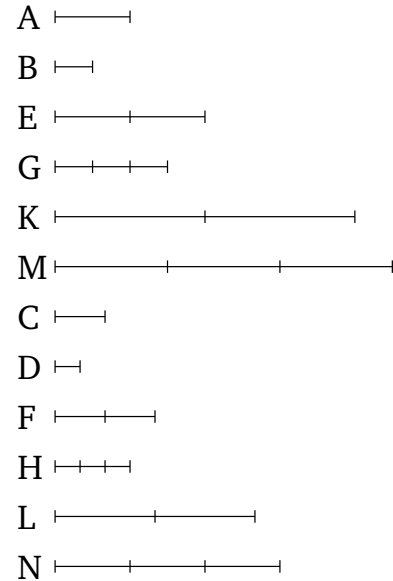


Εἰλήφθω γὰρ τῶν μὲν E , Z ἰσάκεις πολλαπλάσια τὰ K , Λ , τῶν δὲ H , Θ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ M , N .

[Καί] ἐπεὶ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ μὲν E τοῦ A , τὸ δὲ Z τοῦ Γ , καὶ εἴληπται τῶν E , Z ἰσάκεις πολλαπλάσια τὰ K , Λ , ἰσάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ K τοῦ A καὶ τὸ Λ τοῦ Γ . διὰ τὰ αὐτὰ δὴ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ M τοῦ B καὶ τὸ N τοῦ Δ . καὶ ἐπεὶ ἐστὶν ὡς τὸ A πρὸς τὸ B , οὕτως τὸ Γ πρὸς τὸ Δ , καὶ εἴληπται τῶν μὲν A , Γ ἰσάκεις πολλαπλάσια τὰ K , Λ , τῶν δὲ B , Δ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ M , N , εἰ ἄρα ὑπερέχει τὸ K τοῦ M , ὑπερέχει καὶ τὸ Λ τοῦ N , καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καὶ ἐστὶ τὰ μὲν K , Λ τῶν E , Z ἰσάκεις πολλαπλάσια, τὰ δὲ M , N τῶν H , Θ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ E πρὸς τὸ H , οὕτως τὸ Z πρὸς τὸ Θ .

Ἐὰν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχη λόγον καὶ τρίτον πρὸς τέταρτον, καὶ τὰ ἰσάκεις πολλαπλάσια τοῦ τε πρώτου καὶ τρίτου πρὸς τὰ ἰσάκεις πολλαπλάσια τοῦ δευτέρου καὶ τετάρτου τὸν αὐτὸν ἔξει λόγον καθ' ὅποιονοῦν πολλαπλασιασμὸν ληφθέντα κατὰλληλα· ὅπερ ἔδει δεῖξαι.

H of B and D (respectively). I say that as E (is) to G , so F (is) to H .



For let equal multiples K and L have been taken of E and F (respectively), and other random equal multiples M and N of G and H (respectively).

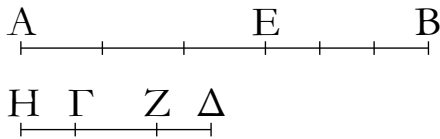
[And] since E and F are equal multiples of A and C (respectively), and the equal multiples K and L have been taken of E and F (respectively), K and L are thus equal multiples of A and C (respectively) [Prop. 5.3]. So, for the same (reasons), M and N are equal multiples of B and D (respectively). And since as A is to B , so C (is) to D , and the equal multiples K and L have been taken of A and C (respectively), and the other random equal multiples M and N of B and D (respectively), then if K exceeds M then L also exceeds N , and if (K is) equal (to M then L is also) equal (to N), and if (K is) less (than M then L is also) less (than N) [Def. 5.5]. And K and L are equal multiples of E and F (respectively), and M and N other random equal multiples of G and H (respectively). Thus, as E (is) to G , so F (is) to H [Def. 5.5].

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth then equal multiples of the first (magnitude) and the third will also have the same ratio to equal multiples of the second and the fourth, being taken in corresponding order, according to any kind of multiplication whatsoever. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $m\alpha : n\beta :: m\gamma : n\delta$, for all m and n .

ε'.

Ἐάν μέγεθος μεγέθους ισάκεις ἢ πολλαπλάσιον, ὅπερ ἀφαιρεθὲν ἀφαιρεθέντος, καὶ τὸ λοιπὸν τοῦ λοιποῦ ισάκεις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστι τὸ ὅλον τοῦ ὅλου.



Μέγεθος γὰρ τὸ AB μεγέθους τοῦ $\Gamma\Delta$ ισάκεις ἔστω πολλαπλάσιον, ὅπερ ἀφαιρεθὲν τὸ AE ἀφαιρεθέντος τοῦ ΓZ λέγω, ὅτι καὶ λοιπὸν τὸ EB λοιποῦ τοῦ $Z\Delta$ ισάκεις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ὅλον τὸ AB ὅλου τοῦ $\Gamma\Delta$.

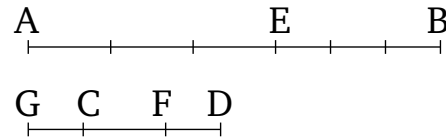
Ὅσαπλάσιον γάρ ἐστι τὸ AE τοῦ ΓZ , τοσαυταπλάσιον γεγονέτω καὶ τὸ EB τοῦ ΓH .

Καὶ ἐπεὶ ισάκεις ἐστὶ πολλαπλάσιον τὸ AE τοῦ ΓZ καὶ τὸ EB τοῦ ΓH , ισάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ AE τοῦ ΓZ καὶ τὸ AB τοῦ HZ . κεῖται δὲ ισάκεις πολλαπλάσιον τὸ AE τοῦ ΓZ καὶ τὸ AB τοῦ $\Gamma\Delta$. ισάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ AB ἑκατέρου τῶν HZ , $\Gamma\Delta$. ἴσον ἄρα τὸ HZ τῷ $\Gamma\Delta$. κοινὸν ἀφηρήσθω τὸ ΓZ . λοιπὸν ἄρα τὸ ΓH λοιπῷ τῷ $Z\Delta$ ἴσον ἐστίν. καὶ ἐπεὶ ισάκεις ἐστὶ πολλαπλάσιον τὸ AE τοῦ ΓZ καὶ τὸ EB τοῦ ΓH , ἴσον δὲ τὸ ΓH τῷ $Z\Delta$, ισάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ AE τοῦ ΓZ καὶ τὸ EB τοῦ $Z\Delta$. ισάκεις δὲ ὑπόκειται πολλαπλάσιον τὸ AE τοῦ ΓZ καὶ τὸ AB τοῦ $\Gamma\Delta$. ισάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ EB τοῦ $Z\Delta$ καὶ τὸ AB τοῦ $\Gamma\Delta$. καὶ λοιπὸν ἄρα τὸ EB λοιποῦ τοῦ $Z\Delta$ ισάκεις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ὅλον τὸ AB ὅλου τοῦ $\Gamma\Delta$.

Ἐάν ἄρα μέγεθος μεγέθους ισάκεις ἢ πολλαπλάσιον, ὅπερ ἀφαιρεθὲν ἀφαιρεθέντος, καὶ τὸ λοιπὸν τοῦ λοιποῦ ισάκεις ἔσται πολλαπλάσιον, ὁσαπλάσιόν ἐστι καὶ τὸ ὅλον τοῦ ὅλου· ὅπερ ἔδει δεῖξαι.

Proposition 5[†]

If a magnitude is the same multiple of a magnitude that a (part) taken away (is) of a (part) taken away (respectively) then the remainder will also be the same multiple of the remainder as that which the whole (is) of the whole (respectively).



For let the magnitude AB be the same multiple of the magnitude CD that the (part) taken away AE (is) of the (part) taken away CF (respectively). I say that the remainder EB will also be the same multiple of the remainder FD as that which the whole AB (is) of the whole CD (respectively).

For as many times as AE is (divisible) by CF , so many times let EB also have been made (divisible) by CG .

And since AE and EB are equal multiples of CF and GC (respectively), AE and AB are thus equal multiples of CF and GF (respectively) [Prop. 5.1]. And AE and AB are assumed (to be) equal multiples of CF and CD (respectively). Thus, AB is an equal multiple of each of GF and CD . Thus, GF (is) equal to CD . Let CF have been subtracted from both. Thus, the remainder GC is equal to the remainder FD . And since AE and EB are equal multiples of CF and GC (respectively), and GC (is) equal to DF , AE and EB are thus equal multiples of CF and FD (respectively). And AE and AB are assumed (to be) equal multiples of CF and CD (respectively). Thus, EB and AB are equal multiples of FD and CD (respectively). Thus, the remainder EB will also be the same multiple of the remainder FD as that which the whole AB (is) of the whole CD (respectively).

Thus, if a magnitude is the same multiple of a magnitude that a (part) taken away (is) of a (part) taken away (respectively) then the remainder will also be the same multiple of the remainder as that which the whole (is) of the whole (respectively). (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads $m\alpha - m\beta = m(\alpha - \beta)$.

ς'.

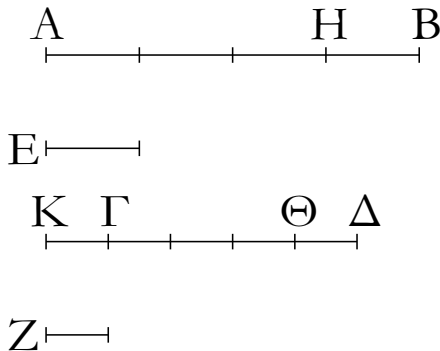
Ἐάν δύο μεγέθη δύο μεγεθῶν ισάκεις ἢ πολλαπλάσια, καὶ ἀφαιρεθέντα τινὰ τῶν αὐτῶν ισάκεις ἢ πολλαπλάσια, καὶ τὰ λοιπὰ τοῖς αὐτοῖς ᾗτοι ἴσα ἐστὶν ἢ ισάκεις αὐτῶν πολλαπλάσια.

Δύο γὰρ μεγέθη τὰ AB , $\Gamma\Delta$ δύο μεγεθῶν τῶν E , Z

Proposition 6[†]

If two magnitudes are equal multiples of two (other) magnitudes, and some (parts) taken away (from the former magnitudes) are equal multiples of the latter (magnitudes, respectively), then the remainders are also either equal to the latter (magnitudes), or (are) equal multiples

ισάκεις ἔστω πολλαπλάσια, καὶ ἀφαιρεθέντα τὰ AH , $\Gamma\Theta$ τῶν αὐτῶν τῶν E , Z ισάκεις ἔστω πολλαπλάσια· λέγω, ὅτι καὶ λοιπὰ τὰ HB , $\Theta\Delta$ τοῖς E , Z ἦτοι ἴσα ἐστὶν ἢ ισάκεις αὐτῶν πολλαπλάσια.



Ἐστω γὰρ πρότερον τὸ HB τῷ E ἴσον· λέγω, ὅτι καὶ τὸ $\Theta\Delta$ τῷ Z ἴσον ἐστίν.

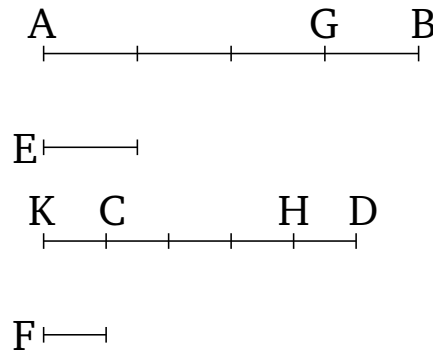
Κείσθω γὰρ τῷ Z ἴσον τὸ ΓK . ἐπεὶ ισάκεις ἐστὶ πολλαπλάσιον τὸ AH τοῦ E καὶ τὸ $\Gamma\Theta$ τοῦ Z , ἴσον δὲ τὸ μὲν HB τῷ E , τὸ δὲ $K\Gamma$ τῷ Z , ισάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ AB τοῦ E καὶ τὸ $K\Theta$ τοῦ Z . ισάκεις δὲ ὑπόκειται πολλαπλάσιον τὸ AB τοῦ E καὶ τὸ $\Gamma\Delta$ τοῦ Z · ἴσάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ $K\Theta$ τοῦ Z καὶ τὸ $\Gamma\Delta$ τοῦ Z . ἐπεὶ οὖν ἐκάτερον τῶν $K\Theta$, $\Gamma\Delta$ τοῦ Z ισάκεις ἐστὶ πολλαπλάσιον, ἴσον ἄρα ἐστὶ τὸ $K\Theta$ τῷ $\Gamma\Delta$. κοινὸν ἀφηρήσθω τὸ $\Gamma\Theta$ · λοιπὸν ἄρα τὸ $K\Gamma$ λοιπῷ τῷ $\Theta\Delta$ ἴσον ἐστίν. ἀλλὰ τὸ Z τῷ $K\Gamma$ ἐστὶν ἴσον· καὶ τὸ $\Theta\Delta$ ἄρα τῷ Z ἴσον ἐστίν. ὥστε εἰ τὸ HB τῷ E ἴσον ἐστίν, καὶ τὸ $\Theta\Delta$ ἴσον ἔσται τῷ Z .

Ὅμοίως δὴ δείξομεν, ὅτι, καὶ πολλαπλάσιον ἢ τὸ HB τοῦ E , τοσαυταπλάσιον ἔσται καὶ τὸ $\Theta\Delta$ τοῦ Z .

Ἐὰν ἄρα δύο μεγέθη δύο μεγεθῶν ισάκεις ἢ πολλαπλάσια, καὶ ἀφαιρεθέντα τινὰ τῶν αὐτῶν ισάκεις ἢ πολλαπλάσια, καὶ τὰ λοιπὰ τοῖς αὐτοῖς ἦτοι ἴσα ἐστὶν ἢ ισάκεις αὐτῶν πολλαπλάσια· ὅπερ ἔδει δεῖξαι.

of them (respectively).

For let two magnitudes AB and CD be equal multiples of two magnitudes E and F (respectively). And let the (parts) taken away (from the former) AG and CH be equal multiples of E and F (respectively). I say that the remainders GB and HD are also either equal to E and F (respectively), or (are) equal multiples of them.



For let GB be, first of all, equal to E . I say that HD is also equal to F .

For let CK be made equal to F . Since AG and CH are equal multiples of E and F (respectively), and GB (is) equal to E , and KC to F , AB and KH are thus equal multiples of E and F (respectively) [Prop. 5.2]. And AB and CD are assumed (to be) equal multiples of E and F (respectively). Thus, KH and CD are equal multiples of F and F (respectively). Therefore, KH and CD are each equal multiples of F . Thus, KH is equal to CD . Let CH have been taken away from both. Thus, the remainder KC is equal to the remainder HD . But, F is equal to KC . Thus, HD is also equal to F . Hence, if GB is equal to E then HD will also be equal to F .

So, similarly, we can show that even if GB is a multiple of E then HD will also be the same multiple of F .

Thus, if two magnitudes are equal multiples of two (other) magnitudes, and some (parts) taken away (from the former magnitudes) are equal multiples of the latter (magnitudes, respectively), then the remainders are also either equal to the latter (magnitudes), or (are) equal multiples of them (respectively). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads $m\alpha - n\alpha = (m - n)\alpha$.

ζ'.

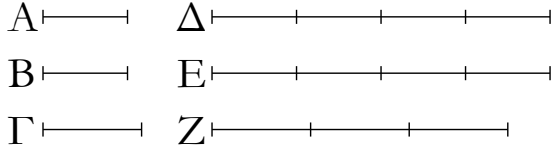
Τὰ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον καὶ τὸ αὐτὸ πρὸς τὰ ἴσα.

Ἐστω ἴσα μεγέθη τὰ A , B , ἄλλο δέ τι, ὃ ἔτυχεν, μέγεθος τὸ Γ · λέγω, ὅτι ἐκάτερον τῶν A , B πρὸς τὸ Γ τὸν αὐτὸν ἔχει λόγον, καὶ τὸ Γ πρὸς ἐκάτερον τῶν A , B .

Proposition 7

Equal (magnitudes) have the same ratio to the same (magnitude), and the latter (magnitude) has the same ratio to the equal (magnitudes).

Let A and B be equal magnitudes, and C some other random magnitude. I say that A and B each have the



Εἰλήφθω γὰρ τῶν μὲν A, B ἰσάκεις πολλαπλάσια τὰ Δ, E , τοῦ δὲ Γ ἄλλο, ὃ ἔτυχεν, πολλαπλάσιον τὸ Z .

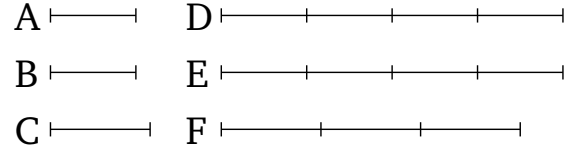
Ἐπεὶ οὖν ἰσάκεις ἐστὶ πολλαπλάσιον τὸ Δ τοῦ A καὶ τὸ E τοῦ B , ἴσον δὲ τὸ A τῷ B , ἴσον ἄρα καὶ τὸ Δ τῷ E . ἄλλο δέ, ὃ ἔτυχεν, τὸ Z . Εἰ ἄρα ὑπερέχει τὸ Δ τοῦ Z , ὑπερέχει καὶ τὸ E τοῦ Z , καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καὶ ἐστὶ τὰ μὲν Δ, E τῶν A, B ἰσάκεις πολλαπλάσια, τὸ δὲ Z τοῦ Γ ἄλλο, ὃ ἔτυχεν, πολλαπλάσιον· ἔστιν ἄρα ὡς τὸ A πρὸς τὸ Γ , οὕτως τὸ B πρὸς τὸ Γ .

Λέγω [δὴ], ὅτι καὶ τὸ Γ πρὸς ἐκάτερον τῶν A, B τὸν αὐτὸν ἔχει λόγον.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δεῖξομεν, ὅτι ἴσον ἐστὶ τὸ Δ τῷ E · ἄλλο δέ τι τὸ Z · εἰ ἄρα ὑπερέχει τὸ Z τοῦ Δ , ὑπερέχει καὶ τοῦ E , καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καὶ ἐστὶ τὸ μὲν Z τοῦ Γ πολλαπλάσιον, τὰ δὲ Δ, E τῶν A, B ἄλλα, ὃ ἔτυχεν, ἰσάκεις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ A , οὕτως τὸ Γ πρὸς τὸ B .

Τὰ ἴσα ἄρα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον καὶ τὸ αὐτὸ πρὸς τὰ ἴσα.

same ratio to C , and (that) C (has the same ratio) to each of A and B .



For let the equal multiples D and E have been taken of A and B (respectively), and the other random multiple F of C .

Therefore, since D and E are equal multiples of A and B (respectively), and A (is) equal to B , D (is) thus also equal to E . And F (is) different, at random. Thus, if D exceeds F then E also exceeds F , and if (D is) equal (to F then E is also) equal (to F), and if (D is) less (than F then E is also) less (than F). And D and E are equal multiples of A and B (respectively), and F another random multiple of C . Thus, as A (is) to C , so B (is) to C [Def. 5.5].

[So] I say that $C^\dagger$ also has the same ratio to each of A and B .

For, similarly, we can show, by the same construction, that D is equal to E . And F (has) some other (value). Thus, if F exceeds D then it also exceeds E , and if (F is) equal (to D then it is also) equal (to E), and if (F is) less (than D then it is also) less (than E). And F is a multiple of C , and D and E other random equal multiples of A and B . Thus, as C (is) to A , so C (is) to B [Def. 5.5].

Thus, equal (magnitudes) have the same ratio to the same (magnitude), and the latter (magnitude has the same ratio) to the equal (magnitudes).

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν μεγέθη τινὰ ἀνάλογον ᾖ, καὶ ἀνάπαλιν ἀνάλογον ἔσται. ὅπερ ἔδει δεῖξαι.

Corollary[†]

So (it is) clear, from this, that if some magnitudes are proportional then they will also be proportional inversely. (Which is) the very thing it was required to show.

[†] The Greek text has " E ", which is obviously a mistake.

[‡] In modern notation, this corollary reads that if $\alpha : \beta :: \gamma : \delta$ then $\beta : \alpha :: \delta : \gamma$.

η'.

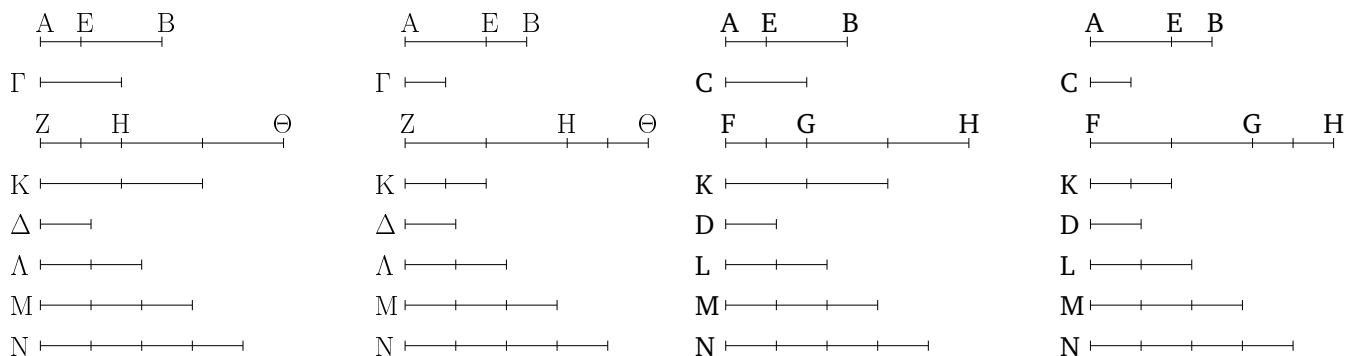
Τῶν ἀνίσων μεγεθῶν τὸ μείζον πρὸς τὸ αὐτὸ μείζονα λόγον ἔχει ἢ πρὸς τὸ ἔλαττον. καὶ τὸ αὐτὸ πρὸς τὸ ἔλαττον μείζονα λόγον ἔχει ἢ πρὸς τὸ μείζον.

Ἐστω ἄνισα μεγέθη τὰ AB, Γ , καὶ ἔστω μείζον τὸ AB , ἄλλο δέ, ὃ ἔτυχεν, τὸ Δ . λέγω, ὅτι τὸ AB πρὸς τὸ Δ μείζονα λόγον ἔχει ἢ πρὸς τὸ Γ πρὸς τὸ Δ , καὶ τὸ Δ πρὸς τὸ Γ μείζονα λόγον ἔχει ἢ πρὸς τὸ AB .

Proposition 8

For unequal magnitudes, the greater (magnitude) has a greater ratio than the lesser to the same (magnitude). And the latter (magnitude) has a greater ratio to the lesser (magnitude) than to the greater.

Let AB and C be unequal magnitudes, and let AB be the greater (of the two), and D another random magnitude. I say that AB has a greater ratio to D than C (has) to D , and (that) D has a greater ratio to C than (it has) to AB .



Ἐπεὶ γὰρ μείζον ἐστὶ τὸ AB τοῦ Γ , κείσθω τῷ Γ ἴσον τὸ BE . τὸ δὲ ἔλασσον τῶν AE , EB πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ Δ μείζον. ἔστω πρότερον τὸ AE ἔλαττον τοῦ EB , καὶ πεπολλαπλασιάσθω τὸ AE , καὶ ἔστω αὐτοῦ πολλαπλάσιον τὸ ZH μείζον ὃν τοῦ Δ , καὶ ὁσαπλάσιόν ἐστὶ τὸ ZH τοῦ AE , τοσαυταπλάσιον γεγονέτω καὶ τὸ μὲν $H\Theta$ τοῦ EB τὸ δὲ K τοῦ Γ . καὶ εἰλήφθω τοῦ Δ διπλάσιον μὲν τὸ Λ , τριπλάσιον δὲ τὸ M , καὶ ἐξῆς ἐνὶ πλεῖον, ἕως ἂν τὸ λαμβανόμενον πολλαπλάσιον μὲν γένηται τοῦ Δ , πρώτως δὲ μείζον τοῦ K . εἰλήφθω, καὶ ἔστω τὸ N τετραπλάσιον μὲν τοῦ Δ , πρώτως δὲ μείζον τοῦ K .

Ἐπεὶ οὖν τὸ K τοῦ N πρώτως ἐστὶν ἔλαττον, τὸ K ἄρα τοῦ M οὐκ ἐστὶν ἔλαττον. καὶ ἐπεὶ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ ZH τοῦ AE καὶ τὸ $H\Theta$ τοῦ EB , ἰσάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ ZH τοῦ AE καὶ τὸ $Z\Theta$ τοῦ AB . ἰσάκεις δὲ ἐστὶ πολλαπλάσιον τὸ ZH τοῦ AE καὶ τὸ K τοῦ Γ . ἰσάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ $Z\Theta$ τοῦ AB καὶ τὸ K τοῦ Γ . τὰ $Z\Theta$, K ἄρα τῶν AB , Γ ἰσάκεις ἐστὶ πολλαπλάσια. πάλιν, ἐπεὶ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ $H\Theta$ τοῦ EB καὶ τὸ K τοῦ Γ , ἴσον δὲ τὸ EB τῷ Γ , ἴσον ἄρα καὶ τὸ $H\Theta$ τῷ K . τὸ δὲ K τοῦ M οὐκ ἐστὶν ἔλαττον· οὐδ' ἄρα τὸ $H\Theta$ τοῦ M ἔλαττον ἐστὶν. μείζον δὲ τὸ ZH τοῦ Δ . ὅλον ἄρα τὸ $Z\Theta$ συναμφοτέρων τῶν Δ , M μείζον ἐστὶν. ἀλλὰ συναμφοτέρα τὰ Δ , M τῷ N ἐστὶν ἴσα, ἐπειδὴ περ τὸ M τοῦ Δ τριπλάσιόν ἐστιν, συναμφοτέρα δὲ τὰ M , Δ τοῦ Δ ἐστὶ τετραπλάσια, ἔστι δὲ καὶ τὸ N τοῦ Δ τετραπλάσιον· συναμφοτέρα ἄρα τὰ M , Δ τῷ N ἴσα ἐστίν. ἀλλὰ τὸ $Z\Theta$ τῶν M , Δ μείζον ἐστὶν· τὸ $Z\Theta$ ἄρα τοῦ N ὑπερέχει· τὸ δὲ K τοῦ N οὐχ ὑπερέχει. καὶ ἐστὶ τὰ μὲν $Z\Theta$, K τῶν AB , Γ ἰσάκεις πολλαπλάσια, τὸ δὲ N τοῦ Δ ἄλλο, ὃ ἔτυχεν, πολλαπλάσιον· τὸ AB ἄρα πρὸς τὸ Δ μείζονα λόγον ἔχει ἥπερ τὸ Γ πρὸς τὸ Δ .

Λέγω δὴ, ὅτι καὶ τὸ Δ πρὸς τὸ Γ μείζονα λόγον ἔχει ἥπερ τὸ Δ πρὸς τὸ AB .

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δεῖξομεν, ὅτι τὸ μὲν N τοῦ K ὑπερέχει, τὸ δὲ N τοῦ $Z\Theta$ οὐχ ὑπερέχει. καὶ ἐστὶ τὸ μὲν N τοῦ Δ πολλαπλάσιον, τὰ δὲ $Z\Theta$, K τῶν AB , Γ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια· τὸ Δ ἄρα πρὸς τὸ Γ μείζονα λόγον ἔχει ἥπερ τὸ Δ πρὸς τὸ AB .

Ἀλλὰ δὴ τὸ AE τοῦ EB μείζον ἔστω. τὸ δὲ ἔλαττον τὸ EB πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ Δ μείζον. πε-

For since AB is greater than C , let BE be made equal to C . So, the lesser of AE and EB , being multiplied, will sometimes be greater than D [Def. 5.4]. First of all, let AE be less than EB , and let AE have been multiplied, and let FG be a multiple of it which (is) greater than D . And as many times as FG is (divisible) by AE , so many times let GH also have become (divisible) by EB , and K by C . And let the double multiple L of D have been taken, and the triple multiple M , and several more, (each increasing) in order by one, until the (multiple) taken becomes the first multiple of D (which is) greater than K . Let it have been taken, and let it also be the quadruple multiple N of D —the first (multiple) greater than K .

Therefore, since K is less than N first, K is thus not less than M . And since FG and GH are equal multiples of AE and EB (respectively), FG and FH are thus equal multiples of AE and AB (respectively) [Prop. 5.1]. And FG and K are equal multiples of AE and C (respectively). Thus, FH and K are equal multiples of AB and C (respectively). Thus, FH , K are equal multiples of AB , C . Again, since GH and K are equal multiples of EB and C , and EB (is) equal to C , GH (is) thus also equal to K . And K is not less than M . Thus, GH not less than M either. And FG (is) greater than D . Thus, the whole of FH is greater than D and M (added) together. But, D and M (added) together is equal to N , inasmuch as M is three times D , and M and D (added) together is four times D , and N is also four times D . Thus, M and D (added) together is equal to N . But, FH is greater than M and D . Thus, FH exceeds N . And K does not exceed N . And FH , K are equal multiples of AB , C , and N another random multiple of D . Thus, AB has a greater ratio to D than C (has) to D [Def. 5.7].

So, I say that D also has a greater ratio to C than D (has) to AB .

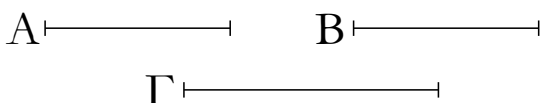
For, similarly, by the same construction, we can show that N exceeds K , and N does not exceed FH . And N is a multiple of D , and FH , K other random equal multiples of AB , C (respectively). Thus, D has a greater

πολλαπλασιάσθω, καὶ ἔστω τὸ $H\Theta$ πολλαπλάσιον μὲν τοῦ EB , μείζον δὲ τοῦ Δ · καὶ ὁσαπλάσιόν ἐστι τὸ $H\Theta$ τοῦ EB , τοσαυταπλάσιον γεγονέντω καὶ τὸ μὲν ZH τοῦ AE , τὸ δὲ K τοῦ Γ . ὁμοίως δὲ δείξομεν, ὅτι τὰ $Z\Theta$, K τῶν AB , Γ ἰσάκεις ἐστὶ πολλαπλάσια· καὶ εἰλήφθω ὁμοίως τὸ N πολλαπλάσιον μὲν τοῦ Δ , πρώτως δὲ μείζον τοῦ ZH · ὥστε πάλιν τὸ ZH τοῦ M οὐκ ἐστὶν ἔλασσον. μείζον δὲ τὸ $H\Theta$ τοῦ Δ · ὅλον ἄρα τὸ $Z\Theta$ τῶν Δ , M , τουτέστι τοῦ N , ὑπερέχει. τὸ δὲ K τοῦ N οὐκ ὑπερέχει, ἐπειδὴ περ καὶ τὸ ZH μείζον ὢν τοῦ $H\Theta$, τουτέστι τοῦ K , τοῦ N οὐκ ὑπερέχει. καὶ ὡσαύτως κατακολουθοῦντες τοῖς ἐπάνω περαίνομεν τὴν ἀπόδειξιν.

Τῶν ἄρα ἀνίσων μεγεθῶν τὸ μείζον πρὸς τὸ αὐτὸ μείζονα λόγον ἔχει ἢ πρὸς τὸ ἔλαττον· καὶ τὸ αὐτὸ πρὸς τὸ ἔλαττον μείζονα λόγον ἔχει ἢ πρὸς τὸ μείζον· ὅπερ ἔδει δεῖξαι.

θ'.

Τὰ πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχοντα λόγον ἴσα ἀλλήλοις ἐστίν· καὶ πρὸς ἃ τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἐκεῖνα ἴσα ἐστίν.



Ἐχέτω γὰρ ἑκάτερον τῶν A , B πρὸς τὸ Γ τὸν αὐτὸν λόγον· λέγω, ὅτι ἴσον ἐστὶ τὸ A τῷ B .

Εἰ γὰρ μή, οὐκ ἂν ἑκάτερον τῶν A , B πρὸς τὸ Γ τὸν αὐτὸν εἶχε λόγον· ἔχει δέ· ἴσον ἄρα ἐστὶ τὸ A τῷ B .

Ἐχέτω δὲ πάλιν τὸ Γ πρὸς ἑκάτερον τῶν A , B τὸν αὐτὸν λόγον· λέγω, ὅτι ἴσον ἐστὶ τὸ A τῷ B .

Εἰ γὰρ μή, οὐκ ἂν τὸ Γ πρὸς ἑκάτερον τῶν A , B τὸν αὐτὸν εἶχε λόγον· ἔχει δέ· ἴσον ἄρα ἐστὶ τὸ A τῷ B .

Τὰ ἄρα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχοντα λόγον ἴσα ἀλλήλοις ἐστίν· καὶ πρὸς ἃ τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἐκεῖνα ἴσα ἐστίν· ὅπερ ἔδει δεῖξαι.

ι'.

Τῶν πρὸς τὸ αὐτὸ λόγον ἔχόντων τὸ μείζονα λόγον ἔχον ἐκεῖνο μείζον ἐστίν· πρὸς δὲ τὸ αὐτὸ μείζονα λόγον

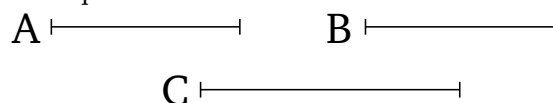
ratio to C than D (has) to AB [Def. 5.5].

And so let AE be greater than EB . So, the lesser, EB , being multiplied, will sometimes be greater than D . Let it have been multiplied, and let GH be a multiple of EB (which is) greater than D . And as many times as GH is (divisible) by EB , so many times let FG also have become (divisible) by AE , and K by C . So, similarly (to the above), we can show that FH and K are equal multiples of AB and C (respectively). And, similarly (to the above), let the multiple N of D , (which is) the first (multiple) greater than FG , have been taken. So, FG is again not less than M . And GH (is) greater than D . Thus, the whole of FH exceeds D and M , that is to say N . And K does not exceed N , inasmuch as FG , which (is) greater than GH —that is to say, K —also does not exceed N . And, following the above (arguments), we (can) complete the proof in the same manner.

Thus, for unequal magnitudes, the greater (magnitude) has a greater ratio than the lesser to the same (magnitude). And the latter (magnitude) has a greater ratio to the lesser (magnitude) than to the greater. (Which is) the very thing it was required to show.

Proposition 9

(Magnitudes) having the same ratio to the same (magnitude) are equal to one another. And those (magnitudes) to which the same (magnitude) has the same ratio are equal.



For let A and B each have the same ratio to C . I say that A is equal to B .

For if not, A and B would not each have the same ratio to C [Prop. 5.8]. But they do. Thus, A is equal to B .

So, again, let C have the same ratio to each of A and B . I say that A is equal to B .

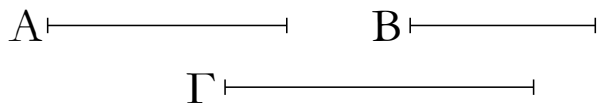
For if not, C would not have the same ratio to each of A and B [Prop. 5.8]. But it does. Thus, A is equal to B .

Thus, (magnitudes) having the same ratio to the same (magnitude) are equal to one another. And those (magnitudes) to which the same (magnitude) has the same ratio are equal. (Which is) the very thing it was required to show.

Proposition 10

For (magnitudes) having a ratio to the same (magnitude), that (magnitude which) has the greater ratio is

ἔχει, ἐκεῖνο ἑλαττόν ἐστιν.



Ἐχέτω γὰρ τὸ A πρὸς τὸ Γ μείζονα λόγον ἥπερ τὸ B πρὸς τὸ Γ . λέγω, ὅτι μείζον ἐστὶ τὸ A τοῦ B .

Εἰ γὰρ μή, ἦτοι ἴσον ἐστὶ τὸ A τῷ B ἢ ἑλασσον. ἴσον μὲν οὖν οὐκ ἐστὶ τὸ A τῷ B . ἐκάτερον γὰρ ἂν τῶν A , B πρὸς τὸ Γ τὸν αὐτὸν εἶχε λόγον. οὐκ ἔχει δέ· οὐκ ἄρα ἴσον ἐστὶ τὸ A τῷ B . οὐδὲ μὴν ἑλασσόν ἐστι τὸ A τοῦ B . τὸ A γὰρ ἂν πρὸς τὸ Γ ἐλάσσονα λόγον εἶχεν ἥπερ τὸ B πρὸς τὸ Γ . οὐκ ἔχει δέ· οὐκ ἄρα ἑλασσόν ἐστι τὸ A τοῦ B . ἐδείχθη δὲ οὐδὲ ἴσον· μείζον ἄρα ἐστὶ τὸ A τοῦ B .

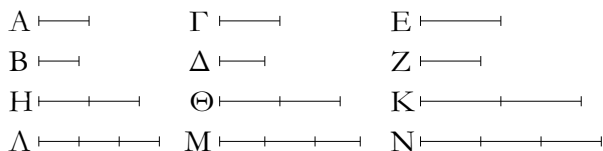
Ἐχέτω δὴ πάλιν τὸ Γ πρὸς τὸ B μείζονα λόγον ἥπερ τὸ Γ πρὸς τὸ A . λέγω, ὅτι ἑλασσόν ἐστι τὸ B τοῦ A .

Εἰ γὰρ μή, ἦτοι ἴσον ἐστὶν ἢ μείζον. ἴσον μὲν οὖν οὐκ ἐστὶ τὸ B τῷ A . τὸ Γ γὰρ ἂν πρὸς ἐκάτερον τῶν A , B τὸν αὐτὸν εἶχε λόγον. οὐκ ἔχει δέ· οὐκ ἄρα ἴσον ἐστὶ τὸ A τῷ B . οὐδὲ μὴν μείζον ἐστὶ τὸ B τοῦ A . τὸ Γ γὰρ ἂν πρὸς τὸ B ἐλάσσονα λόγον εἶχεν ἥπερ πρὸς τὸ A . οὐκ ἔχει δέ· οὐκ ἄρα μείζον ἐστὶ τὸ B τοῦ A . ἐδείχθη δέ, ὅτι οὐδὲ ἴσον· ἑλαττον ἄρα ἐστὶ τὸ B τοῦ A .

Τῶν ἄρα πρὸς τὸ αὐτὸ λόγον ἐχόντων τὸ μείζονα λόγον ἔχον μείζον ἐστίν· καὶ πρὸς ὃ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκεῖνο ἑλαττόν ἐστιν· ὅπερ ἔδει δεῖξαι.

ια'.

Οἱ τῷ αὐτῷ λόγῳ οἱ αὐτοὶ καὶ ἀλλήλοις εἰσὶν οἱ αὐτοί.

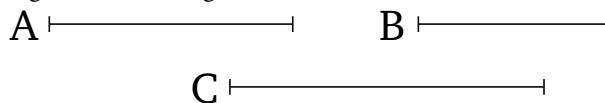


Ἔστωσαν γὰρ ὡς μὲν τὸ A πρὸς τὸ B , οὕτως τὸ Γ πρὸς τὸ Δ , ὡς δὲ τὸ Γ πρὸς τὸ Δ , οὕτως τὸ E πρὸς τὸ Z . λέγω, ὅτι ἐστὶν ὡς τὸ A πρὸς τὸ B , οὕτως τὸ E πρὸς τὸ Z .

Εἰλήφθω γὰρ τῶν A , Γ , E ἰσάκεις πολλαπλάσια τὰ H , Θ , K , τῶν δὲ B , Δ , Z ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ Λ , M , N .

Καὶ ἐπεὶ ἐστὶν ὡς τὸ A πρὸς τὸ B , οὕτως τὸ Γ πρὸς τὸ Δ , καὶ εἰληπται τῶν μὲν A , Γ ἰσάκεις πολλαπλάσια τὰ H , Θ , τῶν δὲ B , Δ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ Λ , M , εἰ ἄρα ὑπερέχει τὸ H τοῦ Λ , ὑπερέχει καὶ τὸ Θ τοῦ M , καὶ εἰ ἴσον ἐστίν, ἴσον, καὶ εἰ ἐλλείπει, ἐλλείπει. πάλιν, ἐπεὶ ἐστὶν

(the) greater. And that (magnitude) to which the latter (magnitude) has a greater ratio is (the) lesser.



For let A have a greater ratio to C than B (has) to C . I say that A is greater than B .

For if not, A is surely either equal to or less than B . In fact, A is not equal to B . For (then) A and B would each have the same ratio to C [Prop. 5.7]. But they do not. Thus, A is not equal to B . Neither, indeed, is A less than B . For (then) A would have a lesser ratio to C than B (has) to C [Prop. 5.8]. But it does not. Thus, A is not less than B . And it was shown not (to be) equal either. Thus, A is greater than B .

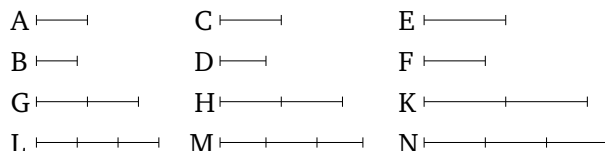
So, again, let C have a greater ratio to B than C (has) to A . I say that B is less than A .

For if not, (it is) surely either equal or greater. In fact, B is not equal to A . For (then) C would have the same ratio to each of A and B [Prop. 5.7]. But it does not. Thus, A is not equal to B . Neither, indeed, is B greater than A . For (then) C would have a lesser ratio to B than (it has) to A [Prop. 5.8]. But it does not. Thus, B is not greater than A . And it was shown that (it is) not equal (to A) either. Thus, B is less than A .

Thus, for (magnitudes) having a ratio to the same (magnitude), that (magnitude which) has the greater ratio is (the) greater. And that (magnitude) to which the latter (magnitude) has a greater ratio is (the) lesser. (Which is) the very thing it was required to show.

Proposition 11[†]

(Ratios which are) the same with the same ratio are also the same with one another.



For let it be that as A (is) to B , so C (is) to D , and as C (is) to D , so E (is) to F . I say that as A is to B , so E (is) to F .

For let the equal multiples G , H , K have been taken of A , C , E (respectively), and the other random equal multiples L , M , N of B , D , F (respectively).

And since as A is to B , so C (is) to D , and the equal multiples G and H have been taken of A and C (respectively), and the other random equal multiples L and M of B and D (respectively), thus if G exceeds L then H also exceeds M , and if (G is) equal (to L then H is also)

ὥς τὸ Γ πρὸς τὸ Δ, οὕτως τὸ Ε πρὸς τὸ Ζ, καὶ εἴληπται τῶν Γ, Ε ἰσάκεις πολλαπλάσια τὰ Θ, Κ, τῶν δὲ Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ Μ, Ν, εἰ ἄρα ὑπερέχει τὸ Θ τοῦ Μ, ὑπερέχει καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλλαττον, ἔλλαττον. ἀλλὰ εἰ ὑπερεῖχε τὸ Θ τοῦ Μ, ὑπερεῖχε καὶ τὸ Η τοῦ Α, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλλαττον, ἔλλαττον· ὥστε καὶ εἰ ὑπερέχει τὸ Η τοῦ Α, ὑπερέχει καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλλαττον, ἔλλαττον. καί ἐστι τὰ μὲν Η, Κ τῶν Α, Ε ἰσάκεις πολλαπλάσια, τὰ δὲ Α, Ν τῶν Β, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ.

Οἱ ἄρα τῷ αὐτῷ λόγῳ οἱ αὐτοὶ καὶ ἀλλήλοις εἰσὶν οἱ αὐτοί· ὅπερ ἔδει δεῖξαι.

equal (to M), and if (G is) less (than L then H is also) less (than M) [Def. 5.5]. Again, since as C is to D , so E (is) to F , and the equal multiples H and K have been taken of C and E (respectively), and the other random equal multiples M and N of D and F (respectively), thus if H exceeds M then K also exceeds N , and if (H is) equal (to M then K is also) equal (to N), and if (H is) less (than M then K is also) less (than N) [Def. 5.5]. But (we saw that) if H was exceeding M then G was also exceeding L , and if (H was) equal (to M then G was also) equal (to L), and if (H was) less (than M then G was also) less (than L). And, hence, if G exceeds L then K also exceeds N , and if (G is) equal (to L then K is also) equal (to N), and if (G is) less (than L then K is also) less (than N). And G and K are equal multiples of A and E (respectively), and L and N other random equal multiples of B and F (respectively). Thus, as A is to B , so E (is) to F [Def. 5.5].

Thus, (ratios which are) the same with the same ratio are also the same with one another. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ and $\gamma : \delta :: \epsilon : \zeta$ then $\alpha : \beta :: \epsilon : \zeta$.

ιβ'.

Ἐὰν ἡ ὁποσαοῦν μεγέθη ἀνάλογον, ἔσται ὡς ἐν τῶν ἡγουμένων πρὸς ἐν τῶν ἐπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἐπόμενα.

A ——— Γ ——— E ———
B ——— Δ ——— Z ———

H ——— Λ ———
Θ ——— Μ ———
Κ ——— Ν ———

Ἐστωσαν ὁποσαοῦν μεγέθη ἀνάλογον τὰ Α, Β, Γ, Δ, Ε, Ζ, ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ τὸ Ε πρὸς τὸ Ζ· λέγω, ὅτι ἐστὶν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὰ Α, Γ, Ε πρὸς τὰ Β, Δ, Ζ.

Εἰλήφθω γὰρ τῶν μὲν Α, Γ, Ε ἰσάκεις πολλαπλάσια τὰ Η, Θ, Κ, τῶν δὲ Β, Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ Λ, Μ, Ν.

Καὶ ἐπεὶ ἐστὶν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ τὸ Ε πρὸς τὸ Ζ, καὶ εἴληπται τῶν μὲν Α, Γ, Ε ἰσάκεις πολλαπλάσια τὰ Η, Θ, Κ τῶν δὲ Β, Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ Λ, Μ, Ν, εἰ ἄρα ὑπερέχει τὸ Η τοῦ Λ, ὑπερέχει καὶ τὸ Θ τοῦ Μ, καὶ τὸ Κ τοῦ Ν, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλλαττον, ἔλλαττον. ὥστε καὶ εἰ ὑπερέχει τὸ Η τοῦ Λ,

Proposition 12†

If there are any number of magnitudes whatsoever (which are) proportional then as one of the leading (magnitudes is) to one of the following, so will all of the leading (magnitudes) be to all of the following.

A ——— C ——— E ———
B ——— D ——— F ———

G ——— L ———
H ——— Μ ———
K ——— Ν ———

Let there be any number of magnitudes whatsoever, A, B, C, D, E, F , (which are) proportional, (so that) as A (is) to B , so C (is) to D , and E to F . I say that as A is to B , so A, C, E (are) to B, D, F .

For let the equal multiples G, H, K have been taken of A, C, E (respectively), and the other random equal multiples L, M, N of B, D, F (respectively).

And since as A is to B , so C (is) to D , and E to F , and the equal multiples G, H, K have been taken of A, C, E (respectively), and the other random equal multiples L, M, N of B, D, F (respectively), thus if G exceeds L then H also exceeds M , and K (exceeds) N , and if (G is) equal (to L then H is also) equal (to M , and K to N),

ὑπερέχει καὶ τὰ Η, Θ, Κ τῶν Α, Μ, Ν, καὶ εἰ ἴσον, ἴσα, καὶ εἰ ἔλαττον, ἔλαττονα. καὶ ἐστὶ τὸ μὲν Η καὶ τὰ Η, Θ, Κ τοῦ Α καὶ τῶν Α, Γ, Ε ἰσάκεις πολλαπλάσια, ἐπειδὴ ἑπὶ ἑκάστου ὅποσαοῦν μεγέθη ὅποσωνοῦν μεγεθῶν ἴσων τὸ πλῆθος ἑκάστων ἑκάστου ἰσάκεις πολλαπλάσιον, ὁσαπλάσιόν ἐστιν ἐν τῶν μεγεθῶν ἐνός, τοσαυταπλάσια ἔσται καὶ τὰ πάντα τῶν πάντων. διὰ τὰ αὐτὰ δὴ καὶ τὸ Α καὶ τὰ Α, Μ, Ν τοῦ Β καὶ τῶν Β, Δ, Ζ ἰσάκεις ἐστὶ πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὰ Α, Γ, Ε πρὸς τὰ Β, Δ, Ζ.

Ἐὰν ἄρα ἡ ὁποσαοῦν μεγέθη ἀνάλογον, ἔσται ὡς ἐν τῶν ἡγουμένων πρὸς ἐν τῶν ἐπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἐπόμενα· ὅπερ ἔδει δεῖξαι.

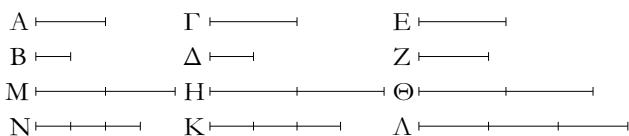
and if (G is) less (than L then H is also) less (than M , and K than N) [Def. 5.5]. And, hence, if G exceeds L then G , H , K also exceed L , M , N , and if (G is) equal (to L then G , H , K are also) equal (to L , M , N) and if (G is) less (than L then G , H , K are also) less (than L , M , N). And G and G , H , K are equal multiples of A and A , C , E (respectively), inasmuch as if there are any number of magnitudes whatsoever (which are) equal multiples, respectively, of some (other) magnitudes, of equal number (to them), then as many times as one of the (first) magnitudes is (divisible) by one (of the second), so many times will all (of the first magnitudes) also (be divisible) by all (of the second) [Prop. 5.1]. So, for the same (reasons), L and L , M , N are also equal multiples of B and B , D , F (respectively). Thus, as A is to B , so A , C , E (are) to B , D , F (respectively).

Thus, if there are any number of magnitudes whatsoever (which are) proportional then as one of the leading (magnitudes is) to one of the following, so will all of the leading (magnitudes) be to all of the following. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \alpha' :: \beta : \beta' :: \gamma : \gamma'$ etc. then $\alpha : \alpha' :: (\alpha + \beta + \gamma + \dots) : (\alpha' + \beta' + \gamma' + \dots)$.

ιγ'.

Ἐὰν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχη λόγον καὶ τρίτον πρὸς τέταρτον, τρίτον δὲ πρὸς τέταρτον μείζονα λόγον ἔχη ἢ πέμπτον πρὸς ἕκτον, καὶ πρῶτον πρὸς δεύτερον μείζονα λόγον ἔξει ἢ πέμπτον πρὸς ἕκτον.

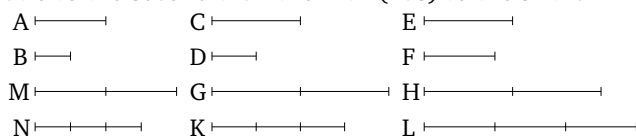


Πρῶτον γάρ τὸ Α πρὸς δεύτερον τὸ Β τὸν αὐτὸν ἔχέτω λόγον καὶ τρίτον τὸ Γ πρὸς τέταρτον τὸ Δ, τρίτον δὲ τὸ Γ πρὸς τέταρτον τὸ Δ μείζονα λόγον ἔχέτω ἢ πέμπτον τὸ Ε πρὸς ἕκτον τὸ Ζ. λέγω, ὅτι καὶ πρῶτον τὸ Α πρὸς δεύτερον τὸ Β μείζονα λόγον ἔξει ἢ πεμπτον τὸ Ε πρὸς ἕκτον τὸ Ζ.

Ἐπεὶ γὰρ ἐστὶ τινὰ τῶν μὲν Γ, Ε ἰσάκεις πολλαπλάσια, τῶν δὲ Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια, καὶ τὸ μὲν τοῦ Γ πολλαπλάσιον τοῦ τοῦ Δ πολλαπλάσιον ὑπερέχει, τὸ δὲ τοῦ Ε πολλαπλάσιον τοῦ τοῦ Ζ πολλαπλάσιον οὐχ ὑπερέχει, εἰλήφθω, καὶ ἔστω τῶν μὲν Γ, Ε ἰσάκεις πολλαπλάσια τὰ Η, Θ, τῶν δὲ Δ, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ Κ, Λ, ὥστε τὸ μὲν Η τοῦ Κ ὑπερέχειν, τὸ δὲ Θ τοῦ Λ μὴ ὑπερέχειν· καὶ ὁσαπλάσιον μὲν ἐστὶ τὸ Η τοῦ Γ, τοσαυταπλάσιον ἔστω καὶ τὸ Μ τοῦ Α, ὁσαπλάσιον δὲ τὸ Κ τοῦ Δ, τοσαυταπλάσιον ἔστω καὶ τὸ Ν τοῦ Β.

Proposition 13[†]

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the third (magnitude) has a greater ratio to the fourth than a fifth (has) to a sixth, then the first (magnitude) will also have a greater ratio to the second than the fifth (has) to the sixth.



For let a first (magnitude) A have the same ratio to a second B that a third C (has) to a fourth D , and let the third (magnitude) C have a greater ratio to the fourth D than a fifth E (has) to a sixth F . I say that the first (magnitude) A will also have a greater ratio to the second B than the fifth E (has) to the sixth F .

For since there are some equal multiples of C and E , and other random equal multiples of D and F , (for which) the multiple of C exceeds the (multiple) of D , and the multiple of E does not exceed the multiple of F [Def. 5.7], let them have been taken. And let G and H be equal multiples of C and E (respectively), and K and L other random equal multiples of D and F (respectively), such that G exceeds K , but H does not exceed L . And as many times as G is (divisible) by C , so many times let M be (divisible) by A . And as many times as K (is divisible)

Καὶ ἐπεὶ ἐστὶν ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ, καὶ εἴληπται τῶν μὲν Α, Γ ἰσάκεις πολλαπλάσια τὰ Μ, Η, τῶν δὲ Β, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ Ν, Κ, εἰ ἄρα ὑπερέχει τὸ Μ τοῦ Ν, ὑπερέχει καὶ τὸ Η τοῦ Κ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. ὑπερέχει δὲ τὸ Η τοῦ Κ· ὑπερέχει ἄρα καὶ τὸ Μ τοῦ Ν. τὸ δὲ Θ τοῦ Α οὐχ ὑπερέχει· καὶ ἐστὶ τὰ μὲν Μ, Θ τῶν Α, Ε ἰσάκεις πολλαπλάσια, τὰ δὲ Ν, Α τῶν Β, Ζ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια· τὸ ἄρα Α πρὸς τὸ Β μείζονα λόγον ἔχει ἢ περ τὸ Ε πρὸς τὸ Ζ.

Ἐάν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχη λόγον καὶ τρίτον πρὸς τέταρτον, τρίτον δὲ πρὸς τέταρτον μείζονα λόγον ἔχη ἢ πέμπτον πρὸς ἕκτον, καὶ πρῶτον πρὸς δεύτερον μείζονα λόγον ἔξει ἢ πέμπτον πρὸς ἕκτον· ὅπερ ἔδει δεῖξαι.

by D , so many times let N be (divisible) by B .

And since as A is to B , so C (is) to D , and the equal multiples M and G have been taken of A and C (respectively), and the other random equal multiples N and K of B and D (respectively), thus if M exceeds N then G exceeds K , and if (M is) equal (to N then G is also) equal (to K), and if (M is) less (than N then G is also) less (than K) [Def. 5.5]. And G exceeds K . Thus, M also exceeds N . And H does not exceeds L . And M and H are equal multiples of A and E (respectively), and N and L other random equal multiples of B and F (respectively). Thus, A has a greater ratio to B than E (has) to F [Def. 5.7].

Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and a third (magnitude) has a greater ratio to a fourth than a fifth (has) to a sixth, then the first (magnitude) will also have a greater ratio to the second than the fifth (has) to the sixth. (Which is) the very thing it was required to show.

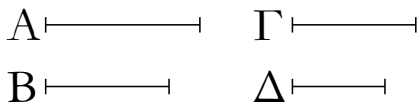
† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ and $\gamma : \delta > \epsilon : \zeta$ then $\alpha : \beta > \epsilon : \zeta$.

ιδ'.

Proposition 14[†]

Ἐάν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχη λόγον καὶ τρίτον πρὸς τέταρτον, τὸ δὲ πρῶτον τοῦ τρίτου μείζον ἢ, καὶ τὸ δεύτερον τοῦ τετάρτου μείζον ἔσται, καὶ ἴσον, ἴσον, καὶ ἔλαττον, ἔλαττον.

If a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the first (magnitude) is greater than the third, then the second will also be greater than the fourth. And if (the first magnitude is) equal (to the third then the second will also be) equal (to the fourth). And if (the first magnitude is) less (than the third then the second will also be) less (than the fourth).

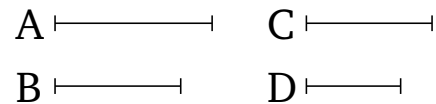


Πρῶτον γὰρ τὸ Α πρὸς δεύτερον τὸ Β αὐτὸν ἔχέτω λόγον καὶ τρίτον τὸ Γ πρὸς τέταρτον τὸ Δ, μείζον δὲ ἔστω τὸ Α τοῦ Γ· λέγω, ὅτι καὶ τὸ Β τοῦ Δ μείζον ἔστιν.

Ἐπεὶ γὰρ τὸ Α τοῦ Γ μείζον ἔστιν, ἄλλο δέ, ὃ ἔτυχεν, [μέγεθος] τὸ Β, τὸ Α ἄρα πρὸς τὸ Β μείζονα λόγον ἔχει ἢ περ τὸ Γ πρὸς τὸ Β. ὡς δὲ τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· καὶ τὸ Γ ἄρα πρὸς τὸ Δ μείζονα λόγον ἔχει ἢ περ τὸ Γ πρὸς τὸ Β. πρὸς ὃ δὲ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκείνο ἔλασσόν ἔστιν· ἔλασσον ἄρα τὸ Δ τοῦ Β· ὥστε μείζον ἔστι τὸ Β τοῦ Δ.

Ὁμοίως δὴ δεῖξομεν, ὅτι καὶ ἴσον ἢ τὸ Α τῷ Γ, ἴσον ἔσται καὶ τὸ Β τῷ Δ, καὶ ἔλασσον ἢ τὸ Α τοῦ Γ, ἔλασσον ἔσται καὶ τὸ Β τοῦ Δ.

Ἐάν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχη λόγον καὶ τρίτον πρὸς τέταρτον, τὸ δὲ πρῶτον τοῦ τρίτου μείζον ἢ, καὶ τὸ δεύτερον τοῦ τετάρτου μείζον ἔσται, καὶ ἴσον, ἴσον, καὶ ἔλαττον, ἔλαττον· ὅπερ ἔδει δεῖξαι.



For let a first (magnitude) A have the same ratio to a second B that a third C (has) to a fourth D . And let A be greater than C . I say that B is also greater than D .

For since A is greater than C , and B (is) another random [magnitude], A thus has a greater ratio to B than C (has) to B [Prop. 5.8]. And as A (is) to B , so C (is) to D . Thus, C also has a greater ratio to D than C (has) to B . And that (magnitude) to which the same (magnitude) has a greater ratio is the lesser [Prop. 5.10]. Thus, D (is) less than B . Hence, B is greater than D .

So, similarly, we can show that even if A is equal to C then B will also be equal to D , and even if A is less than C then B will also be less than D .

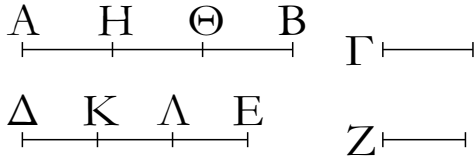
Thus, if a first (magnitude) has the same ratio to a second that a third (has) to a fourth, and the first (magnitude) is greater than the third, then the second will also be greater than the fourth. And if (the first magnitude is)

equal (to the third then the second will also be) equal (to the fourth). And if (the first magnitude is) less (than the third then the second will also be) less (than the fourth). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha \geq \gamma$ as $\beta \geq \delta$.

ιε'.

Τὰ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον ληφθέντα κατάλληλα.



Ἐστω γὰρ ἰσάκεις πολλαπλάσιον τὸ AB τοῦ Γ καὶ τὸ ΔΕ τοῦ Ζ· λέγω, ὅτι ἐστὶν ὡς τὸ Γ πρὸς τὸ Ζ, οὕτως τὸ AB πρὸς τὸ ΔΕ.

Ἐπεὶ γὰρ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ AB τοῦ Γ καὶ τὸ ΔΕ τοῦ Ζ, ὅσα ἄρα ἐστὶν ἐν τῷ AB μεγέθη ἴσα τῷ Γ, τοσαῦτα καὶ ἐν τῷ ΔΕ ἴσα τῷ Ζ. διηρήσθω τὸ μὲν AB εἰς τὰ τῷ Γ ἴσα τὰ ΑΗ, ΗΘ, ΘΒ, τὸ δὲ ΔΕ εἰς τὰ τῷ Ζ ἴσα τὰ ΔΚ, ΚΛ, ΛΕ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΑΗ, ΗΘ, ΘΒ τῷ πλῆθει τῶν ΔΚ, ΚΛ, ΛΕ. καὶ ἐπεὶ ἴσα ἐστὶ τὰ ΑΗ, ΗΘ, ΘΒ ἀλλήλοις, ἔστι δὲ καὶ τὰ ΔΚ, ΚΛ, ΛΕ ἴσα ἀλλήλοις, ἔστιν ἄρα ὡς τὸ ΑΗ πρὸς τὸ ΔΚ, οὕτως τὸ ΗΘ πρὸς τὸ ΚΛ, καὶ τὸ ΘΒ πρὸς τὸ ΛΕ. ἔσται ἄρα καὶ ὡς ἐν τῶν ἡγουμένων πρὸς ἐν τῶν ἐπομένων, οὕτως ἅπαντα τὰ ἡγουμένα πρὸς ἅπαντα τὰ ἐπόμενα· ἔστιν ἄρα ὡς τὸ ΑΗ πρὸς τὸ ΔΚ, οὕτως τὸ AB πρὸς τὸ ΔΕ. ἴσον δὲ τὸ μὲν ΑΗ τῷ Γ, τὸ δὲ ΔΚ τῷ Ζ· ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Ζ οὕτως τὸ AB πρὸς τὸ ΔΕ.

Τὰ ἄρα μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον ληφθέντα κατάλληλα· ὅπερ ἔδει δεῖξαι.

† In modern notation, this proposition reads that $\alpha : \beta :: m\alpha : m\beta$.

ις'.

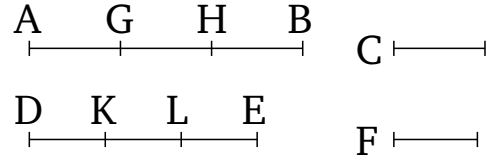
Ἐὰν τέσσαρα μεγέθη ἀνάλογον ᾗ, καὶ ἐναλλάξ ἀνάλογον ἔσται.

Ἐστω τέσσαρα μεγέθη ἀνάλογον τὰ Α, Β, Γ, Δ, ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· λέγω, ὅτι καὶ ἐναλλάξ [ἀνάλογον] ἔσται, ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Β πρὸς τὸ Δ.

Εἰλήφθω γὰρ τῶν μὲν Α, Β ἰσάκεις πολλαπλάσια τὰ Ε, Ζ, τῶν δὲ Γ, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ Η, Θ.

Proposition 15†

Parts have the same ratio as similar multiples, taken in corresponding order.



For let AB and DE be equal multiples of C and F (respectively). I say that as C is to F , so AB (is) to DE .

For since AB and DE are equal multiples of C and F (respectively), thus as many magnitudes as there are in AB equal to C , so many (are there) also in DE equal to F . Let AB have been divided into (magnitudes) AG , GH , HB , equal to C , and DE into (magnitudes) DK , KL , LE , equal to F . So, the number of (magnitudes) AG , GH , HB will equal the number of (magnitudes) DK , KL , LE . And since AG , GH , HB are equal to one another, and DK , KL , LE are also equal to one another, thus as AG is to DK , so GH (is) to KL , and HB to LE [Prop. 5.7]. And, thus (for proportional magnitudes), as one of the leading (magnitudes) will be to one of the following, so all of the leading (magnitudes will be) to all of the following [Prop. 5.12]. Thus, as AG is to DK , so AB (is) to DE . And AG is equal to C , and DK to F . Thus, as C is to F , so AB (is) to DE .

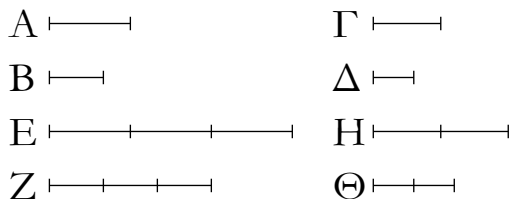
Thus, parts have the same ratio as similar multiples, taken in corresponding order. (Which is) the very thing it was required to show.

Proposition 16†

If four magnitudes are proportional then they will also be proportional alternately.

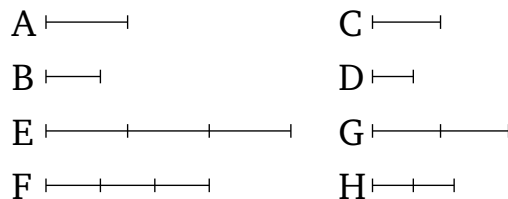
Let A , B , C and D be four proportional magnitudes, (such that) as A (is) to B , so C (is) to D . I say that they will also be [proportional] alternately, (so that) as A (is) to C , so B (is) to D .

For let the equal multiples E and F have been taken of A and B (respectively), and the other random equal multiples G and H of C and D (respectively).



Καὶ ἐπεὶ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ Ε τοῦ Α καὶ τὸ Ζ τοῦ Β, τὰ δὲ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Β, οὕτως τὸ Ε πρὸς τὸ Ζ. ὡς δὲ τὸ Α πρὸς τὸ Β, οὕτως τὸ Γ πρὸς τὸ Δ· καὶ ὡς ἄρα τὸ Γ πρὸς τὸ Δ, οὕτως τὸ Ε πρὸς τὸ Ζ. πάλιν, ἐπεὶ τὰ Η, Θ τῶν Γ, Δ ἰσάκεις ἐστὶ πολλαπλάσια, ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Δ, οὕτως τὸ Η πρὸς τὸ Θ. ὡς δὲ τὸ Γ πρὸς τὸ Δ, [οὕτως] τὸ Ε πρὸς τὸ Ζ· καὶ ὡς ἄρα τὸ Ε πρὸς τὸ Ζ, οὕτως τὸ Η πρὸς τὸ Θ. ἐὰν δὲ τέσσαρα μεγέθη ἀνάλογον ᾗ, τὸ δὲ πρῶτον τοῦ τρίτου μεῖζον ᾗ, καὶ τὸ δεύτερον τοῦ τετάρτου μεῖζον ἔσται, καὶ ἴσον, ἴσον, καὶ ἔλαττον, ἔλαττον. εἰ ἄρα ὑπερέχει τὸ Ε τοῦ Η, ὑπερέχει καὶ τὸ Ζ τοῦ Θ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καὶ ἐστὶ τὰ μὲν Ε, Ζ τῶν Α, Β ἰσάκεις πολλαπλάσια, τὰ δὲ Η, Θ τῶν Γ, Δ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια· ἔστιν ἄρα ὡς τὸ Α πρὸς τὸ Γ, οὕτως τὸ Β πρὸς τὸ Δ.

Ἐὰν ἄρα τέσσαρα μεγέθη ἀνάλογον ᾗ, καὶ ἐναλλάξ ἀνάλογον ἔσται· ὅπερ ἔδει δεῖξαι.



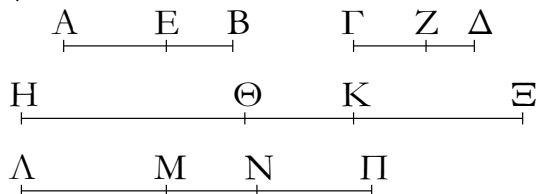
And since E and F are equal multiples of A and B (respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as A is to B , so E (is) to F . But as A (is) to B , so C (is) to D . And, thus, as C (is) to D , so E (is) to F [Prop. 5.11]. Again, since G and H are equal multiples of C and D (respectively), thus as C is to D , so G (is) to H [Prop. 5.15]. But as C (is) to D , [so] E (is) to F . And, thus, as E (is) to F , so G (is) to H [Prop. 5.11]. And if four magnitudes are proportional, and the first is greater than the third then the second will also be greater than the fourth, and if (the first is) equal (to the third then the second will also be) equal (to the fourth), and if (the first is) less (than the third then the second will also be) less (than the fourth) [Prop. 5.14]. Thus, if E exceeds G then F also exceeds H , and if (E is) equal (to G then F is also) equal (to H), and if (E is) less (than G then F is also) less (than H). And E and F are equal multiples of A and B (respectively), and G and H other random equal multiples of C and D (respectively). Thus, as A is to C , so B (is) to D [Def. 5.5].

Thus, if four magnitudes are proportional then they will also be proportional alternately. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha : \gamma :: \beta : \delta$.

ιζ'.

Ἐὰν συγκείμενα μεγέθη ἀνάλογον ᾗ, καὶ διαιρεθέντα ἀνάλογον ἔσται.



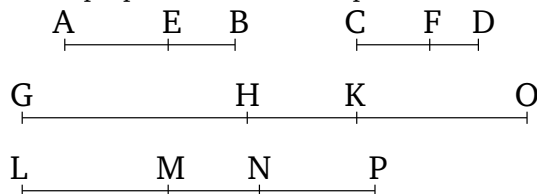
Ἐστω συγκείμενα μεγέθη ἀνάλογον τὰ ΑΒ, ΒΕ, ΓΔ, ΔΖ, ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΔΖ· λέγω, ὅτι καὶ διαιρεθέντα ἀνάλογον ἔσται, ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΖ πρὸς τὸ ΔΖ.

Εἰλήφθω γὰρ τῶν μὲν ΑΕ, ΕΒ, ΓΖ, ΖΔ ἰσάκεις πολλαπλάσια τὰ ΗΘ, ΘΚ, ΑΜ, ΜΝ, τῶν δὲ ΕΒ, ΖΔ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ ΚΞ, ΝΠ.

Καὶ ἐπεὶ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ ΗΘ τοῦ ΑΕ καὶ τὸ ΘΚ τοῦ ΕΒ, ἰσάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ ΗΘ τοῦ

Proposition 17†

If composed magnitudes are proportional then they will also be proportional (when) separated.



Let AB , BE , CD , and DF be composed magnitudes (which are) proportional, (so that) as AB (is) to BE , so CD (is) to DF . I say that they will also be proportional (when) separated, (so that) as AE (is) to EB , so CF (is) to DF .

For let the equal multiples GH , HK , LM , and MN have been taken of AE , EB , CF , and FD (respectively), and the other random equal multiples KO and NP of EB and FD (respectively).

ΑΕ καὶ τὸ ΗΚ τοῦ ΑΒ. ἰσάκεις δὲ ἐστὶ πολλαπλάσιον τὸ ΗΘ τοῦ ΑΕ καὶ τὸ ΑΜ τοῦ ΓΖ· ἰσάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ ΗΚ τοῦ ΑΒ καὶ τὸ ΑΜ τοῦ ΓΖ. πάλιν, ἐπεὶ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ ΑΜ τοῦ ΓΖ καὶ τὸ ΜΝ τοῦ ΖΔ, ἰσάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ ΑΜ τοῦ ΓΖ καὶ τὸ ΑΝ τοῦ ΓΔ. ἰσάκεις δὲ ἦν πολλαπλάσιον τὸ ΑΜ τοῦ ΓΖ καὶ τὸ ΗΚ τοῦ ΑΒ· ἰσάκεις ἄρα ἐστὶ πολλαπλάσιον τὸ ΗΚ τοῦ ΑΒ καὶ τὸ ΑΝ τοῦ ΓΔ. τὰ ΗΚ, ΑΝ ἄρα τῶν ΑΒ, ΓΔ ἰσάκεις ἐστὶ πολλαπλάσια. πάλιν, ἐπεὶ ἰσάκεις ἐστὶ πολλαπλάσιον τὸ ΘΚ τοῦ ΕΒ καὶ τὸ ΜΝ τοῦ ΖΔ, ἐστὶ δὲ καὶ τὸ ΚΞ τοῦ ΕΒ ἰσάκεις πολλαπλάσιον καὶ τὸ ΝΠ τοῦ ΖΔ, καὶ συντεθέν τὸ ΘΞ τοῦ ΕΒ ἰσάκεις ἐστὶ πολλαπλάσιον καὶ τὸ ΜΠ τοῦ ΖΔ. καὶ ἐπεὶ ἐστὶν ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΔΖ, καὶ εἴληπται τῶν μὲν ΑΒ, ΓΔ ἰσάκεις πολλαπλάσια τὰ ΗΚ, ΑΝ, τῶν δὲ ΕΒ, ΖΔ ἰσάκεις πολλαπλάσια τὰ ΘΞ, ΜΠ, εἰ ἄρα ὑπερέχει τὸ ΗΚ τοῦ ΘΞ, ὑπερέχει καὶ τὸ ΑΝ τοῦ ΜΠ, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. ὑπερεχέτω δὴ τὸ ΗΚ τοῦ ΘΞ, καὶ κοινοῦ ἀφαιρεθέντος τοῦ ΘΚ ὑπερέχει ἄρα καὶ τὸ ΗΘ τοῦ ΚΞ. ἄλλα εἰ ὑπερεῖχε τὸ ΗΚ τοῦ ΘΞ ὑπερεῖχε καὶ τὸ ΑΝ τοῦ ΜΠ· ὑπερέχει ἄρα καὶ τὸ ΑΝ τοῦ ΜΠ, καὶ κοινοῦ ἀφαιρεθέντος τοῦ ΜΝ ὑπερέχει καὶ τὸ ΑΜ τοῦ ΝΠ· ὥστε εἰ ὑπερέχει τὸ ΗΘ τοῦ ΚΞ, ὑπερέχει καὶ τὸ ΑΜ τοῦ ΝΠ. ὁμοίως δὴ δεῖξομεν, ὅτι καὶ ἴσον ἢ τὸ ΗΘ τῷ ΚΞ, ἴσον ἔσται καὶ τὸ ΑΜ τῷ ΝΠ, καὶ ἔλαττον, ἔλαττον. καὶ ἐστὶ τὰ μὲν ΗΘ, ΑΜ τῶν ΑΕ, ΓΖ ἰσάκεις πολλαπλάσια, τὰ δὲ ΚΞ, ΝΠ τῶν ΕΒ, ΖΔ ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια· ἐστὶν ἄρα ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ.

Ἐὰν ἄρα συγκείμενα μεγέθη ἀνάλογον ᾖ, καὶ διαιρεθέντα ἀνάλογον ἔσται· ὅπερ εἶδει δεῖξαι.

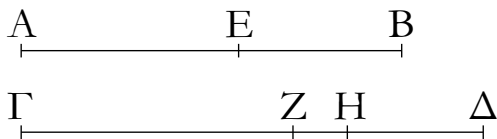
And since GH and HK are equal multiples of AE and EB (respectively), GH and GK are thus equal multiples of AE and AB (respectively) [Prop. 5.1]. But GH and LM are equal multiples of AE and CF (respectively). Thus, GK and LM are equal multiples of AB and CF (respectively). Again, since LM and MN are equal multiples of CF and FD (respectively), LM and LN are thus equal multiples of CF and CD (respectively) [Prop. 5.1]. And LM and GK were equal multiples of CF and AB (respectively). Thus, GK and LN are equal multiples of AB and CD (respectively). Thus, GK , LN are equal multiples of AB , CD . Again, since HK and MN are equal multiples of EB and FD (respectively), and KO and NP are also equal multiples of EB and FD (respectively), then, added together, HO and MP are also equal multiples of EB and FD (respectively) [Prop. 5.2]. And since as AB (is) to BE , so CD (is) to DF , and the equal multiples GK , LN have been taken of AB , CD , and the equal multiples HO , MP of EB , FD , thus if GK exceeds HO then LN also exceeds MP , and if (GK is) equal (to HO then LN is also) equal (to MP), and if (GK is) less (than HO then LN is also) less (than MP) [Def. 5.5]. So let GK exceed HO , and thus, HK being taken away from both, GH exceeds KO . But (we saw that) if GK was exceeding HO then LN was also exceeding MP . Thus, LN also exceeds MP , and, MN being taken away from both, LM also exceeds NP . Hence, if GH exceeds KO then LM also exceeds NP . So, similarly, we can show that even if GH is equal to KO then LM will also be equal to NP , and even if (GH is) less (than KO then LM will also be) less (than NP). And GH , LM are equal multiples of AE , CF , and KO , NP other random equal multiples of EB , FD . Thus, as AE is to EB , so CF (is) to FD [Def. 5.5].

Thus, if composed magnitudes are proportional then they will also be proportional (when) separated. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha + \beta : \beta :: \gamma + \delta : \delta$ then $\alpha : \beta :: \gamma : \delta$.

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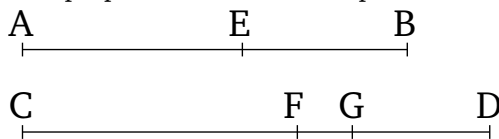
Ἐὰν διηρημένα μεγέθη ἀνάλογον ᾖ, καὶ συντεθέντα ἀνάλογον ἔσται.



Ἐστω διηρημένα μεγέθη ἀνάλογον τὰ ΑΕ, ΕΒ, ΓΖ, ΖΔ, ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ· λέγω, ὅτι καὶ συντεθέντα ἀνάλογον ἔσται, ὡς τὸ ΑΒ πρὸς τὸ ΒΕ,

Proposition 18†

If separated magnitudes are proportional then they will also be proportional (when) composed.



Let AE , EB , CF , and FD be separated magnitudes (which are) proportional, (so that) as AE (is) to EB , so CF (is) to FD . I say that they will also be proportional

οὕτως τὸ ΓΔ πρὸς τὸ ΖΔ.

Εἰ γὰρ μὴ ἔστιν ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΔΖ, ἔσται ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ ἤτοι πρὸς ἑλασσόν τι τοῦ ΔΖ ἢ πρὸς μείζον.

Ἐστω πρότερον πρὸς ἑλασσόν τὸ ΔΗ. καὶ ἐπεὶ ἔστιν ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς τὸ ΔΗ, συγκείμενα μεγέθη ἀνάλογόν ἐστιν· ὥστε καὶ διαιρεθέντα ἀνάλογον ἔσται. ἔστιν ἄρα ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΗ πρὸς τὸ ΗΔ. ὑπόκειται δὲ καὶ ὡς τὸ ΑΕ πρὸς τὸ ΕΒ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ. καὶ ὡς ἄρα τὸ ΓΗ πρὸς τὸ ΗΔ, οὕτως τὸ ΓΖ πρὸς τὸ ΖΔ. μείζον δὲ τὸ πρῶτον τὸ ΓΗ τοῦ τρίτου τοῦ ΓΖ· μείζον ἄρα καὶ τὸ δεύτερον τὸ ΗΔ τοῦ τετάρτου τοῦ ΖΔ. ἀλλὰ καὶ ἑλαττον· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα ἔστιν ὡς τὸ ΑΒ πρὸς τὸ ΒΕ, οὕτως τὸ ΓΔ πρὸς ἑλασσόν τοῦ ΖΔ. ὁμοίως δὲ δείξομεν, ὅτι οὐδὲ πρὸς μείζον· πρὸς αὐτὸ ἄρα.

Ἐάν ἄρα διηρημένα μεγέθη ἀνάλογον ᾖ, καὶ συντεθέντα ἀνάλογον ἔσται· ὅπερ ἔδει δείξαι.

(when) composed, (so that) as AB (is) to BE , so CD (is) to FD .

For if (it is) not (the case that) as AB is to BE , so CD (is) to FD , then it will surely be (the case that) as AB (is) to BE , so CD is either to some (magnitude) less than FD , or (some magnitude) greater (than FD).[†]

Let it, first of all, be to (some magnitude) less (than FD), (namely) DG . And since composed magnitudes are proportional, (so that) as AB is to BE , so CD (is) to DG , they will thus also be proportional (when) separated [Prop. 5.17]. Thus, as AE is to EB , so CG (is) to GD . But it was also assumed that as AE (is) to EB , so CF (is) to FD . Thus, (it is) also (the case that) as CG (is) to GD , so CF (is) to FD [Prop. 5.11]. And the first (magnitude) CG (is) greater than the third CF . Thus, the second (magnitude) GD (is) also greater than the fourth FD [Prop. 5.14]. But (it is) also less. The very thing is impossible. Thus, (it is) not (the case that) as AB is to BE , so CD (is) to less than FD . Similarly, we can show that neither (is it the case) to greater (than FD). Thus, (it is the case) to the same (as FD).

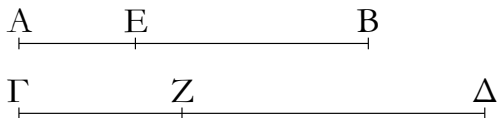
Thus, if separated magnitudes are proportional then they will also be proportional (when) composed. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha + \beta : \beta :: \gamma + \delta : \delta$.

[‡] Here, Euclid assumes, without proof, that a fourth magnitude proportional to three given magnitudes can always be found.

ιθ'.

Ἐάν ᾗ ὡς ὅλον πρὸς ὅλον, οὕτως ἀφαιρεθὲν πρὸς ἀφαιρεθὲν, καὶ τὸ λοιπὸν πρὸς τὸ λοιπὸν ἔσται ὡς ὅλον πρὸς ὅλον.



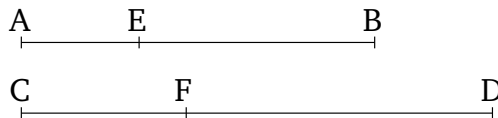
Ἐστω γὰρ ὡς ὅλον πρὸς ὅλον τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως ἀφαιρεθὲν τὸ ΑΕ πρὸς ἀφαιρεθὲν τὸ ΓΖ· λέγω, ὅτι καὶ λοιπὸν τὸ ΕΒ πρὸς λοιπὸν τὸ ΖΔ ἔσται ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ.

Ἐπεὶ γὰρ ἔστιν ὡς τὸ ΑΒ πρὸς τὸ ΓΔ, οὕτως τὸ ΑΕ πρὸς τὸ ΓΖ, καὶ ἐναλλάξ ὡς τὸ ΒΑ πρὸς τὸ ΑΕ, οὕτως τὸ ΔΓ πρὸς τὸ ΓΖ. καὶ ἐπεὶ συγκείμενα μεγέθη ἀνάλογόν ἐστιν, καὶ διαιρεθέντα ἀνάλογον ἔσται, ὡς τὸ ΒΕ πρὸς τὸ ΕΑ, οὕτως τὸ ΔΖ πρὸς τὸ ΖΓ· καὶ ἐναλλάξ, ὡς τὸ ΒΕ πρὸς τὸ ΔΖ, οὕτως τὸ ΕΑ πρὸς τὸ ΖΓ. ὡς δὲ τὸ ΑΕ πρὸς τὸ ΓΖ, οὕτως ὑπόκειται ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ. καὶ λοιπὸν ἄρα τὸ ΕΒ πρὸς λοιπὸν τὸ ΖΔ ἔσται ὡς ὅλον τὸ ΑΒ πρὸς ὅλον τὸ ΓΔ.

Ἐάν ἄρα ᾗ ὡς ὅλον πρὸς ὅλον, οὕτως ἀφαιρεθὲν πρὸς

Proposition 19[†]

If as the whole is to the whole so the (part) taken away is to the (part) taken away then the remainder to the remainder will also be as the whole (is) to the whole.



For let the whole AB be to the whole CD as the (part) taken away AE (is) to the (part) taken away CF . I say that the remainder EB to the remainder FD will also be as the whole AB (is) to the whole CD .

For since as AB is to CD , so AE (is) to CF , (it is) also (the case), alternately, (that) as BA (is) to AE , so DC (is) to CF [Prop. 5.16]. And since composed magnitudes are proportional then they will also be proportional (when) separated, (so that) as BE (is) to EA , so DF (is) to CF [Prop. 5.17]. Also, alternately, as BE (is) to DF , so EA (is) to FC [Prop. 5.16]. And it was assumed that as AE (is) to CF , so the whole AB (is) to the whole CD . And, thus, as the remainder EB (is) to the remainder FD , so the whole AB will be to the whole CD .

ἀφαιρεθέν, καὶ τὸ λοιπὸν πρὸς τὸ λοιπὸν ἔσται ὡς ὅλον πρὸς ὅλον [ὅπερ ἔδει δεῖξαι].

[Καὶ ἐπεὶ ἐδείχθη ὡς τὸ AB πρὸς τὸ ΓΔ, οὕτως τὸ EB πρὸς τὸ ZΔ, καὶ ἐναλλάξ ὡς τὸ AB πρὸς τὸ BE οὕτως τὸ ΓΔ πρὸς τὸ ZΔ, συγκείμενα ἄρα μεγέθη ἀνάλογόν ἐστιν· ἐδείχθη δὲ ὡς τὸ BA πρὸς τὸ AE, οὕτως τὸ ΔΓ πρὸς τὸ ΓΖ· καὶ ἐστὶν ἀναστρέψαντι].

Πόρισμα.

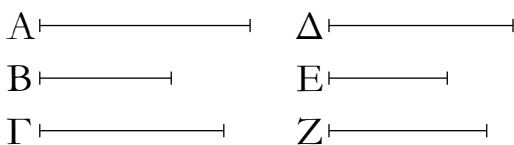
Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν συγκείμενα μεγέθη ἀνάλογον ᾖ, καὶ ἀναστρέψαντι ἀνάλογον ἔσται· ὅπερ ἔδει δεῖξαι.

† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha : \beta :: \alpha - \gamma : \beta - \delta$.

‡ In modern notation, this corollary reads that if $\alpha : \beta :: \gamma : \delta$ then $\alpha : \alpha - \beta :: \gamma : \gamma - \delta$.

κ'.

Ἐὰν ᾖ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδου λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, δι' ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μεῖζον ᾖ, καὶ τὸ τέταρτον τοῦ ἕκτου μεῖζον ἔσται, καὶ ἴσον, ἴσον, καὶ ἑλάττω, ἑλάττω.



Ἐστω τρία μεγέθη τὰ A, B, Γ, καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Δ, E, Ζ, σύνδου λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ὡς μὲν τὸ A πρὸς τὸ B, οὕτως τὸ Δ πρὸς τὸ E, ὡς δὲ τὸ B πρὸς τὸ Γ, οὕτως τὸ E πρὸς τὸ Ζ, δι' ἴσου δὲ μεῖζον ἔστω τὸ A τοῦ Γ· λέγω, ὅτι καὶ τὸ Δ τοῦ Ζ μεῖζον ἔσται, καὶ ἴσον, ἴσον, καὶ ἑλάττω, ἑλάττω.

Ἐπεὶ γὰρ μεῖζόν ἐστι τὸ A τοῦ Γ, ἄλλο δέ τι τὸ B, τὸ δὲ μεῖζον πρὸς τὸ αὐτὸ μεῖζονα λόγον ἔχει ἥπερ τὸ ἑλάττω, τὸ A ἄρα πρὸς τὸ B μεῖζονα λόγον ἔχει ἥπερ τὸ Γ πρὸς τὸ B. ἀλλ' ὡς μὲν τὸ A πρὸς τὸ B [οὕτως] τὸ Δ πρὸς τὸ E, ὡς δὲ τὸ Γ πρὸς τὸ B, ἀνάπαλιν οὕτως τὸ Ζ πρὸς τὸ E· καὶ τὸ Δ ἄρα πρὸς τὸ E μεῖζονα λόγον ἔχει ἥπερ τὸ Ζ πρὸς τὸ E. τῶν δὲ πρὸς τὸ αὐτὸ λόγον ἐχόντων τὸ μεῖζονα λόγον ἔχον μεῖζόν ἐστιν. μεῖζον ἄρα τὸ Δ τοῦ Ζ. ὁμοίως δὲ δείξομεν, ὅτι καὶ ἴσον ᾖ τὸ A τῷ Γ, ἴσον ἔσται καὶ τὸ Δ τῷ Ζ, καὶ

Thus, if as the whole is to the whole so the (part) taken away is to the (part) taken away then the remainder to the remainder will also be as the whole (is) to the whole. [(Which is) the very thing it was required to show.]

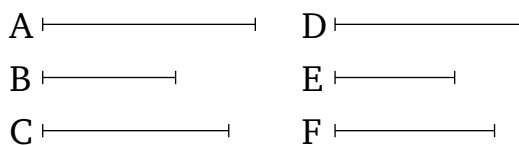
[And since it was shown (that) as AB (is) to CD , so EB (is) to FD , (it is) also (the case), alternately, (that) as AB (is) to BE , so CD (is) to FD . Thus, composed magnitudes are proportional. And it was shown (that) as BA (is) to AE , so DC (is) to CF . And (the latter) is converted (from the former).]

Corollary[†]

So (it is) clear, from this, that if composed magnitudes are proportional then they will also be proportional (when) converted. (Which is) the very thing it was required to show.

Proposition 20[†]

If there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if), via equality, the first is greater than the third then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth).



Let A , B , and C be three magnitudes, and D , E , F other (magnitudes) of equal number to them, (being) in the same ratio taken two by two, (so that) as A (is) to B , so D (is) to E , and as B (is) to C , so E (is) to F . And let A be greater than C , via equality. I say that D will also be greater than F . And if (A is) equal (to C then D will also be) equal (to F). And if (A is) less (than C then D will also be) less (than F).

For since A is greater than C , and B some other (magnitude), and the greater (magnitude) has a greater ratio than the lesser to the same (magnitude) [Prop. 5.8], A thus has a greater ratio to B than C (has) to B . But as A (is) to B , [so] D (is) to E . And, inversely, as C (is) to B , so F (is) to E [Prop. 5.7 corr.]. Thus, D also has a greater ratio to E than F (has) to E [Prop. 5.13]. And for (mag-

ἐλαττον, ἐλαττον.

Ἐάν ἄρα ᾗ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, δι' ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μείζον ᾗ, καὶ τὸ τέταρτον τοῦ ἕκτου μείζον ἔσται, καὶ ἴσον, ἴσον, καὶ ἐλαττον, ἐλαττον· ὅπερ ἔδει δεῖξαι.

nitudes) having a ratio to the same (magnitude), that having the greater ratio is greater [Prop. 5.10]. Thus, D (is) greater than F . Similarly, we can show that even if A is equal to C then D will also be equal to F , and even if $(A$ is) less (than C then D will also be) less (than F).

Thus, if there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if), via equality, the first is greater than the third, then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And (if the first is) less (than the third then the fourth will also be) less (than the sixth). (Which is) the very thing it was required to show.

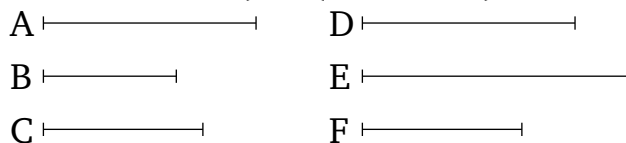
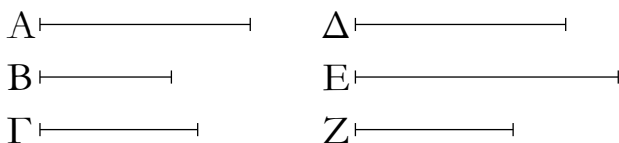
† In modern notation, this proposition reads that if $\alpha : \beta :: \delta : \epsilon$ and $\beta : \gamma :: \epsilon : \zeta$ then $\alpha \gtrless \gamma$ as $\delta \gtrless \zeta$.

κα'.

Proposition 21[†]

Ἐάν ᾗ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, ᾗ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, δι' ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μείζον ᾗ, καὶ τὸ τέταρτον τοῦ ἕκτου μείζον ἔσται, καὶ ἴσον, ἴσον, καὶ ἐλαττον, ἐλαττον.

If there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if) their proportion (is) perturbed, and (if), via equality, the first is greater than the third then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth).



Ἐστω τρία μεγέθη τὰ A, B, Γ καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Δ, E, Z , σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, ἔστω δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, ὥς μὲν τὸ A πρὸς τὸ B , οὕτως τὸ E πρὸς τὸ Z , ὥς δὲ τὸ B πρὸς τὸ Γ , οὕτως τὸ Δ πρὸς τὸ E , δι' ἴσου δὲ τὸ A τοῦ Γ μείζον ἔστω· λέγω, ὅτι καὶ τὸ Δ τοῦ Z μείζον ἔσται, καὶ ἴσον, ἴσον, καὶ ἐλαττον, ἐλαττον.

Let A, B , and C be three magnitudes, and D, E, F other (magnitudes) of equal number to them, (being) in the same ratio taken two by two. And let their proportion be perturbed, (so that) as A (is) to B , so E (is) to F , and as B (is) to C , so D (is) to E . And let A be greater than C , via equality. I say that D will also be greater than F . And if $(A$ is) equal (to C then D will also be) equal (to F). And if $(A$ is) less (than C then D will also be) less (than F).

Ἐπεὶ γὰρ μείζον ἐστὶ τὸ A τοῦ Γ , ἄλλο δὲ τι τὸ B , τὸ A ἄρα πρὸς τὸ B μείζονα λόγον ἔχει ἢ περ τὸ Γ πρὸς τὸ B . ἀλλ' ὥς μὲν τὸ A πρὸς τὸ B , οὕτως τὸ E πρὸς τὸ Z , ὥς δὲ τὸ Γ πρὸς τὸ B , ἀνάπαλιν οὕτως τὸ E πρὸς τὸ Δ . καὶ τὸ E ἄρα πρὸς τὸ Z μείζονα λόγον ἔχει ἢ περ τὸ E πρὸς τὸ Δ . πρὸς δὲ τὸ αὐτὸ μείζονα λόγον ἔχει, ἐκεῖνο ἔλασσόν ἐστιν· ἔλασσον ἄρα ἐστὶ τὸ Z τοῦ Δ · μείζον ἄρα ἐστὶ τὸ Δ τοῦ Z . ὁμοίως δὲ δεῖξομεν, ὅτι καὶ ἴσον ᾗ τὸ A τῷ Γ , ἴσον ἔσται καὶ τὸ Δ τῷ Z , καὶ ἐλαττον, ἐλαττον.

For since A is greater than C , and B some other (magnitude), A thus has a greater ratio to B than C (has) to B [Prop. 5.8]. But as A (is) to B , so E (is) to F . And, inversely, as C (is) to B , so E (is) to D [Prop. 5.7 corr.]. Thus, E also has a greater ratio to F than E (has) to D [Prop. 5.13]. And that (magnitude) to which the same (magnitude) has a greater ratio is (the) lesser (magnitude) [Prop. 5.10]. Thus, F is less than D . Thus, D is greater than F . Similarly, we can show that even if A is equal to C then D will also be equal to F , and even if $(A$ is) less (than C then D will also be) less (than F).

Ἐάν ἄρα ᾗ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, ᾗ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, δι' ἴσου δὲ τὸ πρῶτον τοῦ τρίτου μείζον ᾗ, καὶ τὸ τέταρτον τοῦ ἕκτου μείζον ἔσται, καὶ ἴσον,

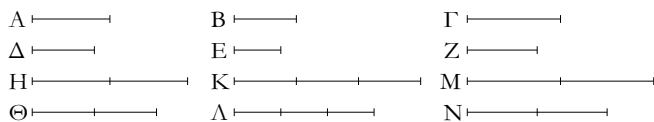
ἴσον, καὶ ἑλάττων, ἑλάττων· ὅπερ ἔδει δεῖξαι.

Thus, if there are three magnitudes, and others of equal number to them, (being) also in the same ratio taken two by two, and (if) their proportion (is) perturbed, and (if), via equality, the first is greater than the third then the fourth will also be greater than the sixth. And if (the first is) equal (to the third then the fourth will also be) equal (to the sixth). And if (the first is) less (than the third then the fourth will also be) less (than the sixth). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \epsilon : \zeta$ and $\beta : \gamma :: \delta : \epsilon$ then $\alpha \gtrless \gamma$ as $\delta \gtrless \zeta$.

κβ'.

Ἐὰν ᾗ ὁποσαοῦν μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται.



Ἐστω ὁποσαοῦν μεγέθη τὰ A, B, Γ καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Δ, E, Z, σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ὡς μὲν τὸ A πρὸς τὸ B, οὕτως τὸ Δ πρὸς τὸ E, ὡς δὲ τὸ B πρὸς τὸ Γ, οὕτως τὸ E πρὸς τὸ Z· λέγω, ὅτι καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται.

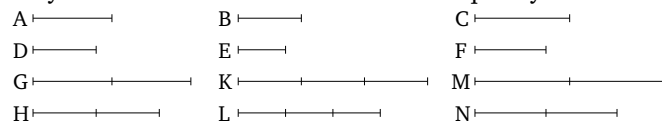
Εἰληφθῶ γὰρ τῶν μὲν A, Δ ἰσάκεις πολλαπλάσια τὰ H, Θ, τῶν δὲ B, E ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ K, Λ, καὶ ἔτι τῶν Γ, Z ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ M, N.

Καὶ ἐπεὶ ἔστιν ὡς τὸ A πρὸς τὸ B, οὕτως τὸ Δ πρὸς τὸ E, καὶ εἰληπται τῶν μὲν A, Δ ἰσάκεις πολλαπλάσια τὰ H, Θ, τῶν δὲ B, E ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ K, Λ, ἔστιν ἄρα ὡς τὸ H πρὸς τὸ K, οὕτως τὸ Θ πρὸς τὸ Λ. διὰ τὰ αὐτὰ δὴ καὶ ὡς τὸ K πρὸς τὸ M, οὕτως τὸ Λ πρὸς τὸ N. ἐπεὶ οὖν τρία μεγέθη ἐστὶ τὰ H, K, M, καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ Θ, Λ, N, σύνδυο λαμβανόμενα καὶ ἐν τῷ αὐτῷ λόγῳ, δι' ἴσου ἄρα, εἰ ὑπερέχει τὸ H τοῦ M, ὑπερέχει καὶ τὸ Θ τοῦ N, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἑλάττων, ἑλάττων. καὶ ἐστὶ τὰ μὲν H, Θ τῶν A, Δ ἰσάκεις πολλαπλάσια, τὰ δὲ M, N τῶν Γ, Z ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια. ἔστιν ἄρα ὡς τὸ A πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Z.

Ἐὰν ἄρα ᾗ ὁποσαοῦν μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος, σύνδυο λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται· ὅπερ ἔδει δεῖξαι.

Proposition 22†

If there are any number of magnitudes whatsoever, and (some) other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, then they will also be in the same ratio via equality.



Let there be any number of magnitudes whatsoever, A, B, C, and (some) other (magnitudes), D, E, F, of equal number to them, (which are) in the same ratio taken two by two, (so that) as A (is) to B, so D (is) to E, and as B (is) to C, so E (is) to F. I say that they will also be in the same ratio via equality. (That is, as A is to C, so D is to F.)

For let the equal multiples G and H have been taken of A and D (respectively), and the other random equal multiples K and L of B and E (respectively), and the yet other random equal multiples M and N of C and F (respectively).

And since as A is to B, so D (is) to E, and the equal multiples G and H have been taken of A and D (respectively), and the other random equal multiples K and L of B and E (respectively), thus as G is to K, so H (is) to L [Prop. 5.4]. And, so, for the same (reasons), as K (is) to M, so L (is) to N. Therefore, since G, K, and M are three magnitudes, and H, L, and N other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by two, thus, via equality, if G exceeds M then H also exceeds N, and if (G is) equal (to M then H is also) equal (to N), and if (G is) less (than M then H is also) less (than N) [Prop. 5.20]. And G and H are equal multiples of A and D (respectively), and M and N other random equal multiples of C and F (respectively). Thus, as A is to C, so D (is) to F [Def. 5.5].

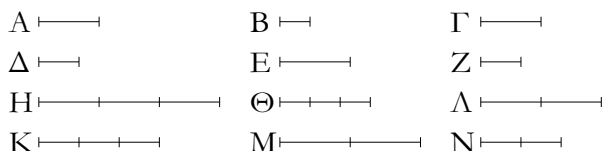
Thus, if there are any number of magnitudes whatsoever, and (some) other (magnitudes) of equal number to them, (which are) also in the same ratio taken two by

two, then they will also be in the same ratio via equality.
(Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \epsilon : \zeta$ and $\beta : \gamma :: \zeta : \eta$ and $\gamma : \delta :: \eta : \theta$ then $\alpha : \delta :: \epsilon : \theta$.

κγ'.

Ἐάν ᾗ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδου λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ᾗ δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται.



Ἐστω τρία μεγέθη τὰ A, B, Γ καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδου λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ τὰ Δ, E, Z, ἔστω δὲ τεταραγμένη αὐτῶν ἡ ἀναλογία, ὡς μὲν τὸ A πρὸς τὸ B, οὕτως τὸ E πρὸς τὸ Z, ὡς δὲ τὸ B πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ E· λέγω, ὅτι ἔστιν ὡς τὸ A πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Z.

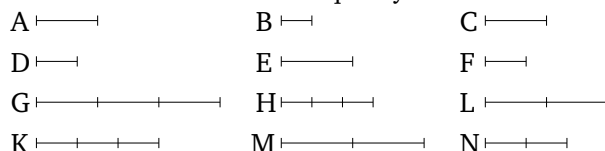
Εἰλήφθω τῶν μὲν A, B, Δ ἰσάκεις πολλαπλάσια τὰ H, Θ, K, τῶν δὲ Γ, E, Z ἄλλα, ἃ ἔτυχεν, ἰσάκεις πολλαπλάσια τὰ Λ, M, N.

Καὶ ἐπεὶ ἰσάκεις ἐστὶ πολλαπλάσια τὰ H, Θ τῶν A, B, τὰ δὲ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ A πρὸς τὸ B, οὕτως τὸ H πρὸς τὸ Θ. διὰ τὰ αὐτὰ δὴ καὶ ὡς τὸ E πρὸς τὸ Z, οὕτως τὸ M πρὸς τὸ N· καὶ ἔστιν ὡς τὸ A πρὸς τὸ B, οὕτως τὸ E πρὸς τὸ Z· καὶ ὡς ἄρα τὸ H πρὸς τὸ Θ, οὕτως τὸ M πρὸς τὸ N. καὶ ἐπεὶ ἔστιν ὡς τὸ B πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ E, καὶ ἐναλλάξ ὡς τὸ B πρὸς τὸ Δ, οὕτως τὸ Γ πρὸς τὸ E. καὶ ἐπεὶ τὰ Θ, K τῶν B, Δ ἰσάκεις ἐστὶ πολλαπλάσια, τὰ δὲ μέρη τοῖς ἰσάκεις πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ B πρὸς τὸ Δ, οὕτως τὸ Θ πρὸς τὸ K. ἀλλ' ὡς τὸ B πρὸς τὸ Δ, οὕτως τὸ Γ πρὸς τὸ E· καὶ ὡς ἄρα τὸ Θ πρὸς τὸ K, οὕτως τὸ Γ πρὸς τὸ E. πάλιν, ἐπεὶ τὰ Λ, M τῶν Γ, E ἰσάκεις ἐστὶ πολλαπλάσια, ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ E, οὕτως τὸ Λ πρὸς τὸ M. ἀλλ' ὡς τὸ Γ πρὸς τὸ E, οὕτως τὸ Θ πρὸς τὸ K· καὶ ὡς ἄρα τὸ Θ πρὸς τὸ K, οὕτως τὸ Λ πρὸς τὸ M, καὶ ἐναλλάξ ὡς τὸ Θ πρὸς τὸ Λ, τὸ K πρὸς τὸ M. ἐδείχθη δὲ καὶ ὡς τὸ H πρὸς τὸ Θ, οὕτως τὸ M πρὸς τὸ N. ἐπεὶ οὖν τρία μεγέθη ἐστὶ τὰ H, Θ, Λ, καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος τὰ K, M, N σύνδου λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, καὶ ἔστιν αὐτῶν τεταραγμένη ἡ ἀναλογία, δι' ἴσου ἄρα, εἰ ὑπερέχει τὸ H τοῦ Λ, ὑπερέχει καὶ τὸ K τοῦ N, καὶ εἰ ἴσον, ἴσον, καὶ εἰ ἔλαττον, ἔλαττον. καὶ ἐστὶ τὰ μὲν H, K τῶν A, Δ ἰσάκεις πολλαπλάσια, τὰ δὲ Λ, N τῶν Γ, Z. ἔστιν ἄρα ὡς τὸ A πρὸς τὸ Γ, οὕτως τὸ Δ πρὸς τὸ Z.

Ἐάν ἄρα ᾗ τρία μεγέθη καὶ ἄλλα αὐτοῖς ἴσα τὸ πλῆθος σύνδου λαμβανόμενα ἐν τῷ αὐτῷ λόγῳ, ᾗ δὲ τεταραγμένη

Proposition 23[†]

If there are three magnitudes, and others of equal number to them, (being) in the same ratio taken two by two, and (if) their proportion is perturbed, then they will also be in the same ratio via equality.



Let A, B, and C be three magnitudes, and D, E and F other (magnitudes) of equal number to them, (being) in the same ratio taken two by two. And let their proportion be perturbed, (so that) as A (is) to B, so E (is) to F, and as B (is) to C, so D (is) to E. I say that as A is to C, so D (is) to F.

Let the equal multiples G, H, and K have been taken of A, B, and D (respectively), and the other random equal multiples L, M, and N of C, E, and F (respectively).

And since G and H are equal multiples of A and B (respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as A (is) to B, so G (is) to H. And, so, for the same (reasons), as E (is) to F, so M (is) to N. And as A is to B, so E (is) to F. And, thus, as G (is) to H, so M (is) to N [Prop. 5.11]. And since as B is to C, so D (is) to E, also, alternately, as B (is) to D, so C (is) to E [Prop. 5.16]. And since H and K are equal multiples of B and D (respectively), and parts have the same ratio as similar multiples [Prop. 5.15], thus as B is to D, so H (is) to K. But, as B (is) to D, so C (is) to E. And, thus, as H (is) to K, so C (is) to E [Prop. 5.11]. Again, since L and M are equal multiples of C and E (respectively), thus as C is to E, so L (is) to M [Prop. 5.15]. But, as C (is) to E, so H (is) to K. And, thus, as H (is) to K, so L (is) to M [Prop. 5.11]. Also, alternately, as H (is) to L, so K (is) to M [Prop. 5.16]. And it was also shown (that) as G (is) to H, so M (is) to N. Therefore, since G, H, and L are three magnitudes, and K, M, and N other (magnitudes) of equal number to them, (being) in the same ratio taken two by two, and their proportion is perturbed, thus, via equality, if G exceeds L then K also exceeds N, and if (G is) equal (to L then K is also) equal (to N), and if (G is) less (than L then K is also) less (than N) [Prop. 5.21]. And G and K are equal multiples of A and D (respectively), and L and N of C and

αὐτῶν ἡ ἀναλογία, καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσται· ὅπερ ἔδει δεῖξαι.

F (respectively). Thus, as A (is) to C , so D (is) to F [Def. 5.5].

Thus, if there are three magnitudes, and others of equal number to them, (being) in the same ratio taken two by two, and (if) their proportion is perturbed, then they will also be in the same ratio via equality. (Which is) the very thing it was required to show.

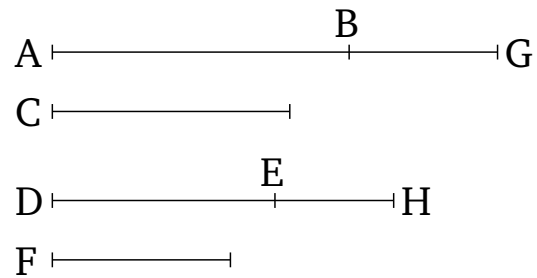
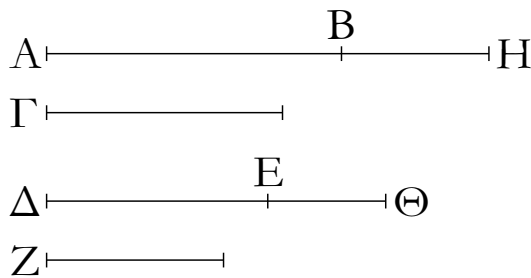
† In modern notation, this proposition reads that if $\alpha : \beta :: \epsilon : \zeta$ and $\beta : \gamma :: \delta : \epsilon$ then $\alpha : \gamma :: \delta : \zeta$.

κδ'.

Proposition 24†

Ἐάν πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, ἔχῃ δὲ καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν λόγον καὶ ἕκτον πρὸς τέταρτον, καὶ συντεθὲν πρῶτον καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν ἔξει λόγον καὶ τρίτον καὶ ἕκτον πρὸς τέταρτον.

If a first (magnitude) has to a second the same ratio that third (has) to a fourth, and a fifth (magnitude) also has to the second the same ratio that a sixth (has) to the fourth, then the first (magnitude) and the fifth, added together, will also have the same ratio to the second that the third (magnitude) and sixth (added together, have) to the fourth.



Πρῶτον γάρ τὸ AB πρὸς δεύτερον τὸ Γ τὸν αὐτὸν ἔχέτω λόγον καὶ τρίτον τὸ ΔE πρὸς τέταρτον τὸ Z , ἔχέτω δὲ καὶ πέμπτον τὸ BH πρὸς δεύτερον τὸ Γ τὸν αὐτὸν λόγον καὶ ἕκτον τὸ $E\Theta$ πρὸς τέταρτον τὸ Z . λέγω, ὅτι καὶ συντεθὲν πρῶτον καὶ πέμπτον τὸ AH πρὸς δεύτερον τὸ Γ τὸν αὐτὸν ἔξει λόγον, καὶ τρίτον καὶ ἕκτον τὸ $\Delta\Theta$ πρὸς τέταρτον τὸ Z .

For let a first (magnitude) AB have the same ratio to a second C that a third DE (has) to a fourth F . And let a fifth (magnitude) BG also have the same ratio to the second C that a sixth EH (has) to the fourth F . I say that the first (magnitude) and the fifth, added together, AG , will also have the same ratio to the second C that the third (magnitude) and the sixth, (added together), DH , (has) to the fourth F .

Ἐπεὶ γὰρ ἐστὶν ὡς τὸ BH πρὸς τὸ Γ , οὕτως τὸ $E\Theta$ πρὸς τὸ Z , ἀνάπαλιν ἄρα ὡς τὸ Γ πρὸς τὸ BH , οὕτως τὸ Z πρὸς τὸ $E\Theta$. ἐπεὶ οὖν ἐστὶν ὡς τὸ AB πρὸς τὸ Γ , οὕτως τὸ ΔE πρὸς τὸ Z , ὡς δὲ τὸ Γ πρὸς τὸ BH , οὕτως τὸ Z πρὸς τὸ $E\Theta$, δι' ἴσου ἄρα ἐστὶν ὡς τὸ AB πρὸς τὸ BH , οὕτως τὸ ΔE πρὸς τὸ $E\Theta$. καὶ ἐπεὶ διηρημένα μεγέθη ἀνάλογόν ἐστιν, καὶ συντεθέντα ἀνάλογον ἔσται· ἔστιν ἄρα ὡς τὸ AH πρὸς τὸ HB , οὕτως τὸ $\Delta\Theta$ πρὸς τὸ ΘE . ἔστι δὲ καὶ ὡς τὸ BH πρὸς τὸ Γ , οὕτως τὸ $E\Theta$ πρὸς τὸ Z · δι' ἴσου ἄρα ἐστὶν ὡς τὸ AH πρὸς τὸ Γ , οὕτως τὸ $\Delta\Theta$ πρὸς τὸ Z .

For since as BG is to C , so EH (is) to F , thus, inversely, as C (is) to BG , so F (is) to EH [Prop. 5.7 corr.]. Therefore, since as AB is to C , so DE (is) to F , and as C (is) to BG , so F (is) to EH , thus, via equality, as AB is to BG , so DE (is) to EH [Prop. 5.22]. And since separated magnitudes are proportional then they will also be proportional (when) composed [Prop. 5.18]. Thus, as AG is to GB , so DH (is) to HE . And, also, as BG is to C , so EH (is) to F . Thus, via equality, as AG is to C , so DH (is) to F [Prop. 5.22].

Ἐάν ἄρα πρῶτον πρὸς δεύτερον τὸν αὐτὸν ἔχῃ λόγον καὶ τρίτον πρὸς τέταρτον, ἔχῃ δὲ καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν λόγον καὶ ἕκτον πρὸς τέταρτον, καὶ συντεθὲν πρῶτον καὶ πέμπτον πρὸς δεύτερον τὸν αὐτὸν ἔξει λόγον καὶ τρίτον καὶ ἕκτον πρὸς τέταρτον· ὅπερ ἔδει δεῖξαι.

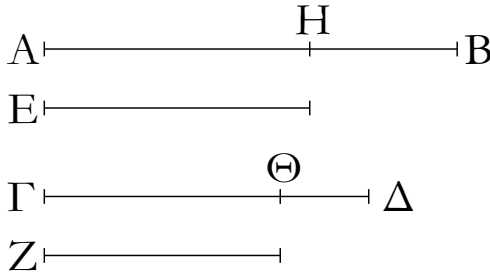
Thus, if a first (magnitude) has to a second the same ratio that a third (has) to a fourth, and a fifth (magnitude) also has to the second the same ratio that a sixth (has) to the fourth, then the first (magnitude) and the fifth, added together, will also have the same ratio to the second that the third (magnitude) and the sixth (added

together, have) to the fourth. (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$ and $\epsilon : \beta :: \zeta : \delta$ then $\alpha + \epsilon : \beta :: \gamma + \zeta : \delta$.

κε'.

Ἐάν τέσσαρα μεγέθη ἀνάλογον ᾗ, τὸ μέγιστον [αὐτῶν] καὶ τὸ ἐλάχιστον δύο τῶν λοιπῶν μείζονά ἐστιν.



Ἐστω τέσσαρα μεγέθη ἀνάλογον τὰ AB, ΓΔ, E, Z, ὡς τὸ AB πρὸς τὸ ΓΔ, οὕτως τὸ E πρὸς τὸ Z, ἔστω δὲ μέγιστον μὲν αὐτῶν τὸ AB, ἐλάχιστον δὲ τὸ Z· λέγω, ὅτι τὰ AB, Z τῶν ΓΔ, E μείζονά ἐστιν.

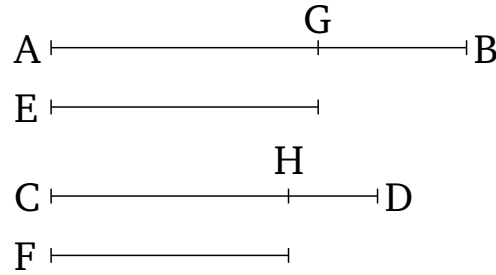
Κείσθω γὰρ τῷ μὲν E ἴσον τὸ AH, τῷ δὲ Z ἴσον τὸ ΓΘ.

Ἐπεὶ [οὖν] ἐστὶν ὡς τὸ AB πρὸς τὸ ΓΔ, οὕτως τὸ E πρὸς τὸ Z, ἴσον δὲ τὸ μὲν E τῷ AH, τὸ δὲ Z τῷ ΓΘ, ἔστιν ἄρα ὡς τὸ AB πρὸς τὸ ΓΔ, οὕτως τὸ AH πρὸς τὸ ΓΘ. καὶ ἐπεὶ ἐστὶν ὡς ὅλον τὸ AB πρὸς ὅλον τὸ ΓΔ, οὕτως ἀφαιρεθὲν τὸ AH πρὸς ἀφαιρεθὲν τὸ ΓΘ, καὶ λοιπὸν ἄρα τὸ HB πρὸς λοιπὸν τὸ ΘΔ ἔσται ὡς ὅλον τὸ AB πρὸς ὅλον τὸ ΓΔ. μείζον δὲ τὸ AB τοῦ ΓΔ· μείζον ἄρα καὶ τὸ HB τοῦ ΘΔ. καὶ ἐπεὶ ἴσον ἐστὶ τὸ μὲν AH τῷ E, τὸ δὲ ΓΘ τῷ Z, τὰ ἄρα AH, Z ἴσα ἐστὶ τοῖς ΓΘ, E. καὶ [ἐπεὶ] ἐὰν [ἀνίσοις] ἴσα προστεθῇ, τὰ ὅλα ἀνισά ἐστιν, ἐὰν ἄρα] τῶν HB, ΘΔ ἀνίσων ὄντων καὶ μείζονος τοῦ HB τῷ μὲν HB προστεθῇ τὰ AH, Z, τῷ δὲ ΘΔ προστεθῇ τὰ ΓΘ, E, συνάγεται τὰ AB, Z μείζονα τῶν ΓΔ, E.

Ἐὰν ἄρα τέσσαρα μεγέθη ἀνάλογον ᾗ, τὸ μέγιστον αὐτῶν καὶ τὸ ἐλάχιστον δύο τῶν λοιπῶν μείζονά ἐστιν. ὅπερ ἔδει δεῖξαι.

Proposition 25[†]

If four magnitudes are proportional then the (sum of the) largest and the smallest [of them] is greater than the (sum of the) remaining two (magnitudes).



Let AB , CD , E , and F be four proportional magnitudes, (such that) as AB (is) to CD , so E (is) to F . And let AB be the greatest of them, and F the least. I say that AB and F is greater than CD and E .

For let AG be made equal to E , and CH equal to F .

[In fact,] since as AB is to CD , so E (is) to F , and E (is) equal to AG , and F to CH , thus as AB is to CD , so AG (is) to CH . And since the whole AB is to the whole CD as the (part) taken away AG (is) to the (part) taken away CH , thus the remainder GB will also be to the remainder HD as the whole AB (is) to the whole CD [Prop. 5.19]. And AB (is) greater than CD . Thus, GB (is) also greater than HD . And since AG is equal to E , and CH to F , thus AG and F is equal to CH and E . And [since] if [equal (magnitudes) are added to unequal (magnitudes) then the wholes are unequal, thus if] AG and F are added to GB , and CH and E to HD — GB and HD being unequal, and GB greater—it is inferred that AB and F (is) greater than CD and E .

Thus, if four magnitudes are proportional then the (sum of the) largest and the smallest of them is greater than the (sum of the) remaining two (magnitudes). (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $\alpha : \beta :: \gamma : \delta$, and α is the greatest and δ the least, then $\alpha + \delta > \beta + \gamma$.

ELEMENTS BOOK 6

Similar Figures

Ὅροι.

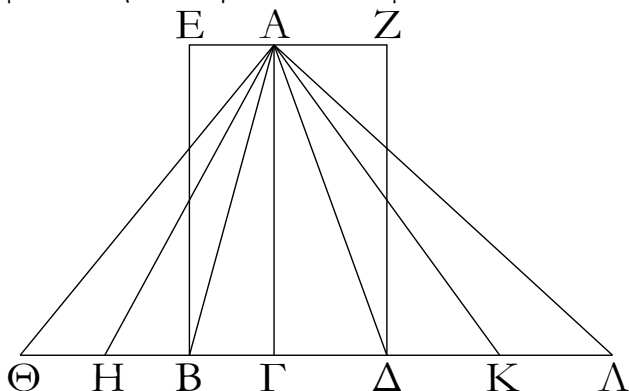
α'. Ὅμοια σχήματα εὐθύγραμμά ἐστιν, ὅσα τὰς τε γωνίας ἴσας ἔχει κατὰ μίαν καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον.

β'. Ἀκρον καὶ μέσον λόγον εὐθεῖα τετυγῆσθαι λέγεται, ὅταν ἢ ὡς ἡ ὅλη πρὸς τὸ μείζον τμήμα, οὕτως τὸ μείζον πρὸς τὸ ἐλάττω.

γ'. Ὑψος ἐστὶ πάντος σχήματος ἢ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν βάσιν κάθετος ἀγομένη.

α'.

Τὰ τρίγωνα καὶ τὰ παραλληλόγραμμα τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα πρὸς ἀλλήλα ἐστὶν ὡς αἱ βάσεις.



Ἐστω τρίγωνα μὲν τὰ ΑΒΓ, ΑΓΔ, παραλληλόγραμμα δὲ τὰ ΕΓ, ΓΖ ὑπὸ τὸ αὐτὸ ὕψος τὸ ΑΓ· λέγω, ὅτι ἐστὶν ὡς ἡ ΒΓ βάσις πρὸς τὴν ΓΔ βάσιν, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΓΔ τρίγωνον, καὶ τὸ ΕΓ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον.

Ἐκβεβλήσθω γὰρ ἡ ΒΔ ἐφ' ἐκάτερα τὰ μέρη ἐπὶ τὰ Θ, Λ σημεῖα, καὶ κείσθωσαν τῇ μὲν ΒΓ βάσει ἴσαι [ὁσαιδηποτοῦν] αἱ ΒΗ, ΗΘ, τῇ δὲ ΓΔ βάσει ἴσαι ὁσαιδηποτοῦν αἱ ΔΚ, ΚΛ, καὶ ἐπεζεύχθωσαν αἱ ΑΗ, ΑΘ, ΑΚ, ΑΛ.

Καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΓΒ, ΒΗ, ΗΘ ἀλλήλαις, ἴσα ἐστὶ καὶ τὰ ΑΘΗ, ΑΗΒ, ΑΒΓ τρίγωνα ἀλλήλοις. ὁσαπλασίον ἄρα ἐστὶν ἡ ΘΓ βάσις τῆς ΒΓ βάσεως, τοσαυταπλασίον ἐστὶ καὶ τὸ ΑΘΓ τρίγωνον τοῦ ΑΒΓ τριγώνου. διὰ τὰ αὐτὰ δὴ ὁσαπλασίον ἐστὶν ἡ ΑΓ βάσις τῆς ΓΔ βάσεως, τοσαυταπλασίον ἐστὶ καὶ τὸ ΑΑΓ τρίγωνον τοῦ ΑΓΔ τριγώνου· καὶ εἰ ἴση ἐστὶν ἡ ΘΓ βάσις τῇ ΓΔ βάσει, ἴσον ἐστὶ καὶ τὸ ΑΘΓ τρίγωνον τῷ ΑΓΔ τριγώνῳ, καὶ εἰ ὑπερέχει ἡ ΘΓ βάσις τῆς ΓΔ βάσεως, ὑπερέχει καὶ τὸ ΑΘΓ τρίγωνον τοῦ ΑΓΔ τριγώνου, καὶ εἰ ἐλάσσων, ἐλασσον. τεσσάρων δὲ ὄντων μεγεθῶν δύο μὲν βάσεων τῶν ΒΓ, ΓΔ, δύο δὲ τριγώνων τῶν ΑΒΓ, ΑΓΔ εἴληπται ἰσάκεις πολλαπλασία τῆς μὲν ΒΓ βάσεως καὶ τοῦ ΑΒΓ τριγώνου ἢ τε ΘΓ βάσις καὶ τὸ ΑΘΓ τρίγωνον, τῆς δὲ ΓΔ βάσεως καὶ τοῦ ΑΓΔ τριγώνου ἄλλα,

Definitions

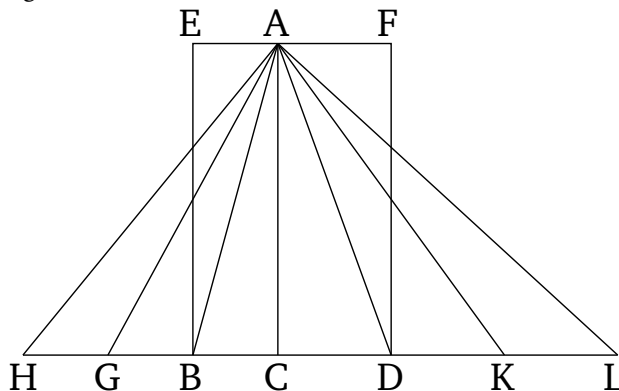
1. Similar rectilinear figures are those (which) have (their) angles separately equal and the (corresponding) sides about the equal angles proportional.

2. A straight-line is said to have been cut in extreme and mean ratio when as the whole is to the greater segment so the greater (segment is) to the lesser.

3. The height of any figure is the (straight-line) drawn from the vertex perpendicular to the base.

Proposition 1[†]

Triangles and parallelograms which are of the same height are to one another as their bases.



Let ABC and ACD be triangles, and EC and CF parallelograms, of the same height AC . I say that as base BC is to base CD , so triangle ABC (is) to triangle ACD , and parallelogram EC to parallelogram CF .

For let the (straight-line) BD have been produced in each direction to points H and L , and let [any number] (of straight-lines) BG and GH be made equal to base BC , and any number (of straight-lines) DK and KL equal to base CD . And let AG , AH , AK , and AL have been joined.

And since CB , BG , and GH are equal to one another, triangles AHG , AGB , and ABC are also equal to one another [Prop. 1.38]. Thus, as many times as base HC is (divisible by) base BC , so many times is triangle AHC also (divisible by) triangle ABC . So, for the same (reasons), as many times as base LC is (divisible by) base CD , so many times is triangle ALC also (divisible by) triangle ACD . And if base HC is equal to base CL then triangle AHC is also equal to triangle ALC [Prop. 1.38]. And if base HC exceeds base CL then triangle AHC also exceeds triangle ALC .[‡] And if (HC is) less (than CL then AHC is also) less (than ALC). So, their being four magnitudes, two bases, BC and CD , and two trian-

ἂ ἔτυχεν, ἰσάκεις πολλαπλάσια ἢ τε $\Lambda\Gamma$ βάσις καὶ τὸ $\Lambda\Lambda\Gamma$ τρίγωνον· καὶ δέδεικται, ὅτι, εἰ ὑπερέχει ἡ $\Theta\Gamma$ βάσις τῆς $\Gamma\Lambda$ βάσεως, ὑπερέχει καὶ τὸ $\Lambda\Theta\Gamma$ τρίγωνον τοῦ $\Lambda\Lambda\Gamma$ τριγώνου, καὶ εἰ ἴση, ἴσον, καὶ εἰ ἔλασσον, ἔλασσον· ἔστιν ἄρα ὡς ἡ $B\Gamma$ βάσις πρὸς τὴν $\Gamma\Delta$ βάσιν, οὕτως τὸ $\Lambda B\Gamma$ τρίγωνον πρὸς τὸ $\Lambda\Gamma\Delta$ τρίγωνον.

Καὶ ἐπεὶ τοῦ μὲν $\Lambda B\Gamma$ τριγώνου διπλάσιόν ἐστι τὸ $E\Gamma$ παραλληλόγραμμον, τοῦ δὲ $\Lambda\Gamma\Delta$ τριγώνου διπλάσιόν ἐστι τὸ $Z\Gamma$ παραλληλόγραμμον, τὰ δὲ μέρη τοῖς ὡσαύτως πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ $\Lambda B\Gamma$ τρίγωνον πρὸς τὸ $\Lambda\Gamma\Delta$ τρίγωνον, οὕτως τὸ $E\Gamma$ παραλληλόγραμμον πρὸς τὸ $Z\Gamma$ παραλληλόγραμμον. ἐπεὶ οὖν ἐδείχθη, ὡς μὲν ἡ $B\Gamma$ βάσις πρὸς τὴν $\Gamma\Delta$, οὕτως τὸ $\Lambda B\Gamma$ τρίγωνον πρὸς τὸ $\Lambda\Gamma\Delta$ τρίγωνον, ὡς δὲ τὸ $\Lambda B\Gamma$ τρίγωνον πρὸς τὸ $\Lambda\Gamma\Delta$ τρίγωνον, οὕτως τὸ $E\Gamma$ παραλληλόγραμμον πρὸς τὸ ΓZ παραλληλόγραμμον, καὶ ὡς ἄρα ἡ $B\Gamma$ βάσις πρὸς τὴν $\Gamma\Delta$ βάσιν, οὕτως τὸ $E\Gamma$ παραλληλόγραμμον πρὸς τὸ $Z\Gamma$ παραλληλόγραμμον.

Τὰ ἄρα τρίγωνα καὶ τὰ παραλληλόγραμμα τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

gles, ABC and ACD , equal multiples have been taken of base BC and triangle ABC —(namely), base HC and triangle AHC —and other random equal multiples of base CD and triangle ADC —(namely), base LC and triangle ALC . And it has been shown that if base HC exceeds base CL then triangle AHC also exceeds triangle ALC , and if (HC is) equal (to CL then AHC is also) equal (to ALC), and if (HC is) less (than CL then AHC is also) less (than ALC). Thus, as base BC is to base CD , so triangle ABC (is) to triangle ACD [Def. 5.5]. And since parallelogram EC is double triangle ABC , and parallelogram FC is double triangle ACD [Prop. 1.34], and parts have the same ratio as similar multiples [Prop. 5.15], thus as triangle ABC is to triangle ACD , so parallelogram EC (is) to parallelogram FC . In fact, since it was shown that as base BC (is) to CD , so triangle ABC (is) to triangle ACD , and as triangle ABC (is) to triangle ACD , so parallelogram EC (is) to parallelogram CF , thus, also, as base BC (is) to base CD , so parallelogram EC (is) to parallelogram FC [Prop. 5.11].

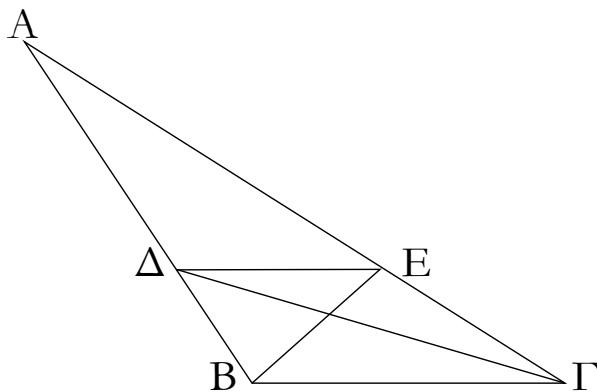
Thus, triangles and parallelograms which are of the same height are to one another as their bases. (Which is) the very thing it was required to show.

† As is easily demonstrated, this proposition holds even when the triangles, or parallelograms, do not share a common side, and/or are not right-angled.

‡ This is a straight-forward generalization of Prop. 1.38.

β'.

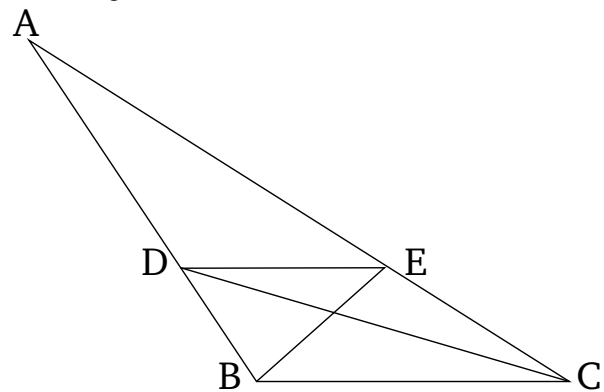
Ἐὰν τριγώνου παρὰ μίαν τῶν πλευρῶν ἀχθῇ τις εὐθεῖα, ἀνάλογον τεμεῖ τὰς τοῦ τριγώνου πλευράς· καὶ ἐὰν αἱ τοῦ τριγώνου πλευραὶ ἀνάλογον τμηθῶσιν, ἡ ἐπὶ τὰς τομὰς ἐπιζευγνυμένη εὐθεῖα παρὰ τὴν λοιπὴν ἔσται τοῦ τριγώνου πλευράν.



Τριγώνου γὰρ τοῦ $\Lambda B\Gamma$ παράλληλος μιᾶ τῶν πλευρῶν τῇ $B\Gamma$ ἢ χθῶ η ΔE · λέγω, ὅτι ἐστὶν ὡς ἡ $B\Delta$ πρὸς τὴν ΔA , οὕτως ἡ ΓE πρὸς τὴν $E A$.

Proposition 2

If some straight-line is drawn parallel to one of the sides of a triangle then it will cut the (other) sides of the triangle proportionally. And if (two of) the sides of a triangle are cut proportionally then the straight-line joining the cutting (points) will be parallel to the remaining side of the triangle.



For let DE have been drawn parallel to one of the sides BC of triangle ABC . I say that as BD is to DA , so CE (is) to EA .

Ἐπεξεύχθωσαν γὰρ αἱ BE , $\Gamma\Delta$.

Ἰσον ἄρα ἐστὶ τὸ $B\Delta E$ τρίγωνον τῷ $\Gamma\Delta E$ τριγώνῳ· ἐπὶ γὰρ τῆς αὐτῆς βάσεως ἐστὶ τῆς ΔE καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς ΔE , $B\Gamma$ · ἄλλο δέ τι τὸ $A\Delta E$ τρίγωνον. τὰ δὲ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον· ἐστὶν ἄρα ὡς τὸ $B\Delta E$ τρίγωνον πρὸς τὸ $A\Delta E$ [τρίγωνον], οὕτως τὸ $\Gamma\Delta E$ τρίγωνον πρὸς τὸ $A\Delta E$ τρίγωνον. ἀλλ' ὡς μὲν τὸ $B\Delta E$ τρίγωνον πρὸς τὸ $A\Delta E$, οὕτως ἡ $B\Delta$ πρὸς τὴν ΔA · ὑπὸ γὰρ τὸ αὐτὸ ὕψος ὄντα τὴν ἀπὸ τοῦ E ἐπὶ τὴν AB κάθετον ἀγομένην πρὸς ἄλληλά εἰσιν ὡς αἱ βάσεις. διὰ τὰ αὐτὰ δὴ ὡς τὸ $\Gamma\Delta E$ τρίγωνον πρὸς τὸ $A\Delta E$, οὕτως ἡ ΓE πρὸς τὴν EA · καὶ ὡς ἄρα ἡ $B\Delta$ πρὸς τὴν ΔA , οὕτως ἡ ΓE πρὸς τὴν EA .

Ἀλλὰ δὴ αἱ τοῦ $AB\Gamma$ τριγώνου πλευραὶ αἱ AB , $A\Gamma$ ἀνάλογον τετμήσθωσαν, ὡς ἡ $B\Delta$ πρὸς τὴν ΔA , οὕτως ἡ ΓE πρὸς τὴν EA , καὶ ἐπεξεύχθω ἡ ΔE · λέγω, ὅτι παράλληλός ἐστιν ἡ ΔE τῇ $B\Gamma$.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἐστὶν ὡς ἡ $B\Delta$ πρὸς τὴν ΔA , οὕτως ἡ ΓE πρὸς τὴν EA , ἀλλ' ὡς μὲν ἡ $B\Delta$ πρὸς τὴν ΔA , οὕτως τὸ $B\Delta E$ τρίγωνον πρὸς τὸ $A\Delta E$ τρίγωνον, ὡς δὲ ἡ ΓE πρὸς τὴν EA , οὕτως τὸ $\Gamma\Delta E$ τρίγωνον πρὸς τὸ $A\Delta E$ τρίγωνον, καὶ ὡς ἄρα τὸ $B\Delta E$ τρίγωνον πρὸς τὸ $A\Delta E$ τρίγωνον, οὕτως τὸ $\Gamma\Delta E$ τρίγωνον πρὸς τὸ $A\Delta E$ τρίγωνον. ἐκάτερον ἄρα τῶν $B\Delta E$, $\Gamma\Delta E$ τριγώνων πρὸς τὸ $A\Delta E$ τὸν αὐτὸν ἔχει λόγον. ἴσον ἄρα ἐστὶ τὸ $B\Delta E$ τρίγωνον τῷ $\Gamma\Delta E$ τριγώνῳ· καὶ εἰσιν ἐπὶ τῆς αὐτῆς βάσεως τῆς ΔE . τὰ δὲ ἴσα τρίγωνα καὶ ἐπὶ τῆς αὐτῆς βάσεως ὄντα καὶ ἐν ταῖς αὐταῖς παραλλήλοις ἐστίν. παράλληλος ἄρα ἐστὶν ἡ ΔE τῇ $B\Gamma$.

Ἐὰν ἄρα τριγώνου παρὰ μίαν τῶν πλευρῶν ἀχθῇ τις εὐθεῖα, ἀνάλογον τεμεῖ τὰς τοῦ τριγώνου πλευράς· καὶ ἐὰν αἱ τοῦ τριγώνου πλευραὶ ἀνάλογον τμηθῶσιν, ἡ ἐπὶ τὰς τομὰς ἐπιζευγνυμένη εὐθεῖα παρὰ τὴν λοιπὴν ἔσται τοῦ τριγώνου πλευράν· ὅπερ ἔδει δεῖξαι.

Υ'.

Ἐὰν τριγώνου ἡ γωνία δίχα τμηθῇ, ἡ δὲ τέμνουσα τὴν γωνίαν εὐθεῖα τέμνη καὶ τὴν βάσιν, τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἔξει λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς· καὶ ἐὰν τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἔχῃ λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς, ἡ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν τομὴν ἐπιζευγνυμένη εὐθεῖα δίχα τεμεῖ τὴν τοῦ τριγώνου γωνίαν.

Ἐστω τρίγωνον τὸ $AB\Gamma$, καὶ τετμήσθω ἡ ὑπὸ $BA\Gamma$ γωνία δίχα ὑπὸ τῆς AD εὐθείας· λέγω, ὅτι ἐστὶν ὡς ἡ $B\Delta$ πρὸς τὴν $\Gamma\Delta$, οὕτως ἡ BA πρὸς τὴν $A\Gamma$.

Ἦχθω γὰρ διὰ τοῦ Γ τῇ ΔA παράλληλος ἡ ΓE , καὶ διαχθεῖσα ἡ BA συμπιπτεῖ αὐτῇ κατὰ τὸ E .

For let BE and CD have been joined.

Thus, triangle BDE is equal to triangle CDE . For they are on the same base DE and between the same parallels DE and BC [Prop. 1.38]. And ADE is some other triangle. And equal (magnitudes) have the same ratio to the same (magnitude) [Prop. 5.7]. Thus, as triangle BDE is to [triangle] ADE , so triangle CDE (is) to triangle ADE . But, as triangle BDE (is) to triangle ADE , so (is) BD to DA . For, having the same height—(namely), the (straight-line) drawn from E perpendicular to AB —they are to one another as their bases [Prop. 6.1]. So, for the same (reasons), as triangle CDE (is) to ADE , so CE (is) to EA . And, thus, as BD (is) to DA , so CE (is) to EA [Prop. 5.11].

And so, let the sides AB and AC of triangle ABC have been cut proportionally (such that) as BD (is) to DA , so CE (is) to EA . And let DE have been joined. I say that DE is parallel to BC .

For, by the same construction, since as BD is to DA , so CE (is) to EA , but as BD (is) to DA , so triangle BDE (is) to triangle ADE , and as CE (is) to EA , so triangle CDE (is) to triangle ADE [Prop. 6.1], thus, also, as triangle BDE (is) to triangle ADE , so triangle CDE (is) to triangle ADE [Prop. 5.11]. Thus, triangles BDE and CDE each have the same ratio to ADE . Thus, triangle BDE is equal to triangle CDE [Prop. 5.9]. And they are on the same base DE . And equal triangles, which are also on the same base, are also between the same parallels [Prop. 1.39]. Thus, DE is parallel to BC .

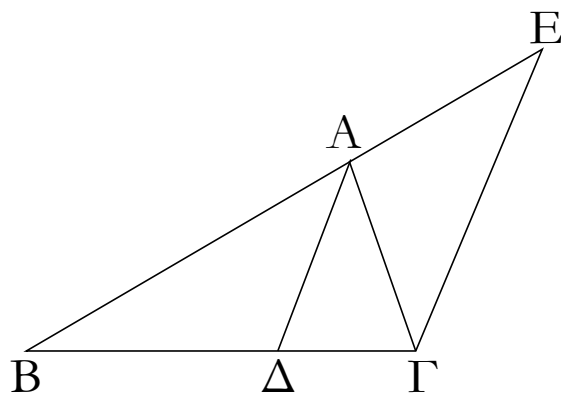
Thus, if some straight-line is drawn parallel to one of the sides of a triangle, then it will cut the (other) sides of the triangle proportionally. And if (two of) the sides of a triangle are cut proportionally, then the straight-line joining the cutting (points) will be parallel to the remaining side of the triangle. (Which is) the very thing it was required to show.

Proposition 3

If an angle of a triangle is cut in half, and the straight-line cutting the angle also cuts the base, then the segments of the base will have the same ratio as the remaining sides of the triangle. And if the segments of the base have the same ratio as the remaining sides of the triangle, then the straight-line joining the vertex to the cutting (point) will cut the angle of the triangle in half.

Let ABC be a triangle. And let the angle BAC have been cut in half by the straight-line AD . I say that as BD is to CD , so BA (is) to AC .

For let CE have been drawn through (point) C parallel to DA . And, BA being drawn through, let it meet



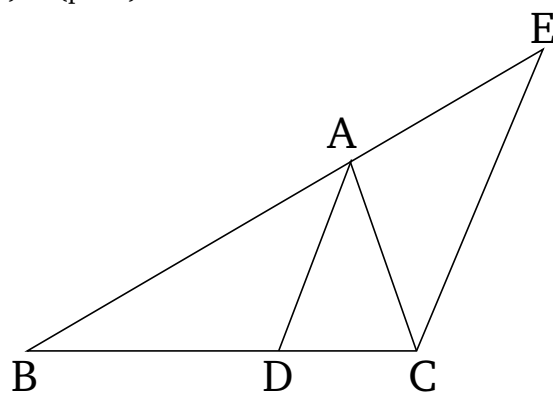
Καὶ ἐπεὶ εἰς παραλλήλους τὰς AD , EG εὐθεῖα ἐνέπεσεν ἡ AG , ἡ ἄρα ὑπὸ AGE γωνία ἴση ἐστὶ τῇ ὑπὸ $ΓAD$. ἀλλ' ἡ ὑπὸ $ΓAD$ τῇ ὑπὸ $BAΔ$ ὑπόκειται ἴση· καὶ ἡ ὑπὸ $BAΔ$ ἄρα τῇ ὑπὸ AGE ἐστὶν ἴση. πάλιν, ἐπεὶ εἰς παραλλήλους τὰς AD , EG εὐθεῖα ἐνέπεσεν ἡ BAE , ἡ ἐκτὸς γωνία ἡ ὑπὸ $BAΔ$ ἴση ἐστὶ τῇ ἐντὸς τῇ ὑπὸ $AEΓ$. ἐδείχθη δὲ καὶ ἡ ὑπὸ AGE τῇ ὑπὸ $BAΔ$ ἴση· καὶ ἡ ὑπὸ AGE ἄρα γωνία τῇ ὑπὸ $AEΓ$ ἐστὶν ἴση· ὥστε καὶ πλευρὰ ἡ AE πλευρᾷ τῇ AG ἐστὶν ἴση. καὶ ἐπεὶ τριγώνου τοῦ BGE παρὰ μίαν τῶν πλευρῶν τὴν EG ἥκται ἡ AD , ἀνάλογον ἄρα ἐστὶν ὡς ἡ BD πρὸς τὴν $ΔΓ$, οὕτως ἡ BA πρὸς τὴν AE . ἴση δὲ ἡ AE τῇ AG · ὡς ἄρα ἡ BD πρὸς τὴν $ΔΓ$, οὕτως ἡ BA πρὸς τὴν AG .

Ἀλλὰ δὴ ἔστω ὡς ἡ BD πρὸς τὴν $ΔΓ$, οὕτως ἡ BA πρὸς τὴν AG , καὶ ἐπεζεύχθω ἡ AD . λέγω, ὅτι δίχα τέμνεται ἡ ὑπὸ $BAΓ$ γωνία ὑπὸ τῆς AD εὐθείας.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἐστὶν ὡς ἡ BD πρὸς τὴν $ΔΓ$, οὕτως ἡ BA πρὸς τὴν AG , ἀλλὰ καὶ ὡς ἡ BD πρὸς τὴν $ΔΓ$, οὕτως ἐστὶν ἡ BA πρὸς τὴν AE . τριγώνου γὰρ τοῦ BGE παρὰ μίαν τὴν EG ἥκται ἡ AD . καὶ ὡς ἄρα ἡ BA πρὸς τὴν AG , οὕτως ἡ BA πρὸς τὴν AE . ἴση ἄρα ἡ AG τῇ AE . ὥστε καὶ γωνία ἡ ὑπὸ $AEΓ$ τῇ ὑπὸ AGE ἐστὶν ἴση. ἀλλ' ἡ μὲν ὑπὸ $AEΓ$ τῇ ἐκτὸς τῇ ὑπὸ $BAΔ$ [ἐστὶν] ἴση, ἡ δὲ ὑπὸ AGE τῇ ἐναλλάξ τῇ ὑπὸ $ΓAD$ ἐστὶν ἴση· καὶ ἡ ὑπὸ $BAΔ$ ἄρα τῇ ὑπὸ $ΓAD$ ἐστὶν ἴση. ἡ ἄρα ὑπὸ $BAΓ$ γωνία δίχα τέμνεται ὑπὸ τῆς AD εὐθείας.

Ἐὰν ἄρα τριγώνου ἡ γωνία δίχα τμηθῇ, ἡ δὲ τέμνουσα τὴν γωνίαν εὐθεῖα τέμνη καὶ τὴν βάσιν, τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἔξει λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς· καὶ ἐὰν τὰ τῆς βάσεως τμήματα τὸν αὐτὸν ἔχη λόγον ταῖς λοιπαῖς τοῦ τριγώνου πλευραῖς, ἡ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν τομὴν ἐπιζευγνυμένη εὐθεῖα δίχα τέμνει τὴν τοῦ τριγώνου γωνίαν· ὅπερ ἔδει δεῖξαι.

(CE) at (point) E .[†]



And since the straight-line AC falls across the parallel (straight-lines) AD and EC , angle ACE is thus equal to CAD [Prop. 1.29]. But, (angle) CAD is assumed (to be) equal to BAD . Thus, (angle) BAD is also equal to ACE . Again, since the straight-line BAE falls across the parallel (straight-lines) AD and EC , the external angle BAD is equal to the internal (angle) AEC [Prop. 1.29]. And (angle) ACE was also shown (to be) equal to BAD . Thus, angle ACE is also equal to AEC . And, hence, side AE is equal to side AC [Prop. 1.6]. And since AD has been drawn parallel to one of the sides EC of triangle BCE , thus, proportionally, as BD is to DC , so BA (is) to AE [Prop. 6.2]. And AE (is) equal to AC . Thus, as BD (is) to DC , so BA (is) to AC .

And so, let BD be to DC , as BA (is) to AC . And let AD have been joined. I say that angle BAC has been cut in half by the straight-line AD .

For, by the same construction, since as BD is to DC , so BA (is) to AC , then also as BD (is) to DC , so BA is to AE . For AD has been drawn parallel to one (of the sides) EC of triangle BCE [Prop. 6.2]. Thus, also, as BA (is) to AC , so BA (is) to AE [Prop. 5.11]. Thus, AC (is) equal to AE [Prop. 5.9]. And, hence, angle AEC is equal to ACE [Prop. 1.5]. But, AEC [is] equal to the external (angle) BAD , and ACE is equal to the alternate (angle) CAD [Prop. 1.29]. Thus, (angle) BAD is also equal to CAD . Thus, angle BAC has been cut in half by the straight-line AD .

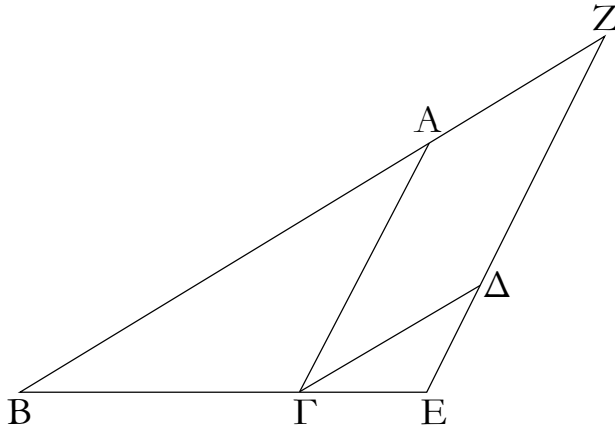
Thus, if an angle of a triangle is cut in half, and the straight-line cutting the angle also cuts the base, then the segments of the base will have the same ratio as the remaining sides of the triangle. And if the segments of the base have the same ratio as the remaining sides of the triangle, then the straight-line joining the vertex to the cutting (point) will cut the angle of the triangle in half. (Which is) the very thing it was required to show.

[†] The fact that the two straight-lines meet follows because the sum of ACE and CAE is less than two right-angles, as can easily be demonstrated.

See Post. 5.

δ'.

Τῶν ἰσογωνίων τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι.



Ἐστω ἰσογώνια τρίγωνα τὰ ABG , DGE ἴσην ἔχοντα τὴν μὲν ὑπὸ ABG γωνίαν τῇ ὑπὸ DGE , τὴν δὲ ὑπὸ BAG τῇ ὑπὸ GDE καὶ ἔτι τὴν ὑπὸ AGB τῇ ὑπὸ GED · λέγω, ὅτι τῶν ABG , DGE τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι.

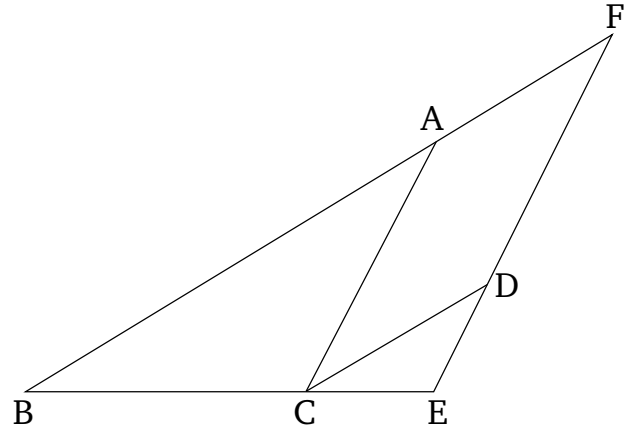
Κεῖσθω γὰρ ἐπ' εὐθείας ἡ BG τῇ GE . καὶ ἐπεὶ αἱ ὑπὸ ABG , AGB γωνίαι δύο ὀρθῶν ἐλάττονές εἰσιν, ἴση δὲ ἡ ὑπὸ AGB τῇ ὑπὸ DEG , αἱ ἄρα ὑπὸ ABG , DEG δύο ὀρθῶν ἐλάττονές εἰσιν· αἱ BA , ED ἄρα ἐκβαλλόμεναι συμπεσοῦνται. ἐκβεβλήσθωσαν καὶ συμπιπτέτωσαν κατὰ τὸ Z .

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ DGE γωνία τῇ ὑπὸ ABG , παράλληλός ἐστιν ἡ BZ τῇ GD . πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ AGB τῇ ὑπὸ DEG , παράλληλός ἐστιν ἡ AG τῇ ZE . παραλληλόγραμμον ἄρα ἐστὶ τὸ $ZAGD$ · ἴση ἄρα ἡ μὲν ZA τῇ GD , ἡ δὲ AG τῇ ZD . καὶ ἐπεὶ τριγώνου τοῦ ZBE παρὰ μίαν τὴν ZE ἦκται ἡ AG , ἐστὶν ἄρα ὡς ἡ BA πρὸς τὴν AZ , οὕτως ἡ BG πρὸς τὴν GE . ἴση δὲ ἡ AZ τῇ GD · ὡς ἄρα ἡ BA πρὸς τὴν GD , οὕτως ἡ BG πρὸς τὴν GE , καὶ ἐναλλάξ ὡς ἡ AB πρὸς τὴν BG , οὕτως ἡ GD πρὸς τὴν GE . πάλιν, ἐπεὶ παράλληλός ἐστιν ἡ GD τῇ BZ , ἐστὶν ἄρα ὡς ἡ BG πρὸς τὴν GE , οὕτως ἡ ZD πρὸς τὴν DE . ἴση δὲ ἡ ZD τῇ AG · ὡς ἄρα ἡ BG πρὸς τὴν GE , οὕτως ἡ AG πρὸς τὴν DE , καὶ ἐναλλάξ ὡς ἡ BG πρὸς τὴν GA , οὕτως ἡ GE πρὸς τὴν ED . ἐπεὶ οὖν ἐδείχθη ὡς μὲν ἡ AB πρὸς τὴν BG , οὕτως ἡ GD πρὸς τὴν GE , ὡς δὲ ἡ BG πρὸς τὴν GA , οὕτως ἡ GE πρὸς τὴν ED , δι' ἴσου ἄρα ὡς ἡ BA πρὸς τὴν AG , οὕτως ἡ GD πρὸς τὴν DE .

Τῶν ἄρα ἰσογωνίων τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσαι· ὅπερ ἔδει δεῖξαι.

Proposition 4

In equiangular triangles the sides about the equal angles are proportional, and those (sides) subtending equal angles correspond.



Let ABC and DCE be equiangular triangles, having angle ABC equal to DCE , and (angle) BAC to CDE , and, further, (angle) ACB to CED . I say that in triangles ABC and DCE the sides about the equal angles are proportional, and those (sides) subtending equal angles correspond.

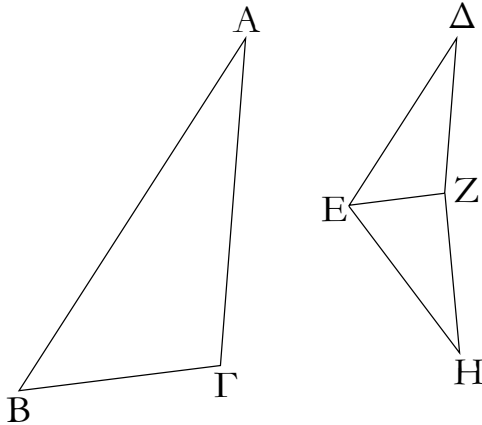
Let BC be placed straight-on to CE . And since angles ABC and ACB are less than two right-angles [Prop. 1.17], and ACB (is) equal to DEC , thus ABC and DEC are less than two right-angles. Thus, BA and ED , being produced, will meet [C.N. 5]. Let them have been produced, and let them meet at (point) F .

And since angle DCE is equal to ABC , BF is parallel to CD [Prop. 1.28]. Again, since (angle) ACB is equal to DEC , AC is parallel to FE [Prop. 1.28]. Thus, $FACD$ is a parallelogram. Thus, FA is equal to DC , and AC to FD [Prop. 1.34]. And since AC has been drawn parallel to one (of the sides) FE of triangle FBE , thus as BA is to AF , so BC (is) to CE [Prop. 6.2]. And AF (is) equal to CD . Thus, as BA (is) to CD , so BC (is) to CE , and, alternately, as AB (is) to BC , so DC (is) to CE [Prop. 5.16]. Again, since CD is parallel to BF , thus as BC (is) to CE , so FD (is) to DE [Prop. 6.2]. And FD (is) equal to AC . Thus, as BC is to CE , so AC (is) to DE , and, alternately, as BC (is) to CA , so CE (is) to ED [Prop. 6.2]. Therefore, since it was shown that as AB (is) to BC , so DC (is) to CE , and as BC (is) to CA , so CE (is) to ED , thus, via equality, as BA (is) to AC , so CD (is) to DE [Prop. 5.22].

Thus, in equiangular triangles the sides about the equal angles are proportional, and those (sides) subtend-

ε'.

Ἐάν δύο τρίγωνα τὰς πλευρὰς ἀνάλογον ἔχῃ, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἔξει τὰς γωνίας, ὅφ' ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν.



Ἐστω δύο τρίγωνα τὰ $AB\Gamma$, ΔEZ τὰς πλευρὰς ἀνάλογον ἔχοντα, ὥς μὲν τὴν AB πρὸς τὴν $B\Gamma$, οὕτως τὴν ΔE πρὸς τὴν EZ , ὥς δὲ τὴν $B\Gamma$ πρὸς τὴν ΓA , οὕτως τὴν EZ πρὸς τὴν $Z\Delta$, καὶ ἔτι ὥς τὴν BA πρὸς τὴν $A\Gamma$, οὕτως τὴν ED πρὸς τὴν DZ . λέγω, ὅτι ἰσογώνιον ἔστι τὸ $AB\Gamma$ τρίγωνον τῷ ΔEZ τριγώνῳ καὶ ἴσας ἔξουσιν τὰς γωνίας, ὅφ' ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν, τὴν μὲν ὑπὸ $AB\Gamma$ τῇ ὑπὸ ΔEZ , τὴν δὲ ὑπὸ $B\Gamma A$ τῇ ὑπὸ $EZ\Delta$ καὶ ἔτι τὴν ὑπὸ $B A \Gamma$ τῇ ὑπὸ $E \Delta Z$.

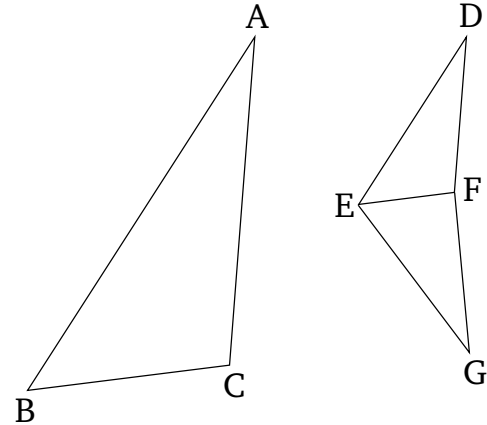
Συνεστάτω γὰρ πρὸς τῇ EZ εὐθείᾳ καὶ τοῖς πρὸς αὐτῇ σημείοις τοῖς E , Z τῇ μὲν ὑπὸ $AB\Gamma$ γωνίᾳ ἴση ἡ ὑπὸ $Z E H$, τῇ δὲ ὑπὸ $A \Gamma B$ ἴση ἡ ὑπὸ $E Z H$. λοιπὴ ἄρα ἡ πρὸς τῷ A λοιπῇ τῇ πρὸς τῷ H ἔστιν ἴση.

Ἰσογώνιον ἄρα ἔστι τὸ $AB\Gamma$ τρίγωνον τῷ $E H Z$ [τριγώνῳ]. τῶν ἄρα $AB\Gamma$, $E H Z$ τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας καὶ ὁμόλογοι αἱ ὑπὸ τὰς ἴσας γωνίας ὑποτείνουσιν· ἔστιν ἄρα ὥς ἡ AB πρὸς τὴν $B\Gamma$, [οὕτως] ἡ HE πρὸς τὴν EZ . ἀλλ' ὥς ἡ AB πρὸς τὴν $B\Gamma$, οὕτως ὑπόκειται ἡ ΔE πρὸς τὴν EZ · ὥς ἄρα ἡ ΔE πρὸς τὴν EZ , οὕτως ἡ HE πρὸς τὴν EZ . ἑκατέρα ἄρα τῶν ΔE , HE πρὸς τὴν EZ τὸν αὐτὸν ἔχει λόγον· ἴση ἄρα ἔστιν ἡ ΔE τῇ HE . διὰ τὰ αὐτὰ δὴ καὶ ἡ ΔZ τῇ HZ ἔστιν ἴση. ἐπεὶ οὖν ἴση ἔστιν ἡ ΔE τῇ HE , κοινὴ δὲ ἡ EZ , δύο δὴ αἱ ΔE , EZ δυοὶ ταῖς HE , EZ ἴσαι εἰσὶν· καὶ βάσεις ἡ ΔZ βάσει τῇ ZH [ἔστιν] ἴση· γωνία ἄρα ἡ ὑπὸ ΔEZ γωνία τῇ ὑπὸ HEZ ἔστιν ἴση, καὶ τὸ ΔEZ τρίγωνον τῷ HEZ τριγώνῳ ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι, ὅφ' ἃς αἱ ἴσαι πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἔστι καὶ ἡ μὲν ὑπὸ ΔZE γωνία τῇ ὑπὸ HZE , ἡ δὲ ὑπὸ $E \Delta Z$ τῇ ὑπὸ $E H Z$. καὶ ἐπεὶ ἡ μὲν ὑπὸ $Z E \Delta$ τῇ ὑπὸ $H E Z$ ἔστιν ἴση, ἀλλ' ἡ ὑπὸ $H E Z$ τῇ ὑπὸ $A B \Gamma$, καὶ ἡ ὑπὸ

ing equal angles correspond. (Which is) the very thing it was required to show.

Proposition 5

If two triangles have proportional sides then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal.



Let ABC and DEF be two triangles having proportional sides, (so that) as AB (is) to BC , so DE (is) to EF , and as BC (is) to CA , so EF (is) to FD , and, further, as BA (is) to AC , so ED (is) to DF . I say that triangle ABC is equiangular to triangle DEF , and (that the triangles) will have the angles which corresponding sides subtend equal. (That is), (angle) ABC (equal) to DEF , BCA to EFD , and, further, BAC to EDF .

For let (angle) FEG , equal to angle ABC , and (angle) EFG , equal to ACB , have been constructed on the straight-line EF at the points E and F on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) at A is equal to the remaining (angle) at G [Prop. 1.32].

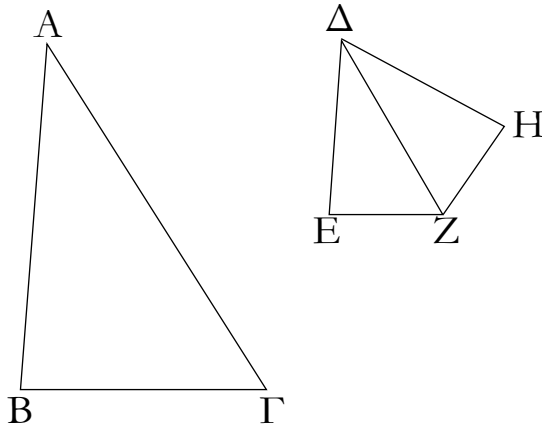
Thus, triangle ABC is equiangular to [triangle] EGF . Thus, for triangles ABC and EGF , the sides about the equal angles are proportional, and (those) sides subtending equal angles correspond [Prop. 6.4]. Thus, as AB is to BC , [so] GE (is) to EF . But, as AB (is) to BC , so, it was assumed, (is) DE to EF . Thus, as DE (is) to EF , so GE (is) to EF [Prop. 5.11]. Thus, DE and GE each have the same ratio to EF . Thus, DE is equal to GE [Prop. 5.9]. So, for the same (reasons), DF is also equal to GF . Therefore, since DE is equal to EG , and EF (is) common, the two (sides) DE , EF are equal to the two (sides) GE , EF (respectively). And base DF [is] equal to base FG . Thus, angle DEF is equal to angle GEF [Prop. 1.8], and triangle DEF (is) equal to triangle GEF , and the remaining angles (are) equal to the remaining angles which the equal sides subtend [Prop. 1.4]. Thus, angle DFE is also equal to GFE , and

ΑΒΓ ἄρα γωνία τῇ ὑπὸ ΔΕΖ ἐστὶν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΖΕ ἐστὶν ἴση, καὶ ἔτι ἡ πρὸς τῷ Α τῇ πρὸς τῷ Δ· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Ἐάν ἄρα δύο τρίγωνα τὰς πλευρὰς ἀνάλογον ἔχῃ, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἔξει τὰς γωνίας, ὅφ' ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

٢'.

Ἐάν δύο τρίγωνα μίαν γωνίαν μιᾶ γωνίᾳ ἴσην ἔχῃ, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἔξει τὰς γωνίας, ὅφ' ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν.



Ἐστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ μίαν γωνίαν τὴν ὑπὸ ΒΑΓ μιᾶ γωνίᾳ τῇ ὑπὸ ΕΔΖ ἴσην ἔχοντα, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ὥς τὴν ΒΑ πρὸς τὴν ΑΓ, οὕτως τὴν ΕΔ πρὸς τὴν ΔΖ· λέγω, ὅτι ἰσογώνιον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ καὶ ἴσην ἔξει τὴν ὑπὸ ΑΒΓ γωνίαν τῇ ὑπὸ ΔΕΖ, τὴν δὲ ὑπὸ ΑΓΒ τῇ ὑπὸ ΔΖΕ.

Συνεστάτω γὰρ πρὸς τῇ ΔΖ εὐθείᾳ καὶ τοῖς πρὸς αὐτῇ σημείοις τοῖς Δ, Ζ ὁποτέρῳ μὲν τῶν ὑπὸ ΒΑΓ, ΕΔΖ ἴση ἡ ὑπὸ ΖΔΗ, τῇ δὲ ὑπὸ ΑΓΒ ἴση ἡ ὑπὸ ΔΖΗ· λοιπὴ ἄρα ἡ πρὸς τῷ Β γωνία λοιπὴ τῇ πρὸς τῷ Η ἴση ἐστίν.

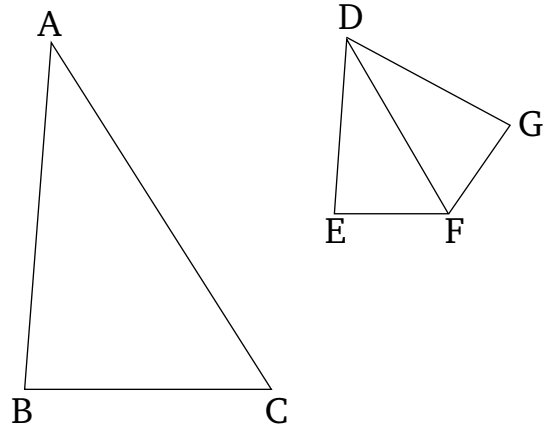
Ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΗΖ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὥς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΗΔ πρὸς τὴν ΔΖ. ὑπόκειται δὲ καὶ ὥς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΕΔ πρὸς τὴν ΔΖ· καὶ ὥς ἄρα ἡ ΕΔ πρὸς τὴν ΔΖ, οὕτως ἡ ΗΔ πρὸς τὴν ΔΖ. ἴση ἄρα ἡ ΕΔ τῇ ΗΔ· καὶ κοινὴ ἡ ΔΖ· δύο δὴ αἱ ΕΔ, ΔΖ δυσὶ ταῖς ΗΔ, ΔΖ ἴσας εἰσίν· καὶ γωνία ἡ ὑπὸ ΕΔΖ γωνία τῇ ὑπὸ ΗΔΖ [ἐστὶν] ἴση· βάσεις ἄρα ἡ ΕΖ βάσει τῇ ΗΖ ἐστὶν ἴση, καὶ τὸ ΔΕΖ τρίγωνον τῷ ΗΔΖ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσας ἔσονται, ὅφ' ἃς ἴσας πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ ΔΖΗ τῇ ὑπὸ ΔΖΕ, ἡ δὲ ὑπὸ ΔΗΖ

(angle) EDF to EGF . And since (angle) FED is equal to GEF , and (angle) GEF to ABC , angle ABC is thus also equal to DEF . So, for the same (reasons), (angle) ACB is also equal to DFE , and, further, the (angle) at A to the (angle) at D . Thus, triangle ABC is equiangular to triangle DEF .

Thus, if two triangles have proportional sides then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal. (Which is) the very thing it was required to show.

Proposition 6

If two triangles have one angle equal to one angle, and the sides about the equal angles proportional, then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal.



Let ABC and DEF be two triangles having one angle, BAC , equal to one angle, EDF (respectively), and the sides about the equal angles proportional, (so that) as BA (is) to AC , so ED (is) to DF . I say that triangle ABC is equiangular to triangle DEF , and will have angle ABC equal to DEF , and (angle) ACB to DFE .

For let (angle) FDG , equal to each of BAC and EDF , and (angle) DFG , equal to ACB , have been constructed on the straight-line AF at the points D and F on it (respectively) [Prop. 1.23]. Thus, the remaining angle at B is equal to the remaining angle at G [Prop. 1.32].

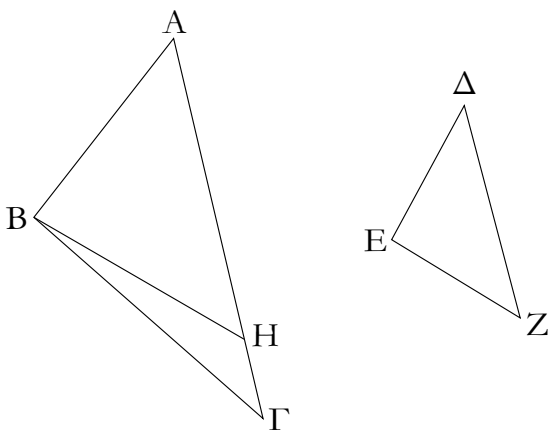
Thus, triangle ABC is equiangular to triangle DGF . Thus, proportionally, as BA (is) to AC , so GD (is) to DF [Prop. 6.4]. And it was also assumed that as BA (is) to AC , so ED (is) to DF . And, thus, as ED (is) to DF , so GD (is) to DF [Prop. 5.11]. Thus, ED (is) equal to DG [Prop. 5.9]. And DF (is) common. So, the two (sides) ED , DF are equal to the two (sides) GD , DF (respectively). And angle EDF [is] equal to angle GDF . Thus, base EF is equal to base GF , and triangle DEF is equal to triangle GDF , and the remaining angles

τῇ ὑπὸ ΔΕΖ. ἀλλ' ἡ ὑπὸ ΔΖΗ τῇ ὑπὸ ΑΓΒ ἐστὶν ἴση· καὶ ἡ ὑπὸ ΑΓΒ ἄρα τῇ ὑπὸ ΔΖΕ ἐστὶν ἴση. ὑπόκειται δὲ καὶ ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ ἴση· καὶ λοιπὴ ἄρα ἡ πρὸς τῷ Β λοιπῇ τῇ πρὸς τῷ Ε ἴση ἐστίν· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ.

Ἐὰν ἄρα δύο τρίγωνα μίαν γωνίαν μιᾶ γωνία ἴσην ἔχῃ, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἔξει τὰς γωνίας, ὅφ' ἃς αἱ ὁμόλογοι πλευραὶ ὑποτείνουσιν· ὅπερ ἔδει δεῖξαι.

ζ'.

Ἐὰν δύο τρίγωνα μίαν γωνίαν μιᾶ γωνία ἴσην ἔχῃ, περὶ δὲ ἄλλας γωνίας τὰς πλευρὰς ἀνάλογον, τῶν δὲ λοιπῶν ἑκατέραν ἅμα ἤτοι ἐλάσσονα ἢ μὴ ἐλάσσονα ὀρθῆς, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἔξει τὰς γωνίας, περὶ ἃς ἀνάλογόν εἰσιν αἱ πλευραί.



Ἐστω δύο τρίγωνα τὰ ΑΒΓ, ΔΕΖ μίαν γωνίαν μιᾶ γωνία ἴσην ἔχοντα τὴν ὑπὸ ΒΑΓ τῇ ὑπὸ ΕΔΖ, περὶ δὲ ἄλλας γωνίας τὰς ὑπὸ ΑΒΓ, ΔΕΖ τὰς πλευρὰς ἀνάλογον, ὥς τὴν ΑΒ πρὸς τὴν ΒΓ, οὕτως τὴν ΔΕ πρὸς τὴν ΕΖ, τῶν δὲ λοιπῶν τῶν πρὸς τοῖς Γ, Ζ πρότερον ἑκατέραν ἅμα ἐλάσσονα ὀρθῆς· λέγω, ὅτι ἰσογώνιον ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΔΕΖ τριγώνῳ, καὶ ἴση ἔσται ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ, καὶ λοιπὴ δηλονότι ἡ πρὸς τῷ Γ λοιπῇ τῇ πρὸς τῷ Ζ ἴση.

Εἰ γὰρ ἄνισός ἐστὶν ἡ ὑπὸ ΑΒΓ γωνία τῇ ὑπὸ ΔΕΖ, μία αὐτῶν μείζων ἐστίν. ἔστω μείζων ἡ ὑπὸ ΑΒΓ. καὶ συνεστήτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Β τῇ ὑπὸ ΔΕΖ γωνία ἴση ἡ ὑπὸ ΑΒΗ.

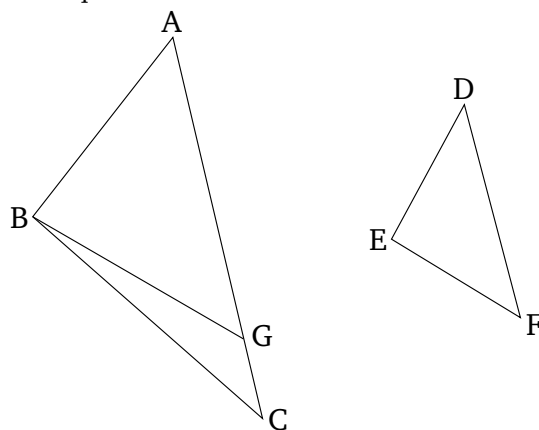
Καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν Α γωνία τῇ Δ, ἡ δὲ ὑπὸ ΑΒΗ τῇ ὑπὸ ΔΕΖ, λοιπὴ ἄρα ἡ ὑπὸ ΑΗΒ λοιπῇ τῇ ὑπὸ ΔΖΕ ἐστὶν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΗ τρίγωνον τῷ ΔΕΖ

will be equal to the remaining angles which the equal sides subtend [Prop. 1.4]. Thus, (angle) DFG is equal to DFE , and (angle) DGF to DEF . But, (angle) DFG is equal to ACB . Thus, (angle) ACB is also equal to DFE . And (angle) BAC was also assumed (to be) equal to EDF . Thus, the remaining (angle) at B is equal to the remaining (angle) at E [Prop. 1.32]. Thus, triangle ABC is equiangular to triangle DEF .

Thus, if two triangles have one angle equal to one angle, and the sides about the equal angles proportional, then the triangles will be equiangular, and will have the angles which corresponding sides subtend equal. (Which is) the very thing it was required to show.

Proposition 7

If two triangles have one angle equal to one angle, and the sides about other angles proportional, and the remaining angles either both less than, or both not less than, right-angles, then the triangles will be equiangular, and will have the angles about which the sides are proportional equal.



Let ABC and DEF be two triangles having one angle, BAC , equal to one angle, EDF (respectively), and the sides about (some) other angles, ABC and DEF (respectively), proportional, (so that) as AB (is) to BC , so DE (is) to EF , and the remaining (angles) at C and F , first of all, both less than right-angles. I say that triangle ABC is equiangular to triangle DEF , and (that) angle ABC will be equal to DEF , and (that) the remaining (angle) at C (will be) manifestly equal to the remaining (angle) at F .

For if angle ABC is not equal to (angle) DEF then one of them is greater. Let ABC be greater. And let (angle) ABG , equal to (angle) DEF , have been constructed on the straight-line AB at the point B on it [Prop. 1.23].

And since angle A is equal to (angle) D , and (angle) ABG to DEF , the remaining (angle) AGB is thus equal

τριγώνω. ἔστιν ἄρα ὡς ἡ AB πρὸς τὴν BH , οὕτως ἡ ΔE πρὸς τὴν EZ . ὡς δὲ ἡ ΔE πρὸς τὴν EZ , [οὕτως] ὑπόκειται ἡ AB πρὸς τὴν $B\Gamma$. ἡ AB ἄρα πρὸς ἑκατέραν τῶν $B\Gamma$, BH τὸν αὐτὸν ἔχει λόγον· ἴση ἄρα ἡ $B\Gamma$ τῇ BH . ὥστε καὶ γωνία ἡ πρὸς τῷ Γ γωνία τῇ ὑπὸ $BH\Gamma$ ἐστὶν ἴση. ἐλάττων δὲ ὀρθῆς ὑπόκειται ἡ πρὸς τῷ Γ . ἐλάττων ἄρα ἐστὶν ὀρθῆς καὶ ὑπὸ $BH\Gamma$. ὥστε ἡ ἐφεξῆς αὐτῇ γωνία ἡ ὑπὸ AHB μείζων ἐστὶν ὀρθῆς. καὶ ἐδείχθη ἴση οὕσα τῇ πρὸς τῷ Z . καὶ ἡ πρὸς τῷ Z ἄρα μείζων ἐστὶν ὀρθῆς. ὑπόκειται δὲ ἐλάσσων ὀρθῆς· ὅπερ ἐστὶν ἀτοπον. οὐκ ἄρα ἄνισός ἐστιν ἡ ὑπὸ $AB\Gamma$ γωνία τῇ ὑπὸ ΔEZ . ἴση ἄρα. ἔστι δὲ καὶ ἡ πρὸς τῷ A ἴση τῇ πρὸς τῷ Δ . καὶ λοιπὴ ἄρα ἡ πρὸς τῷ Γ λοιπῇ τῇ πρὸς τῷ Z ἴση ἐστίν. ἰσογώνιον ἄρα ἐστὶ τὸ $AB\Gamma$ τρίγωνον τῷ ΔEZ τριγώνω.

Ἀλλὰ δὴ πάλιν ὑποκείσθω ἑκατέρα τῶν πρὸς τοῖς Γ , Z μὴ ἐλάσσων ὀρθῆς· λέγω πάλιν, ὅτι καὶ οὕτως ἐστὶν ἰσογώνιον τὸ $AB\Gamma$ τρίγωνον τῷ ΔEZ τριγώνω.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δεῖξομεν, ὅτι ἴση ἐστὶν ἡ $B\Gamma$ τῇ BH . ὥστε καὶ γωνία ἡ πρὸς τῷ Γ τῇ ὑπὸ $BH\Gamma$ ἴση ἐστίν. οὐκ ἐλάττων δὲ ὀρθῆς ἡ πρὸς τῷ Γ . οὐκ ἐλάττων ἄρα ὀρθῆς οὐδὲ ἡ ὑπὸ $BH\Gamma$. τριγώνου δὲ τοῦ $BH\Gamma$ αἱ δύο γωνίαι δύο ὀρθῶν οὐκ εἰσιν ἐλάττονες· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα πάλιν ἄνισός ἐστιν ἡ ὑπὸ $AB\Gamma$ γωνία τῇ ὑπὸ ΔEZ . ἴση ἄρα. ἔστι δὲ καὶ ἡ πρὸς τῷ A τῇ πρὸς τῷ Δ ἴση· λοιπὴ ἄρα ἡ πρὸς τῷ Γ λοιπῇ τῇ πρὸς τῷ Z ἴση ἐστίν. ἰσογώνιον ἄρα ἐστὶ τὸ $AB\Gamma$ τρίγωνον τῷ ΔEZ τριγώνω.

Ἐὰν ἄρα δύο τρίγωνα μίαν γωνίαν μιᾶ γωνία ἴσην ἔχῃ, περὶ δὲ ἄλλας γωνίας τὰς πλευρὰς ἀνάλογον, τῶν δὲ λοιπῶν ἑκατέραν ἅμα ἐλάττονα ἢ μὴ ἐλάττονα ὀρθῆς, ἰσογώνια ἔσται τὰ τρίγωνα καὶ ἴσας ἔξει τὰς γωνίας, περὶ αἷς ἀνάλογόν εἰσιν αἱ πλευραὶ· ὅπερ ἔδει δεῖξαι.

η'.

Ἐὰν ἐν ὀρθογώνιῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἀχθῇ, τὰ πρὸς τῇ καθέτῳ τρίγωνα ὁμοία ἐστὶ τῷ τε ὅλῳ καὶ ἀλλήλοις.

Ἐστω τρίγωνον ὀρθογώνιον τὸ $AB\Gamma$ ὀρθὴν ἔχον τὴν ὑπὸ $BA\Gamma$ γωνίαν, καὶ ἦχθω ἀπὸ τοῦ A ἐπὶ τὴν $B\Gamma$ κάθετος ἡ AD . λέγω, ὅτι ὁμοίον ἐστὶν ἑκάτερον τῶν $AB\Delta$, $AD\Gamma$

to the remaining (angle) DFE [Prop. 1.32]. Thus, triangle ABG is equiangular to triangle DEF . Thus, as AB is to BG , so DE (is) to EF [Prop. 6.4]. And as DE (is) to EF , [so] it was assumed (is) AB to BC . Thus, AB has the same ratio to each of BC and BG [Prop. 5.11]. Thus, BC (is) equal to BG [Prop. 5.9]. And, hence, the angle at C is equal to angle BGC [Prop. 1.5]. And the angle at C was assumed (to be) less than a right-angle. Thus, (angle) BGC is also less than a right-angle. Hence, the adjacent angle to it, AGB , is greater than a right-angle [Prop. 1.13]. And (AGB) was shown to be equal to the (angle) at F . Thus, the (angle) at F is also greater than a right-angle. But it was assumed (to be) less than a right-angle. The very thing is absurd. Thus, angle ABC is not unequal to (angle) DEF . Thus, (it is) equal. And the (angle) at A is also equal to the (angle) at D . And thus the remaining (angle) at C is equal to the remaining (angle) at F [Prop. 1.32]. Thus, triangle ABC is equiangular to triangle DEF .

But, again, let each of the (angles) at C and F be assumed (to be) not less than a right-angle. I say, again, that triangle ABC is equiangular to triangle DEF in this case also.

For, with the same construction, we can similarly show that BC is equal to BG . Hence, also, the angle at C is equal to (angle) BGC . And the (angle) at C (is) not less than a right-angle. Thus, BGC (is) not less than a right-angle either. So, in triangle BGC the (sum of) two angles is not less than two right-angles. The very thing is impossible [Prop. 1.17]. Thus, again, angle ABC is not unequal to DEF . Thus, (it is) equal. And the (angle) at A is also equal to the (angle) at D . Thus, the remaining (angle) at C is equal to the remaining (angle) at F [Prop. 1.32]. Thus, triangle ABC is equiangular to triangle DEF .

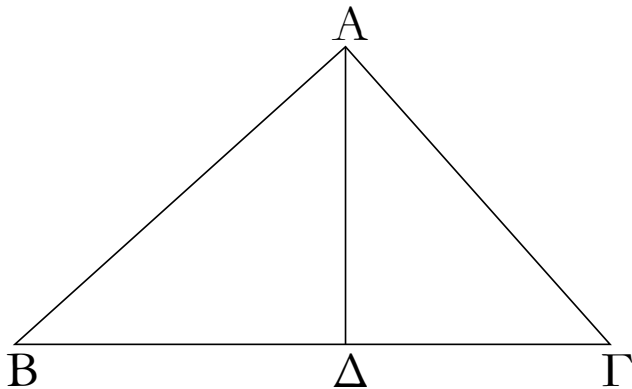
Thus, if two triangles have one angle equal to one angle, and the sides about other angles proportional, and the remaining angles both less than, or both not less than, right-angles, then the triangles will be equiangular, and will have the angles about which the sides (are) proportional equal. (Which is) the very thing it was required to show.

Proposition 8

If, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base then the triangles around the perpendicular are similar to the whole (triangle), and to one another.

Let ABC be a right-angled triangle having the angle BAC a right-angle, and let AD have been drawn from

τριγώνων ὅλῳ τῷ $AB\Gamma$ καὶ ἔτι ἀλλήλοις.



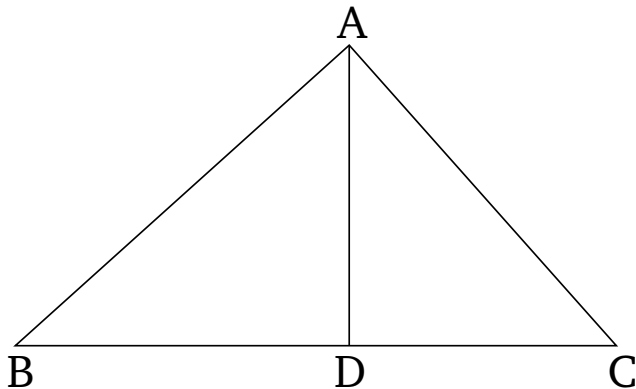
Ἐπεὶ γὰρ ἴση ἐστὶν ἡ ὑπὸ $BA\Gamma$ τῇ ὑπὸ $A\Delta B$ · ὀρθὴ γὰρ ἑκατέρα· καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε $AB\Gamma$ καὶ τοῦ $AB\Delta$ ἡ πρὸς τῷ B , λοιπὴ ἄρα ἡ ὑπὸ AGB λοιπὴ τῇ ὑπὸ $BA\Delta$ ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ $AB\Gamma$ τρίγωνον τῷ $AB\Delta$ τριγώνῳ. ἔστιν ἄρα ὡς ἡ $B\Gamma$ ὑποτείνουσα τὴν ὀρθὴν τοῦ $AB\Gamma$ τριγώνου πρὸς τὴν BA ὑποτείνουσαν τὴν ὀρθὴν τοῦ $AB\Delta$ τριγώνου, οὕτως αὐτὴ ἡ AB ὑποτείνουσα τὴν πρὸς τῷ Γ γωνίαν τοῦ $AB\Gamma$ τριγώνου πρὸς τὴν $B\Delta$ ὑποτείνουσαν τὴν ἴσην τὴν ὑπὸ $BA\Delta$ τοῦ $AB\Delta$ τριγώνου, καὶ ἔτι ἡ AG πρὸς τὴν $A\Delta$ ὑποτείνουσαν τὴν πρὸς τῷ B γωνίαν κοινήν τῶν δύο τριγώνων. τὸ $AB\Gamma$ ἄρα τρίγωνον τῷ $AB\Delta$ τριγώνῳ ἰσογώνιον τέ ἐστι καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει. ὅμοιον ἄρα [ἐστὶ] τὸ $AB\Gamma$ τρίγωνον τῷ $AB\Delta$ τριγώνῳ. ὁμοίως δὲ δείξομεν, ὅτι καὶ τῷ $A\Delta\Gamma$ τριγώνῳ ὅμοιον ἐστὶ τὸ $AB\Gamma$ τρίγωνον· ἑκάτερον ἄρα τῶν $AB\Delta$, $A\Delta\Gamma$ [τριγώνων] ὅμοιον ἐστὶν ὅλῳ τῷ $AB\Gamma$.

Λέγω δὴ, ὅτι καὶ ἀλλήλοις ἐστὶν ὅμοια τὰ $AB\Delta$, $A\Delta\Gamma$ τρίγωνα.

Ἐπεὶ γὰρ ὀρθὴ ἡ ὑπὸ $B\Delta A$ ὀρθὴ τῇ ὑπὸ $A\Delta\Gamma$ ἐστὶν ἴση, ἀλλὰ μὴν καὶ ἡ ὑπὸ $BA\Delta$ τῇ πρὸς τῷ Γ ἐδείχθη ἴση, καὶ λοιπὴ ἄρα ἡ πρὸς τῷ B λοιπὴ τῇ ὑπὸ $\Delta A\Gamma$ ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ $AB\Delta$ τρίγωνον τῷ $A\Delta\Gamma$ τριγώνῳ. ἔστιν ἄρα ὡς ἡ $B\Delta$ τοῦ $AB\Delta$ τριγώνου ὑποτείνουσα τὴν ὑπὸ $BA\Delta$ πρὸς τὴν ΔA τοῦ $A\Delta\Gamma$ τριγώνου ὑποτείνουσαν τὴν πρὸς τῷ Γ ἴσην τῇ ὑπὸ $BA\Delta$, οὕτως αὐτὴ ἡ $A\Delta$ τοῦ $AB\Delta$ τριγώνου ὑποτείνουσα τὴν πρὸς τῷ B γωνίαν πρὸς τὴν $\Delta\Gamma$ ὑποτείνουσαν τὴν ὑπὸ $\Delta A\Gamma$ τοῦ $A\Delta\Gamma$ τριγώνου ἴσην τῇ πρὸς τῷ B , καὶ ἔτι ἡ BA πρὸς τὴν AG ὑποτείνουσαι τὰς ὀρθάς· ὅμοιον ἄρα ἐστὶ τὸ $AB\Delta$ τρίγωνον τῷ $A\Delta\Gamma$ τριγώνῳ.

Ἐὰν ἄρα ἐν ὀρθογώνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βᾶσιν κάθετος ἀχθῇ, τὰ πρὸς τῇ καθέτῳ τρίγωνα ὁμοιά ἐστι τῷ τε ὅλῳ καὶ ἀλλήλοις [ὅπερ ἔδει δεῖξαι].

A , perpendicular to BC [Prop. 1.12]. I say that triangles ABD and ADC are each similar to the whole (triangle) ABC and, further, to one another.



For since (angle) BAC is equal to ADB —for each (are) right-angles—and the (angle) at B (is) common to the two triangles ABC and ABD , the remaining (angle) ACB is thus equal to the remaining (angle) BAD [Prop. 1.32]. Thus, triangle ABC is equiangular to triangle ABD . Thus, as BC , subtending the right-angle in triangle ABC , is to BA , subtending the right-angle in triangle ABD , so the same AB , subtending the angle at C in triangle ABC , (is) to BD , subtending the equal (angle) BAD in triangle ABD , and, further, (so is) AC to AD , (both) subtending the angle at B common to the two triangles [Prop. 6.4]. Thus, triangle ABC is equiangular to triangle ABD , and has the sides about the equal angles proportional. Thus, triangle ABC [is] similar to triangle ABD [Def. 6.1]. So, similarly, we can show that triangle ABC is also similar to triangle ADC . Thus, [triangles] ABD and ADC are each similar to the whole (triangle) ABC .

So I say that triangles ABD and ADC are also similar to one another.

For since the right-angle BDA is equal to the right-angle ADC , and, indeed, (angle) BAD was also shown (to be) equal to the (angle) at C , thus the remaining (angle) at B is also equal to the remaining (angle) DAC [Prop. 1.32]. Thus, triangle ABD is equiangular to triangle ADC . Thus, as BD , subtending (angle) BAD in triangle ABD , is to DA , subtending the (angle) at C in triangle ADC , (which is) equal to (angle) BAD , so (is) the same AD , subtending the angle at B in triangle ABD , to DC , subtending (angle) DAC in triangle ADC , (which is) equal to the (angle) at B , and, further, (so is) BA to AC , (each) subtending right-angles [Prop. 6.4]. Thus, triangle ABD is similar to triangle ADC [Def. 6.1].

Thus, if, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base

then the triangles around the perpendicular are similar to the whole (triangle), and to one another. [(Which is) the very thing it was required to show.]

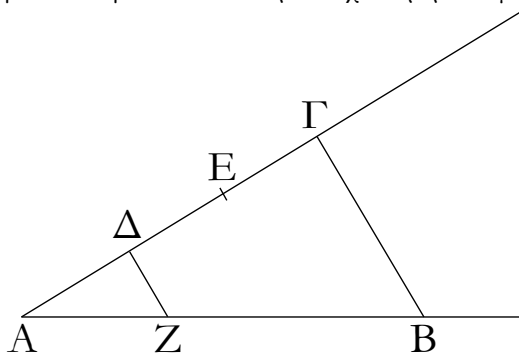
Πόρισμα.

Ἐκ δὴ τοῦτου φανερόν, ὅτι ἐὰν ἐν ὀρθογωνίῳ τριγῶνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἀχθῇ, ἡ ἀχθεῖσα τῶν τῆς βάσεως τμημάτων μέση ἀνάλογόν ἐστιν ὅπερ ἔδει δεῖξαι.

[†] In other words, the perpendicular is the geometric mean of the pieces.

 ϑ'

Τῆς δοθείσης εὐθείας τὸ προσταχθὲν μέρος ἀφελεῖν.



Ἐστω ἡ δοθεῖσα εὐθεῖα ἡ AB . δεῖ δὴ τῆς AB τὸ προ-
σταγθὲν μέρος ἀφελεῖν.

Ἐπιτετάχθω δὴ τὸ τρίτον. [καὶ] διηθῶ τις ἀπὸ τοῦ Α εὐθεῖα ἡ ΑΓ γωνίαν περιέχουσα μετὰ τῆς ΑΒ τυχοῦσαν· καὶ εἰληφθῶ τυχὸν σημεῖον ἐπὶ τῆς ΑΓ τὸ Δ, καὶ κείσθωσαν τῇ ΑΔ ἴσαι αἱ ΔΕ, ΕΓ. καὶ ἐπεζεύχθω ἡ ΒΓ, καὶ διὰ τοῦ Δ παράλληλος αὐτῇ ᾗχθω ἡ ΔΖ.

Ἐπεὶ οὖν τριγώνου τοῦ ΑΒΓ παρὰ μίαν τῶν πλευρῶν τὴν ΒΓ ἤκται ἡ ΖΔ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΓΔ πρὸς τὴν ΔΑ, οὕτως ἡ ΒΖ πρὸς τὴν ΖΑ. διπλῇ δὲ ἡ ΓΔ τῆς ΔΑ· διπλῇ ἄρα καὶ ἡ ΒΖ τῆς ΖΑ· τριπλῇ ἄρα ἡ ΒΑ τῆς ΑΖ.

Τῆς ἄρα δοθείσης εὐθείας τῆς ΑΒ τὸ ἐπιταχθὲν τρίτον μέρος ἀφήρηται τὸ ΑΖ· ὅπερ ἔδει ποιῆσαι.

6.

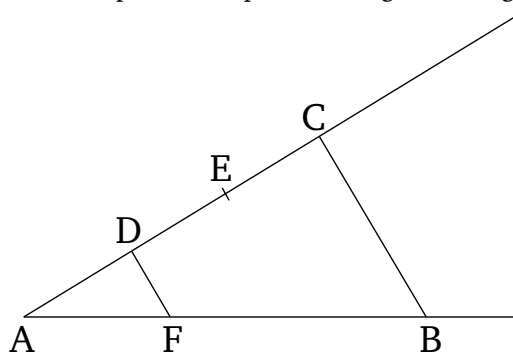
Τὴν δοθεῖσαν εὐθεῖαν ἄτητον τῇ δοθείσῃ τετμημένη
ὁμοίως τεμεῖν.

Corollary

So (it is) clear, from this, that if, in a right-angled triangle, a (straight-line) is drawn from the right-angle perpendicular to the base then the (straight-line so) drawn is in mean proportion to the pieces of the base.[†] (Which is) the very thing it was required to show.

Proposition 9

To cut off a prescribed part from a given straight-line.



Let AB be the given straight-line. So it is required to cut off a prescribed part from AB .

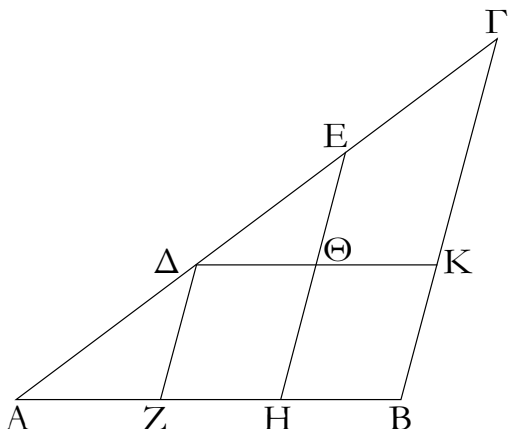
So let a third (part) have been prescribed. [And] let some straight-line AC have been drawn from (point) A , encompassing a random angle with AB . And let a random point D have been taken on AC . And let DE and EC be made equal to AD [Prop. 1.3]. And let BC have been joined. And let DF have been drawn through D parallel to it [Prop. 1.31].

Therefore, since FD has been drawn parallel to one of the sides, BC , of triangle ABC , then, proportionally, as CD is to DA , so BF (is) to FA [Prop. 6.2]. And CD (is) double DA . Thus, BF (is) also double FA . Thus, BA (is) triple AF .

Thus, the prescribed third part, AF , has been cut off from the given straight-line, AB . (Which is) the very thing it was required to do.

Proposition 10

To cut a given uncut straight-line similarly to a given cut (straight-line).



Ἐστω ἡ μὲν δοθεῖσα εὐθεῖα ἀτμητος ἡ AB, ἡ δὲ τετμημένη ἡ AG κατὰ τὰ Δ, E σημεία, καὶ κείσθωσαν ὥστε γωνίαν τυχούσαν περιέχειν, καὶ ἐπεζεύχθω ἡ GB, καὶ διὰ τῶν Δ, E τῇ BG παράλληλοι ῥηθῶσαν αἱ ΔZ, EH, διὰ δὲ τοῦ Δ τῇ AB παράλληλος ῥηθῶ ἡ ΔΘK.

Παράλληλόγραμμον ἄρα ἐστὶν ἐκάτερον τῶν ZΘ, ΘB· ἴση ἄρα ἡ μὲν ΔΘ τῇ ZH, ἡ δὲ ΘK τῇ HB. καὶ ἐπεὶ τριγώνου τοῦ ΔKΓ παρὰ μίαν τῶν πλευρῶν τὴν KΓ εὐθεῖα ῥηται ἡ ΘE, ἀνάλογον ἄρα ἐστὶν ὡς ἡ GE πρὸς τὴν EΔ, οὕτως ἡ KΘ πρὸς τὴν ΘΔ. ἴση δὲ ἡ μὲν KΘ τῇ BH, ἡ δὲ ΘΔ τῇ HZ. ἐστὶν ἄρα ὡς ἡ GE πρὸς τὴν EΔ, οὕτως ἡ BH πρὸς τὴν HZ. πάλιν, ἐπεὶ τριγώνου τοῦ AHE παρὰ μίαν τῶν πλευρῶν τὴν HE ῥηται ἡ ZΔ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ EΔ πρὸς τὴν ΔA, οὕτως ἡ HZ πρὸς τὴν ZA. ἐδείχθη δὲ καὶ ὡς ἡ GE πρὸς τὴν EΔ, οὕτως ἡ BH πρὸς τὴν HZ· ἐστὶν ἄρα ὡς μὲν ἡ GE πρὸς τὴν EΔ, οὕτως ἡ BH πρὸς τὴν HZ, ὡς δὲ ἡ EΔ πρὸς τὴν ΔA, οὕτως ἡ HZ πρὸς τὴν ZA.

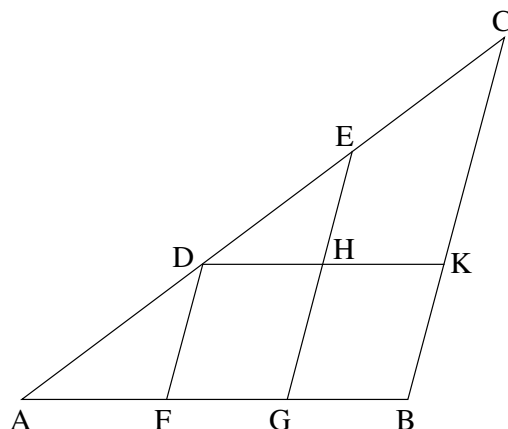
Ἡ ἄρα δοθεῖσα εὐθεῖα ἀτμητος ἡ AB τῇ δοθείσῃ εὐθείᾳ τετμημένῃ τῇ AG ὁμοίως τέτμηται· ὅπερ ἔδει ποιῆσαι.

ια'.

Δύο δοθεισῶν εὐθειῶν τρίτην ἀνάλογον προσευρεῖν.

Ἐστώσαν αἱ δοθεῖσαι [δύο εὐθεῖαι] αἱ BA, AG καὶ κείσθωσαν γωνίαν περιέχουσαι τυχούσαν. δεῖ δὴ τῶν BA, AG τρίτην ἀνάλογον προσευρεῖν. ἐκβεβλήσθωσαν γὰρ ἐπὶ τὰ Δ, E σημεία, καὶ κείσθω τῇ AG ἴση ἡ BΔ, καὶ ἐπεζεύχθω ἡ BG, καὶ διὰ τοῦ Δ παράλληλος αὐτῇ ῥηθῶ ἡ ΔE.

Ἐπεὶ οὖν τριγώνου τοῦ AΔE παρὰ μίαν τῶν πλευρῶν τὴν ΔE ῥηται ἡ BΓ, ἀνάλογόν ἐστιν ὡς ἡ AB πρὸς τὴν BΔ, οὕτως ἡ AG πρὸς τὴν GE. ἴση δὲ ἡ BΔ τῇ AG. ἐστὶν ἄρα ὡς ἡ AB πρὸς τὴν AG, οὕτως ἡ AG πρὸς τὴν GE.



Let AB be the given uncut straight-line, and AC a (straight-line) cut at points D and E, and let (AC) be laid down so as to encompass a random angle (with AB). And let CB have been joined. And let DF and EG have been drawn through (points) D and E (respectively), parallel to BC, and let DHK have been drawn through (point) D, parallel to AB [Prop. 1.31].

Thus, FH and HB are each parallelograms. Thus, DH (is) equal to FG, and HK to GB [Prop. 1.34]. And since the straight-line HE has been drawn parallel to one of the sides, KC, of triangle DKC, thus, proportionally, as CE is to ED, so KH (is) to HD [Prop. 6.2]. And KH (is) equal to BG, and HD to GF. Thus, as CE is to ED, so BG (is) to GF. Again, since FD has been drawn parallel to one of the sides, GE, of triangle AGE, thus, proportionally, as ED is to DA, so GF (is) to FA [Prop. 6.2]. And it was also shown that as CE (is) to ED, so BG (is) to GF. Thus, as CE is to ED, so BG (is) to GF, and as ED (is) to DA, so GF (is) to FA.

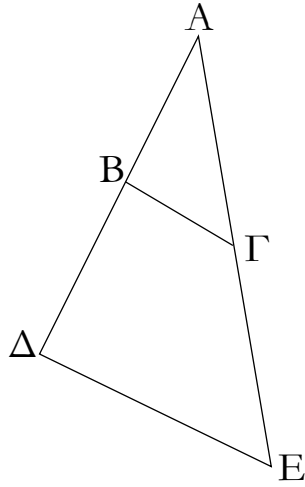
Thus, the given uncut straight-line, AB, has been cut similarly to the given cut straight-line, AC. (Which is) the very thing it was required to do.

Proposition 11

To find a third (straight-line) proportional to two given straight-lines.

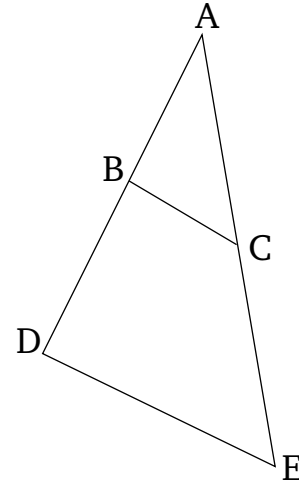
Let BA and AC be the [two] given [straight-lines], and let them be laid down encompassing a random angle. So it is required to find a third (straight-line) proportional to BA and AC. For let (BA and AC) have been produced to points D and E (respectively), and let BD be made equal to AC [Prop. 1.3]. And let BC have been joined. And let DE have been drawn through (point) D parallel to it [Prop. 1.31].

Therefore, since BC has been drawn parallel to one of the sides DE of triangle ADE, proportionally, as AB is to BD, so AC (is) to CE [Prop. 6.2]. And BD (is) equal



Δύο ἄρα δοθεισῶν εὐθειῶν τῶν AB , AG τρίτη ἀνάλογον αὐταῖς προσεύρηται ἡ GE · ὅπερ ἔδει ποιῆσαι.

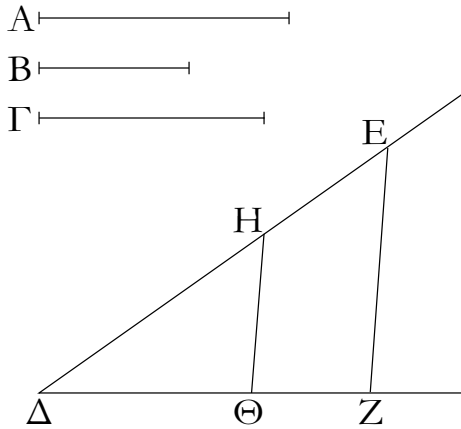
to AC . Thus, as AB is to AC , so AC (is) to CE .



Thus, a third (straight-line), CE , has been found (which is) proportional to the two given straight-lines, AB and AC . (Which is) the very thing it was required to do.

ιβ'.

Τριῶν δοθεισῶν εὐθειῶν τετάρτην ἀνάλογον προσευρεῖν.



Ἐστωσαν αἱ δοθεῖσαι τρεῖς εὐθεῖαι αἱ A , B , Γ · δεῖ δὴ τῶν A , B , Γ τετάρτην ἀνάλογον προσευρεῖν.

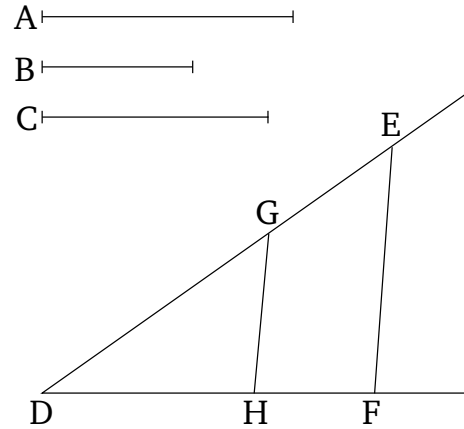
Ἐκκείσθωσαν δύο εὐθεῖαι αἱ DE , DZ γωνίαν περιέχουσai [τυχοῦσαν] τὴν ὑπὸ EDZ · καὶ κείσθω τῇ μὲν A ἴση ἡ DH , τῇ δὲ B ἴση ἡ HE , καὶ ἔτι τῇ Γ ἴση ἡ $D\Theta$ · καὶ ἐπιζευχθείσης τῆς $H\Theta$ παράλληλος αὐτῇ ἦχθω διὰ τοῦ E ἡ EZ .

Ἐπεὶ οὖν τριγώνου τοῦ DEZ παρὰ μίαν τὴν EZ ἦκται ἡ $H\Theta$, ἔστιν ἄρα ὡς ἡ DH πρὸς τὴν HE , οὕτως ἡ $D\Theta$ πρὸς τὴν ΘZ . ἴση δὲ ἡ μὲν DH τῇ A , ἡ δὲ HE τῇ B , ἡ δὲ $D\Theta$ τῇ Γ · ἔστιν ἄρα ὡς ἡ A πρὸς τὴν B , οὕτως ἡ Γ πρὸς τὴν ΘZ .

Τριῶν ἄρα δοθεισῶν εὐθειῶν τῶν A , B , Γ τετάρτη ἀνάλογον προσεύρηται ἡ ΘZ · ὅπερ ἔδει ποιῆσαι.

Proposition 12

To find a fourth (straight-line) proportional to three given straight-lines.



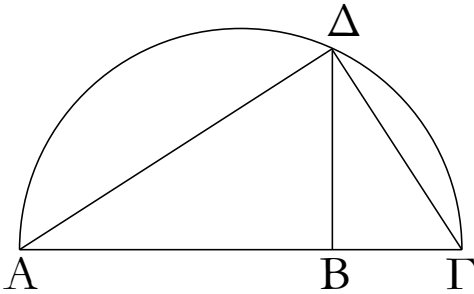
Let A , B , and C be the three given straight-lines. So it is required to find a fourth (straight-line) proportional to A , B , and C .

Let the two straight-lines DE and DF be set out encompassing the [random] angle EDF . And let DG be made equal to A , and GE to B , and, further, DH to C [Prop. 1.3]. And GH being joined, let EF have been drawn through (point) E parallel to it [Prop. 1.31].

Therefore, since GH has been drawn parallel to one of the sides EF of triangle DEF , thus as DG is to GE , so DH (is) to HF [Prop. 6.2]. And DG (is) equal to A , and GE to B , and DH to C . Thus, as A is to B , so C (is)

ιγ'.

Δύο δοθεισῶν εὐθειῶν μέσην ἀνάλογον προσευρεῖν.



Ἐστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ AB , $BΓ$. δεῖ δὴ τῶν AB , $BΓ$ μέσην ἀνάλογον προσευρεῖν.

Κεῖσθωσαν ἐπ' εὐθείας, καὶ γεγράφθω ἐπὶ τῆς AG ἡμικύκλιον τὸ $ADΓ$, καὶ ἤχθω ἀπὸ τοῦ B σημείου τῇ AG εὐθείᾳ πρὸς ὀρθὰς ἡ BA , καὶ ἐπεζεύχθωσαν αἱ AD , $DΓ$.

Ἐπεὶ ἐν ἡμικυκλίῳ γωνία ἐστὶν ἡ ὑπὸ $ADΓ$, ὀρθή ἐστιν. καὶ ἐπεὶ ἐν ὀρθογωνίῳ τριγώνῳ τῷ $ADΓ$ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἤκται ἡ DB , ἡ DB ἄρα τῶν τῆς βάσεως τμημάτων τῶν AB , $BΓ$ μέση ἀνάλογόν ἐστιν.

Δύο ἄρα δοθεισῶν εὐθειῶν τῶν AB , $BΓ$ μέση ἀνάλογον προσεύρηται ἡ DB . ὅπερ ἔδει ποιῆσαι.

† In other words, to find the geometric mean of two given straight-lines.

ιδ'.

Τῶν ἴσων τε καὶ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ὧν ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἕκαστα.

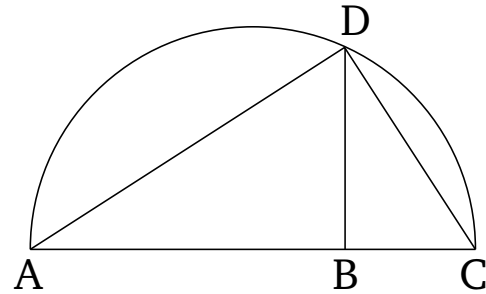
Ἐστω ἴσα τε καὶ ἰσογώνια παραλληλόγραμμα τὰ AB , $BΓ$ ἴσας ἔχοντα τὰς πρὸς τῷ B γωνίας, καὶ κεῖσθωσαν ἐπ' εὐθείας αἱ $ΔB$, BE . ἐπ' εὐθείας ἄρα εἰσὶ καὶ αἱ ZB , BH . λέγω, ὅτι τῶν AB , $BΓ$ ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, τουτέστιν, ὅτι ἐστὶν ὡς ἡ $ΔB$ πρὸς τὴν BE , οὕτως ἡ HB πρὸς τὴν BZ .

to HF .

Thus, a fourth (straight-line), HF , has been found (which is) proportional to the three given straight-lines, A , B , and C . (Which is) the very thing it was required to do.

Proposition 13

To find the (straight-line) in mean proportion to two given straight-lines.[†]



Let AB and BC be the two given straight-lines. So it is required to find the (straight-line) in mean proportion to AB and BC .

Let (AB and BC) be laid down straight-on (with respect to one another), and let the semi-circle ADC have been drawn on AC [Prop. 1.10]. And let BD have been drawn from (point) B , at right-angles to AC [Prop. 1.11]. And let AD and DC have been joined.

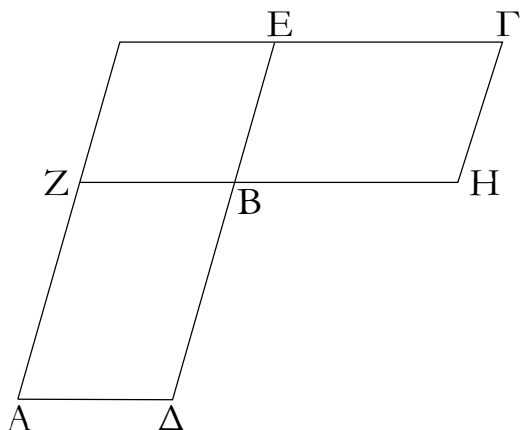
And since ADC is an angle in a semi-circle, it is a right-angle [Prop. 3.31]. And since, in the right-angled triangle ADC , the (straight-line) DB has been drawn from the right-angle perpendicular to the base, DB is thus the mean proportional to the pieces of the base, AB and BC [Prop. 6.8 corr.].

Thus, DB has been found (which is) in mean proportion to the two given straight-lines, AB and BC . (Which is) the very thing it was required to do.

Proposition 14

In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional. And those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.

Let AB and BC be equal and equiangular parallelograms having the angles at B equal. And let DB and BE be laid down straight-on (with respect to one another). Thus, FB and BG are also straight-on (with respect to one another) [Prop. 1.14]. I say that the sides of AB and



Συμπεπληρώσθω γὰρ τὸ ZE παραλληλόγραμμον. ἐπεὶ οὖν ἴσον ἐστὶ τὸ AB παραλληλόγραμμον τῷ BG παραλληλογράμῳ, ἄλλο δέ τι τὸ ZE, ἔστιν ἄρα ὡς τὸ AB πρὸς τὸ ZE, οὕτως τὸ BG πρὸς τὸ ZE. ἀλλ' ὡς μὲν τὸ AB πρὸς τὸ ZE, οὕτως ἡ ΔB πρὸς τὴν BE, ὡς δὲ τὸ BG πρὸς τὸ ZE, οὕτως ἡ HB πρὸς τὴν BZ· καὶ ὡς ἄρα ἡ ΔB πρὸς τὴν BE, οὕτως ἡ HB πρὸς τὴν BZ. τῶν ἄρα AB, BG παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας.

Ἀλλὰ δὴ ἔστω ὡς ἡ ΔB πρὸς τὴν BE, οὕτως ἡ HB πρὸς τὴν BZ· λέγω, ὅτι ἴσον ἐστὶ τὸ AB παραλληλόγραμμον τῷ BG παραλληλογράμῳ.

Ἐπεὶ γάρ ἐστιν ὡς ἡ ΔB πρὸς τὴν BE, οὕτως ἡ HB πρὸς τὴν BZ, ἀλλ' ὡς μὲν ἡ ΔB πρὸς τὴν BE, οὕτως τὸ AB παραλληλόγραμμον πρὸς τὸ ZE παραλληλόγραμμον, ὡς δὲ ἡ HB πρὸς τὴν BZ, οὕτως τὸ BG παραλληλόγραμμον πρὸς τὸ ZE παραλληλόγραμμον, καὶ ὡς ἄρα τὸ AB πρὸς τὸ ZE, οὕτως τὸ BG πρὸς τὸ ZE· ἴσον ἄρα ἐστὶ τὸ AB παραλληλόγραμμον τῷ BG παραλληλογράμῳ.

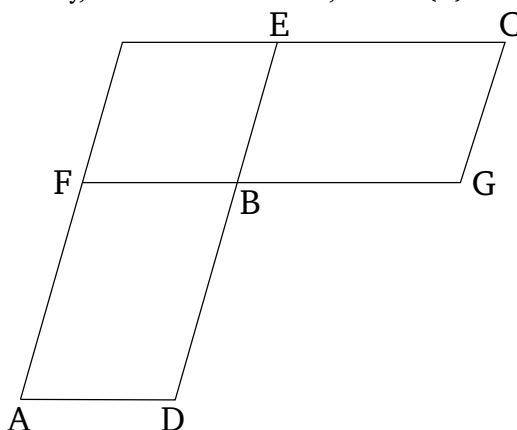
Τῶν ἄρα ἴσων τε καὶ ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ὧν ἰσογωνίων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα· ὅπερ ἔδει δεῖξαι.

ιε'.

Τῶν ἴσων καὶ μίαν μὲν ἴσην ἔχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ὧν μίαν μὲν ἴσην ἔχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα.

Ἐστω ἴσα τρίγωνα τὰ ABΓ, AΔE μίαν μὲν ἴσην ἔχοντα γωνίαν τὴν ὑπὸ BΑΓ τῇ ὑπὸ ΔAE· λέγω, ὅτι τῶν ABΓ, AΔE τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, τουτέστιν, ὅτι ἐστὶν ὡς ἡ ΓA πρὸς τὴν AΔ, οὕτως

BC about the equal angles are reciprocally proportional, that is to say, that as DB is to BE, so GB (is) to BF.



For let the parallelogram FE have been completed. Therefore, since parallelogram AB is equal to parallelogram BC, and FE (is) some other (parallelogram), thus as (parallelogram) AB is to FE, so (parallelogram) BC (is) to FE [Prop. 5.7]. But, as (parallelogram) AB (is) to FE, so DB (is) to BE, and as (parallelogram) BC (is) to FE, so GB (is) to BF [Prop. 6.1]. Thus, also, as DB (is) to BE, so GB (is) to BF. Thus, in parallelograms AB and BC the sides about the equal angles are reciprocally proportional.

And so, let DB be to BE, as GB (is) to BF. I say that parallelogram AB is equal to parallelogram BC.

For since as DB is to BE, so GB (is) to BF, but as DB (is) to BE, so parallelogram AB (is) to parallelogram FE, and as GB (is) to BF, so parallelogram BC (is) to parallelogram FE [Prop. 6.1], thus, also, as (parallelogram) AB (is) to FE, so (parallelogram) BC (is) to FE [Prop. 5.11]. Thus, parallelogram AB is equal to parallelogram BC [Prop. 5.9].

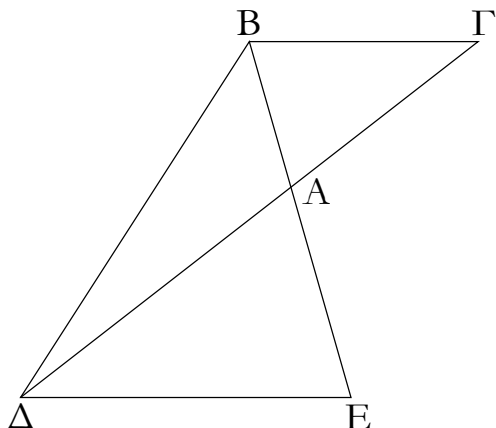
Thus, in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional. And those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal. (Which is) the very thing it was required to show.

Proposition 15

In equal triangles also having one angle equal to one (angle) the sides about the equal angles are reciprocally proportional. And those triangles having one angle equal to one angle for which the sides about the equal angles (are) reciprocally proportional are equal.

Let ABC and ADE be equal triangles having one angle equal to one (angle), (namely) BAC (equal) to DAE. I say that, in triangles ABC and ADE, the sides about the

ἡ EA πρὸς τὴν AB .



Κεῖσθω γὰρ ὥστε ἐπ' εὐθείας εἶναι τὴν GA τῇ AD · ἐπ' εὐθείας ἄρα ἐστὶ καὶ ἡ EA τῇ AB . καὶ ἐπεζεύχθω ἡ BD .

Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ABG τρίγωνον τῷ ADE τριγώνῳ, ἄλλο δέ τι τὸ $BA\Delta$, ἔστιν ἄρα ὡς τὸ ΓAB τρίγωνον πρὸς τὸ $BA\Delta$ τρίγωνον, οὕτως τὸ $EA\Delta$ τρίγωνον πρὸς τὸ $BA\Delta$ τρίγωνον. ἀλλ' ὡς μὲν τὸ ΓAB πρὸς τὸ $BA\Delta$, οὕτως ἡ ΓA πρὸς τὴν AD , ὡς δὲ τὸ $EA\Delta$ πρὸς τὸ $BA\Delta$, οὕτως ἡ EA πρὸς τὴν AB . καὶ ὡς ἄρα ἡ ΓA πρὸς τὴν AD , οὕτως ἡ EA πρὸς τὴν AB . τῶν ABG , ADE ἄρα τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας.

Ἀλλὰ δὴ ἀντιπεπονηθέντων αἱ πλευραὶ τῶν ABG , ADE τριγώνων, καὶ ἔστω ὡς ἡ ΓA πρὸς τὴν AD , οὕτως ἡ EA πρὸς τὴν AB · λέγω, ὅτι ἴσον ἐστὶ τὸ ABG τρίγωνον τῷ ADE τριγώνῳ.

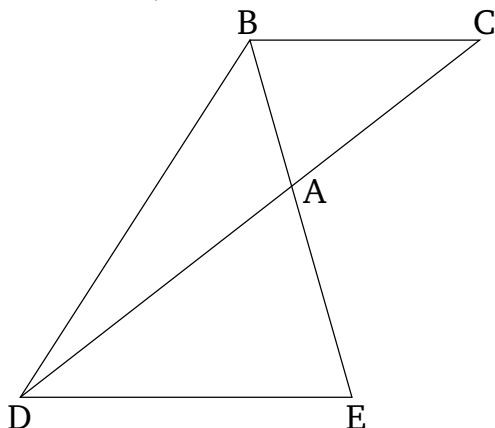
Ἐπιζευχθείσης γὰρ πάλιν τῆς BD , ἐπεὶ ἐστὶν ὡς ἡ ΓA πρὸς τὴν AD , οὕτως ἡ EA πρὸς τὴν AB , ἀλλ' ὡς μὲν ἡ ΓA πρὸς τὴν AD , οὕτως τὸ ABG τρίγωνον πρὸς τὸ $BA\Delta$ τρίγωνον, ὡς δὲ ἡ EA πρὸς τὴν AB , οὕτως τὸ $EA\Delta$ τρίγωνον πρὸς τὸ $BA\Delta$ τρίγωνον, ὡς ἄρα τὸ ABG τρίγωνον πρὸς τὸ $BA\Delta$ τρίγωνον, οὕτως τὸ $EA\Delta$ τρίγωνον πρὸς τὸ $BA\Delta$ τρίγωνον. ἐκάτερον ἄρα τῶν ABG , $EA\Delta$ πρὸς τὸ $BA\Delta$ τὸν αὐτὸν ἔχει λόγον. ἴσων ἄρα ἐστὶ τὸ ABG [τρίγωνον] τῷ $EA\Delta$ τριγώνῳ.

Τῶν ἄρα ἴσων καὶ μίαν μιᾷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· καὶ ὡς μίαν μιᾷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἐκεῖνα ἴσα ἐστὶν· ὅπερ ἔδει δεῖξαι.

١٧'.

Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ᾖσιν, τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογώνῳ· καὶ τὸ ὑπὸ τῶν ἄκρων

equal angles are reciprocally proportional, that is to say, that as CA is to AD , so EA (is) to AB .



For let CA be laid down so as to be straight-on (with respect) to AD . Thus, EA is also straight-on (with respect) to AB [Prop. 1.14]. And let BD have been joined.

Therefore, since triangle ABC is equal to triangle ADE , and BAD (is) some other (triangle), thus as triangle CAB is to triangle BAD , so triangle EAD (is) to triangle BAD [Prop. 5.7]. But, as (triangle) CAB (is) to BAD , so CA (is) to AD , and as (triangle) EAD (is) to BAD , so EA (is) to AB [Prop. 6.1]. And thus, as CA (is) to AD , so EA (is) to AB . Thus, in triangles ABC and ADE the sides about the equal angles (are) reciprocally proportional.

And so, let the sides of triangles ABC and ADE be reciprocally proportional, and (thus) let CA be to AD , as EA (is) to AB . I say that triangle ABC is equal to triangle ADE .

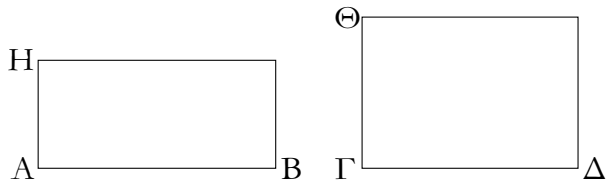
For, BD again being joined, since as CA is to AD , so EA (is) to AB , but as CA (is) to AD , so triangle ABC (is) to triangle BAD , and as EA (is) to AB , so triangle EAD (is) to triangle BAD [Prop. 6.1], thus as triangle ABC (is) to triangle BAD , so triangle EAD (is) to triangle BAD . Thus, (triangles) ABC and EAD each have the same ratio to BAD . Thus, [triangle] ABC is equal to triangle EAD [Prop. 5.9].

Thus, in equal triangles also having one angle equal to one (angle) the sides about the equal angles (are) reciprocally proportional. And those triangles having one angle equal to one angle for which the sides about the equal angles (are) reciprocally proportional are equal. (Which is) the very thing it was required to show.

Proposition 16

If four straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the rectangle contained by the middle (two). And if the rect-

περιεχόμενον ὀρθογώνιον ἴσον ἢ τῷ ὑπὸ τῶν μέσων περιε-
χομένῳ ὀρθογώνιῳ, αἱ τέσσαρες εὐθεῖαι ἀνάλογον ἔσονται.



Ἐστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ AB, ΓΔ, E, Z,
ὥς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ E πρὸς τὴν Z· λέγω, ὅτι
τὸ ὑπὸ τῶν AB, Z περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ
ὑπὸ τῶν ΓΔ, E περιεχομένῳ ὀρθογώνιῳ.

Ἦχθωσαν [γὰρ] ἀπὸ τῶν A, Γ σημείων ταῖς AB, ΓΔ
εὐθείαις πρὸς ὀρθὰς αἱ AH, ΓΘ, καὶ κείσθω τῇ μὲν Z ἴση
ἡ AH, τῇ δὲ E ἴση ἡ ΓΘ. καὶ συμπληρώσθω τὰ BH, ΔΘ
παραλληλόγραμμα.

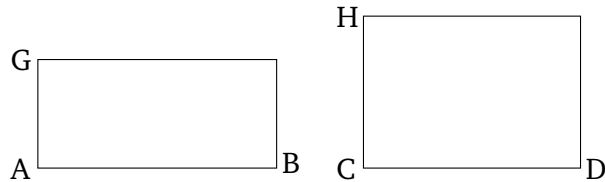
Καὶ ἐπεὶ ἐστὶν ὥς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ E πρὸς
τὴν Z, ἴση δὲ ἡ μὲν E τῇ ΓΘ, ἡ δὲ Z τῇ AH, ἐστὶν ἄρα ὥς
ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ ΓΘ πρὸς τὴν AH. τῶν BH,
ΔΘ ἄρα παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ
περὶ τὰς ἴσας γωνίας. ὣν δὲ ἰσογώνιων παραλληλογράμμων
ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν
ἐκεῖνα· ἴσον ἄρα ἐστὶ τὸ BH παραλληλόγραμμον τῷ ΔΘ
παραλληλόγραμμῳ. καὶ ἐστὶ τὸ μὲν BH τὸ ὑπὸ τῶν AB, Z·
ἴση γὰρ ἡ AH τῇ Z· τὸ δὲ ΔΘ τὸ ὑπὸ τῶν ΓΔ, E· ἴση γὰρ ἡ
E τῇ ΓΘ· τὸ ἄρα ὑπὸ τῶν AB, Z περιεχόμενον ὀρθογώνιον
ἴσον ἐστὶ τῷ ὑπὸ τῶν ΓΔ, E περιεχομένῳ ὀρθογώνιῳ.

Ἀλλὰ δὴ τὸ ὑπὸ τῶν AB, Z περιεχόμενον ὀρθογώνιον
ἴσον ἔστω τῷ ὑπὸ τῶν ΓΔ, E περιεχομένῳ ὀρθογώνιῳ.
λέγω, ὅτι αἱ τέσσαρες εὐθεῖαι ἀνάλογον ἔσονται, ὥς ἡ AB
πρὸς τὴν ΓΔ, οὕτως ἡ E πρὸς τὴν Z.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ τὸ ὑπὸ τῶν
AB, Z ἴσον ἐστὶ τῷ ὑπὸ τῶν ΓΔ, E, καὶ ἐστὶ τὸ μὲν ὑπὸ
τῶν AB, Z τὸ BH· ἴση γὰρ ἐστὶν ἡ AH τῇ Z· τὸ δὲ ὑπὸ τῶν
ΓΔ, E τὸ ΔΘ· ἴση γὰρ ἡ ΓΘ τῇ E· τὸ ἄρα BH ἴσον ἐστὶ
τῷ ΔΘ. καὶ ἐστὶν ἰσογώνια. τῶν δὲ ἴσων καὶ ἰσογώνιων
παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς
ἴσας γωνίας. ἐστὶν ἄρα ὥς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ ΓΘ
πρὸς τὴν AH. ἴση δὲ ἡ μὲν ΓΘ τῇ E, ἡ δὲ AH τῇ Z· ἐστὶν
ἄρα ὥς ἡ AB πρὸς τὴν ΓΔ, οὕτως ἡ E πρὸς τὴν Z.

Ἐὰν ἄρα τέσσαρες εὐθεῖαι ἀνάλογον ᾖσιν, τὸ ὑπὸ τῶν
ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ὑπὸ τῶν
μέσων περιεχομένῳ ὀρθογώνιῳ· καὶ τὸ ὑπὸ τῶν ἄκρων
περιεχόμενον ὀρθογώνιον ἴσον ἢ τῷ ὑπὸ τῶν μέσων περιε-
χομένῳ ὀρθογώνιῳ, αἱ τέσσαρες εὐθεῖαι ἀνάλογον ἔσονται·
ὅπερ εἶδει δεῖξαι.

angle contained by the (two) outermost is equal to the
rectangle contained by the middle (two) then the four
straight-lines will be proportional.



Let AB, CD, E, and F be four proportional straight-
lines, (such that) as AB (is) to CD, so E (is) to F. I say
that the rectangle contained by AB and F is equal to the
rectangle contained by CD and E.

[For] let AG and CH have been drawn from points
A and C at right-angles to the straight-lines AB and CD
(respectively) [Prop. 1.11]. And let AG be made equal to
F, and CH to E [Prop. 1.3]. And let the parallelograms
BG and DH have been completed.

And since as AB is to CD, so E (is) to F, and E
(is) equal CH, and F to AG, thus as AB is to CD, so
CH (is) to AG. Thus, in the parallelograms BG and DH
the sides about the equal angles are reciprocally propor-
tional. And those equiangular parallelograms in which
the sides about the equal angles are reciprocally propor-
tional are equal [Prop. 6.14]. Thus, parallelogram BG
is equal to parallelogram DH. And BG is the (rectangle
contained) by AB and F. For AG (is) equal to F. And
DH (is) the (rectangle contained) by CD and E. For E
(is) equal to CH. Thus, the rectangle contained by AB
and F is equal to the rectangle contained by CD and E.

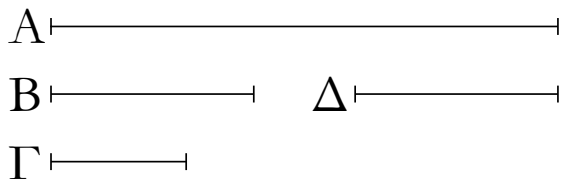
And so, let the rectangle contained by AB and F be
equal to the rectangle contained by CD and E. I say that
the four straight-lines will be proportional, (so that) as
AB (is) to CD, so E (is) to F.

For, with the same construction, since the (rectangle
contained) by AB and F is equal to the (rectangle con-
tained) by CD and E. And BG is the (rectangle con-
tained) by AB and F. For AG is equal to F. And DH
(is) the (rectangle contained) by CD and E. For CH
(is) equal to E. BG is thus equal to DH. And they
are equiangular. And in equal and equiangular parallel-
ograms the sides about the equal angles are reciprocally
proportional [Prop. 6.14]. Thus, as AB is to CD, so CH
(is) to AG. And CH (is) equal to E, and AG to F. Thus,
as AB is to CD, so E (is) to F.

Thus, if four straight-lines are proportional then the
rectangle contained by the (two) outermost is equal to
the rectangle contained by the middle (two). And if the
rectangle contained by the (two) outermost is equal to

ιζ'.

Ἐάν τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης τετραγώνῳ· καὶ τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ἢ τῷ ἀπὸ τῆς μέσης τετραγώνῳ, αἱ τρεῖς εὐθεῖαι ἀνάλογον ἔσσονται.



Ἐστωσαν τρεῖς εὐθεῖαι ἀνάλογον αἱ A, B, Γ , ὡς ἡ A πρὸς τὴν B , οὕτως ἡ B πρὸς τὴν Γ . λέγω, ὅτι τὸ ὑπὸ τῶν A, Γ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς B τετραγώνῳ.

Κείσθω τῇ B ἴση ἡ Δ .

Καὶ ἐπεὶ ἐστὶν ὡς ἡ A πρὸς τὴν B , οὕτως ἡ B πρὸς τὴν Γ , ἴση δὲ ἡ B τῇ Δ , ἔστιν ἄρα ὡς ἡ A πρὸς τὴν B , ἡ Δ πρὸς τὴν Γ . ἐὰν δὲ τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων περιεχόμενον [ὀρθογώνιον] ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων περιεχομένῳ ὀρθογώνιῳ. τὸ ἄρα ὑπὸ τῶν A, Γ ἴσον ἐστὶ τῷ ὑπὸ τῶν B, Δ . ἀλλὰ τὸ ὑπὸ τῶν B, Δ τὸ ἀπὸ τῆς B ἐστίν· ἴση γὰρ ἡ B τῇ Δ . τὸ ἄρα ὑπὸ τῶν A, Γ περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς B τετραγώνῳ.

Ἀλλὰ δὴ τὸ ὑπὸ τῶν A, Γ ἴσον ἔστω τῷ ἀπὸ τῆς B . λέγω, ὅτι ἐστὶν ὡς ἡ A πρὸς τὴν B , οὕτως ἡ B πρὸς τὴν Γ .

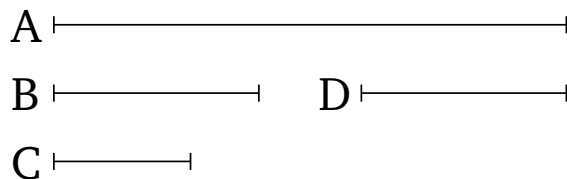
Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ τὸ ὑπὸ τῶν A, Γ ἴσον ἐστὶ τῷ ἀπὸ τῆς B , ἀλλὰ τὸ ἀπὸ τῆς B τὸ ὑπὸ τῶν B, Δ ἐστίν· ἴση γὰρ ἡ B τῇ Δ . τὸ ἄρα ὑπὸ τῶν A, Γ ἴσον ἐστὶ τῷ ὑπὸ τῶν B, Δ . ἐὰν δὲ τὸ ὑπὸ τῶν ἄκρων ἴσον ἢ τῷ ὑπὸ τῶν μέσων, αἱ τέσσαρες εὐθεῖαι ἀνάλογον εἰσιν. ἔστιν ἄρα ὡς ἡ A πρὸς τὴν B , οὕτως ἡ Δ πρὸς τὴν Γ . ἴση δὲ ἡ B τῇ Δ . ὡς ἄρα ἡ A πρὸς τὴν B , οὕτως ἡ B πρὸς τὴν Γ .

Ἐὰν ἄρα τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης τετραγώνῳ· καὶ τὸ ὑπὸ τῶν ἄκρων περιεχόμενον ὀρθογώνιον ἴσον ἢ τῷ ἀπὸ τῆς μέσης τετραγώνῳ, αἱ τρεῖς εὐθεῖαι ἀνάλογον ἔσσονται· ὅπερ ἔδει δεῖξαι.

the rectangle contained by the middle (two) then the four straight-lines will be proportional. (Which is) the very thing it was required to show.

Proposition 17

If three straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the square on the middle (one). And if the rectangle contained by the (two) outermost is equal to the square on the middle (one) then the three straight-lines will be proportional.



Let A, B and C be three proportional straight-lines, (such that) as A (is) to B , so B (is) to C . I say that the rectangle contained by A and C is equal to the square on B .

Let D be made equal to B [Prop. 1.3].

And since as A is to B , so B (is) to C , and B (is) equal to D , thus as A is to B , (so) D (is) to C . And if four straight-lines are proportional then the [rectangle] contained by the (two) outermost is equal to the rectangle contained by the middle (two) [Prop. 6.16]. Thus, the (rectangle contained) by A and C is equal to the (rectangle contained) by B and D . But, the (rectangle contained) by B and D is the (square) on B . For B (is) equal to D . Thus, the rectangle contained by A and C is equal to the square on B .

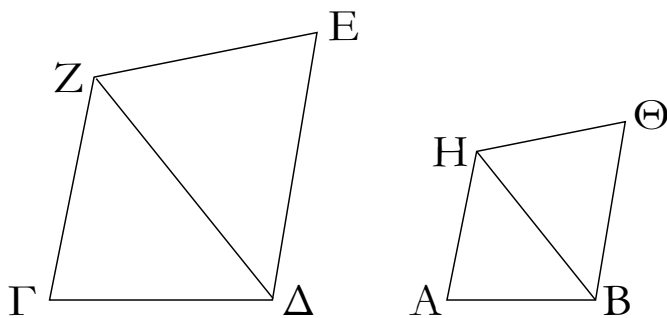
And so, let the (rectangle contained) by A and C be equal to the (square) on B . I say that as A is to B , so B (is) to C .

For, with the same construction, since the (rectangle contained) by A and C is equal to the (square) on B . But, the (square) on B is the (rectangle contained) by B and D . For B (is) equal to D . The (rectangle contained) by A and C is thus equal to the (rectangle contained) by B and D . And if the (rectangle contained) by the (two) outermost is equal to the (rectangle contained) by the middle (two) then the four straight-lines are proportional [Prop. 6.16]. Thus, as A is to B , so D (is) to C . And B (is) equal to D . Thus, as A (is) to B , so B (is) to C .

Thus, if three straight-lines are proportional then the rectangle contained by the (two) outermost is equal to the square on the middle (one). And if the rectangle contained by the (two) outermost is equal to the square on the middle (one) then the three straight-lines will be proportional. (Which is) the very thing it was required to

ιη'.

Ἀπὸ τῆς δοθείσης εὐθείας τῷ δοθέντι εὐθυγράμμῳ ὁμοίον τε καὶ ὁμοίως κείμενον εὐθυγράμμον ἀναγράψαι.



Ἐστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ ΑΒ, τὸ δὲ δοθὲν εὐθυγράμμον τὸ ΓΕ· δεῖ δὲ ἀπὸ τῆς ΑΒ εὐθείας τῷ ΓΕ εὐθυγράμμῳ ὁμοίον τε καὶ ὁμοίως κείμενον εὐθυγράμμον ἀναγράψαι.

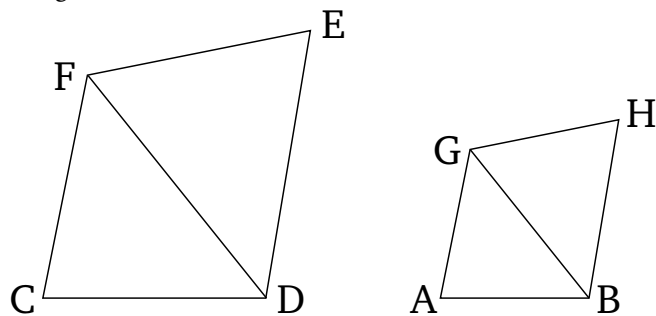
Ἐπεζεύχθω ἡ ΔΖ, καὶ συνεστώτω πρὸς τῇ ΑΒ εὐθείᾳ καὶ τοῖς πρὸς αὐτῇ σημείοις τοῖς Α, Β τῇ μὲν πρὸς τῷ Γ γωνία ἴση ἢ ὑπὸ ΗΑΒ, τῇ δὲ ὑπὸ ΓΔΖ ἴση ἢ ὑπὸ ΑΒΗ. λοιπὴ ἄρα ἡ ὑπὸ ΓΖΔ τῇ ὑπὸ ΑΗΒ ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΖΓΔ τρίγωνον τῷ ΗΑΒ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΖΔ πρὸς τὴν ΗΒ, οὕτως ἡ ΖΓ πρὸς τὴν ΗΑ, καὶ ἡ ΓΔ πρὸς τὴν ΑΒ. πάλιν συνεστώτω πρὸς τῇ ΒΗ εὐθείᾳ καὶ τοῖς πρὸς αὐτῇ σημείοις τοῖς Β, Η τῇ μὲν ὑπὸ ΔΖΕ γωνία ἴση ἢ ὑπὸ ΒΗΘ, τῇ δὲ ὑπὸ ΖΔΕ ἴση ἢ ὑπὸ ΗΒΘ. λοιπὴ ἄρα ἡ πρὸς τῷ Ε λοιπῇ τῇ πρὸς τῷ Θ ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΖΔΕ τρίγωνον τῷ ΗΘΒ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΖΔ πρὸς τὴν ΗΒ, οὕτως ἡ ΖΕ πρὸς τὴν ΗΘ καὶ ἡ ΕΔ πρὸς τὴν ΘΒ. ἐδείχθη δὲ καὶ ὡς ἡ ΖΔ πρὸς τὴν ΗΒ, οὕτως ἡ ΖΓ πρὸς τὴν ΗΑ καὶ ἡ ΓΔ πρὸς τὴν ΑΒ· καὶ ὡς ἄρα ἡ ΖΓ πρὸς τὴν ΑΗ, οὕτως ἡ τε ΓΔ πρὸς τὴν ΑΒ καὶ ἡ ΖΕ πρὸς τὴν ΗΘ καὶ ἔτι ἡ ΕΑ πρὸς τὴν ΘΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ὑπὸ ΓΖΔ γωνία τῇ ὑπὸ ΑΗΒ, ἡ δὲ ὑπὸ ΔΖΕ τῇ ὑπὸ ΒΗΘ, ὅλη ἄρα ἡ ὑπὸ ΓΖΕ ὅλη τῇ ὑπὸ ΑΗΘ ἐστὶν ἴση. διὰ τὰ αὐτὰ δὲ καὶ ἡ ὑπὸ ΓΔΕ τῇ ὑπὸ ΑΒΘ ἐστὶν ἴση. ἔστι δὲ καὶ ἡ μὲν πρὸς τῷ Γ τῇ πρὸς τῷ Α ἴση, ἡ δὲ πρὸς τῷ Ε τῇ πρὸς τῷ Θ. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΘ τῷ ΓΕ· καὶ τὰς περὶ τὰς ἴσας γωνίας αὐτῶν πλευρὰς ἀνάλογον ἔχει· ὁμοιον ἄρα ἐστὶ τὸ ΑΘ εὐθυγράμμον τῷ ΓΕ εὐθυγράμμῳ.

Ἀπὸ τῆς δοθείσης ἄρα εὐθείας τῆς ΑΒ τῷ δοθέντι εὐθυγράμμῳ τῷ ΓΕ ὁμοίον τε καὶ ὁμοίως κείμενον εὐθυγράμμον ἀναγράφεται τὸ ΑΘ· ὅπερ ἔδει ποιῆσαι.

show.

Proposition 18

To describe a rectilinear figure similar, and similarly laid down, to a given rectilinear figure on a given straight-line.



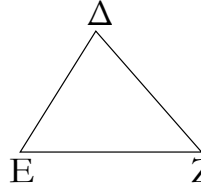
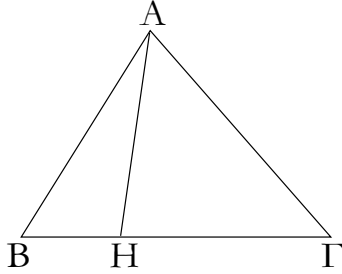
Let ΑΒ be the given straight-line, and CΕ the given rectilinear figure. So it is required to describe a rectilinear figure similar, and similarly laid down, to the rectilinear figure CΕ on the straight-line ΑΒ.

Let DF have been joined, and let GAB, equal to the angle at C, and ABG, equal to (angle) CDF, have been constructed on the straight-line ΑΒ at the points Α and Β on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) CFD is equal to AGB [Prop. 1.32]. Thus, triangle FCD is equiangular to triangle GAB. Thus, proportionally, as FD is to GB, so FC (is) to GA, and CD to AB [Prop. 6.4]. Again, let BGH, equal to angle DFE, and GBH equal to (angle) FDE, have been constructed on the straight-line BG at the points G and B on it (respectively) [Prop. 1.23]. Thus, the remaining (angle) at Ε is equal to the remaining (angle) at Η [Prop. 1.32]. Thus, triangle FDE is equiangular to triangle GHB. Thus, proportionally, as FD is to GB, so FE (is) to GH, and ED to HB [Prop. 6.4]. And it was also shown (that) as FD (is) to GB, so FC (is) to GA, and CD to AB. Thus, also, as FC (is) to AG, so CD (is) to AB, and FE to GH, and, further, ED to HB. And since angle CFD is equal to AGB, and DFE to BGH, thus the whole (angle) CFE is equal to the whole (angle) AGH. So, for the same (reasons), (angle) CDE is also equal to ABH. And the (angle) at C is also equal to the (angle) at Α, and the (angle) at Ε to the (angle) at Η. Thus, (figure) AH is equiangular to CΕ. And (the two figures) have the sides about their equal angles proportional. Thus, the rectilinear figure AH is similar to the rectilinear figure CΕ [Def. 6.1].

Thus, the rectilinear figure AH, similar, and similarly laid down, to the given rectilinear figure CΕ has been constructed on the given straight-line ΑΒ. (Which is) the

10'.

Τὰ ὅμοια τρίγωνα πρὸς ἀλλήλα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν.



Ἐστω ὅμοια τρίγωνα τὰ ABG , ΔEZ ἴσην ἔχοντα τὴν πρὸς τῷ B γωνίαν τῇ πρὸς τῷ E , ὡς δὲ τὴν AB πρὸς τὴν BG , οὕτως τὴν ΔE πρὸς τὴν EZ , ὥστε ὁμόλογον εἶναι τὴν BG τῇ EZ : λέγω, ὅτι τὸ ABG τρίγωνον πρὸς τὸ ΔEZ τρίγωνον διπλασίονα λόγον ἔχει ἥπερ ἡ BG πρὸς τὴν EZ .

Εἰλήφθω γὰρ τῶν BG , EZ τρίτη ἀνάλογον ἡ BH , ὥστε εἶναι ὡς τὴν BG πρὸς τὴν EZ , οὕτως τὴν EZ πρὸς τὴν BH : καὶ ἐπεζεύχθω ἡ AH .

Ἐπεὶ οὖν ἐστὶν ὡς ἡ AB πρὸς τὴν BG , οὕτως ἡ ΔE πρὸς τὴν EZ , ἐναλλάξ ἄρα ἐστὶν ὡς ἡ AB πρὸς τὴν ΔE , οὕτως ἡ BG πρὸς τὴν EZ . ἀλλ' ὡς ἡ BG πρὸς EZ , οὕτως ἐστὶν ἡ EZ πρὸς BH . καὶ ὡς ἄρα ἡ AB πρὸς ΔE , οὕτως ἡ EZ πρὸς BH : τῶν ABH , ΔEZ ἄρα τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωναίας. ὦν δὲ μίαν μὲν ἴσην ἔχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωναίας, ἴσα ἐστὶν ἕκαστα. ἴσον ἄρα ἐστὶ τὸ ABH τρίγωνον τῷ ΔEZ τριγώνῳ. καὶ ἐπεὶ ἐστὶν ὡς ἡ BG πρὸς τὴν EZ , οὕτως ἡ EZ πρὸς τὴν BH , ἐὰν δὲ τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, ἡ πρώτη πρὸς τὴν τρίτην διπλασίονα λόγον ἔχει ἥπερ πρὸς τὴν δευτέραν, ἡ BG ἄρα πρὸς τὴν BH διπλασίονα λόγον ἔχει ἥπερ ἡ GB πρὸς τὴν EZ . ὡς δὲ ἡ GB πρὸς τὴν BH , οὕτως τὸ ABG τρίγωνον πρὸς τὸ ABH τρίγωνον: καὶ τὸ ABG ἄρα τρίγωνον πρὸς τὸ ΔEZ τρίγωνον διπλασίονα λόγον ἔχει ἥπερ ἡ BG πρὸς τὴν EZ .

Τὰ ἄρα ὅμοια τρίγωνα πρὸς ἀλλήλα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν. [ὅπερ ἔδει δεῖξαι.]

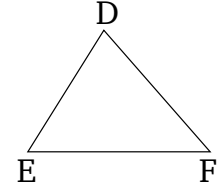
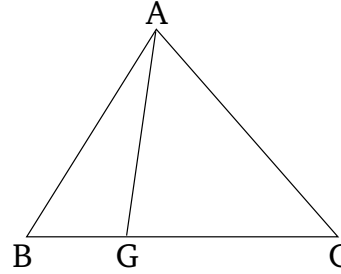
Πόρισμα.

Ἐκ δὲ τούτου φανερόν, ὅτι, ἐὰν τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, ἐστὶν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ

very thing it was required to do.

Proposition 19

Similar triangles are to one another in the squared[†] ratio of (their) corresponding sides.



Let ABC and DEF be similar triangles having the angle at B equal to the (angle) at E , and AB to BC , as DE (is) to EF , such that BC corresponds to EF . I say that triangle ABC has a squared ratio to triangle DEF with respect to (that side) BC (has) to EF .

For let a third (straight-line), BG , have been taken (which is) proportional to BC and EF , so that as BC (is) to EF , so EF (is) to BG [Prop. 6.11]. And let AG have been joined.

Therefore, since as AB is to BC , so DE (is) to EF , thus, alternately, as AB is to DE , so BC (is) to EF [Prop. 5.16]. But, as BC (is) to EF , so EF is to BG . And, thus, as AB (is) to DE , so EF (is) to BG . Thus, for triangles ABG and DEF , the sides about the equal angles are reciprocally proportional. And those triangles having one (angle) equal to one (angle) for which the sides about the equal angles are reciprocally proportional are equal [Prop. 6.15]. Thus, triangle ABG is equal to triangle DEF . And since as BC (is) to EF , so EF (is) to BG , and if three straight-lines are proportional then the first has a squared ratio to the third with respect to the second [Def. 5.9], BC thus has a squared ratio to BG with respect to (that) CB (has) to EF . And as CB (is) to BG , so triangle ABC (is) to triangle ABG [Prop. 6.1]. Thus, triangle ABC also has a squared ratio to (triangle) ABG with respect to (that side) BC (has) to EF . And triangle ABG (is) equal to triangle DEF . Thus, triangle ABC also has a squared ratio to triangle DEF with respect to (that side) BC (has) to EF .

Thus, similar triangles are to one another in the squared ratio of (their) corresponding sides. [(Which is) the very thing it was required to show].

Corollary

So it is clear, from this, that if three straight-lines are proportional, then as the first is to the third, so the figure

τῆς πρώτης εἶδος πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον. ὅπερ ἔδει δεῖξαι.

(described) on the first (is) to the similar, and similarly described, (figure) on the second. (Which is) the very thing it was required to show.

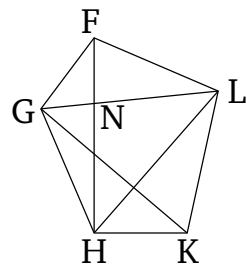
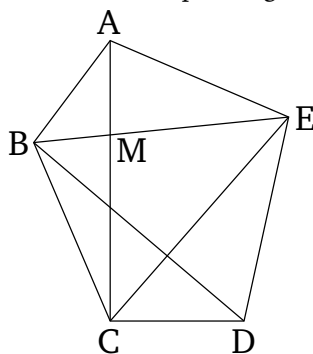
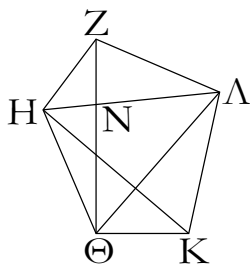
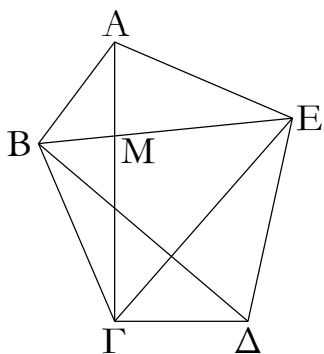
† Literally, “double”.

κ'.

Proposition 20

Τὰ ὅμοια πολύγωνα εἰς τε ὅμοια τρίγωνα διαιρεῖται καὶ εἰς ἴσα τὸ πλῆθος καὶ ὁμόλογα τοῖς ὅλοις, καὶ τὸ πολύγωνον πρὸς τὸ πολύγωνον διπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν.

Similar polygons can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and one polygon has to the (other) polygon a squared ratio with respect to (that) a corresponding side (has) to a corresponding side.



Ἐστω ὅμοια πολύγωνα τὰ ABΓΔΕ, ΖΗΘΚΛ, ὁμόλογος δὲ ἔστω ἡ AB τῇ ΖΗ· λέγω, ὅτι τὰ ABΓΔΕ, ΖΗΘΚΛ πολύγωνα εἰς τε ὅμοια τρίγωνα διαιρεῖται καὶ εἰς ἴσα τὸ πλῆθος καὶ ὁμόλογα τοῖς ὅλοις, καὶ τὸ ABΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον διπλασίονα λόγον ἔχει ἥπερ ἡ AB πρὸς τὴν ΖΗ.

Ἐπεξεύχθωσαν αἱ BE, ΕΓ, ΗΛ, ΛΘ.

Καὶ ἐπεὶ ὁμοίον ἐστὶ τὸ ABΓΔΕ πολύγωνον τῷ ΖΗΘΚΛ πολυγώνῳ, ἴση ἐστὶν ἡ ὑπὸ BAE γωνία τῇ ὑπὸ HZΛ. καὶ ἐστὶν ὡς ἡ BA πρὸς AE, οὕτως ἡ HZ πρὸς ZΛ. ἐπεὶ οὖν δύο τρίγωνά ἐστι τὰ ABE, ZHL μίαν γωνίαν μὲν γωνία ἴσην ἔχοντα, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ἰσογώνιον ἄρα ἐστὶ τὸ ABE τρίγωνον τῷ ZHL τριγώνῳ· ὥστε καὶ ὁμοίον· ἴση ἄρα ἐστὶν ἡ ὑπὸ ABE γωνία τῇ ὑπὸ ZHL. ἔστι δὲ καὶ ὅλη ἡ ὑπὸ ABΓ ὅλη τῇ ὑπὸ ZHΘ ἴση διὰ τὴν ὁμοιότητα τῶν πολυγώνων· λοιπὴ ἄρα ἡ ὑπὸ EBF γωνία τῇ ὑπὸ LHΘ ἐστὶν ἴση. καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν ABE, ZHL τριγώνων ἐστὶν ὡς ἡ EB πρὸς BA, οὕτως ἡ LH πρὸς HZ, ἀλλὰ μὴν καὶ διὰ τὴν ὁμοιότητα τῶν πολυγώνων ἐστὶν ὡς ἡ AB πρὸς BΓ, οὕτως ἡ ZH πρὸς HΘ, δι' ἴσου ἄρα ἐστὶν ὡς ἡ EB πρὸς BΓ, οὕτως ἡ LH πρὸς HΘ, καὶ περὶ τὰς ἴσας γωνίας τὰς ὑπὸ EBF, LHΘ αἱ πλευραὶ ἀνάλογόν εἰσιν· ἰσογώνιον ἄρα ἐστὶ τὸ EBF τρίγωνον τῷ LHΘ τριγώνῳ· ὥστε καὶ ὁμοίον ἐστὶ τὸ EBF τρίγωνον τῷ LHΘ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ EΓΔ τρίγωνον ὁμοίον ἐστὶ τῷ ΛΘΚ τριγώνῳ. τὰ ἄρα ὅμοια πολύγωνα τὰ ABΓΔΕ, ΖΗΘΚΛ εἰς τε ὅμοια τρίγωνα διήρηται καὶ εἰς ἴσα

Let $ABCDE$ and $FGHKL$ be similar polygons, and let AB correspond to FG . I say that polygons $ABCDE$ and $FGHKL$ can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and (that) polygon $ABCDE$ has a squared ratio to polygon $FGHKL$ with respect to that AB (has) to FG .

Let BE , EC , GL , and LH have been joined.

And since polygon $ABCDE$ is similar to polygon $FGHKL$, angle BAE is equal to angle GFL , and as BA is to AE , so GF (is) to FL [Def. 6.1]. Therefore, since ABE and FGL are two triangles having one angle equal to one angle and the sides about the equal angles proportional, triangle ABE is thus equiangular to triangle FGL [Prop. 6.6]. Hence, (they are) also similar [Prop. 6.4, Def. 6.1]. Thus, angle ABE is equal to (angle) FGL . And the whole (angle) ABC is equal to the whole (angle) FGH , on account of the similarity of the polygons. Thus, the remaining angle EBC is equal to LGH . And since, on account of the similarity of triangles ABE and FGL , as EB is to BA , so LG (is) to GF , but also, on account of the similarity of the polygons, as AB is to BC , so FG (is) to GH , thus, via equality, as EB is to BC , so LG (is) to GH [Prop. 5.22], and the sides about the equal angles, EBC and LGH , are proportional. Thus, triangle EBC is equiangular to triangle LGH [Prop. 6.6]. Hence, triangle EBC is also similar to triangle LGH [Prop. 6.4, Def. 6.1]. So, for the same (reasons), triangle ECD is also similar

τὸ πλῆθος.

Λέγω, ὅτι καὶ ὁμόλογα τοῖς ὅλοις, τούτέστιν ὥστε ἀνάλογον εἶναι τὰ τρίγωνα, καὶ ἡγούμενα μὲν εἶναι τὰ ABE , $EBΓ$, $ΕΓΔ$, ἐπόμενα δὲ αὐτῶν τὰ ZHA , $ΛΗΘ$, $ΛΘΚ$, καὶ ὅτι τὸ $ABΓΔΕ$ πολύγωνον πρὸς τὸ $ZHΘΚΛ$ πολύγωνον διπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τούτέστιν ἡ AB πρὸς τὴν ZH .

Ἐπεξεύχθησαν γάρ αἱ $ΑΓ$, $ΖΘ$. καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν πολυγώνων ἴση ἐστὶν ἡ ὑπὸ $ABΓ$ γωνία τῇ ὑπὸ $ZHΘ$, καὶ ἐστὶν ὡς ἡ AB πρὸς $BΓ$, οὕτως ἡ ZH πρὸς $HΘ$, ἰσογώνιον ἐστὶ τὸ $ABΓ$ τρίγωνον τῷ $ZHΘ$ τριγώνῳ· ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ $BAΓ$ γωνία τῇ ὑπὸ $HΖΘ$, ἡ δὲ ὑπὸ $BΓA$ τῇ ὑπὸ $HΘZ$. καὶ ἐπεὶ ἴση ἐστὶν ἡ ὑπὸ BAM γωνία τῇ ὑπὸ HZN , ἐστὶ δὲ καὶ ἡ ὑπὸ ABM τῇ ὑπὸ ZHN ἴση, καὶ λοιπὴ ἄρα ἡ ὑπὸ AMB λοιπὴ τῇ ὑπὸ ZNH ἴση ἐστὶν· ἰσογώνιον ἄρα ἐστὶ τὸ ABM τρίγωνον τῷ ZHN τριγώνῳ. ὁμοίως δὲ δεῖξομεν, ὅτι καὶ τὸ $BMΓ$ τρίγωνον ἰσογώνιον ἐστὶ τῷ $HNΘ$ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν, ὡς μὲν ἡ AM πρὸς MB , οὕτως ἡ ZN πρὸς NH , ὡς δὲ ἡ BM πρὸς $ΜΓ$, οὕτως ἡ HN πρὸς $NΘ$. ὥστε καὶ δι' ἴσου, ὡς ἡ AM πρὸς $ΜΓ$, οὕτως ἡ ZN πρὸς $NΘ$. ἀλλ' ὡς ἡ AM πρὸς $ΜΓ$, οὕτως τὸ ABM [τρίγωνον] πρὸς τὸ $MBΓ$, καὶ τὸ AME πρὸς τὸ $EMΓ$ · πρὸς ἀλλήλα γάρ εἰσιν ὡς αἱ βάσεις. καὶ ὡς ἄρα ἐν τῶν ἡγουμένων πρὸς ἐν τῶν ἐπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἐπόμενα· ὡς ἄρα τὸ AMB τρίγωνον πρὸς τὸ $BMΓ$, οὕτως τὸ ABE πρὸς τὸ $ΓBE$. ἀλλ' ὡς τὸ AMB πρὸς τὸ $BMΓ$, οὕτως ἡ AM πρὸς $ΜΓ$ · καὶ ὡς ἄρα ἡ AM πρὸς $ΜΓ$, οὕτως τὸ ABE τρίγωνον πρὸς τὸ $EBΓ$ τρίγωνον. διὰ τὰ αὐτὰ δὲ καὶ ὡς ἡ ZN πρὸς $NΘ$, οὕτως τὸ ZHA τρίγωνον πρὸς τὸ $HAΘ$ τρίγωνον. καὶ ἐστὶν ὡς ἡ AM πρὸς $ΜΓ$, οὕτως ἡ ZN πρὸς $NΘ$ · καὶ ὡς ἄρα τὸ ABE τρίγωνον πρὸς τὸ $BEΓ$ τρίγωνον, οὕτως τὸ ZHA τρίγωνον πρὸς τὸ $HAΘ$ τρίγωνον, καὶ ἐναλλάξ ὡς τὸ ABE τρίγωνον πρὸς τὸ ZHA τρίγωνον, οὕτως τὸ $BEΓ$ τρίγωνον πρὸς τὸ $HAΘ$ τρίγωνον. ὁμοίως δὲ δεῖξομεν ἐπιzeugηθεῖσιν τῶν $BΔ$, HK , ὅτι καὶ ὡς τὸ $BEΓ$ τρίγωνον πρὸς τὸ $ΛΗΘ$ τρίγωνον, οὕτως τὸ $ΕΓΔ$ τρίγωνον πρὸς τὸ $ΛΘΚ$ τρίγωνον. καὶ ἐπεὶ ἐστὶν ὡς τὸ ABE τρίγωνον πρὸς τὸ ZHA τρίγωνον, οὕτως τὸ $EBΓ$ πρὸς τὸ $ΛΗΘ$, καὶ ἔτι τὸ $ΕΓΔ$ πρὸς τὸ $ΛΘΚ$, καὶ ὡς ἄρα ἐν τῶν ἡγουμένων πρὸς ἐν τῶν ἐπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἐπόμενα· ἐστὶν ἄρα ὡς τὸ ABE τρίγωνον πρὸς τὸ ZHA τρίγωνον, οὕτως τὸ $ABΓΔΕ$ πολύγωνον πρὸς τὸ $ZHΘΚΛ$ πολύγωνον. ἀλλὰ τὸ ABE τρίγωνον πρὸς τὸ ZHA τρίγωνον διπλασίονα λόγον ἔχει ἥπερ ἡ AB ὁμόλογος πλευρὰ πρὸς τὴν ZH ὁμόλογον πλευράν· τὰ γὰρ ὅμοια τρίγωνα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν. καὶ τὸ $ABΓΔΕ$ ἄρα πολύγωνον πρὸς τὸ $ZHΘΚΛ$ πολύγωνον διπλασίονα λόγον ἔχει ἥπερ ἡ AB ὁμόλογος πλευρὰ πρὸς τὴν ZH ὁμόλογον πλευράν.

Τὰ ἄρα ὅμοια πολύγωνα εἰς τε ὅμοια τρίγωνα διαιρεῖται καὶ εἰς ἴσα τὸ πλῆθος καὶ ὁμόλογα τοῖς ὅλοις, καὶ τὸ

to triangle LHK . Thus, the similar polygons $ABCDE$ and $FGHKL$ have been divided into equal numbers of similar triangles.

I also say that (the triangles) correspond (in proportion) to the wholes. That is to say, the triangles are proportional: ABE , EBC , and ECD are the leading (magnitudes), and their (associated) following (magnitudes are) FGL , LGH , and LHK (respectively). (I) also (say) that polygon $ABCDE$ has a squared ratio to polygon $FGHKL$ with respect to (that) a corresponding side (has) to a corresponding side—that is to say, (side) AB to FG .

For let AC and FH have been joined. And since angle ABC is equal to FGH , and as AB is to BC , so FG (is) to GH , on account of the similarity of the polygons, triangle ABC is equiangular to triangle FGH [Prop. 6.6]. Thus, angle BAC is equal to GFH , and (angle) BCA to GHF . And since angle BAM is equal to GFN , and (angle) ABM is also equal to FGN (see earlier), the remaining (angle) AMB is thus also equal to the remaining (angle) FNG [Prop. 1.32]. Thus, triangle ABM is equiangular to triangle FGN . So, similarly, we can show that triangle BMC is also equiangular to triangle GNH . Thus, proportionally, as AM is to MB , so FN (is) to NG , and as BM (is) to MC , so GN (is) to NH [Prop. 6.4]. Hence, also, via equality, as AM (is) to MC , so FN (is) to NH [Prop. 5.22]. But, as AM (is) to MC , so [triangle] ABM is to MBC , and AME to EMC . For they are to one another as their bases [Prop. 6.1]. And as one of the leading (magnitudes) is to one of the following (magnitudes), so (the sum of) all the leading (magnitudes) is to (the sum of) all the following (magnitudes) [Prop. 5.12]. Thus, as triangle AMB (is) to BMC , so (triangle) ABE (is) to CBE . But, as (triangle) AMB (is) to BMC , so AM (is) to MC . Thus, also, as AM (is) to MC , so triangle ABE (is) to triangle EBC . And so, for the same (reasons), as FN (is) to NH , so triangle FGL (is) to triangle GLH . And as AM is to MC , so FN (is) to NH . Thus, also, as triangle ABE (is) to triangle BEC , so triangle FGL (is) to triangle GLH , and, alternately, as triangle ABE (is) to triangle FGL , so triangle BEC (is) to triangle GLH [Prop. 5.16]. So, similarly, we can also show, by joining BD and GK , that as triangle BEC (is) to triangle LGH , so triangle ECD (is) to triangle LHK . And since as triangle ABE is to triangle FGL , so (triangle) EBC (is) to LGH , and, further, (triangle) ECD to LHK , and also as one of the leading (magnitudes) is to one of the following, so (the sum of) all the leading (magnitudes) is to (the sum of) all the following [Prop. 5.12], thus as triangle ABE is to triangle FGL , so polygon $ABCDE$ (is) to polygon $FGHKL$. But, triangle ABE has a squared ratio

πολύγωνον πρὸς τὸ πολύγωνον διπλασίονα λόγον ἔχει ἢ περ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν [ὅπερ ἔδει δεῖξαι].

to triangle FGL with respect to (that) the corresponding side AB (has) to the corresponding side FG . For, similar triangles are in the squared ratio of corresponding sides [Prop. 6.14]. Thus, polygon $ABCDE$ also has a squared ratio to polygon $FGHKL$ with respect to (that) the corresponding side AB (has) to the corresponding side FG .

Thus, similar polygons can be divided into equal numbers of similar triangles corresponding (in proportion) to the wholes, and one polygon has to the (other) polygon a squared ratio with respect to (that) a corresponding side (has) to a corresponding side. [(Which is) the very thing it was required to show].

Πόρισμα.

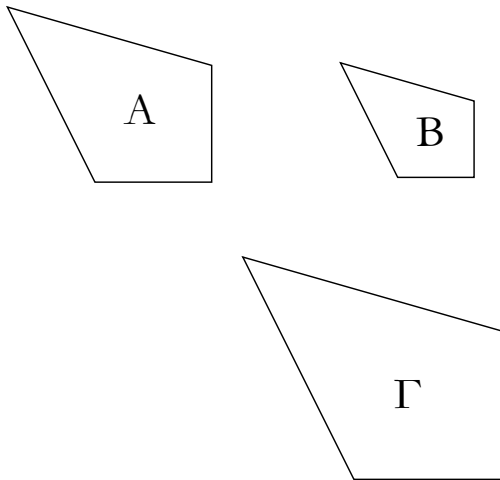
Ὡσαύτως δὲ καὶ ἐπὶ τῶν [ὁμοίων] τετραπλεύρων δειχθήσεται, ὅτι ἐν διπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. ἐδείχθη δὲ καὶ ἐπὶ τῶν τριγώνων· ὥστε καὶ καθόλου τὰ ὅμοια εὐθύγραμμα σχήματα πρὸς ἄλληλα ἐν διπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. ὅπερ ἔδει δεῖξαι.

Corollary

And, in the same manner, it can also be shown for [similar] quadrilaterals that they are in the squared ratio of (their) corresponding sides. And it was also shown for triangles. Hence, in general, similar rectilinear figures are also to one another in the squared ratio of (their) corresponding sides. (Which is) the very thing it was required to show.

κα'.

Τὰ τῶ αὐτῶ εὐθυγράμμω ὅμοια καὶ ἀλλήλοις ἐστὶν ὅμοια.

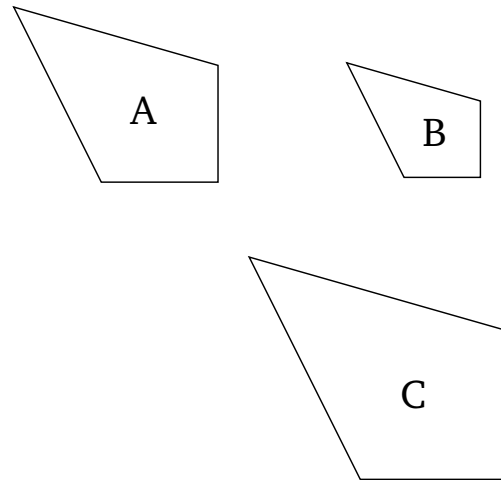


Ἐστω γὰρ ἑκάτερον τῶν A , B εὐθυγράμμων τῶ Γ ὅμοιον· λέγω, ὅτι καὶ τὸ A τῶ B ἐστὶν ὅμοιον.

Ἐπεὶ γὰρ ὅμοιον ἐστὶ τὸ A τῶ Γ , ἰσογώνιον τέ ἐστὶν αὐτῶ καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει. πάλιν, ἐπεὶ ὅμοιον ἐστὶ τὸ B τῶ Γ , ἰσογώνιον τέ ἐστὶν αὐτῶ καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει. ἑκάτερον ἄρα τῶν A , B τῶ Γ ἰσογώνιον τέ ἐστὶ καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον ἔχει [ὥστε καὶ τὸ A τῶ B ἰσογώνιον τέ ἐστὶ καὶ τὰς περὶ τὰς ἴσας γωνίας

Proposition 21

(Rectilinear figures) similar to the same rectilinear figure are also similar to one another.



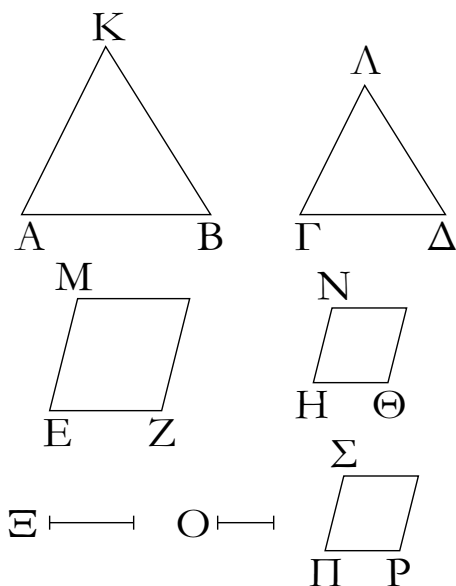
Let each of the rectilinear figures A and B be similar to (the rectilinear figure) C . I say that A is also similar to B .

For since A is similar to C , (A) is equiangular to (C), and has the sides about the equal angles proportional [Def. 6.1]. Again, since B is similar to C , (B) is equiangular to (C), and has the sides about the equal angles proportional [Def. 6.1]. Thus, A and B are each equiangular to C , and have the sides about the equal angles

πλευρὰς ἀνάλογον ἔχει]. ὁμοιον ἄρα ἐστὶ τὸ A τῷ B · ὅπερ ἔδει δεῖξαι.

κβ'.

Ἐάν τέσσαρες εὐθεῖαι ἀνάλογον ᾧσιν, καὶ τὰ ἀπ' αὐτῶν εὐθύγραμμα ὁμοία τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ἔσται· καὶ τὰ ἀπ' αὐτῶν εὐθύγραμμα ὁμοία τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ἦ, καὶ αὐτὰ αἱ εὐθεῖαι ἀνάλογον ἔσονται.



Ἐστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ AB , $\Gamma\Delta$, EZ , $H\Theta$, ὡς ἡ AB πρὸς τὴν $\Gamma\Delta$, οὕτως ἡ EZ πρὸς τὴν $H\Theta$, καὶ ἀναγεγράφθωσαν ἀπὸ μὲν τῶν AB , $\Gamma\Delta$ ὁμοία τε καὶ ὁμοίως κείμενα εὐθύγραμμα τὰ KAB , $\Lambda\Gamma\Delta$, ἀπὸ δὲ τῶν EZ , $H\Theta$ ὁμοία τε καὶ ὁμοίως κείμενα εὐθύγραμμα τὰ MZ , $N\Theta$ · λέγω, ὅτι ἐστὶν ὡς τὸ KAB πρὸς τὸ $\Lambda\Gamma\Delta$, οὕτως τὸ MZ πρὸς τὸ $N\Theta$.

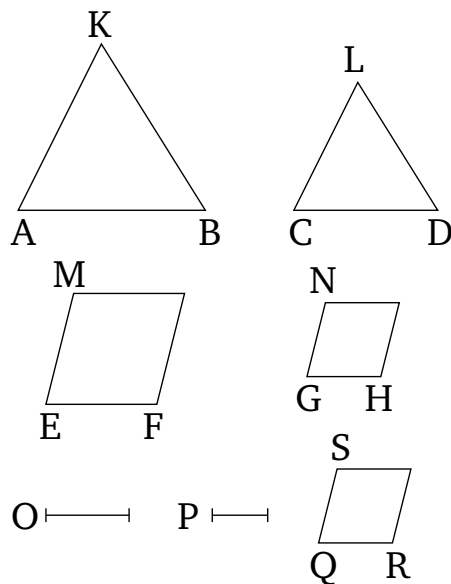
Εἰλήφθω γὰρ τῶν μὲν AB , $\Gamma\Delta$ τρίτη ἀνάλογον ἡ Ξ , τῶν δὲ EZ , $H\Theta$ τρίτη ἀνάλογον ἡ O . καὶ ἐπεὶ ἐστὶν ὡς μὲν ἡ AB πρὸς τὴν $\Gamma\Delta$, οὕτως ἡ EZ πρὸς τὴν $H\Theta$, ὡς δὲ ἡ $\Gamma\Delta$ πρὸς τὴν Ξ , οὕτως ἡ $H\Theta$ πρὸς τὴν O , δι' ἴσου ἄρα ἐστὶν ὡς ἡ AB πρὸς τὴν Ξ , οὕτως ἡ EZ πρὸς τὴν O . ἀλλ' ὡς μὲν ἡ AB πρὸς τὴν Ξ , οὕτως [καὶ] τὸ KAB πρὸς τὸ $\Lambda\Gamma\Delta$, ὡς δὲ ἡ EZ πρὸς τὴν O , οὕτως τὸ MZ πρὸς τὸ $N\Theta$ · καὶ ὡς ἄρα τὸ KAB πρὸς τὸ $\Lambda\Gamma\Delta$, οὕτως τὸ MZ πρὸς τὸ $N\Theta$.

Ἀλλὰ δὴ ἔστω ὡς τὸ KAB πρὸς τὸ $\Lambda\Gamma\Delta$, οὕτως τὸ MZ πρὸς τὸ $N\Theta$ · λέγω, ὅτι ἐστὶ καὶ ὡς ἡ AB πρὸς τὴν $\Gamma\Delta$, οὕτως ἡ EZ πρὸς τὴν $H\Theta$. εἰ γὰρ μὴ ἐστὶν, ὡς ἡ AB πρὸς τὴν $\Gamma\Delta$, οὕτως ἡ EZ πρὸς τὴν $H\Theta$, ἔστω ὡς ἡ AB πρὸς τὴν $\Gamma\Delta$, οὕτως ἡ EZ πρὸς τὴν $\Pi\Gamma$, καὶ ἀναγεγράφθω ἀπὸ τῆς

proportional [hence, A is also equiangular to B , and has the sides about the equal angles proportional]. Thus, A is similar to B [Def. 6.1]. (Which is) the very thing it was required to show.

Proposition 22

If four straight-lines are proportional then similar, and similarly described, rectilinear figures (drawn) on them will also be proportional. And if similar, and similarly described, rectilinear figures (drawn) on them are proportional then the straight-lines themselves will also be proportional.



Let AB , CD , EF , and GH be four proportional straight-lines, (such that) as AB (is) to CD , so EF (is) to GH . And let the similar, and similarly laid out, rectilinear figures KAB and LCD have been described on AB and CD (respectively), and the similar, and similarly laid out, rectilinear figures ME and NH on EF and GH (respectively). I say that as KAB is to LCD , so ME (is) to NH .

For let a third (straight-line) O have been taken (which is) proportional to AB and CD , and a third (straight-line) P proportional to EF and GH [Prop. 6.11]. And since as AB is to CD , so EF (is) to GH , and as CD (is) to O , so GH (is) to P , thus, via equality, as AB is to O , so EF (is) to P [Prop. 5.22]. But, as AB (is) to O , so [also] KAB (is) to LCD , and as EF (is) to P , so ME (is) to NH [Prop. 5.19 corr.]. And, thus, as KAB (is) to LCD , so ME (is) to NH .

And so let KAB be to LCD , as ME (is) to NH . I say also that as AB is to CD , so EF (is) to GH . For if as AB is to CD , so EF (is) not to GH , let AB be to CD , as EF

ΠΡ ὁποτέρῳ τῶν ΜΖ, ΝΘ ὁμοίον τε καὶ ὁμοίως κείμενον εὐθύγραμμον τὸ ΣΡ.

Ἐπεὶ οὖν ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΠΡ, καὶ ἀναγέγραπται ἀπὸ μὲν τῶν ΑΒ, ΓΔ ὁμοιά τε καὶ ὁμοίως κείμενα τὰ ΚΑΒ, ΛΓΔ, ἀπὸ δὲ τῶν ΕΖ, ΠΡ ὁμοιά τε καὶ ὁμοίως κείμενα τὰ ΜΖ, ΣΡ, ἔστιν ἄρα ὡς τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, οὕτως τὸ ΜΖ πρὸς τὸ ΣΡ. ὑπόκειται δὲ καὶ ὡς τὸ ΚΑΒ πρὸς τὸ ΛΓΔ, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ· καὶ ὡς ἄρα τὸ ΜΖ πρὸς τὸ ΣΡ, οὕτως τὸ ΜΖ πρὸς τὸ ΝΘ. τὸ ΜΖ ἄρα πρὸς ἐκάτερον τῶν ΝΘ, ΣΡ τὸν αὐτὸν ἔχει λόγον· ἴσον ἄρα ἐστὶ τὸ ΝΘ τῷ ΣΡ. ἔστι δὲ αὐτῷ καὶ ὁμοιον καὶ ὁμοίως κείμενον· ἴση ἄρα ἡ ΗΘ τῇ ΠΡ. καὶ ἐπεὶ ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΠΡ, ἴση δὲ ἡ ΠΡ τῇ ΗΘ, ἔστιν ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἡ ΕΖ πρὸς τὴν ΗΘ.

Ἐάν ἄρα τέσσαρες εὐθεῖαι ἀνάλογον ᾧσιν, καὶ τὰ ἀπ' αὐτῶν εὐθύγραμμα ὁμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ἔσται· καθ' ὅτι τὰ ἀπ' αὐτῶν εὐθύγραμμα ὁμοιά τε καὶ ὁμοίως ἀναγεγραμμένα ἀνάλογον ᾧ, καὶ αὐταὶ αἱ εὐθεῖαι ἀνάλογον ἔσονται· ὅπερ ἔδει δεῖξαι.

(is) to QR [Prop. 6.12]. And let the rectilinear figure SR , similar, and similarly laid down, to either of MF or NH , have been described on QR [Props. 6.18, 6.21].

Therefore, since as AB is to CD , so EF (is) to QR , and the similar, and similarly laid out, (rectilinear figures) KAB and LCD have been described on AB and CD (respectively), and the similar, and similarly laid out, (rectilinear figures) MF and SR on EF and QR (respectively), thus as KAB is to LCD , so MF (is) to SR (see above). And it was also assumed that as KAB (is) to LCD , so MF (is) to NH . Thus, also, as MF (is) to SR , so MF (is) to NH [Prop. 5.11]. Thus, MF has the same ratio to each of NH and SR . Thus, NH is equal to SR [Prop. 5.9]. And it is also similar, and similarly laid out, to it. Thus, GH (is) equal to QR .[†] And since AB is to CD , as EF (is) to QR , and QR (is) equal to GH , thus as AB is to CD , so EF (is) to GH .

Thus, if four straight-lines are proportional, then similar, and similarly described, rectilinear figures (drawn) on them will also be proportional. And if similar, and similarly described, rectilinear figures (drawn) on them are proportional then the straight-lines themselves will also be proportional. (Which is) the very thing it was required to show.

[†] Here, Euclid assumes, without proof, that if two similar figures are equal then any pair of corresponding sides is also equal.

κγ'.

Τὰ ἰσογώνια παραλληλόγραμμα πρὸς ἄλληλα λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν.

Ἐστω ἰσογώνια παραλληλόγραμμα τὰ ΑΓ, ΓΖ ἴσην ἔχοντα τὴν ὑπὸ ΒΓΔ γωνίαν τῇ ὑπὸ ΕΓΗ· λέγω, ὅτι τὸ ΑΓ παραλληλόγραμμον πρὸς τὸ ΓΖ παραλληλόγραμμον λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν.

Κείσθω γὰρ ὥστε ἐπ' εὐθείας εἶναι τὴν ΒΓ τῇ ΓΗ· ἐπ' εὐθείας ἄρα ἐστὶ καὶ ἡ ΔΓ τῇ ΓΕ. καὶ συμπληρώσθω τὸ ΔΗ παραλληλόγραμμον, καὶ ἐκκείσθω τις εὐθεῖα ἡ Κ, καὶ γεγονέτω ὡς μὲν ἡ ΒΓ πρὸς τὴν ΓΗ, οὕτως ἡ Κ πρὸς τὴν Λ, ὡς δὲ ἡ ΔΓ πρὸς τὴν ΓΕ, οὕτως ἡ Λ πρὸς τὴν Μ.

Οἱ ἄρα λόγοι τῆς τε Κ πρὸς τὴν Λ καὶ τῆς Λ πρὸς τὴν Μ οἱ αὐτοὶ εἰσι τοῖς λόγοις τῶν πλευρῶν, τῆς τε ΒΓ πρὸς τὴν ΓΗ καὶ τῆς ΔΓ πρὸς τὴν ΓΕ. ἀλλ' ὁ τῆς Κ πρὸς Μ λόγος σύγκειται ἐκ τε τοῦ τῆς Κ πρὸς Λ λόγου καὶ τοῦ τῆς Λ πρὸς Μ· ὥστε καὶ ἡ Κ πρὸς τὴν Μ λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν. καὶ ἐπεὶ ἐστὶν ὡς ἡ ΒΓ πρὸς τὴν ΓΗ, οὕτως τὸ ΑΓ παραλληλόγραμμον πρὸς τὸ ΓΘ, ἀλλ' ὡς ἡ ΒΓ πρὸς τὴν ΓΗ, οὕτως ἡ Κ πρὸς τὴν Λ, καὶ ὡς ἄρα ἡ Κ πρὸς τὴν Λ, οὕτως τὸ ΑΓ πρὸς τὸ ΓΘ. πάλιν, ἐπεὶ ἐστὶν ὡς ἡ ΔΓ πρὸς τὴν ΓΕ, οὕτως τὸ ΓΘ παραλληλόγραμμον πρὸς τὸ ΓΖ, ἀλλ' ὡς ἡ ΔΓ πρὸς τὴν ΓΕ,

Proposition 23

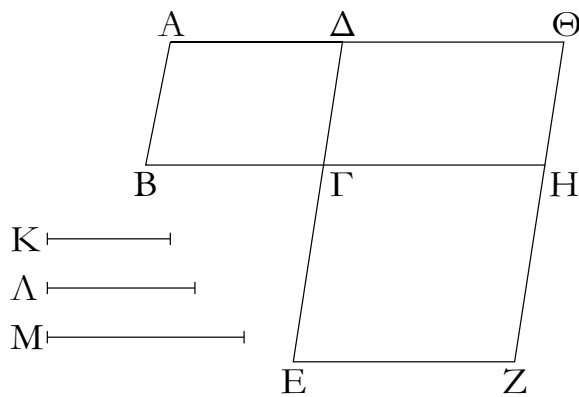
Equiangular parallelograms have to one another the ratio compounded[†] out of (the ratios of) their sides.

Let AC and CF be equiangular parallelograms having angle BCD equal to ECG . I say that parallelogram AC has to parallelogram CF the ratio compounded out of (the ratios of) their sides.

For let BC be laid down so as to be straight-on to CG . Thus, DC is also straight-on to CE [Prop. 1.14]. And let the parallelogram DG have been completed. And let some straight-line K have been laid down. And let it be contrived that as BC (is) to CG , so K (is) to L , and as DC (is) to CE , so L (is) to M [Prop. 6.12].

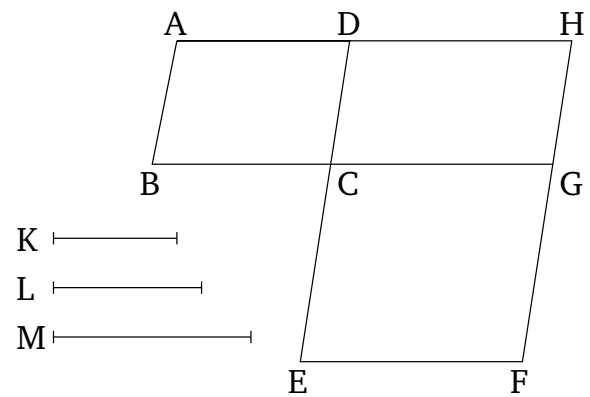
Thus, the ratios of K to L and of L to M are the same as the ratios of the sides, (namely), BC to CG and DC to CE (respectively). But, the ratio of K to M is compounded out of the ratio of K to L and (the ratio) of L to M . Hence, K also has to M the ratio compounded out of (the ratios of) the sides (of the parallelograms). And since as BC is to CG , so parallelogram AC (is) to CH [Prop. 6.1], but as BC (is) to CG , so K (is) to L , thus, also, as K (is) to L , so (parallelogram) AC (is) to CH . Again, since as DC (is) to CE , so parallelogram

οὕτως ἡ Λ πρὸς τὴν M , καὶ ὥς ἄρα ἡ Λ πρὸς τὴν M , οὕτως τὸ $\Gamma\Theta$ παραλληλόγραμμον πρὸς τὸ ΓZ παραλληλόγραμμον. ἐπεὶ οὖν ἐδείχθη, ὡς μὲν ἡ K πρὸς τὴν Λ , οὕτως τὸ $ΑΓ$ παραλληλόγραμμον πρὸς τὸ $\Gamma\Theta$ παραλληλόγραμμον, ὡς δὲ ἡ Λ πρὸς τὴν M , οὕτως τὸ $\Gamma\Theta$ παραλληλόγραμμον πρὸς τὸ ΓZ παραλληλόγραμμον, δι' ἴσου ἄρα ἐστὶν ὡς ἡ K πρὸς τὴν M , οὕτως τὸ $ΑΓ$ πρὸς τὸ ΓZ παραλληλόγραμμον. ἡ δὲ K πρὸς τὴν M λόγον ἔχει τὸν συγχείμενον ἐκ τῶν πλευρῶν· καὶ τὸ $ΑΓ$ ἄρα πρὸς τὸ ΓZ λόγον ἔχει τὸν συγχείμενον ἐκ τῶν πλευρῶν.



Τὰ ἄρα ἰσογώνια παραλληλόγραμμα πρὸς ἀλλήλα λόγον ἔχει τὸν συγχείμενον ἐκ τῶν πλευρῶν· ὅπερ ἔδει δεῖξαι.

CH (is) to CF [Prop. 6.1], but as DC (is) to CE , so L (is) to M , thus, also, as L (is) to M , so parallelogram CH (is) to parallelogram CF . Therefore, since it was shown that as K (is) to L , so parallelogram AC (is) to parallelogram CH , and as L (is) to M , so parallelogram CH (is) to parallelogram CF , thus, via equality, as K is to M , so (parallelogram) AC (is) to parallelogram CF [Prop. 5.22]. And K has to M the ratio compounded out of (the ratios of) the sides (of the parallelograms). Thus, (parallelogram) AC also has to (parallelogram) CF the ratio compounded out of (the ratio of) their sides.



Thus, equiangular parallelograms have to one another the ratio compounded out of (the ratio of) their sides. (Which is) the very thing it was required to show.

† In modern terminology, if two ratios are “compounded” then they are multiplied together.

κδ'.

Proposition 24

Παντὸς παραλληλογράμμου τὰ περὶ τὴν διάμετρον παραλληλόγραμμα ὁμοία ἐστὶ τῷ τε ὅλῳ καὶ ἀλλήλοις.

Ἐστω παραλληλόγραμμον τὸ $ΑΒΓΔ$, διάμετρος δὲ αὐτοῦ ἡ $ΑΓ$, περὶ δὲ τὴν $ΑΓ$ παραλληλόγραμμα ἔστω τὰ $ΕΗ$, $ΘΚ$. λέγω, ὅτι ἐκάτερον τῶν $ΕΗ$, $ΘΚ$ παραλληλογράμμων ὁμοιόν ἐστὶ ὅλῳ τῷ $ΑΒΓΔ$ καὶ ἀλλήλοις.

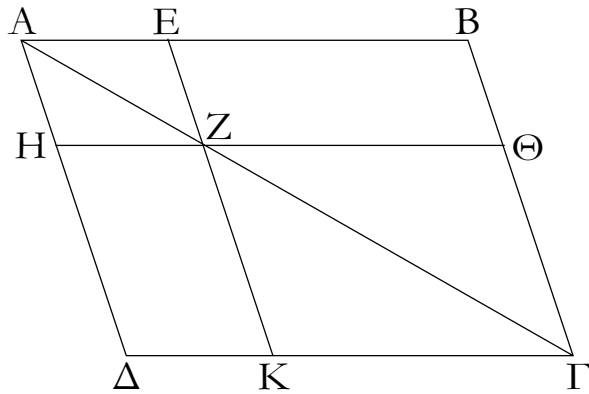
Ἐπεὶ γὰρ τριγώνου τοῦ $ΑΒΓ$ παρὰ μίαν τῶν πλευρῶν τὴν $ΒΓ$ ῥίπται ἡ $ΕΖ$, ἀνάλογόν ἐστιν ὡς ἡ $ΒΕ$ πρὸς τὴν $ΕΑ$, οὕτως ἡ $ΓΖ$ πρὸς τὴν $ΖΑ$. πάλιν, ἐπεὶ τριγώνου τοῦ $ΑΓΔ$ παρὰ μίαν τὴν $ΓΔ$ ῥίπται ἡ $ΖΗ$, ἀνάλογόν ἐστιν ὡς ἡ $ΓΖ$ πρὸς τὴν $ΖΑ$, οὕτως ἡ $ΔΗ$ πρὸς τὴν $ΗΑ$. ἀλλ' ὡς ἡ $ΓΖ$ πρὸς τὴν $ΖΑ$, οὕτως ἐδείχθη καὶ ἡ $ΒΕ$ πρὸς τὴν $ΕΑ$ · καὶ ὥς ἄρα ἡ $ΒΕ$ πρὸς τὴν $ΕΑ$, οὕτως ἡ $ΔΗ$ πρὸς τὴν $ΗΑ$, καὶ συνθέντι ἄρα ὡς ἡ $ΒΑ$ πρὸς $ΑΕ$, οὕτως ἡ $ΔΑ$ πρὸς $ΑΗ$, καὶ ἐναλλάξ ὡς ἡ $ΒΑ$ πρὸς τὴν $ΑΔ$, οὕτως ἡ $ΕΑ$ πρὸς τὴν $ΑΗ$. τῶν ἄρα $ΑΒΓΔ$, $ΕΗ$ παραλληλογράμμων ἀνάλογόν εἰσιν αἱ πλευραὶ αἱ περὶ τὴν κοινὴν γωνίαν τὴν ὑπὸ $ΒΑΔ$. καὶ ἐπεὶ παράλληλός ἐστιν ἡ $ΗΖ$ τῇ $ΔΓ$, ἴση ἐστὶν ἡ μὲν ὑπὸ $ΑΖΗ$ γωνία τῇ ὑπὸ $ΔΓΑ$ · καὶ κοινὴ τῶν δύο

In any parallelogram the parallelograms about the diagonal are similar to the whole, and to one another.

Let $ABCD$ be a parallelogram, and AC its diagonal. And let EG and HK be parallelograms about AC . I say that the parallelograms EG and HK are each similar to the whole (parallelogram) $ABCD$, and to one another.

For since EF has been drawn parallel to one of the sides BC of triangle ABC , proportionally, as BE is to EA , so CF (is) to FA [Prop. 6.2]. Again, since FG has been drawn parallel to one (of the sides) CD of triangle ACD , proportionally, as CF is to FA , so DG (is) to GA [Prop. 6.2]. But, as CF (is) to FA , so it was also shown (is) BE to EA . And thus as BE (is) to EA , so DG (is) to GA . And, thus, compounding, as BA (is) to AE , so DA (is) to AG [Prop. 5.18]. And, alternately, as BA (is) to AD , so EA (is) to AG [Prop. 5.16]. Thus, in parallelograms $ABCD$ and EG the sides about the common angle BAD are proportional. And since GF is parallel to DC , angle AFG is equal to DCA [Prop. 1.29].

τριγώνων τῶν $\triangle A\Delta\Gamma$, $\triangle AHZ$ ἡ ὑπὸ $\triangle A\Gamma$ γωνία· ἰσογώνιον ἄρα ἐστὶ τὸ $\triangle A\Delta\Gamma$ τριγώνον τῷ $\triangle AHZ$ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ $\triangle A\Gamma B$ τριγώνον ἰσογώνιον ἐστὶ τῷ $\triangle AZE$ τριγώνῳ, καὶ ὅλον τὸ $\triangle AB\Gamma$ παραλληλόγραμμον τῷ $\triangle EH$ παραλληλογράμμῳ ἰσογώνιον ἐστίν. ἀνάλογον ἄρα ἐστὶν ὡς ἡ $A\Delta$ πρὸς τὴν $\Delta\Gamma$, οὕτως ἡ AH πρὸς τὴν HZ , ὡς δὲ ἡ $\Delta\Gamma$ πρὸς τὴν ΓA , οὕτως ἡ HZ πρὸς τὴν ZA , ὡς δὲ ἡ $A\Gamma$ πρὸς τὴν ΓB , οὕτως ἡ AZ πρὸς τὴν ZE , καὶ ἔτι ὡς ἡ ΓB πρὸς τὴν BA , οὕτως ἡ ZE πρὸς τὴν EA . καὶ ἐπεὶ ἐδείχθη ὡς μὲν ἡ $\Delta\Gamma$ πρὸς τὴν ΓA , οὕτως ἡ HZ πρὸς τὴν ZA , ὡς δὲ ἡ $A\Gamma$ πρὸς τὴν ΓB , οὕτως ἡ AZ πρὸς τὴν ZE , δι' ἴσου ἄρα ἐστὶν ὡς ἡ $\Delta\Gamma$ πρὸς τὴν ΓB , οὕτως ἡ HZ πρὸς τὴν ZE . τῶν ἄρα $\triangle AB\Gamma$, $\triangle EH$ παραλληλογράμμων ἀνάλογον εἰσὶν αἱ πλευраὶ αἱ περὶ τὰς ἴσας γωνίας· ὁμοιον ἄρα ἐστὶ τὸ $\triangle AB\Gamma$ παραλληλόγραμμον τῷ $\triangle EH$ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ τὸ $\triangle AB\Gamma$ παραλληλόγραμμον καὶ τῷ $\triangle K\Theta$ παραλληλογράμμῳ ὁμοιον ἐστίν· ἐκάτερον ἄρα τῶν $\triangle EH$, $\triangle K\Theta$ παραλληλογράμμων τῷ $\triangle AB\Gamma$ [παραλληλογράμμῳ] ὁμοιον ἐστίν. τὰ δὲ τῷ αὐτῷ εὐθυγράμμῳ ὁμοία καὶ ἀλλήλοις ἐστὶν ὁμοία· καὶ τὸ $\triangle EH$ ἄρα παραλληλόγραμμον τῷ $\triangle K\Theta$ παραλληλογράμμῳ ὁμοιον ἐστίν.

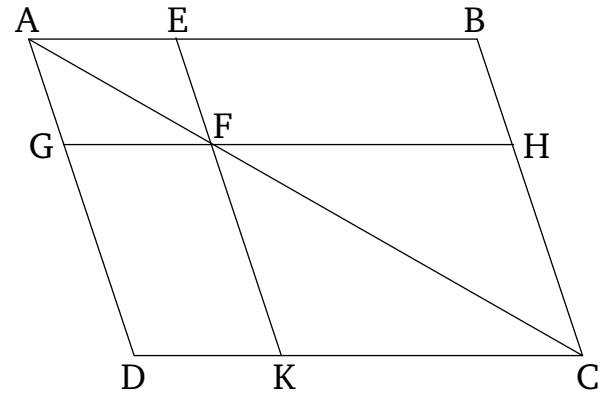


Παντὸς ἄρα παραλληλογράμμου τὰ περὶ τὴν διάμετρον παραλληλόγραμμα ὁμοία ἐστὶ τῷ τε ὅλῳ καὶ ἀλλήλοις· ὅπερ ἔδει δεῖξαι.

κε'.

Τῷ δοθέντι εὐθυγράμμῳ ὁμοιον καὶ ἄλλῳ τῷ δοθέντι ἴσον τὸ αὐτὸ συστήσασθαι.

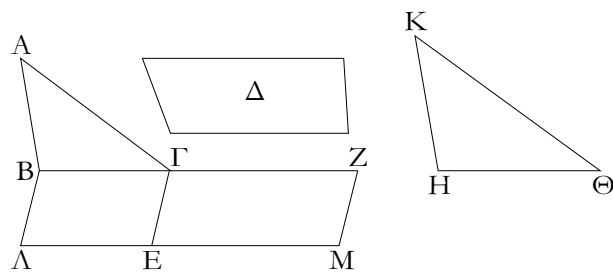
And angle $\angle DAC$ (is) common to the two triangles $\triangle ADC$ and $\triangle AGF$. Thus, triangle $\triangle ADC$ is equiangular to triangle $\triangle AGF$ [Prop. 1.32]. So, for the same (reasons), triangle $\triangle ACB$ is equiangular to triangle $\triangle AFE$, and the whole parallelogram $ABCD$ is equiangular to parallelogram EG . Thus, proportionally, as AD (is) to DC , so AG (is) to GF , and as DC (is) to CA , so GF (is) to FA , and as AC (is) to CB , so AF (is) to FE , and, further, as CB (is) to BA , so FE (is) to EA [Prop. 6.4]. And since it was shown that as DC is to CA , so GF (is) to FA , and as AC (is) to CB , so AF (is) to FE , thus, via equality, as DC is to CB , so GF (is) to FE [Prop. 5.22]. Thus, in parallelograms $ABCD$ and EG the sides about the equal angles are proportional. Thus, parallelogram $ABCD$ is similar to parallelogram EG [Def. 6.1]. So, for the same (reasons), parallelogram $ABCD$ is also similar to parallelogram KH . Thus, parallelograms EG and KH are each similar to [parallelogram] $ABCD$. And (rectilinear figures) similar to the same rectilinear figure are also similar to one another [Prop. 6.21]. Thus, parallelogram EG is also similar to parallelogram KH .



Thus, in any parallelogram the parallelograms about the diagonal are similar to the whole, and to one another. (Which is) the very thing it was required to show.

Proposition 25

To construct a single (rectilinear figure) similar to a given rectilinear figure, and equal to a different given rectilinear figure.



Ἐστω τὸ μὲν δοθὲν εὐθύγραμμον, ᾧ δεῖ ὁμοιον συστήσασθαι, τὸ $AB\Gamma$, ᾧ δὲ δεῖ ἴσον, τὸ Δ . δεῖ δὴ τῶ μὲν $AB\Gamma$ ὁμοιον, τῶ δὲ Δ ἴσον τὸ αὐτὸ συστήσασθαι.

Παραβεβλήσθω γὰρ παρὰ μὲν τὴν $B\Gamma$ τῶ $AB\Gamma$ τριγώνω ἴσον παραλληλόγραμμον τὸ BE , παρὰ δὲ τὴν GE τῶ Δ ἴσον παραλληλόγραμμον τὸ GM ἐν γωνίᾳ τῇ ὑπὸ ZGE , ἣ ἐστὶν ἴση τῇ ὑπὸ GBA . ἐπ' εὐθείας ἄρα ἐστὶν ἡ μὲν $B\Gamma$ τῇ $ΓZ$, ἡ δὲ AE τῇ EM . καὶ εἰλήφθω τῶν $B\Gamma$, $ΓZ$ μέση ἀνάλογον ἡ $H\Theta$, καὶ ἀναγεγράφθω ἀπὸ τῆς $H\Theta$ τῶ $AB\Gamma$ ὁμοιόν τε καὶ ὁμοίως κείμενον τὸ $KH\Theta$.

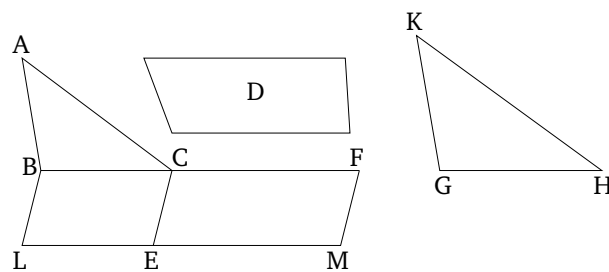
Καὶ ἐπεὶ ἐστὶν ὡς ἡ $B\Gamma$ πρὸς τὴν $H\Theta$, οὕτως ἡ $H\Theta$ πρὸς τὴν $ΓZ$, ἐὰν δὲ τρεῖς εὐθεῖαι ἀνάλογον ᾶσιν, ἐστὶν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης εἶδος πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὁμοιον καὶ ὁμοίως ἀναγεγόμενον, ἐστὶν ἄρα ὡς ἡ $B\Gamma$ πρὸς τὴν $ΓZ$, οὕτως τὸ $AB\Gamma$ τρίγωνον πρὸς τὸ $KH\Theta$ τρίγωνον. ἀλλὰ καὶ ὡς ἡ $B\Gamma$ πρὸς τὴν $ΓZ$, οὕτως τὸ BE παραλληλόγραμμον πρὸς τὸ EZ παραλληλόγραμμον. καὶ ὡς ἄρα τὸ $AB\Gamma$ τρίγωνον πρὸς τὸ $KH\Theta$ τρίγωνον, οὕτως τὸ BE παραλληλόγραμμον πρὸς τὸ EZ παραλληλόγραμμον. ἐναλλάξ ἄρα ὡς τὸ $AB\Gamma$ τρίγωνον πρὸς τὸ BE παραλληλόγραμμον, οὕτως τὸ $KH\Theta$ τρίγωνον πρὸς τὸ EZ παραλληλόγραμμον. ἴσον δὲ τὸ $AB\Gamma$ τρίγωνον τῶ BE παραλληλογράμμῳ· ἴσον ἄρα καὶ τὸ $KH\Theta$ τρίγωνον τῶ EZ παραλληλογράμμῳ. ἀλλὰ τὸ EZ παραλληλόγραμμον τῶ Δ ἐστὶν ἴσον· καὶ τὸ $KH\Theta$ ἄρα τῶ Δ ἐστὶν ἴσον. ἐστὶ δὲ τὸ $KH\Theta$ καὶ τῶ $AB\Gamma$ ὁμοιον.

Τῶ ἄρα δοθέντι εὐθυγράμμῳ τῶ $AB\Gamma$ ὁμοιον καὶ ἄλλω τῶ δοθέντι τῶ Δ ἴσον τὸ αὐτὸ συνέσταται τὸ $KH\Theta$. ὅπερ ἔδει ποιῆσαι.

κς'.

Ἐὰν ἀπὸ παραλληλογράμμου παραλληλόγραμμον ἀφαιρεθῇ ὁμοιόν τε τῶ ὅλῳ καὶ ὁμοίως κείμενον κοινὴν γωνίαν ἔχον αὐτῶ, περὶ τὴν αὐτὴν διάμετρον ἐστί τῶ ὅλῳ.

Ἀπὸ γὰρ παραλληλογράμμου τοῦ $AB\Gamma\Delta$ παραλληλόγραμμον ἀφηρήσθω τὸ AZ ὁμοιον τῶ $AB\Gamma\Delta$ καὶ ὁμοίως κείμενον κοινὴν γωνίαν ἔχον αὐτῶ τὴν ὑπὸ ΔAB . λέγω,



Let ABC be the given rectilinear figure to which it is required to construct a similar (rectilinear figure), and D the (rectilinear figure) to which (the constructed figure) is required (to be) equal. So it is required to construct a single (rectilinear figure) similar to ABC , and equal to D .

For let the parallelogram BE , equal to triangle ABC , have been applied to (the straight-line) BC [Prop. 1.44], and the parallelogram CM , equal to D , (have been applied) to (the straight-line) CE , in the angle FCE , which is equal to CBL [Prop. 1.45]. Thus, BC is straight-on to CF , and LE to EM [Prop. 1.14]. And let the mean proportion GH have been taken of BC and CF [Prop. 6.13]. And let KGH , similar, and similarly laid out, to ABC have been described on GH [Prop. 6.18].

And since as BC is to GH , so GH (is) to CF , and if three straight-lines are proportional then as the first is to the third, so the figure (described) on the first (is) to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.], thus as BC is to CF , so triangle ABC (is) to triangle KGH . But, also, as BC (is) to CF , so parallelogram BE (is) to parallelogram EF [Prop. 6.1]. And, thus, as triangle ABC (is) to triangle KGH , so parallelogram BE (is) to parallelogram EF . Thus, alternately, as triangle ABC (is) to parallelogram BE , so triangle KGH (is) to parallelogram EF [Prop. 5.16]. And triangle ABC (is) equal to parallelogram BE . Thus, triangle KGH (is) also equal to parallelogram EF . But, parallelogram EF is equal to D . Thus, KGH is also equal to D . And KGH is also similar to ABC .

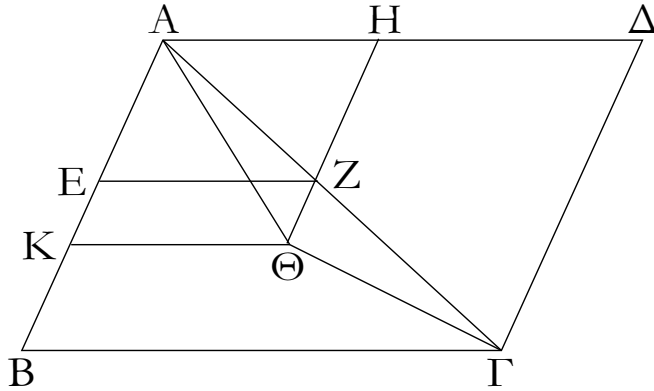
Thus, a single (rectilinear figure) KGH has been constructed (which is) similar to the given rectilinear figure ABC , and equal to a different given (rectilinear figure) D . (Which is) the very thing it was required to do.

Proposition 26

If from a parallelogram a(nother) parallelogram is subtracted (which is) similar, and similarly laid out, to the whole, having a common angle with it, then (the subtracted parallelogram) is about the same diagonal as the whole.

For, from parallelogram $ABCD$, let (parallelogram)

ὅτι περὶ τὴν αὐτὴν διάμετρον ἐστὶ τὸ $AB\Gamma\Delta$ τῷ AZ .



Μὴ γάρ, ἀλλ' εἰ δυνατόν, ἔστω [αὐτῶν] διάμετρος ἡ $A\Theta\Gamma$, καὶ ἐκβληθεῖσα ἡ HZ διήχθω ἐπὶ τὸ Θ , καὶ ἤχθω διὰ τοῦ Θ ὁπορέρα τῶν $A\Delta$, $B\Gamma$ παράλληλος ἡ ΘK .

Ἐπεὶ οὖν περὶ τὴν αὐτὴν διάμετρον ἐστὶ τὸ $AB\Gamma\Delta$ τῷ KH , ἔστιν ἄρα ὡς ἡ ΔA πρὸς τὴν AB , οὕτως ἡ HA πρὸς τὴν AK . ἔστι δὲ καὶ διὰ τὴν ὁμοιότητα τῶν $AB\Gamma\Delta$, EH καὶ ὡς ἡ ΔA πρὸς τὴν AB , οὕτως ἡ HA πρὸς τὴν AE · καὶ ὡς ἄρα ἡ HA πρὸς τὴν AK , οὕτως ἡ HA πρὸς τὴν AE . ἡ HA ἄρα πρὸς ἑκατέραν τῶν AK , AE τὸν αὐτὸν ἔχει λόγον. ἴση ἄρα ἐστὶν ἡ AE τῇ AK ἢ ἐλάττω τῇ μείζονι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οὐκ ἐστὶ περὶ τὴν αὐτὴν διάμετρον τὸ $AB\Gamma\Delta$ τῷ AZ · περὶ τὴν αὐτὴν ἄρα ἐστὶ διάμετρον τὸ $AB\Gamma\Delta$ παραλληλόγραμμον τῷ AZ παραλληλογράμμῳ.

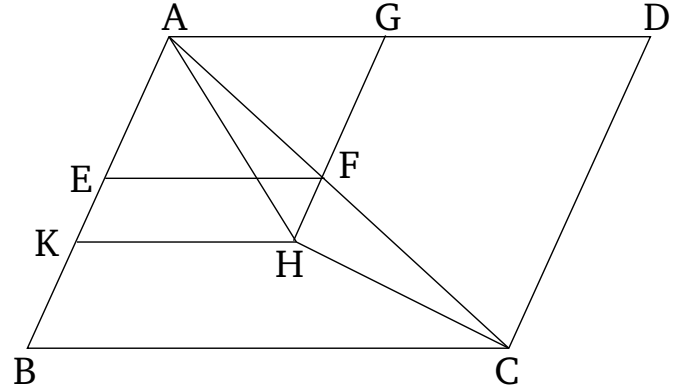
Ἐὰν ἄρα ἀπὸ παραλληλογράμμου παραλληλόγραμμον ἀφαιρεθῇ ὁμοίον τε τῷ ὅλῳ καὶ ὁμοίως κείμενον κοινὴν γωνίαν ἔχον αὐτῷ, περὶ τὴν αὐτὴν διάμετρον ἐστὶ τῷ ὅλῳ· ὅπερ ἔδει δεῖξαι.

κζ'.

Πάντων τῶν παρὰ τὴν αὐτὴν εὐθεῖαν παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἶδεσι παραλληλογράμμοις ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ἀπὸ τῆς ἡμισείας ἀναγραφόμενῳ μέγιστόν ἐστὶ τὸ ἀπὸ τῆς ἡμισείας παραβαλλόμενον [παραλληλόγραμμον] ὁμοίον δὲ τῷ ἐλλείμμενῳ.

Ἐστω εὐθεῖα ἡ AB καὶ τετμήσθω δίχα κατὰ τὸ Γ , καὶ παραβεβλήσθω παρὰ τὴν AB εὐθεῖαν τὸ $A\Delta$ παραλληλόγραμμον ἐλλείπον εἶδει παραλληλογράμμῳ τῷ ΔB ἀναγραφέντι ἀπὸ τῆς ἡμισείας τῆς AB , τουτέστι τῆς ΓB · λέγω, ὅτι πάντων τῶν παρὰ τὴν AB παραβαλλομένων παραλληλογράμμων καὶ ἐλλειπόντων εἶδεσι [παραλληλογράμμοις] ὁμοίοις τε καὶ ὁμοίως κειμένοις τῷ ΔB μέγιστόν ἐστὶ τὸ

AF have been subtracted (which is) similar, and similarly laid out, to $ABCD$, having the common angle DAB with it. I say that $ABCD$ is about the same diagonal as AF .



For (if) not, then, if possible, let AHC be [$ABCD$'s] diagonal. And producing GF , let it have been drawn through (point) H . And let HK have been drawn through (point) H , parallel to either of AD or BC [Prop. 1.31].

Therefore, since $ABCD$ is about the same diagonal as KG , thus as DA is to AB , so GA (is) to AK [Prop. 6.24]. And, on account of the similarity of $ABCD$ and EG , also, as DA (is) to AB , so GA (is) to AE . Thus, also, as GA (is) to AK , so GA (is) to AE . Thus, GA has the same ratio to each of AK and AE . Thus, AE is equal to AK [Prop. 5.9], the lesser to the greater. The very thing is impossible. Thus, $ABCD$ is not not about the same diagonal as AF . Thus, parallelogram $ABCD$ is about the same diagonal as parallelogram AF .

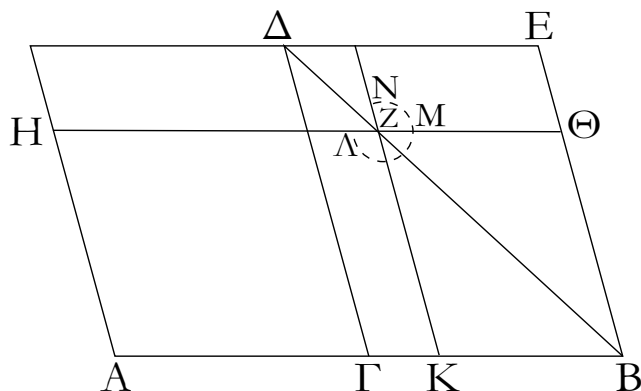
Thus, if from a parallelogram a (nother) parallelogram is subtracted (which is) similar, and similarly laid out, to the whole, having a common angle with it, then (the subtracted parallelogram) is about the same diagonal as the whole. (Which is) the very thing it was required to show.

Proposition 27

Of all the parallelograms applied to the same straight-line, and falling short by parallelogrammic figures similar, and similarly laid out, to the (parallelogram) described on half (the straight-line), the greatest is the [parallelogram] applied to half (the straight-line) which (is) similar to (that parallelogram) by which it falls short.

Let AB be a straight-line, and let it have been cut in half at (point) C [Prop. 1.10]. And let the parallelogram AD have been applied to the straight-line AB , falling short by the parallelogrammic figure DB (which is) applied to half of AB —that is to say, CB . I say that of all the parallelograms applied to AB , and falling short by

ΑΔ. παραβεβλήσθω γὰρ παρὰ τὴν ΑΒ εὐθεῖαν τὸ ΑΖ πα-
ραλληλόγραμμον ἐλλείπον εἶδει παραλληλογράμμω τῷ ΖΒ
ὁμοίω τε καὶ ὁμοίως κεμένω τῷ ΔΒ· λέγω, ὅτι μείζον ἐστι
τὸ ΑΔ τοῦ ΑΖ.



Ἐπεὶ γὰρ ὁμοιὸν ἔστι τὸ ΔΒ παραλληλόγραμμον τῷ ΖΒ παραλληλογράμμῳ, περὶ τὴν αὐτὴν εἰσι διάμετρον. ἤχθῳ αὐτῶν διάμετρος ἡ ΔΒ, καὶ καταγεγράφθῃ τὸ σχῆμα.

Ἐπεὶ οὖν ἴσον ἐστὶ τὸ ΓΖ τῷ ΖΕ, κοινὸν δὲ τὸ ΖΒ, ὅλον ἄρα τὸ ΓΘ ὅλῳ τῷ ΚΕ ἐστὶν ἴσον. ἀλλὰ τὸ ΓΘ τῷ ΓΗ ἐστὶν ἴσον, ἐπεὶ καὶ ἡ ΑΓ τῇ ΓΒ. καὶ τὸ ΗΓ ἄρα τῷ ΕΚ ἐστὶν ἴσον. κοινὸν προσκείσθω τὸ ΓΖ· ὅλον ἄρα τὸ ΑΖ τῷ ΑΜΝ γνώμονι ἐστὶν ἴσον· ὥστε τὸ ΔΒ παραλληλόγραμμον, τοιούτεσι τὸ ΑΔ, τοῦ ΑΖ παραλληλογράμμου μεῖζόν ἐστιν.

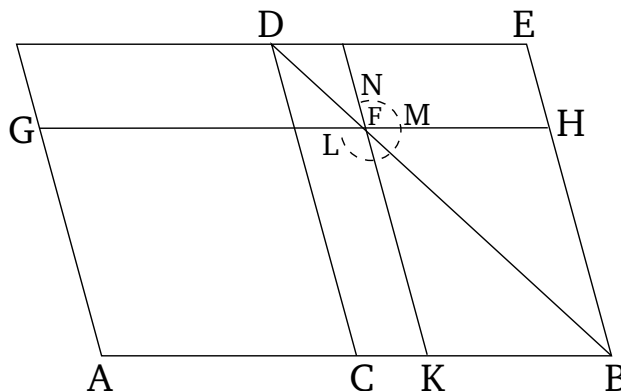
Πάντων ἄρα τῶν παρὰ τὴν αὐτὴν εὐθείαν παραβαλ-
λομένων παραλληλογράμμων καὶ ἑλλειπόντων εἶδεσι παραλ-
ληλογράμμοις ὁμοίοις τε καὶ ὁμοίως χειμένοις τῷ ἀπὸ τῆς
ἡμισείας ἀναγραφομένῳ μέγιστόν ἐστι τὸ ἀπὸ τῆς ἡμισείας
παραβληθέν· ὅπερ ἔδει δεῖξαι.

κη'.

Παρά τὴν δοθεῖσαν εὐθείαν τῷ δοθέντι εὐθυγράμμῳ ἴσον παραλληλόγραμμον παραβαλεῖν ἑλλείπον εἶδει παραλληλογράμμῳ ὁμοίῳ τῷ δοθέντι· δεῖ δὲ τὸ διδόμενον εὐθύγραμμον [ᾧ δεῖ ἴσον παραβαλεῖν] μὴ μείζον εἶναι τοῦ ἀπὸ τῆς ἡμισείας ἀναγραφομένου ὁμοίου τῷ ἐλλείμματι [τοῦ τε ἀπὸ τῆς ἡμισείας καὶ ᾧ δεῖ ὅμοιον ἑλλείπειν].

Ἐστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ AB, τὸ δὲ δοθὲν
εὐθύγραμμον, ᾧ δεῖ ἴσον παρὰ τὴν AB παραβαλεῖν, τὸ Γ μὴ
μειζον [ὄν] τοῦ ἀπὸ τῆς ἡμισείας τῆς AB ἀναγραφομένου
ὁμοίου τῷ ἐλλείμματι, ᾧ δὲ δεῖ ὅμοιον ἐλλείπειν, τὸ Δ· δεῖ δὴ

[parallelogrammic] figures similar, and similarly laid out, to DB , the greatest is AD . For let the parallelogram AF have been applied to the straight-line AB , falling short by the parallelogrammic figure FB (which is) similar, and similarly laid out, to DB . I say that AD is greater than AF .



For since parallelogram DB is similar to parallelogram FB , they are about the same diagonal [Prop. 6.26]. Let their (common) diagonal DB have been drawn, and let the (rest of the) figure have been described.

Therefore, since (complement) CF is equal to (complement) FE [Prop. 1.43], and (parallelogram) FB is common, the whole (parallelogram) CH is thus equal to the whole (parallelogram) KE . But, (parallelogram) CH is equal to CG , since AC (is) also (equal) to CB [Prop. 6.1]. Thus, (parallelogram) GC is also equal to EK . Let (parallelogram) CF have been added to both. Thus, the whole (parallelogram) AF is equal to the gnomon LMN . Hence, parallelogram DB —that is to say, AD —is greater than parallelogram AF .

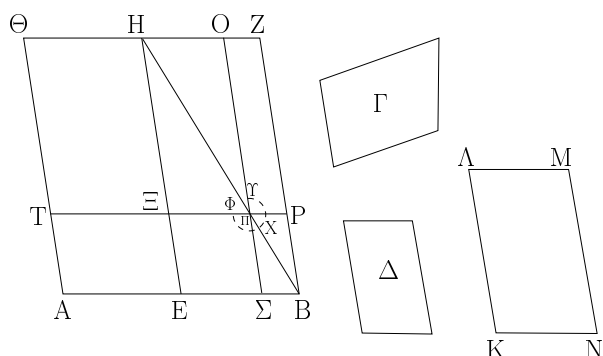
Thus, for all parallelograms applied to the same straight-line, and falling short by a parallelogrammic figure similar, and similarly laid out, to the (parallelogram) described on half (the straight-line), the greatest is the [parallelogram] applied to half (the straight-line). (Which is) the very thing it was required to show.

Proposition 28[†]

To apply a parallelogram, equal to a given rectilinear figure, to a given straight-line, (the applied parallelogram) falling short by a parallelogrammic figure similar to a given (parallelogram). It is necessary for the given rectilinear figure [to which it is required to apply an equal (parallelogram)] not to be greater than the (parallelogram) described on half (of the straight-line) and similar to the deficit.

Let AB be the given straight-line, and C the given rectilinear figure to which the (parallelogram) applied to

παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν AB τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον παραβαλεῖν ἐλλείπον εἶδει παραλληλογράμμῳ ὁμοίῳ ὄντι τῷ Δ .



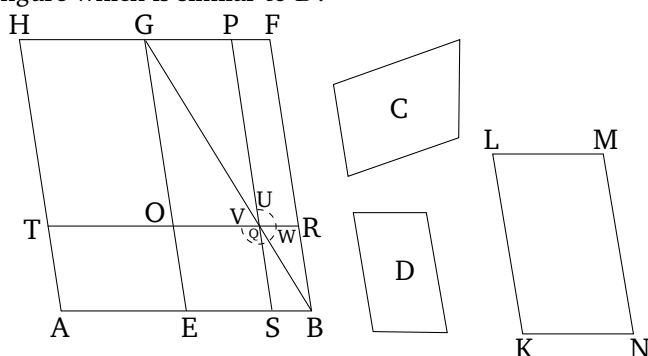
Τετμήσθω ἡ AB δίχα κατὰ τὸ E σημεῖον, καὶ ἀναγεγράφθω ἀπὸ τῆς EB τῷ Δ ὁμοιον καὶ ὁμοίως κείμενον τὸ $EBZH$, καὶ συμπεπληρώσθω τὸ AH παραλληλόγραμμον.

Εἰ μὲν οὖν ἴσον ἐστὶ τὸ AH τῷ Γ , γεγονός δὲ ἂν εἴη τὸ ἐπιταχθέν· παραβέβληται γὰρ παρὰ τὴν δοθεῖσαν εὐθεῖαν τὴν AB τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον τὸ AH ἐλλείπον εἶδει παραλληλογράμμῳ τῷ HB ὁμοίῳ ὄντι τῷ Δ . εἰ δὲ οὐ, μείζον ἔστω τὸ ΘE τοῦ Γ . ἴσον δὲ τὸ ΘE τῷ HB · μείζον ἄρα καὶ τὸ HB τοῦ Γ . ὅ δὲ μείζον ἐστὶ τὸ HB τοῦ Γ , ταύτῃ τῇ ὑπεροχῇ ἴσον, τῷ δὲ Δ ὁμοιον καὶ ὁμοίως κείμενον τὸ αὐτὸ συνεστάτω τὸ $KLMN$. ἀλλὰ τὸ Δ τῷ HB [ἐστίν] ὁμοιον· καὶ τὸ KM ἄρα τῷ HB ἐστὶν ὁμοιον. ἔστω οὖν ὁμόλογος ἡ μὲν KA τῇ HE , ἡ δὲ AM τῇ HZ . καὶ ἐπεὶ ἴσον ἐστὶ τὸ HB τοῖς Γ , KM , μείζον ἄρα ἐστὶ τὸ HB τοῦ KM · μείζων ἄρα ἐστὶ καὶ ἡ μὲν HE τῆς KA , ἡ δὲ HZ τῆς AM . κείσθω τῇ μὲν KA ἴση ἡ HE , τῇ δὲ AM ἴση ἡ HO , καὶ συμπεπληρώσθω τὸ $\Xi H O \Pi$ παραλληλόγραμμον· ἴσον ἄρα καὶ ὁμοιον ἐστὶ [τὸ $H \Pi$] τῷ KM [ἀλλὰ τὸ KM τῷ HB ὁμοιον ἐστίν]. καὶ τὸ $H \Pi$ ἄρα τῷ HB ὁμοιον ἐστίν· περὶ τὴν αὐτὴν ἄρα διάμετρον ἐστὶ τὸ $H \Pi$ τῷ HB . ἔστω αὐτῶν διάμετρος ἡ $H \Pi B$, καὶ καταγεγράφθω τὸ σχῆμα.

Ἐπεὶ οὖν ἴσον ἐστὶ τὸ BH τοῖς Γ , KM , ὣν τὸ $H \Pi$ τῷ KM ἐστὶν ἴσον, λοιπὸς ἄρα ὁ $\Upsilon X \Phi$ γνόμενος λοιπῷ τῷ Γ ἴσος ἐστίν. καὶ ἐπεὶ ἴσον ἐστὶ τὸ OP τῷ $\Xi \Sigma$, κοινὸν προσκείσθω τὸ ΠB · ὅλον ἄρα τὸ OB ὅλῳ τῷ ΞB ἴσον ἐστίν. ἀλλὰ τὸ ΞB τῷ TE ἐστὶν ἴσον, ἐπεὶ καὶ πλευρὰ ἡ AE πλευρᾷ τῇ EB ἐστὶν ἴση· καὶ τὸ TE ἄρα τῷ OB ἐστὶν ἴσον. κοινὸν προσκείσθω τὸ $\Xi \Sigma$ · ὅλον ἄρα τὸ $T \Sigma$ ὅλῳ τῷ $\Phi X \Upsilon$ γνόμενόν ἐστιν ἴσον. ἀλλ' ὁ $\Phi X \Upsilon$ γνόμενος τῷ Γ ἐδείχθη ἴσος· καὶ τὸ $T \Sigma$ ἄρα τῷ Γ ἐστὶν ἴσον.

Παρὰ τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν AB τῷ δοθέντι εὐθυγράμμῳ τῷ Γ ἴσον παραλληλόγραμμον παραβέβληται τὸ ΣT ἐλλείπον εἶδει παραλληλογράμμῳ τῷ ΠB ὁμοίῳ ὄντι

AB is required (to be) equal, [being] not greater than the (parallelogram) described on half of AB and similar to the deficit, and D the (parallelogram) to which the deficit is required (to be) similar. So it is required to apply a parallelogram, equal to the given rectilinear figure C , to the straight-line AB , falling short by a parallelogrammic figure which is similar to D .



Let AB have been cut in half at point E [Prop. 1.10], and let (parallelogram) $EBFG$, (which is) similar, and similarly laid out, to (parallelogram) D , have been described on EB [Prop. 6.18]. And let parallelogram AG have been completed.

Therefore, if AG is equal to C then the thing prescribed has happened. For a parallelogram AG , equal to the given rectilinear figure C , has been applied to the given straight-line AB , falling short by a parallelogrammic figure GB which is similar to D . And if not, let HE be greater than C . And HE (is) equal to GB [Prop. 6.1]. Thus, GB (is) also greater than C . So, let (parallelogram) $KLMN$ have been constructed (so as to be) both similar, and similarly laid out, to D , and equal to the excess by which GB is greater than C [Prop. 6.25]. But, GB [is] similar to D . Thus, KM is also similar to GB [Prop. 6.21]. Therefore, let KL correspond to GE , and LM to GF . And since (parallelogram) GB is equal to (figure) C and (parallelogram) KM , GB is thus greater than KM . Thus, GE is also greater than KL , and GF than LM . Let GO be made equal to KL , and GP to LM [Prop. 1.3]. And let the parallelogram $OGPQ$ have been completed. Thus, [GQ] is equal and similar to KM [but, KM is similar to GB]. Thus, GQ is also similar to GB [Prop. 6.21]. Thus, GQ and GB are about the same diagonal [Prop. 6.26]. Let GQB be their (common) diagonal, and let the (remainder of the) figure have been described.

Therefore, since BG is equal to C and KM , of which GQ is equal to KM , the remaining gnomon UWV is thus equal to the remainder C . And since (the complement) PR is equal to (the complement) OS [Prop. 1.43], let (parallelogram) QB have been added to both. Thus, the whole (parallelogram) PB is equal to the whole (par-

τῷ Δ [ἐπειδήπερ τὸ ΠB τῷ $ΗΠ$ ὁμοίον ἐστίν]· ὅπερ ἔδει ποιῆσαι.

allelogram) OB . But, OB is equal to TE , since side AE is equal to side EB [Prop. 6.1]. Thus, TE is also equal to PB . Let (parallelogram) OS have been added to both. Thus, the whole (parallelogram) TS is equal to the gnomon VWU . But, gnomon VWU was shown (to be) equal to C . Therefore, (parallelogram) TS is also equal to (figure) C .

Thus, the parallelogram ST , equal to the given rectilinear figure C , has been applied to the given straight-line AB , falling short by the parallelogrammic figure QB , which is similar to D [inasmuch as QB is similar to GQ [Prop. 6.24]]. (Which is) the very thing it was required to do.

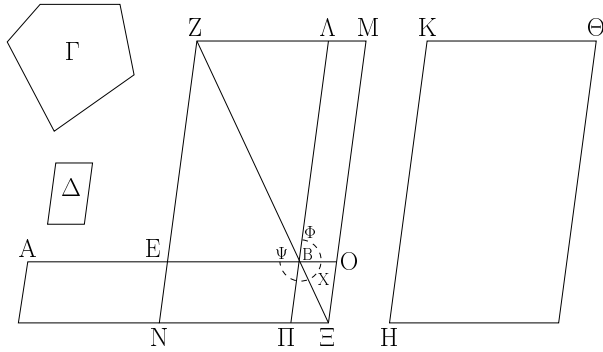
† This proposition is a geometric solution of the quadratic equation $x^2 - \alpha x + \beta = 0$. Here, x is the ratio of a side of the deficit to the corresponding side of figure D , α is the ratio of the length of AB to the length of that side of figure D which corresponds to the side of the deficit running along AB , and β is the ratio of the areas of figures C and D . The constraint corresponds to the condition $\beta < \alpha^2/4$ for the equation to have real roots. Only the smaller root of the equation is found. The larger root can be found by a similar method.

κθ'.

Proposition 29†

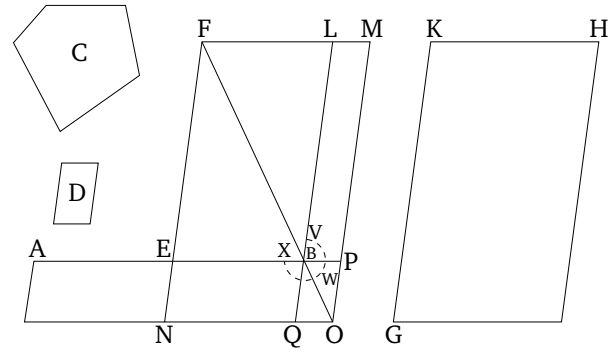
Παρά τὴν δοθεῖσαν εὐθεῖαν τῷ δοθέντι εὐθύγραμμῳ ἴσον παραλληλόγραμμον παραβαλεῖν ὑπερβάλλον εἶδει παραλληλογράμμῳ ὁμοίῳ τῷ δοθέντι.

To apply a parallelogram, equal to a given rectilinear figure, to a given straight-line, (the applied parallelogram) overshooting by a parallelogrammic figure similar to a given (parallelogram).



Ἐστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ AB , τὸ δὲ δοθὲν εὐθύγραμμον, ὃ δεῖ ἴσον παρὰ τὴν AB παραβαλεῖν, τὸ Γ , ὃ δὲ δεῖ ὁμοίον ὑπερβάλλειν, τὸ Δ · δεῖ δὴ παρὰ τὴν AB εὐθεῖαν τῷ Γ εὐθύγραμμῳ ἴσον παραλληλόγραμμον παραβαλεῖν ὑπερβάλλον εἶδει παραλληλογράμμῳ ὁμοίῳ τῷ Δ .

Τετμήσθω ἡ AB δίχα κατὰ τὸ E , καὶ ἀναγεγράφω ἀπὸ τῆς EB τῷ Δ ὁμοίον καὶ ὁμοίως κείμενον παραλληλόγραμμον τὸ BZ , καὶ συναμφοτέροις μὲν τοῖς BZ , Γ ἴσον, τῷ δὲ Δ ὁμοίον καὶ ὁμοίως κείμενον τὸ αὐτὸ συνεστάτω τὸ $H\Theta$. ὁμόλογος δὲ ἔστω ἡ μὲν $K\Theta$ τῇ ZA , ἡ δὲ KH τῇ ZE . καὶ ἐπεὶ μείζον ἐστὶ τὸ $H\Theta$ τοῦ ZB , μείζων ἄρα ἐστὶ καὶ ἡ μὲν $K\Theta$ τῆς ZA , ἡ δὲ KH τῇ ZE . ἐκβεβλήσθωσαν αἱ ZA , ZE , καὶ τῇ μὲν $K\Theta$ ἴση ἔστω ἡ ZAM , τῇ δὲ KH ἴση ἡ ZEN , καὶ συμπεπληρώσθω τὸ MN · τὸ MN ἄρα τῷ $H\Theta$ ἴσον τέ ἐστὶ καὶ ὁμοίον. ἀλλὰ τὸ $H\Theta$ τῷ EL ἐστὶν ὁμοίον·



Let AB be the given straight-line, and C the given rectilinear figure to which the (parallelogram) applied to AB is required (to be) equal, and D the (parallelogram) to which the excess is required (to be) similar. So it is required to apply a parallelogram, equal to the given rectilinear figure C , to the given straight-line AB , overshooting by a parallelogrammic figure similar to D .

Let AB have been cut in half at (point) E [Prop. 1.10], and let the parallelogram BF , (which is) similar, and similarly laid out, to D , have been described on EB [Prop. 6.18]. And let (parallelogram) GH have been constructed (so as to be) both similar, and similarly laid out, to D , and equal to the sum of BF and C [Prop. 6.25]. And let KH correspond to FL , and KG to FE . And since (parallelogram) GH is greater than (parallelogram) FB ,

καὶ τὸ MN ἄρα τῷ $ΕΛ$ ὁμοίον ἐστίν· περὶ τὴν αὐτὴν ἄρα διάμετρον ἐστὶ τὸ $ΕΛ$ τῷ MN . ἤχθω αὐτῶν διάμετρος ἡ $ZΞ$, καὶ καταγεγράφθω τὸ σχῆμα.

Ἐπεὶ ἴσον ἐστὶ τὸ $ΗΘ$ τοῖς $ΕΛ$, $Γ$, ἀλλὰ τὸ $ΗΘ$ τῷ MN ἴσον ἐστίν, καὶ τὸ MN ἄρα τοῖς $ΕΛ$, $Γ$ ἴσον ἐστίν. κοινὸν ἀφηρήσθω τὸ $ΕΛ$ · λοιπὸς ἄρα ὁ $ΨΧΦ$ γνώμων τῷ $Γ$ ἐστὶν ἴσος. καὶ ἐπεὶ ἴση ἐστὶν ἡ $ΑΕ$ τῇ $ΕΒ$, ἴσον ἐστὶ καὶ τὸ $ΑΝ$ τῷ NB , τουτέστι τῷ $ΛΟ$. κοινὸν προσκείσθω τὸ $ΕΞ$ · ὅλον ἄρα τὸ $ΑΞ$ ἴσον ἐστὶ τῷ $ΦΧΨ$ γνώμονι. ἀλλὰ ὁ $ΦΧΨ$ γνώμων τῷ $Γ$ ἴσος ἐστίν· καὶ τὸ $ΑΞ$ ἄρα τῷ $Γ$ ἴσον ἐστίν.

Παρά τὴν δοθεῖσαν ἄρα εὐθεῖαν τὴν $ΑΒ$ τῷ δοθέντι εὐθυγράμμῳ τῷ $Γ$ ἴσον παραλληλόγραμμον παραβέβληται τὸ $ΑΞ$ ὑπερβάλλον εἶδει παραλληλογράμμῳ τῷ $ΠΟ$ ὁμοίῳ ὄντι τῷ $Δ$, ἐπεὶ καὶ τῷ $ΕΛ$ ἐστὶν ὁμοίον τὸ $ΟΠ$ · ὅπερ ἔδει ποιῆσαι.

KH is thus also greater than FL , and KG than FE . Let FL and FE have been produced, and let FLM be (made) equal to KH , and FEN to KG [Prop. 1.3]. And let (parallelogram) MN have been completed. Thus, MN is equal and similar to GH . But, GH is similar to EL . Thus, MN is also similar to EL [Prop. 6.21]. EL is thus about the same diagonal as MN [Prop. 6.26]. Let their (common) diagonal FO have been drawn, and let the (remainder of the) figure have been described.

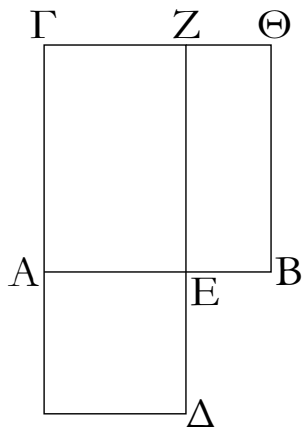
And since (parallelogram) GH is equal to (parallelogram) EL and (figure) C , but GH is equal to (parallelogram) MN , MN is thus also equal to EL and C . Let EL have been subtracted from both. Thus, the remaining gnomon XWV is equal to (figure) C . And since AE is equal to EB , (parallelogram) AN is also equal to (parallelogram) NB [Prop. 6.1], that is to say, (parallelogram) LP [Prop. 1.43]. Let (parallelogram) EO have been added to both. Thus, the whole (parallelogram) AO is equal to the gnomon VWX . But, the gnomon VWX is equal to (figure) C . Thus, (parallelogram) AO is also equal to (figure) C .

Thus, the parallelogram AO , equal to the given rectilinear figure C , has been applied to the given straight-line AB , overshooting by the parallelogrammic figure QP which is similar to D , since PQ is also similar to EL [Prop. 6.24]. (Which is) the very thing it was required to do.

† This proposition is a geometric solution of the quadratic equation $x^2 + \alpha x - \beta = 0$. Here, x is the ratio of a side of the excess to the corresponding side of figure D , α is the ratio of the length of AB to the length of that side of figure D which corresponds to the side of the excess running along AB , and β is the ratio of the areas of figures C and D . Only the positive root of the equation is found.

λ'.

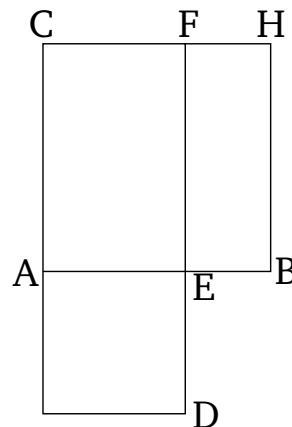
Τὴν δοθεῖσαν εὐθεῖαν πεπερασμένην ἄκρον καὶ μέσον λόγον τεμεῖν.



Ἐστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ $ΑΒ$ · δεῖ δὴ τὴν $ΑΒ$ εὐθεῖαν ἄκρον καὶ μέσον λόγον τεμεῖν.

Proposition 30†

To cut a given finite straight-line in extreme and mean ratio.



Let AB be the given finite straight-line. So it is required to cut the straight-line AB in extreme and mean

Ἀναγεγράφθω ἀπὸ τῆς AB τετράγωνον τὸ $BΓ$, καὶ παραβεβλήσθω παρὰ τὴν $ΑΓ$ τῷ $BΓ$ ἴσον παραλληλόγραμμον τὸ $ΓΔ$ ὑπερβάλλον εἶδει τῷ $ΑΔ$ ὁμοίῳ τῷ $BΓ$.

Τετράγωνον δὲ ἐστὶ τὸ $BΓ$ · τετράγωνον ἄρα ἐστὶ καὶ τὸ $ΑΔ$. καὶ ἐπεὶ ἴσον ἐστὶ τὸ $BΓ$ τῷ $ΓΔ$, κοινὸν ἀφῆρήσθω τὸ $ΓΕ$ · λοιπὸν ἄρα τὸ $BΖ$ λοιπῷ τῷ $ΑΔ$ ἐστὶν ἴσον. ἐστὶ δὲ αὐτῷ καὶ ἰσογώνιον· τῶν $BΖ$, $ΑΔ$ ἄρα ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· ἐστὶν ἄρα ὡς ἡ $ΖΕ$ πρὸς τὴν $ΕΔ$, οὕτως ἡ $ΑΕ$ πρὸς τὴν $ΕΒ$. ἴση δὲ ἡ μὲν $ΖΕ$ τῇ $ΑΒ$, ἡ δὲ $ΕΔ$ τῇ $ΑΕ$. ἐστὶν ἄρα ὡς ἡ $ΒΑ$ πρὸς τὴν $ΑΕ$, οὕτως ἡ $ΑΕ$ πρὸς τὴν $ΕΒ$. μείζων δὲ ἡ $ΑΒ$ τῆς $ΑΕ$ · μείζων ἄρα καὶ ἡ $ΑΕ$ τῆς $ΕΒ$.

Ἡ ἄρα $ΑΒ$ εὐθεῖα ἄκρον καὶ μέσον λόγον τέμνεται κατὰ τὸ $Ε$, καὶ τὸ μείζον αὐτῆς τμήμα ἐστὶ τὸ $ΑΕ$ · ὅπερ ἔδει ποιῆσαι.

ratio.

Let the square BC have been described on AB [Prop. 1.46], and let the parallelogram CD , equal to BC , have been applied to AC , overshooting by the figure AD (which is) similar to BC [Prop. 6.29].

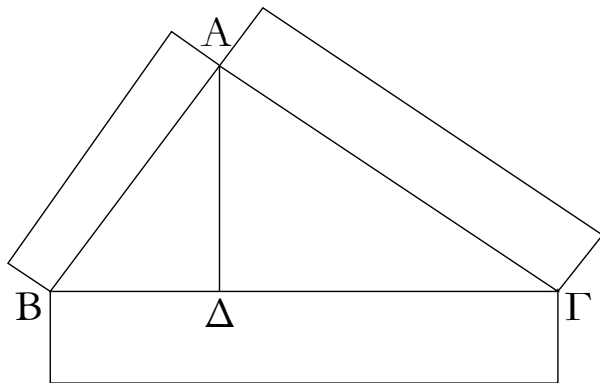
And BC is a square. Thus, AD is also a square. And since BC is equal to CD , let (rectangle) CE have been subtracted from both. Thus, the remaining (rectangle) BF is equal to the remaining (square) AD . And it is also equiangular to it. Thus, the sides of BF and AD about the equal angles are reciprocally proportional [Prop. 6.14]. Thus, as FE is to ED , so AE (is) to EB . And FE (is) equal to AB , and ED to AE . Thus, as BA is to AE , so AE (is) to EB . And AB (is) greater than AE . Thus, AE (is) also greater than EB [Prop. 5.14].

Thus, the straight-line AB has been cut in extreme and mean ratio at E , and AE is its greater piece. (Which is) the very thing it was required to do.

† This method of cutting a straight-line is sometimes called the “Golden Section”—see Prop. 2.11.

λα'.

Ἐν τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτείνουσας πλευρᾶς εἶδος ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν εἶδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφόμενοις.



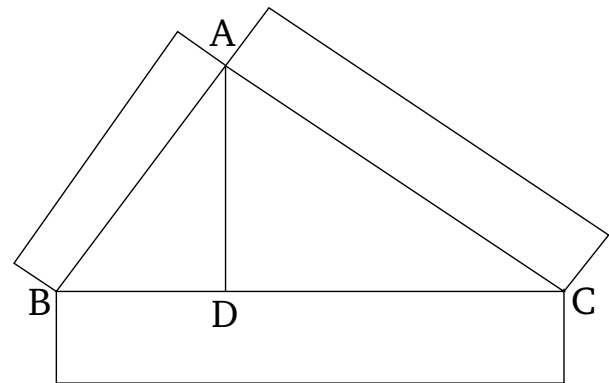
Ἐστω τρίγωνον ὀρθογώνιον τὸ $ΑΒΓ$ ὀρθὴν ἔχον τὴν ὑπὸ $ΒΑΓ$ γωνίαν· λέγω, ὅτι τὸ ἀπὸ τῆς $ΒΓ$ εἶδος ἴσον ἐστὶ τοῖς ἀπὸ τῶν $ΒΑ$, $ΑΓ$ εἶδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφόμενοις.

Ἦχθω κάθετος ἡ $ΑΔ$.

Ἐπεὶ οὖν ἐν ὀρθογωνίῳ τριγώνῳ τῷ $ΑΒΓ$ ἀπὸ τῆς πρὸς τῷ $Α$ ὀρθῆς γωνίας ἐπὶ τὴν $ΒΓ$ βάσιν κάθετος ἦκται ἡ $ΑΔ$, τὰ $ΑΒΔ$, $ΑΔΓ$ πρὸς τῇ καθετῷ τρίγωνα ὁμοιά ἐστὶ τῷ τε ὅλῳ τῷ $ΑΒΓ$ καὶ ἀλλήλοις. καὶ ἐπεὶ ὁμοίον ἐστὶ τὸ $ΑΒΓ$ τῷ $ΑΒΔ$, ἐστὶν ἄρα ὡς ἡ $ΓΒ$ πρὸς τὴν $ΒΑ$, οὕτως ἡ $ΑΒ$ πρὸς τὴν $ΒΔ$. καὶ ἐπεὶ τρεῖς εὐθεῖαι ἀνάλογόν εἰσιν, ἐστὶν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης εἶδος πρὸς

Proposition 31

In right-angled triangles, the figure (drawn) on the side subtending the right-angle is equal to the (sum of the) similar, and similarly described, figures on the sides surrounding the right-angle.



Let ABC be a right-angled triangle having the angle BAC a right-angle. I say that the figure (drawn) on BC is equal to the (sum of the) similar, and similarly described, figures on BA and AC .

Let the perpendicular AD have been drawn [Prop. 1.12].

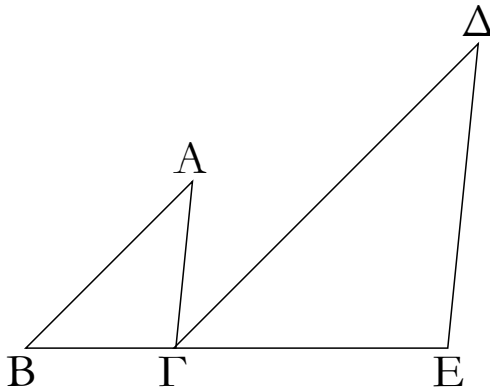
Therefore, since, in the right-angled triangle ABC , the (straight-line) AD has been drawn from the right-angle at A perpendicular to the base BC , the triangles ABD and ADC about the perpendicular are similar to the whole (triangle) ABC , and to one another [Prop. 6.8]. And since ABC is similar to ABD , thus

τὸ ἀπὸ τῆς δευτέρας τὸ ὁμοιον καὶ ὁμοίως ἀναγραφόμενον. ὡς ἄρα ἡ ΓΒ πρὸς τὴν ΒΔ, οὕτως τὸ ἀπὸ τῆς ΓΒ εἶδος πρὸς τὸ ἀπὸ τῆς ΒΑ τὸ ὁμοιον καὶ ὁμοίως ἀναγραφόμενον. διὰ τὰ αὐτὰ δὲ καὶ ὡς ἡ ΒΓ πρὸς τὴν ΓΔ, οὕτως τὸ ἀπὸ τῆς ΒΓ εἶδος πρὸς τὸ ἀπὸ τῆς ΓΑ. ὥστε καὶ ὡς ἡ ΒΓ πρὸς τὰς ΒΔ, ΔΓ, οὕτως τὸ ἀπὸ τῆς ΒΓ εἶδος πρὸς τὰ ἀπὸ τῶν ΒΑ, ΑΓ τὰ ὁμοια καὶ ὁμοίως ἀναγραφόμενα. ἴση δὲ ἡ ΒΓ ταῖς ΒΔ, ΔΓ· ἴσον ἄρα καὶ τὸ ἀπὸ τῆς ΒΓ εἶδος τοῖς ἀπὸ τῶν ΒΑ, ΑΓ εἶδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφόμενοις.

Ἐν ἄρα τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτείνουσας πλευρᾶς εἶδος ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν πλευρῶν εἶδεσι τοῖς ὁμοίοις τε καὶ ὁμοίως ἀναγραφόμενοις· ὅπερ ἔδει δεῖξαι.

λβ'.

Ἐὰν δύο τρίγωνα συντεθῇ κατὰ μίαν γωνίαν τὰς δύο πλευρὰς ταῖς δυσὶ πλευραῖς ἀνάλογον ἔχοντα ὥστε τὰς ὁμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἱ λοιπαὶ τῶν τριγώνων πλευραὶ ἐπ' εὐθείας ἔσσονται.



Ἐστω δύο τρίγωνα τὰ ΑΒΓ, ΔΓΕ τὰς δύο πλευρὰς τὰς ΒΑ, ΑΓ ταῖς δυσὶ πλευραῖς ταῖς ΔΓ, ΔΕ ἀνάλογον ἔχοντα, ὡς μὲν τὴν ΑΒ πρὸς τὴν ΑΓ, οὕτως τὴν ΔΓ πρὸς τὴν ΔΕ, παραλλήλων δὲ τὴν μὲν ΑΒ τῇ ΔΓ, τὴν δὲ ΑΓ τῇ ΔΕ· λέγω, ὅτι ἐπ' εὐθείας ἐστὶν ἡ ΒΓ τῇ ΓΕ.

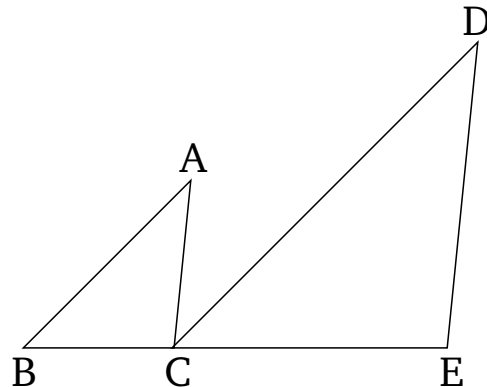
Ἐπεὶ γὰρ παράλληλός ἐστιν ἡ ΑΒ τῇ ΔΓ, καὶ εἰς αὐτὰς ἐμπίπτωκεν εὐθεῖα ἡ ΑΓ, αἱ ἐναλλάξ γωνίαι αἱ ὑπὸ ΒΑΓ, ΑΓΔ ἴσαι ἀλλήλαις εἰσίν. διὰ τὰ αὐτὰ δὲ καὶ ἡ ὑπὸ ΓΔΕ τῇ ὑπὸ ΑΓΔ ἴση ἐστίν. ὥστε καὶ ἡ ὑπὸ ΒΑΓ τῇ ὑπὸ ΓΔΕ ἐστὶν ἴση. καὶ ἐπεὶ δύο τρίγωνα ἐσὶ τὰ ΑΒΓ, ΔΓΕ μίαν γωνίαν τὴν πρὸς τῷ Α μιᾶ γωνίᾳ τῇ πρὸς τῷ Δ ἴσην ἔχοντα, περὶ

as CB is to BA , so AB (is) to BD [Def. 6.1]. And since three straight-lines are proportional, as the first is to the third, so the figure (drawn) on the first is to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.]. Thus, as CB (is) to BD , so the figure (drawn) on CB (is) to the similar, and similarly described, (figure) on BA . And so, for the same (reasons), as BC (is) to CD , so the figure (drawn) on BC (is) to the (figure) on CA . Hence, also, as BC (is) to BD and DC , so the figure (drawn) on BC (is) to the (sum of the) similar, and similarly described, (figures) on BA and AC [Prop. 5.24]. And BC is equal to BD and DC . Thus, the figure (drawn) on BC (is) also equal to the (sum of the) similar, and similarly described, figures on BA and AC [Prop. 5.9].

Thus, in right-angled triangles, the figure (drawn) on the side subtending the right-angle is equal to the (sum of the) similar, and similarly described, figures on the sides surrounding the right-angle. (Which is) the very thing it was required to show.

Proposition 32

If two triangles, having two sides proportional to two sides, are placed together at a single angle such that the corresponding sides are also parallel, then the remaining sides of the triangles will be straight-on (with respect to one another).



Let ABC and DCE be two triangles having the two sides BA and AC proportional to the two sides DC and DE —so that as AB (is) to AC , so DC (is) to DE —and (having side) AB parallel to DC , and AC to DE . I say that (side) BC is straight-on to CE .

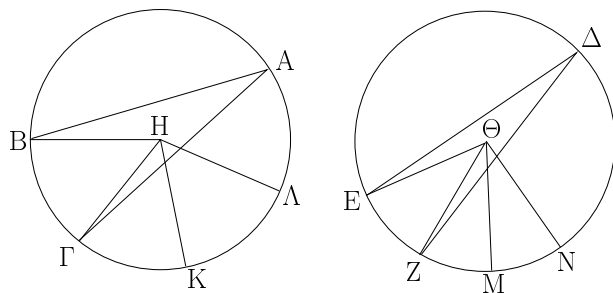
For since AB is parallel to DC , and the straight-line AC has fallen across them, the alternate angles BAC and ACD are equal to one another [Prop. 1.29]. So, for the same (reasons), CDE is also equal to ACD . And, hence, BAC is equal to CDE . And since ABC and DCE are two triangles having the one angle at A equal to the one

δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον, ὥς τὴν BA πρὸς τὴν AG , οὕτως τὴν $ΓΔ$ πρὸς τὴν $ΔΕ$, ἰσογώνιον ἄρα ἐστὶ τὸ $ABΓ$ τρίγωνον τῷ $ΔΓΕ$ τριγώνῳ· ἴση ἄρα ἡ ὑπὸ $ABΓ$ γωνία τῇ ὑπὸ $ΔΓΕ$. ἐδείχθη δὲ καὶ ἡ ὑπὸ $ΑΓΔ$ τῇ ὑπὸ $BAΓ$ ἴση· ὅλη ἄρα ἡ ὑπὸ $ΑΓΕ$ δυσὶ ταῖς ὑπὸ $ABΓ$, $BAΓ$ ἴση ἐστίν. κοινὴ προσκείσθω ἡ ὑπὸ $ΑΓΒ$ · αἱ ἄρα ὑπὸ $ΑΓΕ$, $ΑΓΒ$ ταῖς ὑπὸ $BAΓ$, $ΑΓΒ$, $ΓΒΑ$ ἴσαι εἰσίν. ἀλλ' αἱ ὑπὸ $BAΓ$, $ABΓ$, $ΑΓΒ$ δυσὶν ὀρθαῖς ἴσαι εἰσίν· καὶ αἱ ὑπὸ $ΑΓΕ$, $ΑΓΒ$ ἄρα δυσὶν ὀρθαῖς ἴσαι εἰσίν. πρὸς δὲ τινὶ εὐθείᾳ τῇ $ΑΓ$ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ $Γ$ δύο εὐθεῖαι αἱ $ΒΓ$, $ΓΕ$ μὴ ἐπὶ τὰ αὐτὰ μέρη κείμεναι τὰς ἐφεξῆς γωνίας τὰς ὑπὸ $ΑΓΕ$, $ΑΓΒ$ δυσὶν ὀρθαῖς ἴσας ποιοῦσιν· ἐπ' εὐθείας ἄρα ἐστὶν ἡ $ΒΓ$ τῇ $ΓΕ$.

Ἐὰν ἄρα δύο τρίγωνα συντεθῇ κατὰ μίαν γωνίαν τὰς δύο πλευρὰς ταῖς δυσὶ πλευραῖς ἀνάλογον ἔχοντα ὥστε τὰς ὁμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἱ λοιπαὶ τῶν τριγώνων πλευραὶ ἐπ' εὐθείας ἔσσονται· ὅπερ ἔδει δεῖξαι.

λγ'.

Ἐν τοῖς ἴσοις κύκλοις αἱ γωνίαι τὸν αὐτὸν ἔχουσι λόγον ταῖς περιφερείαις, ἐφ' ὧν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὥσι βεβηκυῖαι.



Ἐστωσαν ἴσοι κύκλοι οἱ $ABΓ$, $ΔΕΖ$, καὶ πρὸς μὲν τοῖς κέντροις αὐτῶν τοῖς H , $Θ$ γωνίαι ἔστωσαν αἱ ὑπὸ $BHΓ$, $EΘΖ$, πρὸς δὲ ταῖς περιφερείαις αἱ ὑπὸ $BAΓ$, $ΕΔΖ$ · λέγω, ὅτι ἐστὶν ὥς ἡ $ΒΓ$ περιφέρεια πρὸς τὴν $ΕΖ$ περιφέρειαν, οὕτως ἡ τε ὑπὸ $BHΓ$ γωνία πρὸς τὴν ὑπὸ $EΘΖ$ καὶ ἡ ὑπὸ $BAΓ$ πρὸς τὴν ὑπὸ $ΕΔΖ$.

Κείσθωσαν γὰρ τῇ μὲν $ΒΓ$ περιφέρειᾳ ἴσαι κατὰ τὸ ἐξῆς ὁσαυδηποτοῦν αἱ $ΓΚ$, $ΚΛ$, τῇ δὲ $ΕΖ$ περιφέρειᾳ ἴσαι ὁσαυδηποτοῦν αἱ $ΖΜ$, $ΜΝ$, καὶ ἐπεζεύχθωσαν αἱ $ΗΚ$, $ΗΛ$, $ΘΜ$, $ΘΝ$.

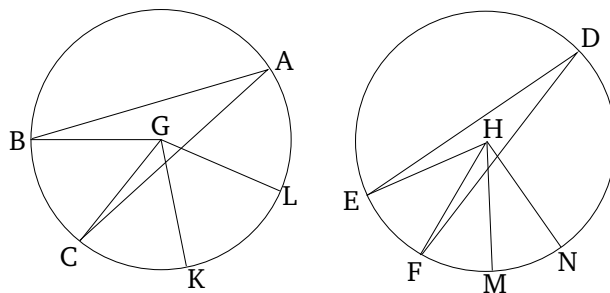
Ἐπεὶ οὖν ἴσαι εἰσὶν αἱ $ΒΓ$, $ΓΚ$, $ΚΛ$ περιφέρειαι ἀλλήλαις, ἴσαι εἰσὶ καὶ αἱ ὑπὸ $BHΓ$, $ΓΗΚ$, $ΚΗΛ$ γωνίαι ἀλλήλαις· ὁσαπλασίων ἄρα ἐστὶν ἡ $ΒΛ$ περιφέρεια τῆς $ΒΓ$, τοσαυταπλασίων ἐστὶ καὶ ἡ ὑπὸ $BHΛ$ γωνία τῆς ὑπὸ $BHΓ$. διὰ τὰ

angle at D , and the sides about the equal angles proportional, (so that) as BA (is) to AC , so CD (is) to DE , triangle ABC is thus equiangular to triangle DCE [Prop. 6.6]. Thus, angle ABC is equal to DCE . And (angle) ACD was also shown (to be) equal to BAC . Thus, the whole (angle) ACE is equal to the two (angles) ABC and BAC . Let ACB have been added to both. Thus, ACE and ACB are equal to BAC , ACB , and CBA . But, BAC , ABC , and ACB are equal to two right-angles [Prop. 1.32]. Thus, ACE and ACB are also equal to two right-angles. Thus, the two straight-lines BC and CE , not lying on the same side, make adjacent angles ACE and ACB (whose sum is) equal to two right-angles with some straight-line AC , at the point C on it. Thus, BC is straight-on to CE [Prop. 1.14].

Thus, if two triangles, having two sides proportional to two sides, are placed together at a single angle such that the corresponding sides are also parallel, then the remaining sides of the triangles will be straight-on (with respect to one another). (Which is) the very thing it was required to show.

Proposition 33

In equal circles, angles have the same ratio as the (ratio of the) circumferences on which they stand, whether they are standing at the centers (of the circles) or at the circumferences.



Let ABC and DEF be equal circles, and let BGC and EHF be angles at their centers, G and H (respectively), and BAC and EDF (angles) at their circumferences. I say that as circumference BC is to circumference EF , so angle BGC (is) to EHF , and (angle) BAC to EDF .

For let any number whatsoever of consecutive (circumferences), CK and KL , be made equal to circumference BC , and any number whatsoever, FM and MN , to circumference EF . And let GK , GL , HM , and HN have been joined.

Therefore, since circumferences BC , CK , and KL are equal to one another, angles BGC , CGK , and KGL are also equal to one another [Prop. 3.27]. Thus, as many times as circumference BL is (divisible) by BC , so many

αὐτὰ δὴ καὶ ὁσαπλασίων ἐστὶν ἡ NE περιφέρεια τῆς EZ , τοσαυταπλασίων ἐστὶ καὶ ἡ ὑπὸ $N\Theta E$ γωνία τῆς ὑπὸ $E\Theta Z$. εἰ ἄρα ἴση ἐστὶν ἡ BA περιφέρεια τῇ EN περιφερείᾳ, ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ BHA τῇ ὑπὸ $E\Theta N$, καὶ εἰ μείζων ἐστὶν ἡ BA περιφέρεια τῆς EN περιφερείας, μείζων ἐστὶ καὶ ἡ ὑπὸ BHA γωνία τῆς ὑπὸ $E\Theta N$, καὶ εἰ ἐλάσσων, ἐλάσσων. τεσσάρων δὴ ὄντων μεγεθῶν, δύο μὲν περιφερειῶν τῶν $B\Gamma$, EZ , δύο δὲ γωνιῶν τῶν ὑπὸ $BH\Gamma$, $E\Theta Z$, εἴληπται τῆς μὲν $B\Gamma$ περιφερείας καὶ τῆς ὑπὸ $BH\Gamma$ γωνίας ἰσάκεις πολλαπλασίων ἢ τε BA περιφέρεια καὶ ἡ ὑπὸ BHA γωνία, τῆς δὲ EZ περιφερείας καὶ τῆς ὑπὸ $E\Theta Z$ γωνίας ἢ τε EN περιφέρεια καὶ ἡ ὑπὸ $E\Theta N$ γωνία. καὶ δέδεικται, ὅτι εἰ ὑπερέχει ἡ BA περιφέρεια τῆς EN περιφερείας, ὑπερέχει καὶ ἡ ὑπὸ BHA γωνία τῆς ὑπὸ $E\Theta N$ γωνίας, καὶ εἰ ἴση, ἴση, καὶ εἰ ἐλάσσων, ἐλάσσων. ἔστιν ἄρα, ὡς ἡ $B\Gamma$ περιφέρεια πρὸς τὴν EZ , οὕτως ἡ ὑπὸ $BH\Gamma$ γωνία πρὸς τὴν ὑπὸ $E\Theta Z$. ἀλλ' ὡς ἡ ὑπὸ $BH\Gamma$ γωνία πρὸς τὴν ὑπὸ $E\Theta Z$, οὕτως ἡ ὑπὸ $BA\Gamma$ πρὸς τὴν ὑπὸ $EA\Delta Z$. διπλασία γὰρ ἑκατέρα ἑκατέρας. καὶ ὡς ἄρα ἡ $B\Gamma$ περιφέρεια πρὸς τὴν EZ περιφέρειαν, οὕτως ἢ τε ὑπὸ $BH\Gamma$ γωνία πρὸς τὴν ὑπὸ $E\Theta Z$ καὶ ἡ ὑπὸ $BA\Gamma$ πρὸς τὴν ὑπὸ $EA\Delta Z$.

Ἐν ἄρα τοῖς ἴσοις κύκλοις αἱ γωνίαι τὸν αὐτὸν ἔχουσι λόγον ταῖς περιφερείαις, ἐφ' ὧν βεβήκασιν, ἐάν τε πρὸς τοῖς κέντροις ἐάν τε πρὸς ταῖς περιφερείαις ὥσι βεβηκῶσι· ὅπερ ἔδει δεῖξαι.

times is angle BGL also (divisible) by BGC . And so, for the same (reasons), as many times as circumference NE is (divisible) by EF , so many times is angle NHE also (divisible) by EHF . Thus, if circumference BL is equal to circumference EN then angle BGL is also equal to EHN [Prop. 3.27], and if circumference BL is greater than circumference EN then angle BGL is also greater than EHN ,[†] and if (BL is) less (than EN then BGL is also) less (than EHN). So there are four magnitudes, two circumferences BC and EF , and two angles BGC and EHF . And equal multiples have been taken of circumference BC and angle BGC , (namely) circumference BL and angle BGL , and of circumference EF and angle EHF , (namely) circumference EN and angle EHN . And it has been shown that if circumference BL exceeds circumference EN then angle BGL also exceeds angle EHN , and if (BL is) equal (to EN then BGL is also) equal (to EHN), and if (BL is) less (than EN then BGL is also) less (than EHN). Thus, as circumference BC (is) to EF , so angle BGC (is) to EHF [Def. 5.5]. But as angle BGC (is) to EHF , so (angle) BAC (is) to EDF [Prop. 5.15]. For the former (are) double the latter (respectively) [Prop. 3.20]. Thus, also, as circumference BC (is) to circumference EF , so angle BGC (is) to EHF , and BAC to EDF .

Thus, in equal circles, angles have the same ratio as the (ratio of the) circumferences on which they stand, whether they are standing at the centers (of the circles) or at the circumferences. (Which is) the very thing it was required to show.

[†] This is a straight-forward generalization of Prop. 3.27

ELEMENTS BOOK 7

Elementary Number Theory[†]

[†]The propositions contained in Books 7–9 are generally attributed to the school of Pythagoras.

Ὅροι.

- α'. Μονάς ἐστίν, καθ' ἣν ἕκαστον τῶν ὄντων ἐν λέγεται.
 β'. Ἀριθμὸς δὲ τὸ ἐκ μονάδων συγχείμενον πλῆθος.
 γ'. Μέρος ἐστίν ἀριθμὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος, ὅταν καταμετρηῇ τὸν μείζονα.
 δ'. Μέρη δέ, ὅταν μὴ καταμετρηῇ.
 ε'. Πολλαπλάσιος δὲ ὁ μείζων τοῦ ἐλάσσονος, ὅταν καταμετρηῇται ὑπὸ τοῦ ἐλάσσονος.
 ς'. Ἄρτιος ἀριθμὸς ἐστίν ὁ δίχα διαιρούμενος.
 ζ'. Περισσὸς δὲ ὁ μὴ διαιρούμενος δίχα ἢ [ὁ] μονάδι διαφέρων ἀρτίου ἀριθμοῦ.
 η'. Ἀρτιάκις ἄρτιος ἀριθμὸς ἐστίν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν.
 θ'. Ἀρτιάκις δὲ περισσὸς ἐστίν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.
 ι'. Περισσάκις δὲ περισσὸς ἀριθμὸς ἐστίν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.
 ια'. Πρῶτος ἀριθμὸς ἐστίν ὁ μονάδι μόνῃ μετρούμενος.
 ιβ'. Πρῶτοι πρὸς ἀλλήλους ἀριθμοὶ εἰσιν οἱ μονάδι μόνῃ μετρούμενοι κοινῷ μέτρῳ.
 ιγ'. Σύνθετος ἀριθμὸς ἐστίν ὁ ἀριθμῷ τινι μετρούμενος.
 ιδ'. Σύνθετοι δὲ πρὸς ἀλλήλους ἀριθμοὶ εἰσιν οἱ ἀριθμῷ τινι μετρούμενοι κοινῷ μέτρῳ.
 ιε'. Ἀριθμὸς ἀριθμὸν πολλαπλασιάζειν λέγεται, ὅταν, ὅσαι εἰσὶν ἐν αὐτῷ μονάδες, τοσαυτάκις συντεθῇ ὁ πολλαπλασιαζόμενος, καὶ γέννηταί τις.
 ις'. Ὅταν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσιν τινα, ὁ γενόμενος ἐπίπεδος καλεῖται, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.
 ιζ'. Ὅταν δὲ τρεῖς ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσιν τινα, ὁ γενόμενος στερεός ἐστίν, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.
 ιη'. Τετράγωνος ἀριθμὸς ἐστίν ὁ ισάκις ἴσος ἢ [ὁ] ὑπὸ δύο ἴσων ἀριθμῶν περιεχόμενος.
 ιθ'. Κύβος δὲ ὁ ισάκις ἴσος ισάκις ἢ [ὁ] ὑπὸ τριῶν ἴσων ἀριθμῶν περιεχόμενος.
 κ'. Ἀριθμοὶ ἀνάλογόν εἰσιν, ὅταν ὁ πρῶτος τοῦ δευτέρου καὶ ὁ τρίτος τοῦ τετάρτου ισάκις ἢ πολλαπλάσιος ἢ τὸ αὐτὸ μέρος ἢ τὰ αὐτὰ μέρη ᾧσιν.
 κα'. Ὅμοιοι ἐπίπεδοι καὶ στερεοὶ ἀριθμοὶ εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς.
 κβ'. Τέλεις ἀριθμὸς ἐστίν ὁ τοῖς ἑαυτοῦ μέρεσιν ἴσος ᾧν.

Definitions

1. A unit is (that) according to which each existing (thing) is said (to be) one.
2. And a number (is) a multitude composed of units.[†]
3. A number is part of a(nother) number, the lesser of the greater, when it measures the greater.[‡]
4. But (the lesser is) parts (of the greater) when it does not measure it.[§]
5. And the greater (number is) a multiple of the lesser when it is measured by the lesser.
6. An even number is one (which can be) divided in half.
7. And an odd number is one (which can)not (be) divided in half, or which differs from an even number by a unit.
8. An even-times-even number is one (which is) measured by an even number according to an even number.[¶]
9. And an even-times-odd number is one (which is) measured by an even number according to an odd number.*
10. And an odd-times-odd number is one (which is) measured by an odd number according to an odd number.[§]
11. A prime^{||} number is one (which is) measured by a unit alone.
12. Numbers prime to one another are those (which are) measured by a unit alone as a common measure.
13. A composite number is one (which is) measured by some number.
14. And numbers composite to one another are those (which are) measured by some number as a common measure.
15. A number is said to multiply a(nother) number when the (number being) multiplied is added (to itself) as many times as there are units in the former (number), and (thereby) some (other number) is produced.
16. And when two numbers multiplying one another make some (other number) then the (number so) created is called plane, and its sides (are) the numbers which multiply one another.
17. And when three numbers multiplying one another make some (other number) then the (number so) created is (called) solid, and its sides (are) the numbers which multiply one another.
18. A square number is an equal times an equal, or (a plane number) contained by two equal numbers.
19. And a cube (number) is an equal times an equal times an equal, or (a solid number) contained by three equal numbers.

20. Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third (is) of the fourth.

21. Similar plane and solid numbers are those having proportional sides.

22. A perfect number is that which is equal to its own parts.^{††}

[†] In other words, a “number” is a positive integer greater than unity.

[‡] In other words, a number a is part of another number b if there exists some number n such that $na = b$.

[§] In other words, a number a is parts of another number b (where $a < b$) if there exist distinct numbers, m and n , such that $na = mb$.

[¶] In other words, an even-times-even number is the product of two even numbers.

^{*} In other words, an even-times-odd number is the product of an even and an odd number.

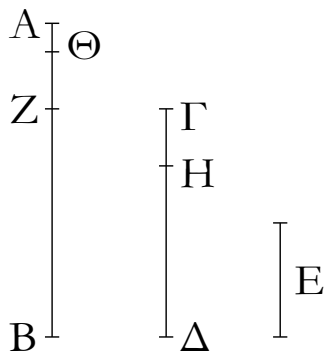
[§] In other words, an odd-times-odd number is the product of two odd numbers.

^{||} Literally, “first”.

^{††} In other words, a perfect number is equal to the sum of its own factors.

α΄.

Δύο ἀριθμῶν ἀνίσων ἐκκειμένων, ἀνθυφαιρουμένου δὲ αἰ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος, ἐὰν ὁ λειπόμενος μηδέποτε καταμετρήῃ τὸν πρὸ ἑαυτοῦ, ἕως οὗ λειφθῇ μονάς, οἱ ἐξ ἀρχῆς ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ἔσονται.



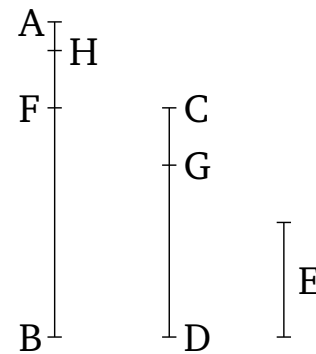
Δύο γὰρ [ἀνίσων] ἀριθμῶν τῶν AB , $\Gamma\Delta$ ἀνθυφαιρουμένου αἰ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος ὁ λειπόμενος μηδέποτε καταμετρεῖται τὸν πρὸ ἑαυτοῦ, ἕως οὗ λειφθῇ μονάς· λέγω, ὅτι οἱ AB , $\Gamma\Delta$ πρῶτοι πρὸς ἀλλήλους εἰσίν, τουτέστιν ὅτι τοὺς AB , $\Gamma\Delta$ μονάς μόνη μετρεῖ.

Εἰ γὰρ μὴ εἰσιν οἱ AB , $\Gamma\Delta$ πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός· μετρεῖται, καὶ ἔστω ὁ E · καὶ ὁ μὲν $\Gamma\Delta$ τὸν BZ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ZA , ὁ δὲ AZ τὸν ΔH μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν $H\Gamma$, ὁ δὲ $H\Gamma$ τὸν $Z\Theta$ μετρῶν λειπέτω μονάδα τὴν ΘA .

Ἐπεὶ οὖν ὁ E τὸν $\Gamma\Delta$ μετρεῖ, ὁ δὲ $\Gamma\Delta$ τὸν BZ μετρεῖ, καὶ ὁ E ἄρα τὸν BZ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν BA · καὶ λοιπὸν ἄρα τὸν AZ μετρήσει. ὁ δὲ AZ τὸν ΔH μετρεῖ· καὶ ὁ E ἄρα τὸν ΔH μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν $\Delta\Gamma$ · καὶ λοιπὸν ἄρα τὸν ΓH μετρήσει. ὁ δὲ ΓH τὸν $Z\Theta$ μετρεῖ·

Proposition 1

Two unequal numbers (being) laid down, and the lesser being continually subtracted, in turn, from the greater, if the remainder never measures the (number) preceding it, until a unit remains, then the original numbers will be prime to one another.



For two [unequal] numbers, AB and CD , the lesser being continually subtracted, in turn, from the greater, let the remainder never measure the (number) preceding it, until a unit remains. I say that AB and CD are prime to one another—that is to say, that a unit alone measures (both) AB and CD .

For if AB and CD are not prime to one another then some number will measure them. Let (some number) measure them, and let it be E . And let CD measuring BF leave FA less than itself, and let AF measuring DG leave GC less than itself, and let GC measuring FH leave a unit, HA .

In fact, since E measures CD , and CD measures BF , E thus also measures BF .[†] And (E) also measures the whole of BA . Thus, (E) will also measure the remainder

καὶ ὁ E ἄρα τὸν $Z\Theta$ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ZA · καὶ λοιπὴν ἄρα τὴν $A\Theta$ μονάδα μετρήσει ἀριθμὸς ὢν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς AB , $\Gamma\Delta$ ἀριθμοὺς μετρήσει τις ἀριθμὸς· οἱ AB , $\Gamma\Delta$ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσὶν· ὅπερ ἔδει δεῖξαι.

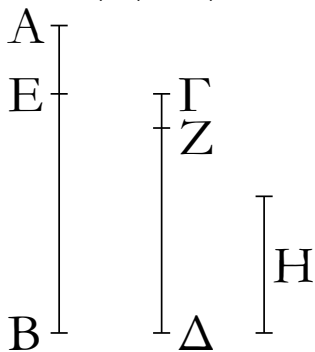
AF .[‡] And AF measures DG . Thus, E also measures DG . And (E) also measures the whole of DC . Thus, (E) will also measure the remainder CG . And CG measures FH . Thus, E also measures FH . And (E) also measures the whole of FA . Thus, (E) will also measure the remaining unit AH , (despite) being a number. The very thing is impossible. Thus, some number does not measure (both) the numbers AB and CD . Thus, AB and CD are prime to one another. (Which is) the very thing it was required to show.

[†] Here, use is made of the unstated common notion that if a measures b , and b measures c , then a also measures c , where all symbols denote numbers.

[‡] Here, use is made of the unstated common notion that if a measures b , and a measures part of b , then a also measures the remainder of b , where all symbols denote numbers.

β'.

Δύο ἀριθμῶν δοθέντων μὴ πρῶτων πρὸς ἀλλήλους τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.



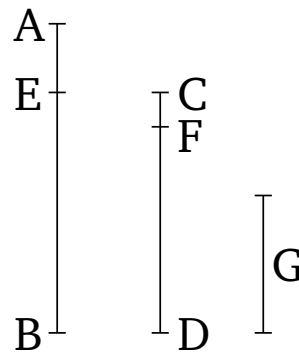
Ἐστωσαν οἱ δοθέντες δύο ἀριθμοὶ μὴ πρῶτοι πρὸς ἀλλήλους οἱ AB , $\Gamma\Delta$. δεῖ δὴ τῶν AB , $\Gamma\Delta$ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

Εἰ μὲν οὖν ὁ $\Gamma\Delta$ τὸν AB μετρεῖ, μετρεῖ δὲ καὶ ἑαυτόν, ὁ $\Gamma\Delta$ ἄρα τῶν $\Gamma\Delta$, AB κοινὸν μέτρον ἐστίν. καὶ φανερόν, ὅτι καὶ μέγιστον· οὐδεὶς γὰρ μείζων τοῦ $\Gamma\Delta$ τὸν $\Gamma\Delta$ μετρήσει.

Εἰ δὲ οὐ μετρεῖ ὁ $\Gamma\Delta$ τὸν AB , τῶν AB , $\Gamma\Delta$ ἀνθυφαίρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος λειψθήσεται τις ἀριθμὸς, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ. μονὰς μὲν γὰρ οὐ λειψθήσεται· εἰ δὲ μή, ἔσσονται οἱ AB , $\Gamma\Delta$ πρῶτοι πρὸς ἀλλήλους· ὅπερ οὐχ ὑπόκειται. λειψθήσεται τις ἄρα ἀριθμὸς, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ. καὶ ὁ μὲν $\Gamma\Delta$ τὸν BE μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν EA , ὁ δὲ EA τὸν ΔZ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν $Z\Gamma$, ὁ δὲ ΓZ τὸν AE μετρεῖτω. ἐπεὶ οὖν ὁ ΓZ τὸν AE μετρεῖ, ὁ δὲ AE τὸν ΔZ μετρεῖ, καὶ ὁ ΓZ ἄρα τὸν ΔZ μετρήσει. μετρεῖ δὲ καὶ ἑαυτόν· καὶ ὅλον ἄρα τὸν $\Gamma\Delta$ μετρήσει. ὁ δὲ $\Gamma\Delta$ τὸν BE μετρεῖ· καὶ ὁ ΓZ ἄρα τὸν BE μετρεῖ· μετρεῖ δὲ καὶ τὸν EA · καὶ ὅλον ἄρα τὸν BA μετρήσει· μετρεῖ δὲ καὶ τὸν $\Gamma\Delta$ · ὁ ΓZ ἄρα τοὺς AB , $\Gamma\Delta$ μετρεῖ. ὁ ΓZ ἄρα τῶν AB , $\Gamma\Delta$ κοινὸν

Proposition 2

To find the greatest common measure of two given numbers (which are) not prime to one another.



Let AB and CD be the two given numbers (which are) not prime to one another. So it is required to find the greatest common measure of AB and CD .

In fact, if CD measures AB , CD is thus a common measure of CD and AB , (since CD) also measures itself. And (it is) manifest that (it is) also the greatest (common measure). For nothing greater than CD can measure CD .

But if CD does not measure AB then some number will remain from AB and CD , the lesser being continually subtracted, in turn, from the greater, which will measure the (number) preceding it. For a unit will not be left. But if not, AB and CD will be prime to one another [Prop. 7.1]. The very opposite thing was assumed. Thus, some number will remain which will measure the (number) preceding it. And let CD measuring BE leave EA less than itself, and let EA measuring DF leave FC less than itself, and let CF measure AE . Therefore, since CF measures AE , and AE measures DF , CF will thus also measure DF . And it also measures itself. Thus, it will

μέτρον ἐστίν. λέγω δὴ, ὅτι καὶ μέγιστον. εἰ γὰρ μὴ ἐστὶν ὁ ΓΖ τῶν ΑΒ, ΓΔ μέγιστον κοινὸν μέτρον, μετρήσει τις τοὺς ΑΒ, ΓΔ ἀριθμοὺς ἀριθμὸς μείζων ὢν τοῦ ΓΖ. μετρείτω, καὶ ἔστω ὁ Η. καὶ ἐπεὶ ὁ Η τὸν ΓΔ μετρεῖ, ὁ δὲ ΓΔ τὸν ΒΕ μετρεῖ, καὶ ὁ Η ἄρα τὸν ΒΕ μετρεῖ· μετρεῖ δὲ καὶ ὅλον τὸν ΒΑ· καὶ λοιπὸν ἄρα τὸν ΑΕ μετρήσει. ὁ δὲ ΑΕ τὸν ΔΖ μετρεῖ· καὶ ὁ Η ἄρα τὸν ΔΖ μετρήσει· μετρεῖ δὲ καὶ ὅλον τὸν ΔΓ· καὶ λοιπὸν ἄρα τὸν ΓΖ μετρήσει ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα τοὺς ΑΒ, ΓΔ ἀριθμοὺς ἀριθμὸς τις μετρήσει μείζων ὢν τοῦ ΓΖ· ὁ ΓΖ ἄρα τῶν ΑΒ, ΓΔ μέγιστόν ἐστι κοινὸν μέτρον [ὅπερ ἔδει δεῖξαι].

also measure the whole of CD . And CD measures BE . Thus, CF also measures BE . And it also measures EA . Thus, it will also measure the whole of BA . And it also measures CD . Thus, CF measures (both) AB and CD . Thus, CF is a common measure of AB and CD . So I say that (it is) also the greatest (common measure). For if CF is not the greatest common measure of AB and CD then some number which is greater than CF will measure the numbers AB and CD . Let it (so) measure (AB and CD), and let it be G . And since G measures CD , and CD measures BE , G thus also measures BE . And it also measures the whole of BA . Thus, it will also measure the remainder AE . And AE measures DF . Thus, G will also measure DF . And it also measures the whole of DC . Thus, it will also measure the remainder CF , the greater (measuring) the lesser. The very thing is impossible. Thus, some number which is greater than CF cannot measure the numbers AB and CD . Thus, CF is the greatest common measure of AB and CD . [(Which is) the very thing it was required to show].

Πόρισμα.

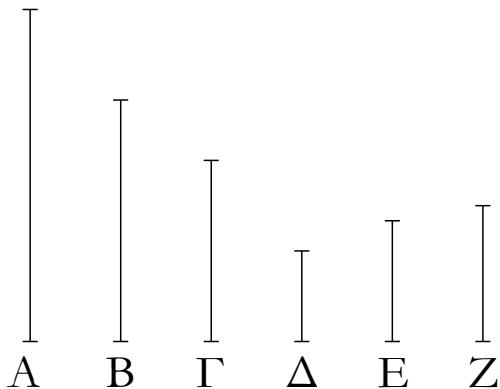
Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν ἀριθμὸς δύο ἀριθμοὺς μετρήῃ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσῃ· ὅπερ ἔδει δεῖξαι.

Corollary

So it is manifest, from this, that if a number measures two numbers then it will also measure their greatest common measure. (Which is) the very thing it was required to show.

Υ΄.

Τριῶν ἀριθμῶν δοθέντων μὴ πρώτων πρὸς ἀλλήλους τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.

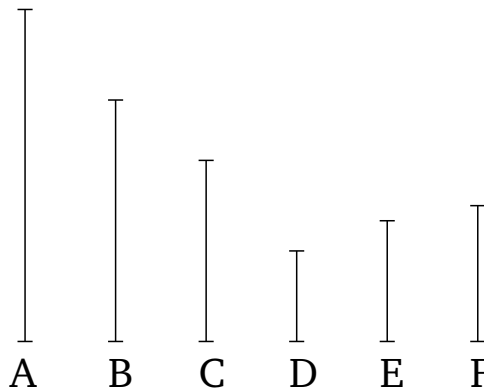


Ἐστωσαν οἱ δοθέντες τρεῖς ἀριθμοὶ μὴ πρώτοι πρὸς ἀλλήλους οἱ Α, Β, Γ· δεῖ δὴ τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

Εἰλήφθω γὰρ δύο τῶν Α, Β τὸ μέγιστον κοινὸν μέτρον ὁ Δ· ὁ δὲ Δ τὸν Γ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρείτω πρότερον· μετρεῖ δὲ καὶ τοὺς Α, Β· ὁ Δ ἄρα τοὺς Α, Β, Γ μετρεῖ· ὁ Δ ἄρα τῶν Α, Β, Γ κοινὸν μέτρον ἐστίν. λέγω δὴ, ὅτι καὶ

Proposition 3

To find the greatest common measure of three given numbers (which are) not prime to one another.



Let A , B , and C be the three given numbers (which are) not prime to one another. So it is required to find the greatest common measure of A , B , and C .

For let the greatest common measure, D , of the two (numbers) A and B have been taken [Prop. 7.2]. So D either measures, or does not measure, C . First of all, let it measure (C). And it also measures A and B . Thus, D

μέγιστον. εἰ γὰρ μὴ ἔστιν ὁ Δ τῶν A, B, Γ μέγιστον κοινὸν μέτρον, μετρήσει τις τοὺς A, B, Γ ἀριθμοὺς ἀριθμὸς μείζων ὢν τοῦ Δ . μετρεῖτω, καὶ ἔστω ὁ E . ἐπεὶ οὖν ὁ E τοὺς A, B, Γ μετρεῖ, καὶ τοὺς A, B ἄρα μετρήσει· καὶ τὸ τῶν A, B ἄρα μέγιστον κοινὸν μέτρον μετρήσει. τὸ δὲ τῶν A, B μέγιστον κοινὸν μέτρον ἔστιν ὁ Δ · ὁ E ἄρα τὸν Δ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἔστιν ἀδύνατον. οὐκ ἄρα τοὺς A, B, Γ ἀριθμοὺς ἀριθμὸς τις μετρήσει μείζων ὢν τοῦ Δ · ὁ Δ ἄρα τῶν A, B, Γ μέγιστόν ἐστι κοινὸν μέτρον.

Μὴ μετρεῖτω δὴ ὁ Δ τὸν Γ · λέγω πρῶτον, ὅτι οἱ Γ, Δ οὐκ εἰσι πρῶτοι πρὸς ἀλλήλους. ἐπεὶ γὰρ οἱ A, B, Γ οὐκ εἰσι πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμὸς. ὁ δὴ τοὺς A, B, Γ μετρῶν καὶ τοὺς A, B μετρήσει, καὶ τὸ τῶν A, B μέγιστον κοινὸν μέτρον τὸν Δ μετρήσει· μετρεῖ δὲ καὶ τὸν Γ · τοὺς Δ, Γ ἄρα ἀριθμοὺς ἀριθμὸς τις μετρήσει· οἱ Δ, Γ ἄρα οὐκ εἰσι πρῶτοι πρὸς ἀλλήλους. εἰλήφθω οὖν αὐτῶν τὸ μέγιστον κοινὸν μέτρον ὁ E . καὶ ἐπεὶ ὁ E τὸν Δ μετρεῖ, ὁ δὲ Δ τοὺς A, B μετρεῖ, καὶ ὁ E ἄρα τοὺς A, B μετρεῖ· μετρεῖ δὲ καὶ τὸν Γ · ὁ E ἄρα τοὺς A, B, Γ μετρεῖ. ὁ E ἄρα τῶν A, B, Γ κοινόν ἐστι μέτρον. λέγω δὴ, ὅτι καὶ μέγιστον. εἰ γὰρ μὴ ἔστιν ὁ E τῶν A, B, Γ τὸ μέγιστον κοινὸν μέτρον, μετρήσει τις τοὺς A, B, Γ ἀριθμοὺς ἀριθμὸς μείζων ὢν τοῦ E . μετρεῖτω, καὶ ἔστω ὁ Z . καὶ ἐπεὶ ὁ Z τοὺς A, B, Γ μετρεῖ, καὶ τοὺς A, B μετρεῖ· καὶ τὸ τῶν A, B ἄρα μέγιστον κοινὸν μέτρον μετρήσει. τὸ δὲ τῶν A, B μέγιστον κοινὸν μέτρον ἔστιν ὁ Δ · ὁ Z ἄρα τὸν Δ μετρεῖ· μετρεῖ δὲ καὶ τὸν Γ · ὁ Z ἄρα τοὺς Δ, Γ μετρεῖ· καὶ τὸ τῶν Δ, Γ ἄρα μέγιστον κοινὸν μέτρον μετρήσει. τὸ δὲ τῶν Δ, Γ μέγιστον κοινὸν μέτρον ἔστιν ὁ E · ὁ Z ἄρα τὸν E μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἔστιν ἀδύνατον. οὐκ ἄρα τοὺς A, B, Γ ἀριθμοὺς ἀριθμὸς τις μετρήσει μείζων ὢν τοῦ E · ὁ E ἄρα τῶν A, B, Γ μέγιστόν ἐστι κοινὸν μέτρον· ὅπερ ἔδει δεῖξαι.

δ΄.

Ἄπας ἀριθμὸς παντὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος ἥτοι μέρος ἔστιν ἢ μέρος.

Ἐστώσαν δύο ἀριθμοὶ οἱ $A, B\Gamma$, καὶ ἔστω ἐλάσσων ὁ $B\Gamma$ · λέγω, ὅτι ὁ $B\Gamma$ τοῦ A ἥτοι μέρος ἔστιν ἢ μέρος.

measures A, B , and C . Thus, D is a common measure of A, B , and C . So I say that (it is) also the greatest (common measure). For if D is not the greatest common measure of A, B , and C then some number greater than D will measure the numbers A, B , and C . Let it (so) measure (A, B , and C), and let it be E . Therefore, since E measures A, B , and C , it will thus also measure A and B . Thus, it will also measure the greatest common measure of A and B [Prop. 7.2 corr.]. And D is the greatest common measure of A and B . Thus, E measures D , the greater (measuring) the lesser. The very thing is impossible. Thus, some number which is greater than D cannot measure the numbers A, B , and C . Thus, D is the greatest common measure of A, B , and C .

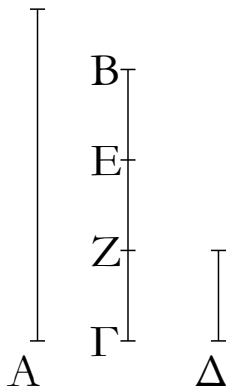
So let D not measure C . I say, first of all, that C and D are not prime to one another. For since A, B, C are not prime to one another, some number will measure them. So the (number) measuring A, B , and C will also measure A and B , and it will also measure the greatest common measure, D , of A and B [Prop. 7.2 corr.]. And it also measures C . Thus, some number will measure the numbers D and C . Thus, D and C are not prime to one another. Therefore, let their greatest common measure, E , have been taken [Prop. 7.2]. And since E measures D , and D measures A and B , E thus also measures A and B . And it also measures C . Thus, E measures A, B , and C . Thus, E is a common measure of A, B , and C . So I say that (it is) also the greatest (common measure). For if E is not the greatest common measure of A, B , and C then some number greater than E will measure the numbers A, B , and C . Let it (so) measure (A, B , and C), and let it be F . And since F measures A, B , and C , it also measures A and B . Thus, it will also measure the greatest common measure of A and B [Prop. 7.2 corr.]. And D is the greatest common measure of A and B . Thus, F measures D . And it also measures C . Thus, F measures D and C . Thus, it will also measure the greatest common measure of D and C [Prop. 7.2 corr.]. And E is the greatest common measure of D and C . Thus, F measures E , the greater (measuring) the lesser. The very thing is impossible. Thus, some number which is greater than E does not measure the numbers A, B , and C . Thus, E is the greatest common measure of A, B , and C . (Which is) the very thing it was required to show.

Proposition 4

Any number is either part or parts of any (other) number, the lesser of the greater.

Let A and BC be two numbers, and let BC be the lesser. I say that BC is either part or parts of A .

Οἱ A , $B\Gamma$ γὰρ ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὐ. ἔστωσαν πρότερον οἱ A , $B\Gamma$ πρῶτοι πρὸς ἀλλήλους. διαιρεθέντος δὴ τοῦ $B\Gamma$ εἰς τὰς ἐν αὐτῷ μονάδας ἔσται ἐκάστη μονὰς τῶν ἐν τῷ $B\Gamma$ μέρος τι τοῦ A . ὥστε μέρη ἐστὶν ὁ $B\Gamma$ τοῦ A .

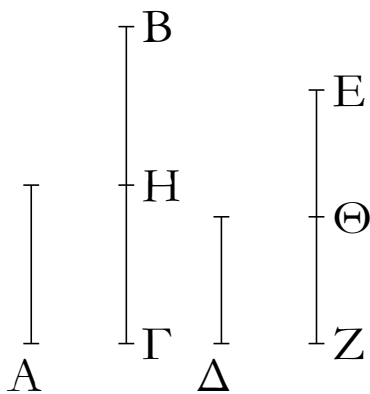


Μὴ ἔστωσαν δὴ οἱ A , $B\Gamma$ πρῶτοι πρὸς ἀλλήλους· ὁ δὴ $B\Gamma$ τὸν A ἤτοι μετρεῖ ἢ οὐ μετρεῖ. εἰ μὲν οὖν ὁ $B\Gamma$ τὸν A μετρεῖ, μέρος ἐστὶν ὁ $B\Gamma$ τοῦ A . εἰ δὲ οὐ, εἰλήφθω τῶν A , $B\Gamma$ μέγιστον κοινὸν μέτρον ὁ Δ , καὶ διηρήσθω ὁ $B\Gamma$ εἰς τοὺς τῷ Δ ἴσους τοὺς BE , EZ , $Z\Gamma$. καὶ ἐπεὶ ὁ Δ τὸν A μετρεῖ, μέρος ἐστὶν ὁ Δ τοῦ A . ἴσος δὲ ὁ Δ ἐκάστῳ τῶν BE , EZ , $Z\Gamma$. καὶ ἕκαστος ἄρα τῶν BE , EZ , $Z\Gamma$ τοῦ A μέρος ἐστίν· ὥστε μέρη ἐστὶν ὁ $B\Gamma$ τοῦ A .

Ἄπας ἄρα ἀριθμὸς παντὸς ἀριθμοῦ ὁ ἐλάσσων τοῦ μείζονος ἤτοι μέρος ἐστὶν ἢ μέρη· ὅπερ ἔδει δεῖξαι.

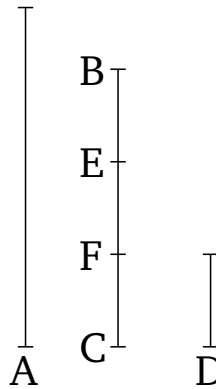
ε΄.

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρος ᾗ, καὶ ἕτερος ἑτέρου τὸ αὐτὸ μέρος ᾗ, καὶ συναμφοτέρους συναμφοτέρου τὸ αὐτὸ μέρος ἔσται, ὅπερ ὁ εἰς τοῦ ἐνός.



Ἀριθμὸς γὰρ ὁ A [ἀριθμοῦ] τοῦ $B\Gamma$ μέρος ἔστω, καὶ

For A and BC are either prime to one another, or not. Let A and BC , first of all, be prime to one another. So separating BC into its constituent units, each of the units in BC will be some part of A . Hence, BC is parts of A .

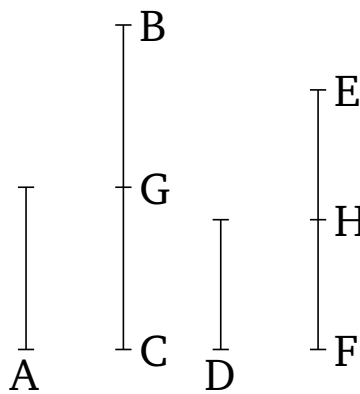


So let A and BC be not prime to one another. So BC either measures, or does not measure, A . Therefore, if BC measures A then BC is part of A . And if not, let the greatest common measure, D , of A and BC have been taken [Prop. 7.2], and let BC have been divided into BE , EF , and FC , equal to D . And since D measures A , D is a part of A . And D is equal to each of BE , EF , and FC . Thus, BE , EF , and FC are also each part of A . Hence, BC is parts of A .

Thus, any number is either part or parts of any (other) number, the lesser of the greater. (Which is) the very thing it was required to show.

Proposition 5[†]

If a number is part of a number, and another (number) is the same part of another, then the sum (of the leading numbers) will also be the same part of the sum (of the following numbers) that one (number) is of another.



For let a number A be part of a [number] BC , and

ἕτερος ὁ Δ ἑτέρου τοῦ EZ τὸ αὐτὸ μέρος, ὅπερ ὁ A τοῦ $B\Gamma$. λέγω, ὅτι καὶ συναμφοτέρος ὁ A , Δ συναμφοτέρου τοῦ $B\Gamma$, EZ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὁ A τοῦ $B\Gamma$.

Ἐπεὶ γάρ, ὁ μέρος ἐστὶν ὁ A τοῦ $B\Gamma$, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Δ τοῦ EZ , ὅσοι ἄρα εἰσὶν ἐν τῷ $B\Gamma$ ἀριθμοὶ ἴσοι τῷ A , τοσοῦτοί εἰσι καὶ ἐν τῷ EZ ἀριθμοὶ ἴσοι τῷ Δ . διηγήσθω ὁ μὲν $B\Gamma$ εἰς τοὺς τῷ A ἴσους τοὺς BH , $H\Gamma$, ὁ δὲ EZ εἰς τοὺς τῷ Δ ἴσους τοὺς $E\Theta$, ΘZ . ἔσται δὴ ἴσον τὸ πλῆθος τῶν BH , $H\Gamma$ τῷ πλῆθει τῶν $E\Theta$, ΘZ . καὶ ἐπεὶ ἴσος ἐστὶν ὁ μὲν BH τῷ A , ὁ δὲ $E\Theta$ τῷ Δ , καὶ οἱ BH , $E\Theta$ ἄρα τοῖς A , Δ ἴσοι. διὰ τὰ αὐτὰ δὴ καὶ οἱ $H\Gamma$, ΘZ τοῖς A , Δ . ὅσοι ἄρα [εἰσὶν] ἐν τῷ $B\Gamma$ ἀριθμοὶ ἴσοι τῷ A , τοσοῦτοί εἰσι καὶ ἐν τοῖς $B\Gamma$, EZ ἴσοι τοῖς A , Δ . ὁσαυταπλασίων ἔστι καὶ συναμφοτέρος ὁ $B\Gamma$, EZ συναμφοτέρου τοῦ A , Δ . ὁ ἄρα μέρος ἐστὶν ὁ A τοῦ $B\Gamma$, τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφοτέρος ὁ A , Δ συναμφοτέρου τοῦ $B\Gamma$, EZ . ὅπερ ἔδει δεῖξαι.

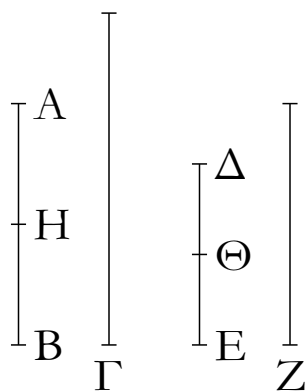
another (number) D (be) the same part of another (number) EF that A (is) of BC . I say that the sum A , D is also the same part of the sum BC , EF that A (is) of BC .

For since which(ever) part A is of BC , D is the same part of EF , thus as many numbers as are in BC equal to A , so many numbers are also in EF equal to D . Let BC have been divided into BG and GC , equal to A , and EF into EH and HF , equal to D . So the multitude of (divisions) BG , GC will be equal to the multitude of (divisions) EH , HF . And since BG is equal to A , and EH to D , thus BG , EH (is) also equal to A , D . So, for the same (reasons), GC , HF (is) also (equal) to A , D . Thus, as many numbers as [are] in BC equal to A , so many are also in BC , EF equal to A , D . Thus, as many times as BC is (divisible) by A , so many times is the sum BC , EF also (divisible) by the sum A , D . Thus, which(ever) part A is of BC , the sum A , D is also the same part of the sum BC , EF . (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a = (1/n)b$ and $c = (1/n)d$ then $(a + c) = (1/n)(b + d)$, where all symbols denote numbers.

ζ΄.

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρη ᾗ, καὶ ἕτερος ἑτέρου τὰ αὐτὰ μέρη ᾗ, καὶ συναμφοτέρος συναμφοτέρου τὰ αὐτὰ μέρη ἔσται, ὅπερ ὁ εἰς τοῦ ἑνός.

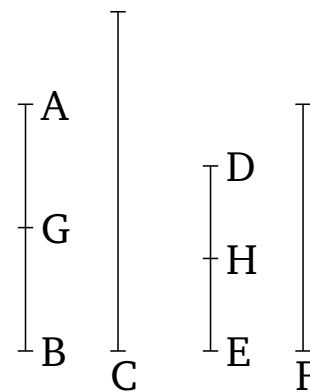


Ἀριθμὸς γάρ ὁ AB ἀριθμοῦ τοῦ Γ μέρη ἔστω, καὶ ἕτερος ὁ ΔE ἑτέρου τοῦ Z τὰ αὐτὰ μέρη, ὅπερ ὁ AB τοῦ Γ . λέγω, ὅτι καὶ συναμφοτέρος ὁ AB , ΔE συναμφοτέρου τοῦ Γ , Z τὰ αὐτὰ μέρη ἐστίν, ὅπερ ὁ AB τοῦ Γ .

Ἐπεὶ γάρ, ἂ μέρη ἐστὶν ὁ AB τοῦ Γ , τὰ αὐτὰ μέρη καὶ ὁ ΔE τοῦ Z , ὅσα ἄρα ἐστὶν ἐν τῷ AB μέρη τοῦ Γ , τοσαῦτά ἐστι καὶ ἐν τῷ ΔE μέρη τοῦ Z . διηγήσθω ὁ μὲν AB εἰς τὰ τοῦ Γ μέρη τὰ AH , HB , ὁ δὲ ΔE εἰς τὰ τοῦ Z μέρη τὰ $\Delta\Theta$, ΘE . ἔσται δὴ ἴσον τὸ πλῆθος τῶν AH , HB τῷ πλῆθει τῶν $\Delta\Theta$, ΘE . καὶ ἐπεὶ, ὁ μέρος ἐστὶν ὁ AH τοῦ Γ , τὸ

Proposition 6†

If a number is parts of a number, and another (number) is the same parts of another, then the sum (of the leading numbers) will also be the same parts of the sum (of the following numbers) that one (number) is of another.



For let a number AB be parts of a number C , and another (number) DE (be) the same parts of another (number) F that AB (is) of C . I say that the sum AB , DE is also the same parts of the sum C , F that AB (is) of C .

For since which(ever) parts AB is of C , DE (is) also the same parts of F , thus as many parts of C as are in AB , so many parts of F are also in DE . Let AB have been divided into the parts of C , AG and GB , and DE into the parts of F , DH and HE . So the multitude of (divisions) AG , GB will be equal to the multitude of (divisions) DH ,

αὐτὸ μέρος ἐστὶ καὶ ὁ $\Delta\Theta$ τοῦ Z , ὃ ἄρα μέρος ἐστὶν ὁ AH τοῦ Γ , τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφοτέρος ὁ AH , $\Delta\Theta$ συναμφοτέρου τοῦ Γ , Z . διὰ τὰ αὐτὰ δὴ καὶ ὁ μέρος ἐστὶν ὁ HB τοῦ Γ , τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφοτέρος ὁ HB , ΘE συναμφοτέρου τοῦ Γ , Z . ἂ ἄρα μέρη ἐστὶν ὁ AB τοῦ Γ , τὰ αὐτὰ μέρη ἐστὶ καὶ συναμφοτέρος ὁ AB , ΔE συναμφοτέρου τοῦ Γ , Z . ὅπερ ἔδει δεῖξαι.

HE . And since which(ever) part AG is of C , DH is also the same part of F , thus which(ever) part AG is of C , the sum AG , DH is also the same part of the sum C , F [Prop. 7.5]. And so, for the same (reasons), which(ever) part GB is of C , the sum GB , HE is also the same part of the sum C , F . Thus, which(ever) parts AB is of C , the sum AB , DE is also the same parts of the sum C , F . (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a = (m/n)b$ and $c = (m/n)d$ then $(a + c) = (m/n)(b + d)$, where all symbols denote numbers.

ζ΄.

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρος ἦ, ὅπερ ἀφαιρεθεὶς ἀφαιρεθέντος, καὶ ὁ λοιπὸς τοῦ λοιποῦ τὸ αὐτὸ μέρος ἔσται, ὅπερ ὁ ὅλος τοῦ ὅλου.

A E B

H Γ Z Δ

Ἀριθμὸς γὰρ ὁ AB ἀριθμοῦ τοῦ $\Gamma\Delta$ μέρος ἔστω, ὅπερ ἀφαιρεθεὶς ὁ AE ἀφαιρεθέντος τοῦ ΓZ λέγω, ὅτι καὶ λοιπὸς ὁ EB λοιποῦ τοῦ $Z\Delta$ τὸ αὐτὸ μέρος ἐστὶν, ὅπερ ὅλος ὁ AB ὅλου τοῦ $\Gamma\Delta$.

Ὁ γὰρ μέρος ἐστὶν ὁ AE τοῦ ΓZ , τὸ αὐτὸ μέρος ἔστω καὶ ὁ EB τοῦ ΓH . καὶ ἐπεὶ, ὁ μέρος ἐστὶν ὁ AE τοῦ ΓZ , τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ EB τοῦ ΓH , ὃ ἄρα μέρος ἐστὶν ὁ AE τοῦ ΓZ , τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ AB τοῦ HZ . ὃ δὲ μέρος ἐστὶν ὁ AE τοῦ ΓZ , τὸ αὐτὸ μέρος ὑπόκειται καὶ ὁ AB τοῦ $\Gamma\Delta$. ὃ ἄρα μέρος ἐστὶ καὶ ὁ AB τοῦ HZ , τὸ αὐτὸ μέρος ἐστὶ καὶ τοῦ $\Gamma\Delta$. ἴσος ἄρα ἐστὶν ὁ HZ τῷ $\Gamma\Delta$. κοινὸς ἀφηρησθῶ ὁ ΓZ . λοιπὸς ἄρα ὁ $H\Gamma$ λοιπῷ τῷ $Z\Delta$ ἐστὶν ἴσος. καὶ ἐπεὶ, ὁ μέρος ἐστὶν ὁ AE τοῦ ΓZ , τὸ αὐτὸ μέρος [ἐστὶ] καὶ ὁ EB τοῦ $H\Gamma$, ἴσος δὲ ὁ $H\Gamma$ τῷ $Z\Delta$, ὃ ἄρα μέρος ἐστὶν ὁ AE τοῦ ΓZ , τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ EB τοῦ $Z\Delta$. ἀλλὰ ὁ μέρος ἐστὶν ὁ AE τοῦ ΓZ , τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ AB τοῦ $\Gamma\Delta$. καὶ λοιπὸς ἄρα ὁ EB λοιποῦ τοῦ $Z\Delta$ τὸ αὐτὸ μέρος ἐστὶν, ὅπερ ὅλος ὁ AB ὅλου τοῦ $\Gamma\Delta$. ὅπερ ἔδει δεῖξαι.

Proposition 7†

If a number is that part of a number that a (part) taken away (is) of a (part) taken away then the remainder will also be the same part of the remainder that the whole (is) of the whole.

A E B

G C F D

For let a number AB be that part of a number CD that a (part) taken away AE (is) of a part taken away CF . I say that the remainder EB is also the same part of the remainder FD that the whole AB (is) of the whole CD .

For which(ever) part AE is of CF , let EB also be the same part of CG . And since which(ever) part AE is of CF , EB is also the same part of CG , thus which(ever) part AE is of CF , AB is also the same part of GF [Prop. 7.5]. And which(ever) part AE is of CF , AB is also assumed (to be) the same part of CD . Thus, also, which(ever) part AB is of GF , (AB) is also the same part of CD . Thus, GF is equal to CD . Let CF have been subtracted from both. Thus, the remainder GC is equal to the remainder FD . And since which(ever) part AE is of CF , EB [is] also the same part of GC , and GC (is) equal to FD , thus which(ever) part AE is of CF , EB is also the same part of FD . But, which(ever) part AE is of CF , AB is also the same part of CD . Thus, the remainder EB is also the same part of the remainder FD that the whole AB (is) of the whole CD . (Which is) the very thing it was required to show.

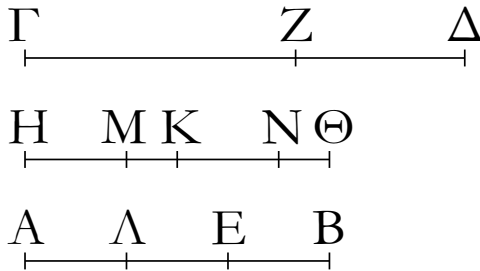
† In modern notation, this proposition states that if $a = (1/n)b$ and $c = (1/n)d$ then $(a - c) = (1/n)(b - d)$, where all symbols denote numbers.

η΄.

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρη ἦ, ἅπερ ἀφαιρεθεὶς ἀφαιρεθέντος, καὶ ὁ λοιπὸς τοῦ λοιποῦ τὰ αὐτὰ μέρη ἔσται, ἅπερ ὁ ὅλος τοῦ ὅλου.

Proposition 8†

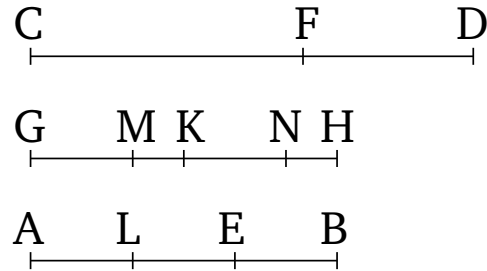
If a number is those parts of a number that a (part) taken away (is) of a (part) taken away then the remainder will also be the same parts of the remainder that the



Ἀριθμὸς γὰρ ὁ ΑΒ ἀριθμοῦ τοῦ ΓΔ μέρη ἔστω, ἅπερ ἀφαιρεθεὶς ὁ ΑΕ ἀφαιρεθέντος τοῦ ΓΖ· λέγω, ὅτι καὶ λοιπὸς ὁ ΕΒ λοιποῦ τοῦ ΖΔ τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὅλος ὁ ΑΒ ὅλου τοῦ ΓΔ.

Κείσθω γὰρ τῷ ΑΒ ἴσος ὁ ΗΘ, ὃ ἄρα μέρη ἐστίν ὁ ΗΘ τοῦ ΓΔ, τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ ΑΕ τοῦ ΓΖ. διηγήσθω ὁ μὲν ΗΘ εἰς τὰ τοῦ ΓΔ μέρη τὰ ΗΚ, ΚΘ, ὁ δὲ ΑΕ εἰς τὰ τοῦ ΓΖ μέρη τὰ ΑΛ, ΛΕ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΗΚ, ΚΘ τῷ πλῆθει τῶν ΑΛ, ΛΕ. καὶ ἐπεὶ, ὃ μέρος ἐστὶν ὁ ΗΚ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΑΛ τοῦ ΓΖ, μείζων δὲ ὁ ΓΔ τοῦ ΓΖ, μείζων ἄρα καὶ ὁ ΗΚ τοῦ ΑΛ. κείσθω τῷ ΑΛ ἴσος ὁ ΗΜ. ὃ ἄρα μέρος ἐστὶν ὁ ΗΚ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΗΜ τοῦ ΓΖ· καὶ λοιπὸς ἄρα ὁ ΜΚ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ ΗΚ ὅλου τοῦ ΓΔ. πάλιν ἐπεὶ, ὃ μέρος ἐστὶν ὁ ΚΘ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΕΛ τοῦ ΓΖ, μείζων δὲ ὁ ΓΔ τοῦ ΓΖ, μείζων ἄρα καὶ ὁ ΚΘ τοῦ ΕΛ. κείσθω τῷ ΕΛ ἴσος ὁ ΚΝ. ὃ ἄρα μέρος ἐστὶν ὁ ΚΘ τοῦ ΓΔ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ ΚΝ τοῦ ΓΖ· καὶ λοιπὸς ἄρα ὁ ΝΘ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὅλος ὁ ΚΘ ὅλου τοῦ ΓΔ. ἐδείχθη δὲ καὶ λοιπὸς ὁ ΜΚ λοιποῦ τοῦ ΖΔ τὸ αὐτὸ μέρος ὄν, ὅπερ ὅλος ὁ ΗΚ ὅλου τοῦ ΓΔ· καὶ συναμφοτέρος ἄρα ὁ ΜΚ, ΝΘ τοῦ ΔΖ τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὅλος ὁ ΘΗ ὅλου τοῦ ΓΔ. ἴσος δὲ συναμφοτέρος μὲν ὁ ΜΚ, ΝΘ τῷ ΕΒ, ὁ δὲ ΘΗ τῷ ΒΑ· καὶ λοιπὸς ἄρα ὁ ΕΒ λοιποῦ τοῦ ΖΔ τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὅλος ὁ ΑΒ ὅλου τοῦ ΓΔ· ὅπερ ἔδει δεῖξαι.

whole (is) of the whole.



For let a number AB be those parts of a number CD that a (part) taken away AE (is) of a (part) taken away CF . I say that the remainder EB is also the same parts of the remainder FD that the whole AB (is) of the whole CD .

For let GH be laid down equal to AB . Thus, which(ever) parts GH is of CD , AE is also the same parts of CF . Let GH have been divided into the parts of CD , GK and KH , and AE into the part of CF , AL and LE . So the multitude of (divisions) GK , KH will be equal to the multitude of (divisions) AL , LE . And since which(ever) part GK is of CD , AL is also the same part of CF , and CD (is) greater than CF , GK (is) thus also greater than AL . Let GM be made equal to AL . Thus, which(ever) part GK is of CD , GM is also the same part of CF . Thus, the remainder MK is also the same part of the remainder FD that the whole GK (is) of the whole CD [Prop. 7.5]. Again, since which(ever) part KH is of CD , EL is also the same part of CF , and CD (is) greater than CF , KH (is) thus also greater than EL . Let KN be made equal to EL . Thus, which(ever) part KH (is) of CD , KN is also the same part of CF . Thus, the remainder NH is also the same part of the remainder FD that the whole KH (is) of the whole CD [Prop. 7.5]. And the remainder MK was also shown to be the same part of the remainder FD that the whole GK (is) of the whole CD . Thus, the sum MK , NH is the same parts of DF that the whole HG (is) of the whole CD . And the sum MK , NH (is) equal to EB , and HG to BA . Thus, the remainder EB is also the same parts of the remainder FD that the whole AB (is) of the whole CD . (Which is) the very thing it was required to show.

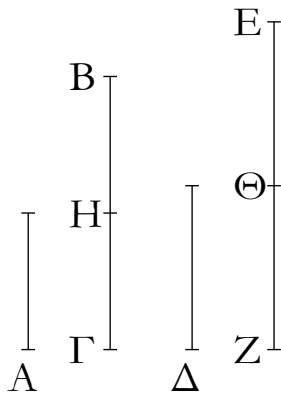
† In modern notation, this proposition states that if $a = (m/n)b$ and $c = (m/n)d$ then $(a - c) = (m/n)(b - d)$, where all symbols denote numbers.

θ'.

Proposition 9†

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρος ᾗ, καὶ ἕτερος ἑτέρου τὸ αὐτὸ μέρος ᾗ, καὶ ἐναλλάξ, ὃ μέρος ἐστὶν ἡ μέρη ὁ πρῶτος τοῦ τρίτου, τὸ αὐτὸ μέρος ἔσται ἡ τὰ αὐτὰ μέρη καὶ ὁ δεῦτερος τοῦ τετάρτου.

If a number is part of a number, and another (number) is the same part of another, also, alternately, which(ever) part, or parts, the first (number) is of the third, the second (number) will also be the same part, or

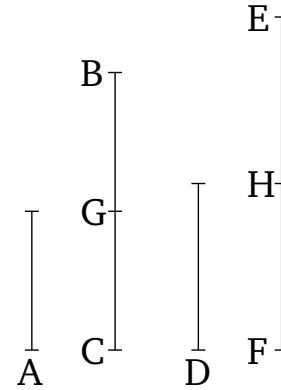


Ἀριθμὸς γὰρ ὁ A ἀριθμοῦ τοῦ $BΓ$ μέρος ἔστω, καὶ ἕτερος ὁ $Δ$ ἐτέρου τοῦ $EΖ$ τὸ αὐτὸ μέρος, ὅπερ ὁ A τοῦ $BΓ$ · λέγω, ὅτι καὶ ἐναλλάξ, ὁ μέρος ἐστὶν ὁ A τοῦ $Δ$ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ $BΓ$ τοῦ $EΖ$ ἢ μέρη.

Ἐπεὶ γὰρ ὁ μέρος ἐστὶν ὁ A τοῦ $BΓ$, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ $Δ$ τοῦ $EΖ$, ὅσοι ἄρα εἰσὶν ἐν τῷ $BΓ$ ἀριθμοὶ ἴσοι τῷ A , τοσοῦτοί εἰσι καὶ ἐν τῷ $EΖ$ ἴσοι τῷ $Δ$. διηγήσθω ὁ μὲν $BΓ$ εἰς τοὺς τῷ A ἴσους τοὺς $BΗ$, $HΓ$, ὁ δὲ $EΖ$ εἰς τοὺς τῷ $Δ$ ἴσους τοὺς $EΘ$, $ΘΖ$ · ἔσται δὴ ἴσον τὸ πλῆθος τῶν $BΗ$, $HΓ$ τῷ πλῆθει τῶν $EΘ$, $ΘΖ$.

Καὶ ἐπεὶ ἴσοι εἰσὶν οἱ $BΗ$, $HΓ$ ἀριθμοὶ ἀλλήλοις, εἰσὶ δὲ καὶ οἱ $EΘ$, $ΘΖ$ ἀριθμοὶ ἴσοι ἀλλήλοις, καὶ ἐστὶν ἴσον τὸ πλῆθος τῶν $BΗ$, $HΓ$ τῷ πλῆθει τῶν $EΘ$, $ΘΖ$, ὃ ἄρα μέρος ἐστὶν ὁ $BΗ$ τοῦ $EΘ$ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ $HΓ$ τοῦ $ΘΖ$ ἢ τὰ αὐτὰ μέρη· ὥστε καὶ ὁ μέρος ἐστὶν ὁ $BΗ$ τοῦ $EΘ$ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ συναμφοτέρως ὁ $BΓ$ συναμφοτέρου τοῦ $EΖ$ ἢ τὰ αὐτὰ μέρη. ἴσος δὲ ὁ μὲν $BΗ$ τῷ A , ὁ δὲ $EΘ$ τῷ $Δ$ · ὃ ἄρα μέρος ἐστὶν ὁ A τοῦ $Δ$ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ $BΓ$ τοῦ $EΖ$ ἢ τὰ αὐτὰ μέρη· ὅπερ εἶδει δεῖξαι.

the same parts, of the fourth.



For let a number A be part of a number BC , and another (number) D (be) the same part of another EF that A (is) of BC . I say that, also, alternately, which(ever) part, or parts, A is of D , BC is also the same part, or parts, of EF .

For since which(ever) part A is of BC , D is also the same part of EF , thus as many numbers as are in BC equal to A , so many are also in EF equal to D . Let BC have been divided into BG and GC , equal to A , and EF into EH and HF , equal to D . So the multitude of (divisions) BG , GC will be equal to the multitude of (divisions) EH , HF .

And since the numbers BG and GC are equal to one another, and the numbers EH and HF are also equal to one another, and the multitude of (divisions) BG , GC is equal to the multitude of (divisions) EH , HC , thus which(ever) part, or parts, BG is of EH , GC is also the same part, or the same parts, of HF . And hence, which(ever) part, or parts, BG is of EH , the sum BC is also the same part, or the same parts, of the sum EF [Props. 7.5, 7.6]. And BG (is) equal to A , and EH to D . Thus, which(ever) part, or parts, A is of D , BC is also the same part, or the same parts, of EF . (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a = (1/n)b$ and $c = (1/n)d$ then if $a = (k/l)c$ then $b = (k/l)d$, where all symbols denote numbers.

ι'.

Ἐὰν ἀριθμὸς ἀριθμοῦ μέρη ᾗ, καὶ ἕτερος ἐτέρου τὰ αὐτὰ μέρη ᾗ, καὶ ἐναλλάξ, ἃ μέρη ἐστὶν ὁ πρῶτος τοῦ τρίτου ἢ μέρος, τὰ αὐτὰ μέρη ἔσται καὶ ὁ δεύτερος τοῦ τετάρτου ἢ τὸ αὐτὸ μέρος.

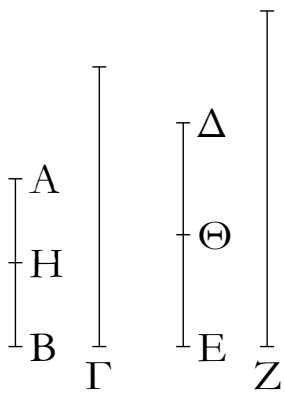
Ἀριθμὸς γὰρ ὁ AB ἀριθμοῦ τοῦ $Γ$ μέρη ἔστω, καὶ ἕτερος ὁ $ΔΕ$ ἐτέρου τοῦ $Ζ$ τὰ αὐτὰ μέρη· λέγω, ὅτι καὶ ἐναλλάξ, ἃ μέρη ἐστὶν ὁ AB τοῦ $ΔΕ$ ἢ μέρος, τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ $Γ$ τοῦ $Ζ$ ἢ τὸ αὐτὸ μέρος.

Proposition 10[†]

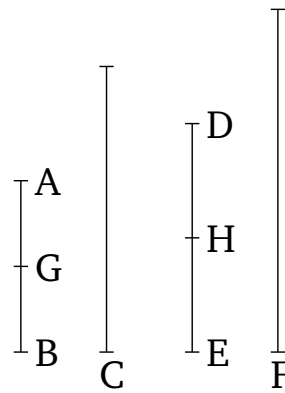
If a number is parts of a number, and another (number) is the same parts of another, also, alternately, which(ever) parts, or part, the first (number) is of the third, the second will also be the same parts, or the same part, of the fourth.

For let a number AB be parts of a number C , and another (number) DE (be) the same parts of another F . I say that, also, alternately, which(ever) parts, or part,

AB is of DE , C is also the same parts, or the same part, of F .



Ἐπεὶ γάρ, ἃ μέρη ἐστὶν ὁ AB τοῦ Γ , τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ ΔE τοῦ Z , ὅσα ἄρα ἐστὶν ἐν τῷ AB μέρη τοῦ Γ , τοσαῦτα καὶ ἐν τῷ ΔE μέρη τοῦ Z . διηρῆσθω ὁ μὲν AB εἰς τὰ τοῦ Γ μέρη τὰ AH , HB , ὁ δὲ ΔE εἰς τὰ τοῦ Z μέρη τὰ $\Delta\Theta$, ΘE . ἔσται δὴ ἴσον τὸ πλῆθος τῶν AH , HB τῷ πλῆθει τῶν $\Delta\Theta$, ΘE . καὶ ἐπεὶ, ὃ μέρος ἐστὶν ὁ AH τοῦ Γ , τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ $\Delta\Theta$ τοῦ Z , καὶ ἐναλλάξ, ὃ μέρος ἐστὶν ὁ AH τοῦ $\Delta\Theta$ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Z ἢ τὰ αὐτὰ μέρη. διὰ τὰ αὐτὰ δὴ καὶ, ὃ μέρος ἐστὶν ὁ HB τοῦ ΘE ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Z ἢ τὰ αὐτὰ μέρη. ὥστε καὶ [ὃ μέρος ἐστὶν ὁ AH τοῦ $\Delta\Theta$ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ HB τοῦ ΘE ἢ τὰ αὐτὰ μέρη· καὶ ὃ ἄρα μέρος ἐστὶν ὁ AH τοῦ $\Delta\Theta$ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ AB τοῦ ΔE ἢ τὰ αὐτὰ μέρη· ἀλλ' ὃ μέρος ἐστὶν ὁ AH τοῦ $\Delta\Theta$ ἢ μέρη, τὸ αὐτὸ μέρος ἐδείχθη καὶ ὁ Γ τοῦ Z ἢ τὰ αὐτὰ μέρη, καὶ] ἃ [ἄρα] μέρη ἐστὶν ὁ AB τοῦ ΔE ἢ μέρος, τὰ αὐτὰ μέρη ἐστὶ καὶ ὁ Γ τοῦ Z ἢ τὸ αὐτὸ μέρος· ὅπερ ἔδει δείξαι.



For since which(ever) parts AB is of C , DE is also the same parts of F , thus as many parts of C as are in AB , so many parts of F (are) also in DE . Let AB have been divided into the parts of C , AG and GB , and DE into the parts of F , DH and HE . So the multitude of (divisions) AG , GB will be equal to the multitude of (divisions) DH , HE . And since which(ever) part AG is of C , DH is also the same part of F , also, alternately, which(ever) part, or parts, AG is of DH , C is also the same part, or the same parts, of F [Prop. 7.9]. And so, for the same (reasons), which(ever) part, or parts, GB is of HE , C is also the same part, or the same parts, of F [Prop. 7.9]. And so [which(ever) part, or parts, AG is of DH , GB is also the same part, or the same parts, of HE . And thus, which(ever) part, or parts, AG is of DH , AB is also the same part, or the same parts, of DE [Props. 7.5, 7.6]. But, which(ever) part, or parts, AG is of DH , C was also shown (to be) the same part, or the same parts, of F . And, thus] which(ever) parts, or part, AB is of DE , C is also the same parts, or the same part, of F . (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a = (m/n)b$ and $c = (m/n)d$ then if $a = (k/l)c$ then $b = (k/l)d$, where all symbols denote numbers.

ια'.

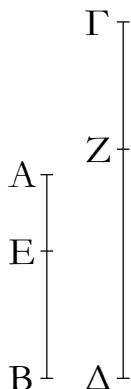
Proposition 11

Ἐὰν ᾗ ὡς ὅλος πρὸς ὅλον, οὕτως ἀφαιρεθεὶς πρὸς ἀφαιρεθέντα, καὶ ὁ λοιπὸς πρὸς τὸν λοιπὸν ἔσται, ὡς ὅλος πρὸς ὅλον.

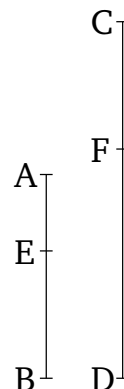
Ἐστω ὡς ὅλος ὁ AB πρὸς ὅλον τὸν $\Gamma\Delta$, οὕτως ἀφαιρεθεὶς ὁ AE πρὸς ἀφαιρεθέντα τὸν ΓZ : λέγω, ὅτι καὶ λοιπὸς ὁ EB πρὸς λοιπὸν τὸν $Z\Delta$ ἐστὶν, ὡς ὅλος ὁ AB πρὸς ὅλον τὸν $\Gamma\Delta$.

If as the whole (of a number) is to the whole (of another), so a (part) taken away (is) to a (part) taken away, then the remainder will also be to the remainder as the whole (is) to the whole.

Let the whole AB be to the whole CD as the (part) taken away AE (is) to the (part) taken away CF . I say that the remainder EB is to the remainder FD as the whole AB (is) to the whole CD .



Ἐπεὶ ἐστὶν ὡς ὁ AB πρὸς τὸν $\Gamma\Delta$, οὕτως ὁ AE πρὸς τὸν ΓZ , ὃ ἄρα μέρος ἐστὶν ὁ AB τοῦ $\Gamma\Delta$ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ AE τοῦ ΓZ ἢ τὰ αὐτὰ μέρη. καὶ λοιπὸς ἄρα ὁ EB λοιποῦ τοῦ $Z\Delta$ τὸ αὐτὸ μέρος ἐστὶν ἢ μέρη, ἅπερ ὁ AB τοῦ $\Gamma\Delta$. ἔστιν ἄρα ὡς ὁ EB πρὸς τὸν $Z\Delta$, οὕτως ὁ AB πρὸς τὸν $\Gamma\Delta$. ὅπερ ἔδει δεῖξαι.

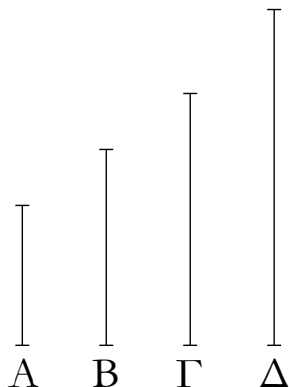


(For) since as AB is to CD , so AE (is) to CF , thus which(ever) part, or parts, AB is of CD , AE is also the same part, or the same parts, of CF [Def. 7.20]. Thus, the remainder EB is also the same part, or parts, of the remainder FD that AB (is) of CD [Props. 7.7, 7.8]. Thus, as EB is to FD , so AB (is) to CD [Def. 7.20]. (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that if $a : b :: c : d$ then $a : b :: a - c : b - d$, where all symbols denote numbers.

ιβ'.

Ἐὰν ὧσιν ὅποιοι οὖν ἀριθμοὶ ἀνάλογον, ἔσται ὡς εἷς τῶν ἡγουμένων πρὸς ἓνα τῶν ἐπομένων, οὕτως ἅπαντες οἱ ἡγούμενοι πρὸς ἅπαντας τοὺς ἐπομένους.

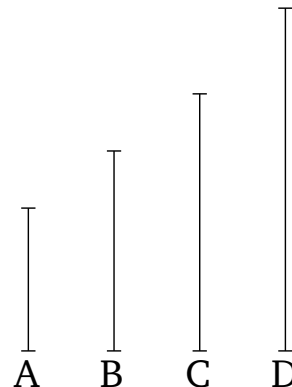


Ἐστωσαν ὅποιοι οὖν ἀριθμοὶ ἀνάλογον οἱ A, B, Γ, Δ , ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Γ πρὸς τὸν Δ . λέγω, ὅτι ἐστὶν ὡς ὁ A πρὸς τὸν B , οὕτως οἱ A, Γ πρὸς τοὺς B, Δ .

Ἐπεὶ γάρ ἐστὶν ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Γ πρὸς τὸν Δ , ὃ ἄρα μέρος ἐστὶν ὁ A τοῦ B ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Δ ἢ μέρη. καὶ συναμφοτέρως ἄρα ὁ A, Γ συναμφοτέρου τοῦ B, Δ τὸ αὐτὸ μέρος ἐστὶν ἢ τὰ αὐτὰ μέρη, ἅπερ ὁ A τοῦ B . ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , οὕτως οἱ A, Γ πρὸς τοὺς B, Δ . ὅπερ ἔδει δεῖξαι.

Proposition 12[†]

If any multitude whatsoever of numbers are proportional then as one of the leading (numbers is) to one of the following so (the sum of) all of the leading (numbers) will be to (the sum of) all of the following.



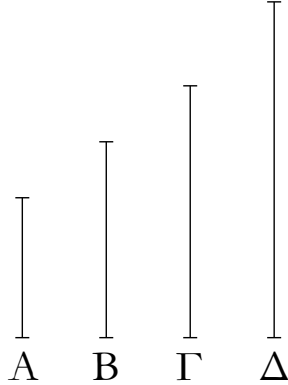
Let any multitude whatsoever of numbers, A, B, C, D , be proportional, (such that) as A (is) to B , so C (is) to D . I say that as A is to B , so A, C (is) to B, D .

For since as A is to B , so C (is) to D , thus which(ever) part, or parts, A is of B , C is also the same part, or parts, of D [Def. 7.20]. Thus, the sum A, C is also the same part, or the same parts, of the sum B, D that A (is) of B [Props. 7.5, 7.6]. Thus, as A is to B , so A, C (is) to B, D [Def. 7.20]. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a : b :: c : d$ then $a : b :: a + c : b + d$, where all symbols denote numbers.

ιγ΄.

Ἐάν τέσσαρες ἀριθμοὶ ἀνάλογον ὦσιν, καὶ ἐναλλάξ ἀνάλογον ἔσονται.

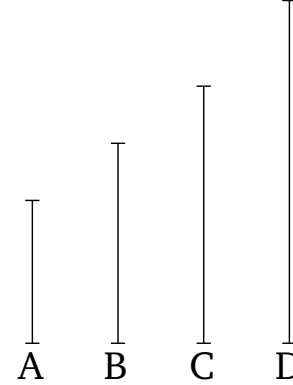


Ἐστωσαν τέσσαρες ἀριθμοὶ ἀνάλογον οἱ A, B, Γ, Δ , ὥς ὁ A πρὸς τὸν B , οὕτως ὁ Γ πρὸς τὸν Δ . λέγω, ὅτι καὶ ἐναλλάξ ἀνάλογον ἔσονται, ὥς ὁ A πρὸς τὸν Γ , οὕτως ὁ B πρὸς τὸν Δ .

Ἐπεὶ γὰρ ἔστιν ὥς ὁ A πρὸς τὸν B , οὕτως ὁ Γ πρὸς τὸν Δ , ὃ ἄρα μέρος ἐστὶν ὁ A τοῦ B ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ Δ ἢ τὰ αὐτὰ μέρη. ἐναλλάξ ἄρα, ὃ μέρος ἐστὶν ὁ A τοῦ Γ ἢ μέρη, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ B τοῦ Δ ἢ τὰ αὐτὰ μέρη. ἔστιν ἄρα ὥς ὁ A πρὸς τὸν Γ , οὕτως ὁ B πρὸς τὸν Δ . ὅπερ ἔδει δεῖξαι.

Proposition 13†

If four numbers are proportional then they will also be proportional alternately.



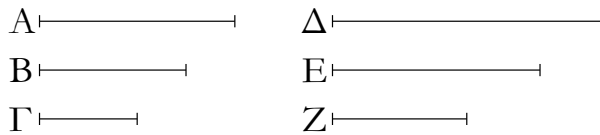
Let the four numbers A, B, C , and D be proportional, (such that) as A (is) to B , so C (is) to D . I say that they will also be proportional alternately, (such that) as A (is) to C , so B (is) to D .

For since as A is to B , so C (is) to D , thus which(ever) part, or parts, A is of B , C is also the same part, or the same parts, of D [Def. 7.20]. Thus, alternately, which(ever) part, or parts, A is of C , B is also the same part, or the same parts, of D [Props. 7.9, 7.10]. Thus, as A is to C , so B (is) to D [Def. 7.20]. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $a : b :: c : d$ then $a : c :: b : d$, where all symbols denote numbers.

ιδ΄.

Ἐάν ὦσιν ὅποσοιοῦν ἀριθμοὶ καὶ ἄλλοι αὐτοῖς ἴσοι τὸ πλῆθος σύνδυο λαμβανόμενοι καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ δι' ἴσου ἐν τῷ αὐτῷ λόγῳ ἔσονται.

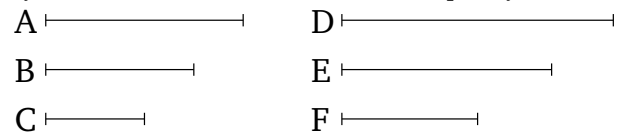


Ἐστωσαν ὅποσοιοῦν ἀριθμοὶ οἱ A, B, Γ καὶ ἄλλοι αὐτοῖς ἴσοι τὸ πλῆθος σύνδυο λαμβανόμενοι ἐν τῷ αὐτῷ λόγῳ οἱ Δ, E, Z , ὥς μὲν ὁ A πρὸς τὸν B , οὕτως ὁ Δ πρὸς τὸν E , ὥς δὲ ὁ B πρὸς τὸν Γ , οὕτως ὁ E πρὸς τὸν Z . λέγω, ὅτι καὶ δι' ἴσου ἔστιν ὥς ὁ A πρὸς τὸν Γ , οὕτως ὁ Δ πρὸς τὸν Z .

Ἐπεὶ γὰρ ἔστιν ὥς ὁ A πρὸς τὸν B , οὕτως ὁ Δ πρὸς τὸν E , ἐναλλάξ ἄρα ἔστιν ὥς ὁ A πρὸς τὸν Δ , οὕτως ὁ B πρὸς τὸν E . πάλιν, ἐπεὶ ἔστιν ὥς ὁ B πρὸς τὸν Γ , οὕτως ὁ

Proposition 14†

If there are any multitude of numbers whatsoever, and (some) other (numbers) of equal multitude to them, (which are) also in the same ratio taken two by two, then they will also be in the same ratio via equality.



Let there be any multitude of numbers whatsoever, A, B, C , and (some) other (numbers), D, E, F , of equal multitude to them, (which are) in the same ratio taken two by two, (such that) as A (is) to B , so D (is) to E , and as B (is) to C , so E (is) to F . I say that also, via equality, as A is to C , so D (is) to F .

For since as A is to B , so D (is) to E , thus, alternately, as A is to D , so B (is) to E [Prop. 7.13]. Again, since as B is to C , so E (is) to F , thus, alternately, as B is

Ε πρὸς τὸν Ζ, ἐναλλάξ ἄρα ἐστὶν ὡς ὁ Β πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Ζ. ὡς δὲ ὁ Β πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Δ· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Δ, οὕτως ὁ Γ πρὸς τὸν Ζ· ἐναλλάξ ἄρα ἐστὶν ὡς ὁ Α πρὸς τὸν Γ, οὕτως ὁ Δ πρὸς τὸν Ζ· ὅπερ ἔδει δεῖξαι.

to E , so C (is) to F [Prop. 7.13]. And as B (is) to E , so A (is) to D . Thus, also, as A (is) to D , so C (is) to F . Thus, alternately, as A is to C , so D (is) to F [Prop. 7.13]. (Which is) the very thing it was required to show.

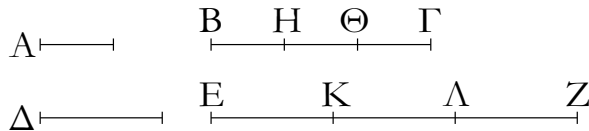
† In modern notation, this proposition states that if $a : b :: d : e$ and $b : c :: e : f$ then $a : c :: d : f$, where all symbols denote numbers.

ιε'.

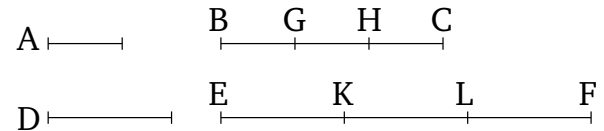
Proposition 15

Ἐάν μονὰς ἀριθμὸν τινα μετρήῃ, ἰσάκεις δὲ ἕτερος ἀριθμὸς ἄλλον τινα ἀριθμὸν μετρήῃ, καὶ ἐναλλάξ ἰσάκεις ἡ μονὰς τὸν τρίτον ἀριθμὸν μετρήσει καὶ ὁ δεύτερος τὸν τέταρτον.

If a unit measures some number, and another number measures some other number as many times, then, also, alternately, the unit will measure the third number as many times as the second (number measures) the fourth.



Μονὰς γὰρ ἡ Α ἀριθμὸν τινα τὸν ΒΓ μετρεῖτω, ἰσάκεις δὲ ἕτερος ἀριθμὸς ὁ Δ ἄλλον τινα ἀριθμὸν τὸν ΕΖ μετρεῖτω· λέγω, ὅτι καὶ ἐναλλάξ ἰσάκεις ἡ Α μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ ΒΓ τὸν ΕΖ.



For let a unit A measure some number BC , and let another number D measure some other number EF as many times. I say that, also, alternately, the unit A also measures the number D as many times as BC (measures) EF .

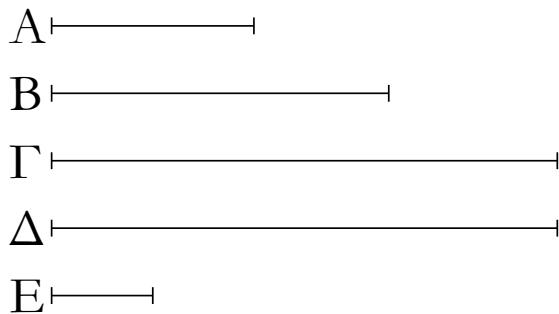
Ἐπεὶ γὰρ ἰσάκεις ἡ Α μονὰς τὸν ΒΓ ἀριθμὸν μετρεῖ καὶ ὁ Δ τὸν ΕΖ, ὅσαι ἄρα εἰσὶν ἐν τῷ ΒΓ μονάδες, τοσοῦτοί εἰσι καὶ ἐν τῷ ΕΖ ἀριθμοὶ ἴσοι τῷ Δ. διηγήσθω ὁ μὲν ΒΓ εἰς τὰς ἐν ἑαυτῷ μονάδας τὰς ΒΗ, ΗΘ, ΘΓ, ὁ δὲ ΕΖ εἰς τοὺς τῷ Δ ἴσους τοὺς ΕΚ, ΚΛ, ΛΖ. ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΘ, ΘΓ τῷ πλῆθει τῶν ΕΚ, ΚΛ, ΛΖ. καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΒΗ, ΗΘ, ΘΓ μονάδες ἀλλήλαις, εἰσὶ δὲ καὶ οἱ ΕΚ, ΚΛ, ΛΖ ἀριθμοὶ ἴσοι ἀλλήλοις, καὶ ἐστὶν ἴσον τὸ πλῆθος τῶν ΒΗ, ΗΘ, ΘΓ μονάδων τῷ πλῆθει τῶν ΕΚ, ΚΛ, ΛΖ ἀριθμῶν, ἔσται ἄρα ὡς ἡ ΒΗ μονὰς πρὸς τὸν ΕΚ ἀριθμὸν, οὕτως ἡ ΗΘ μονὰς πρὸς τὸν ΚΛ ἀριθμὸν καὶ ἡ ΘΓ μονὰς πρὸς τὸν ΛΖ ἀριθμὸν. ἔσται ἄρα καὶ ὡς εἷς τῶν ἡγούμενων πρὸς ἓνα τῶν ἐπομένων, οὕτως ἅπαντες οἱ ἡγούμενοι πρὸς ἅπαντας τοὺς ἐπομένους· ἐστὶν ἄρα ὡς ἡ ΒΗ μονὰς πρὸς τὸν ΕΚ ἀριθμὸν, οὕτως ὁ ΒΓ πρὸς τὸν ΕΖ. ἴση δὲ ἡ ΒΗ μονὰς τῇ Α μονάδι, ὁ δὲ ΕΚ ἀριθμὸς τῷ Δ ἀριθμῷ. ἐστὶν ἄρα ὡς ἡ Α μονὰς πρὸς τὸν Δ ἀριθμὸν, οὕτως ὁ ΒΓ πρὸς τὸν ΕΖ. ἰσάκεις ἄρα ἡ Α μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ ΒΓ τὸν ΕΖ· ὅπερ ἔδει δεῖξαι.

For since the unit A measures the number BC as many times as D (measures) EF , thus as many units as are in BC , so many numbers are also in EF equal to D . Let BC have been divided into its constituent units, BG , GH , and HC , and EF into the (divisions) EK , KL , and LF , equal to D . So the multitude of (units) BG , GH , HC will be equal to the multitude of (divisions) EK , KL , LF . And since the units BG , GH , and HC are equal to one another, and the numbers EK , KL , and LF are also equal to one another, and the multitude of the (units) BG , GH , HC is equal to the multitude of the numbers EK , KL , LF , thus as the unit BG (is) to the number EK , so the unit GH will be to the number KL , and the unit HC to the number LF . And thus, as one of the leading (numbers is) to one of the following, so (the sum of) all of the leading will be to (the sum of) all of the following [Prop. 7.12]. Thus, as the unit BG (is) to the number EK , so BC (is) to EF . And the unit BG (is) equal to the unit A , and the number EK to the number D . Thus, as the unit A is to the number D , so BC (is) to EF . Thus, the unit A measures the number D as many times as BC (measures) EF [Def. 7.20]. (Which is) the very thing it was required to show.

† This proposition is a special case of Prop. 7.9.

ιϛ'.

Εάν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσι
τινας, οἱ γενόμενοι ἐξ αὐτῶν ἴσοι ἀλλήλοις ἔσσονται.

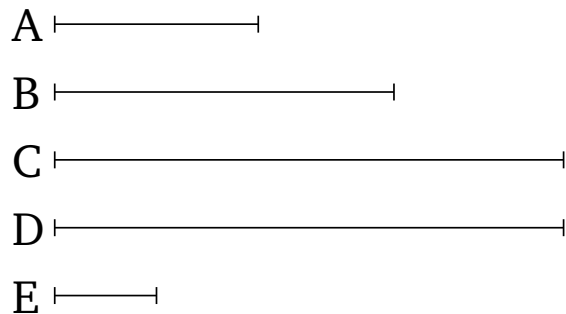


Ἐστωσαν δύο ἀριθμοὶ οἱ A, B , καὶ ὁ μὲν A τὸν B πολλαπλασιάσας τὸν Γ ποιείτω, ὁ δὲ B τὸν A πολλαπλασιάσας τὸν Δ ποιείτω· λέγω, ὅτι ἴσος ἐστὶν ὁ Γ τῷ Δ .

Ἐπεὶ γὰρ ὁ A τὸν B πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ B ἄρα τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ A μονάδας· μετρεῖ δὲ καὶ ἡ E μονὰς τὸν A ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκεις ἄρα ἡ E μονὰς τὸν A ἀριθμὸν μετρεῖ καὶ ὁ B τὸν Γ . ἐναλλάξ ἄρα ἰσάκεις ἡ E μονὰς τὸν B ἀριθμὸν μετρεῖ καὶ ὁ A τὸν Γ . πάλιν, ἐπεὶ ὁ B τὸν A πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ A ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ B μονάδας· μετρεῖ δὲ καὶ ἡ E μονὰς τὸν B κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκεις ἄρα ἡ E μονὰς τὸν B ἀριθμὸν μετρεῖ καὶ ὁ A τὸν Δ . ἰσάκεις δὲ ἡ E μονὰς τὸν B ἀριθμὸν ἐμέτρει καὶ ὁ A τὸν Γ · ἰσάκεις ἄρα ὁ A ἐκάτερον τῶν Γ, Δ μετρεῖ. ἴσος ἄρα ἐστὶν ὁ Γ τῷ Δ · ὅπερ ἔδει δεῖξαι.

Proposition 16[†]

If two numbers multiplying one another make some (numbers) then the (numbers) generated from them will be equal to one another.



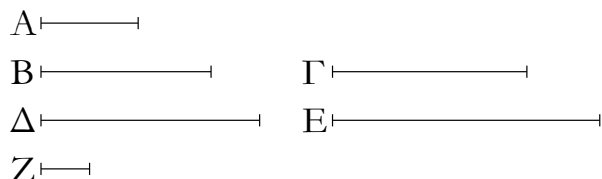
Let A and B be two numbers. And let A make C (by) multiplying B , and let B make D (by) multiplying A . I say that C is equal to D .

For since A has made C (by) multiplying B , B thus measures C according to the units in A [Def. 7.15]. And the unit E also measures the number A according to the units in it. Thus, the unit E measures the number A as many times as B (measures) C . Thus, alternately, the unit E measures the number B as many times as A (measures) C [Prop. 7.15]. Again, since B has made D (by) multiplying A , A thus measures D according to the units in B [Def. 7.15]. And the unit E also measures B according to the units in it. Thus, the unit E measures the number B as many times as A (measures) D . And the unit E was measuring the number B as many times as A (measures) C . Thus, A measures each of C and D an equal number of times. Thus, C is equal to D . (Which is) the very thing it was required to show.

[†] In modern notation, this proposition states that $ab = ba$, where all symbols denote numbers.

ιζ'.

Ἐάν ἀριθμὸς δύο ἀριθμοὺς πολλαπλασιάσας ποιῇ τινας, οἱ γενόμενοι ἐξ αὐτῶν τὸν αὐτὸν ἔξουσι λόγον τοῖς πολλαπλασιασθεῖσιν.

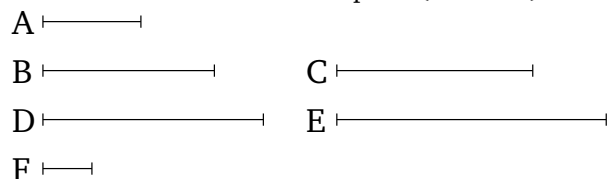


Ἀριθμὸς γὰρ ὁ A δύο ἀριθμοὺς τοὺς B, Γ πολλαπλασιάσας τοὺς Δ, E ποιείτω· λέγω, ὅτι ἐστὶν ὡς ὁ B πρὸς τὸν Γ , οὕτως ὁ Δ πρὸς τὸν E .

Ἐπεὶ γὰρ ὁ A τὸν B πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ B ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ A μονάδας· μετρεῖ

Proposition 17[†]

If a number multiplying two numbers makes some (numbers) then the (numbers) generated from them will have the same ratio as the multiplied (numbers).



For let the number A make (the numbers) D and E (by) multiplying the two numbers B and C (respectively). I say that as B is to C , so D (is) to E .

For since A has made D (by) multiplying B , B thus measures D according to the units in A [Def. 7.15]. And

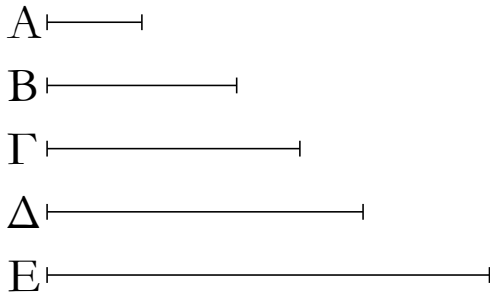
δὲ καὶ ἡ Z μονὰς τὸν A ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκεις ἄρα ἡ Z μονὰς τὸν A ἀριθμὸν μετρεῖ καὶ ὁ B τὸν Δ . ἔστιν ἄρα ὡς ἡ Z μονὰς πρὸς τὸν A ἀριθμὸν, οὕτως ὁ B πρὸς τὸν Δ . διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ Z μονὰς πρὸς τὸν A ἀριθμὸν, οὕτως ὁ Γ πρὸς τὸν E · καὶ ὡς ἄρα ὁ B πρὸς τὸν Δ , οὕτως ὁ Γ πρὸς τὸν E . ἐναλλάξ ἄρα ἐστὶν ὡς ὁ B πρὸς τὸν Γ , οὕτως ὁ Δ πρὸς τὸν E · ὅπερ ἔδει δεῖξαι.

the unit F also measures the number A according to the units in it. Thus, the unit F measures the number A as many times as B (measures) D . Thus, as the unit F is to the number A , so B (is) to D [Def. 7.20]. And so, for the same (reasons), as the unit F (is) to the number A , so C (is) to E . And thus, as B (is) to D , so C (is) to E . Thus, alternately, as B is to C , so D (is) to E [Prop. 7.13]. (Which is) the very thing it was required to show.

† In modern notation, this proposition states that if $d = ab$ and $e = ac$ then $d : e :: b : c$, where all symbols denote numbers.

ιη΄.

Ἐὰν δύο ἀριθμοὶ ἀριθμὸν τινὰ πολλαπλασιάσαντες ποιῶσι τινὰς, οἱ γενόμενοι ἐξ αὐτῶν τὸν αὐτὸν ἔξουσι λόγον τοῖς πολλαπλασιάσαντι.

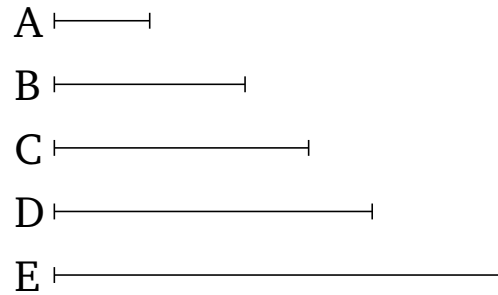


Δύο γὰρ ἀριθμοὶ οἱ A , B ἀριθμὸν τινὰ τὸν Γ πολλαπλασιάσαντες τοὺς Δ , E ποιείτωσαν· λέγω, ὅτι ἐστὶν ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Δ πρὸς τὸν E .

Ἐπεὶ γὰρ ὁ A τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν, καὶ ὁ Γ ἄρα τὸν A πολλαπλασιάσας τὸν Δ πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν B πολλαπλασιάσας τὸν E πεποίηκεν. ἀριθμὸς δὴ ὁ Γ δύο ἀριθμοὺς τοὺς A , B πολλαπλασιάσας τοὺς Δ , E πεποίηκεν. ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Δ πρὸς τὸν E · ὅπερ ἔδει δεῖξαι.

Proposition 18[†]

If two numbers multiplying some number make some (other numbers) then the (numbers) generated from them will have the same ratio as the multiplying (numbers).



For let the two numbers A and B make (the numbers) D and E (respectively, by) multiplying some number C . I say that as A is to B , so D (is) to E .

For since A has made D (by) multiplying C , C has thus also made D (by) multiplying A [Prop. 7.16]. So, for the same (reasons), C has also made E (by) multiplying B . So the number C has made D and E (by) multiplying the two numbers A and B (respectively). Thus, as A is to B , so D (is) to E [Prop. 7.17]. (Which is) the very thing it was required to show.

† In modern notation, this propositions states that if $ac = d$ and $bc = e$ then $a : b :: d : e$, where all symbols denote numbers.

ιθ΄.

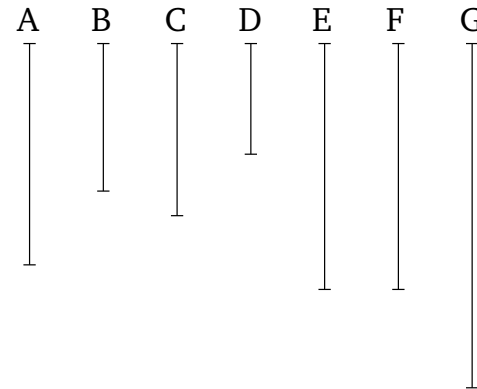
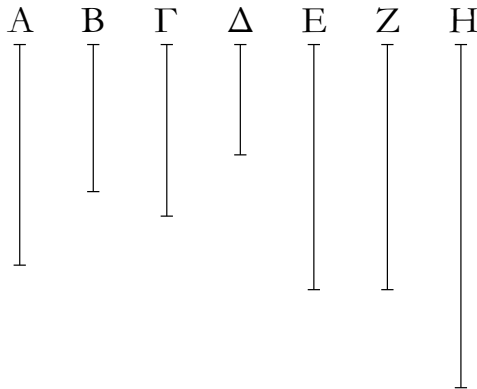
Ἐὰν τέσσαρες ἀριθμοὶ ἀνάλογον ᾧσιν, ὁ ἐκ πρώτου καὶ τετάρτου γενόμενος ἀριθμὸς ἴσος ἔσται τῷ ἐκ δευτέρου καὶ τρίτου γενομένῳ ἀριθμῷ· καὶ ἐὰν ὁ ἐκ πρώτου καὶ τετάρτου γενόμενος ἀριθμὸς ἴσος ᾗ τῷ ἐκ δευτέρου καὶ τρίτου, οἱ τέσσαρες ἀριθμοὶ ἀνάλογον ἔσονται.

Ἐστωσαν τέσσαρες ἀριθμοὶ ἀνάλογον οἱ A , B , Γ , Δ , ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Γ πρὸς τὸν Δ , καὶ ὁ μὲν A τὸν Δ πολλαπλασιάσας τὸν E ποιείτω, ὁ δὲ B τὸν Γ πολλαπλασιάσας τὸν Z ποιείτω· λέγω, ὅτι ἴσος ἐστὶν ὁ E τῷ Z .

Proposition 19[†]

If four number are proportional then the number created from (multiplying) the first and fourth will be equal to the number created from (multiplying) the second and third. And if the number created from (multiplying) the first and fourth is equal to the (number created) from (multiplying) the second and third then the four numbers will be proportional.

Let A , B , C , and D be four proportional numbers, (such that) as A (is) to B , so C (is) to D . And let A make E (by) multiplying D , and let B make F (by) multiplying C . I say that E is equal to F .



Ὅ γὰρ A τὸν Γ πολλαπλασιάσας τὸν H ποιεῖτω. ἐπεὶ οὖν ὁ A τὸν Γ πολλαπλασιάσας τὸν H πεποίηκεν, τὸν δὲ Δ πολλαπλασιάσας τὸν E πεποίηκεν, ἀριθμὸς δὴ ὁ A δύο ἀριθμοὺς τοὺς Γ , Δ πολλαπλασιάσας τοὺς H , E πεποίηκεν. ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ , οὕτως ὁ H πρὸς τὸν E . ἀλλ' ὡς ὁ Γ πρὸς τὸν Δ , οὕτως ὁ A πρὸς τὸν B · καὶ ὡς ἄρα ὁ A πρὸς τὸν B , οὕτως ὁ H πρὸς τὸν E . πάλιν, ἐπεὶ ὁ A τὸν Γ πολλαπλασιάσας τὸν H πεποίηκεν, ἀλλὰ μὴν καὶ ὁ B τὸν Γ πολλαπλασιάσας τὸν Z πεποίηκεν, δύο δὴ ἀριθμοὶ οἱ A , B ἀριθμὸν τινὰ τὸν Γ πολλαπλασιάσαντες τοὺς H , Z πεποίηκασιν. ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , οὕτως ὁ H πρὸς τὸν Z . ἀλλὰ μὴν καὶ ὡς ὁ A πρὸς τὸν B , οὕτως ὁ H πρὸς τὸν E · καὶ ὡς ἄρα ὁ H πρὸς τὸν E , οὕτως ὁ H πρὸς τὸν Z . ὁ H ἄρα πρὸς ἐκάτερον τῶν E , Z τὸν αὐτὸν ἔχει λόγον· ἴσος ἄρα ἐστὶν ὁ E τῷ Z .

Ἔστω δὴ πάλιν ἴσος ὁ E τῷ Z · λέγω, ὅτι ἐστὶν ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Γ πρὸς τὸν Δ .

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἴσος ἐστὶν ὁ E τῷ Z , ἔστιν ἄρα ὡς ὁ H πρὸς τὸν E , οὕτως ὁ H πρὸς τὸν Z . ἀλλ' ὡς μὲν ὁ H πρὸς τὸν E , οὕτως ὁ Γ πρὸς τὸν Δ , ὡς δὲ ὁ H πρὸς τὸν Z , οὕτως ὁ A πρὸς τὸν B . καὶ ὡς ἄρα ὁ A πρὸς τὸν B , οὕτως ὁ Γ πρὸς τὸν Δ · ὅπερ ἔδει δεῖξαι.

For let A make G (by) multiplying C . Therefore, since A has made G (by) multiplying C , and has made E (by) multiplying D , the number A has made G and E by multiplying the two numbers C and D (respectively). Thus, as C is to D , so G (is) to E [Prop. 7.17]. But, as C (is) to D , so A (is) to B . Thus, also, as A (is) to B , so G (is) to E . Again, since A has made G (by) multiplying C , but, in fact, B has also made F (by) multiplying C , the two numbers A and B have made G and F (respectively, by) multiplying some number C . Thus, as A is to B , so G (is) to F [Prop. 7.18]. But, also, as A (is) to B , so G (is) to E . And thus, as G (is) to E , so G (is) to F . Thus, G has the same ratio to each of E and F . Thus, E is equal to F [Prop. 5.9].

So, again, let E be equal to F . I say that as A is to B , so C (is) to D .

For, with the same construction, since E is equal to F , thus as G is to E , so G (is) to F [Prop. 5.7]. But, as G (is) to E , so C (is) to D [Prop. 7.17]. And as G (is) to F , so A (is) to B [Prop. 7.18]. And, thus, as A (is) to B , so C (is) to D . (Which is) the very thing it was required to show.

† In modern notation, this proposition reads that if $a : b :: c : d$ then $ad = bc$, and vice versa, where all symbols denote numbers.

κ'.

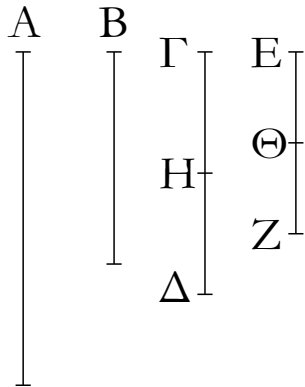
Οἱ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκεις ὁ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα.

Ἔστωσαν γὰρ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A , B οἱ $\Gamma\Delta$, EZ · λέγω, ὅτι ἰσάκεις ὁ $\Gamma\Delta$ τὸν A μετρεῖ καὶ ὁ EZ τὸν B .

Proposition 20

The least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser.

For let CD and EF be the least numbers having the same ratio as A and B (respectively). I say that CD measures A the same number of times as EF (measures) B .



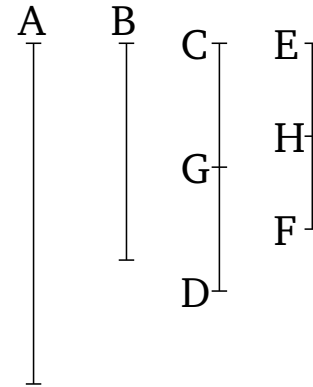
Ὁ ΓΔ γὰρ τοῦ Α οὐκ ἐστὶ μέρος. εἰ γὰρ δυνατόν, ἔστω· καὶ ὁ ΕΖ ἄρα τοῦ Β τὰ αὐτὰ μέρη ἐστίν, ἅπερ ὁ ΓΔ τοῦ Α. ὅσα ἄρα ἐστὶν ἐν τῷ ΓΔ μέρη τοῦ Α, τοσαῦτά ἐστι καὶ ἐν τῷ ΕΖ μέρη τοῦ Β. διηγήσθω ὁ μὲν ΓΔ εἰς τὰ τοῦ Α μέρη τὰ ΓΗ, ΗΔ, ὁ δὲ ΕΖ εἰς τὰ τοῦ Β μέρη τὰ ΕΘ, ΘΖ· ἔσται δὴ ἴσον τὸ πλῆθος τῶν ΓΗ, ΗΔ τῷ πλῆθει τῶν ΕΘ, ΘΖ. καὶ ἐπεὶ ἴσοι εἰσὶν οἱ ΓΗ, ΗΔ ἀριθμοὶ ἀλλήλοις, εἰσὶ δὲ καὶ οἱ ΕΘ, ΘΖ ἀριθμοὶ ἴσοι ἀλλήλοις, καὶ ἐστὶν ἴσον τὸ πλῆθος τῶν ΓΗ, ΗΔ τῷ πλῆθει τῶν ΕΘ, ΘΖ, ἔστιν ἄρα ὡς ὁ ΓΗ πρὸς τὸν ΕΘ, οὕτως ὁ ΗΔ πρὸς τὸν ΘΖ. ἔσται ἄρα καὶ ὡς εἷς τῶν ἡγούμενων πρὸς ἓνα τῶν ἐπομένων, οὕτως ἅπαντες οἱ ἡγούμενοι πρὸς ἅπαντας τοὺς ἐπομένους. ἔστιν ἄρα ὡς ὁ ΓΗ πρὸς τὸν ΕΘ, οὕτως ὁ ΓΔ πρὸς τὸν ΕΖ· οἱ ΓΗ, ΕΘ ἄρα τοῖς ΓΔ, ΕΖ ἐν τῷ αὐτῷ λόγῳ εἰσὶν ἐλάσσονες ὄντες αὐτῶν· ὅπερ ἐστὶν ἀδύνατον· ὑπόκειται γὰρ οἱ ΓΔ, ΕΖ ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς. οὐκ ἄρα μέρη ἐστὶν ὁ ΓΔ τοῦ Α· μέρος ἄρα. καὶ ὁ ΕΖ τοῦ Β τὸ αὐτὸ μέρος ἐστίν, ὅπερ ὁ ΓΔ τοῦ Α· ἰσάκως ἄρα ὁ ΓΔ τὸν Α μετρεῖ καὶ ὁ ΕΖ τὸν Β· ὅπερ ἔδει δεῖξαι.

κα΄.

Οἱ πρῶτοι πρὸς ἀλλήλους ἀριθμοὶ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Ἐστώσαν πρῶτοι πρὸς ἀλλήλους ἀριθμοὶ οἱ Α, Β· λέγω, ὅτι οἱ Α, Β ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Εἰ γὰρ μή, ἔσονται τινες τῶν Α, Β ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β. ἔστωσαν οἱ Γ, Δ.



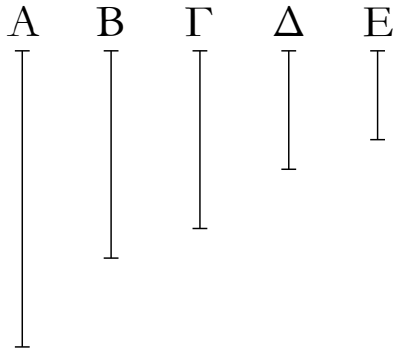
For CD is not parts of A . For, if possible, let it be (parts of A). Thus, EF is also the same parts of B that CD (is) of A [Def. 7.20, Prop. 7.13]. Thus, as many parts of A as are in CD , so many parts of B are also in EF . Let CD have been divided into the parts of A , CG and GD , and EF into the parts of B , EH and HF . So the multitude of (divisions) CG , GD will be equal to the multitude of (divisions) EH , HF . And since the numbers CG and GD are equal to one another, and the numbers EH and HF are also equal to one another, and the multitude of (divisions) CG , GD is equal to the multitude of (divisions) EH , HF , thus as CG is to EH , so GD (is) to HF . Thus, as one of the leading (numbers is) to one of the following, so will (the sum of) all of the leading (numbers) be to (the sum of) all of the following [Prop. 7.12]. Thus, as CG is to EH , so CD (is) to EF . Thus, CG and EH are in the same ratio as CD and EF , being less than them. The very thing is impossible. For CD and EF were assumed (to be) the least of those (numbers) having the same ratio as them. Thus, CD is not parts of A . Thus, (it is) a part (of A) [Prop. 7.4]. And EF is the same part of B that CD (is) of A [Def. 7.20, Prop. 7.13]. Thus, CD measures A the same number of times that EF (measures) B . (Which is) the very thing it was required to show.

Proposition 21

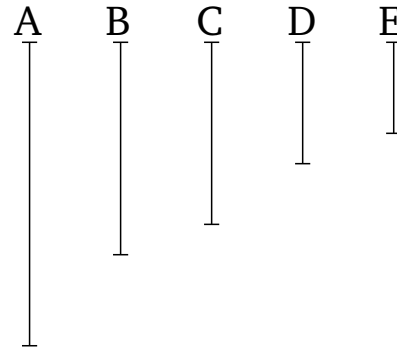
Numbers prime to one another are the least of those (numbers) having the same ratio as them.

Let A and B be numbers prime to one another. I say that A and B are the least of those (numbers) having the same ratio as them.

For if not then there will be some numbers less than A and B which are in the same ratio as A and B . Let them be C and D .



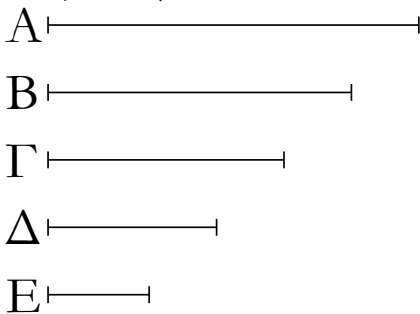
Ἐπεὶ οὖν οἱ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἔχόντων μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκεις ὃ τε μείζων τὸν μείζονα καὶ ὁ ἐλάττων τὸν ἐλάττονα, τουτέστιν ὃ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον, ἰσάκεις ἄρα ὁ Γ τὸν Α μετρεῖ καὶ ὁ Δ τὸν Β. ὁσάκεις δὴ ὁ Γ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε. καὶ ὁ Δ ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας. καὶ ἐπεὶ ὁ Γ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Ε μονάδας, καὶ ὁ Ε ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας. διὰ τὰ αὐτὰ δὴ ὁ Ε καὶ τὸν Β μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας. ὁ Ε ἄρα τοὺς Α, Β μετρεῖ πρῶτους ὄντας πρὸς ἀλλήλους· ὅπερ ἔστιν ἀδύνατον. οὐκ ἄρα ἔσσονται τινες τῶν Α, Β ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς Α, Β. οἱ Α, Β ἄρα ἐλάχιστοι εἰσι τῶν τὸν αὐτὸν λόγον ἔχόντων αὐτοῖς· ὅπερ ἔδει δεῖξαι.



Therefore, since the least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following—*C* thus measures *A* the same number of times that *D* (measures) *B* [Prop. 7.20]. So as many times as *C* measures *A*, so many units let there be in *E*. Thus, *D* also measures *B* according to the units in *E*. And since *C* measures *A* according to the units in *E*, *E* thus also measures *A* according to the units in *C* [Prop. 7.16]. So, for the same (reasons), *E* also measures *B* according to the units in *D* [Prop. 7.16]. Thus, *E* measures *A* and *B*, which are prime to one another. The very thing is impossible. Thus, there cannot be any numbers less than *A* and *B* which are in the same ratio as *A* and *B*. Thus, *A* and *B* are the least of those (numbers) having the same ratio as them. (Which is) the very thing it was required to show.

κβ΄.

Οἱ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἔχόντων αὐτοῖς πρῶτοι πρὸς ἀλλήλους εἰσιν.

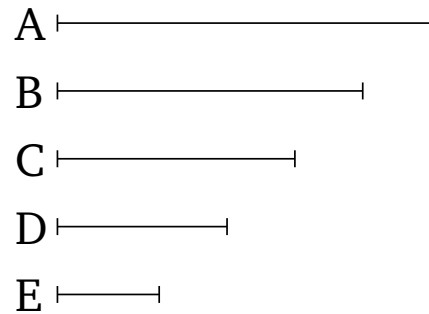


Ἐστωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἔχόντων αὐτοῖς οἱ Α, Β· λέγω, ὅτι οἱ Α, Β πρῶτοι πρὸς ἀλλήλους εἰσιν.

Εἰ γὰρ μὴ εἰσι πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμός. μετρείτω, καὶ ἔστω ὁ Γ. καὶ ὁσάκεις μὲν ὁ Γ τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Δ,

Proposition 22

The least numbers of those (numbers) having the same ratio as them are prime to one another.



Let *A* and *B* be the least numbers of those (numbers) having the same ratio as them. I say that *A* and *B* are prime to one another.

For if they are not prime to one another then some number will measure them. Let it (so measure them), and let it be *C*. And as many times as *C* measures *A*, so

ὁσάκις δὲ ὁ Γ τὸν Β μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ε.

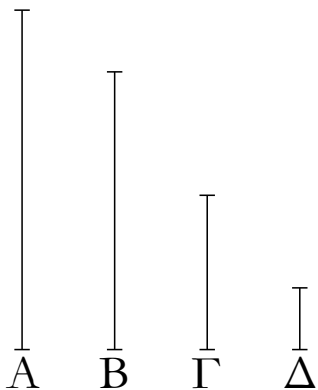
Ἐπεὶ ὁ Γ τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας, ὁ Γ ἄρα τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Ε πολλαπλασιάσας τὸν Β πεποίηκεν. ἀριθμὸς δὴ ὁ Γ δύο ἀριθμοὺς τοὺς Δ, Ε πολλαπλασιάσας τοὺς Α, Β πεποίηκεν· ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Α πρὸς τὸν Β· οἱ Δ, Ε ἄρα τοῖς Α, Β ἐν τῷ αὐτῷ λόγῳ εἰσὶν ἐλάσσονες ὄντες αὐτῶν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Α, Β ἀριθμοὺς ἀριθμὸς τις μετρήσει. οἱ Α, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσὶν· ὅπερ ἔδει δεῖξαι.

many units let there be in D . And as many times as C measures B , so many units let there be in E .

Since C measures A according to the units in D , C has thus made A (by) multiplying D [Def. 7.15]. So, for the same (reasons), C has also made B (by) multiplying E . So the number C has made A and B (by) multiplying the two numbers D and E (respectively). Thus, as D is to E , so A (is) to B [Prop. 7.17]. Thus, D and E are in the same ratio as A and B , being less than them. The very thing is impossible. Thus, some number does not measure the numbers A and B . Thus, A and B are prime to one another. (Which is) the very thing it was required to show.

κγ'.

Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, ὁ τὸν ἓνα αὐτῶν μετρῶν ἀριθμὸς πρὸς τὸν λοιπὸν πρῶτος ἔσται.

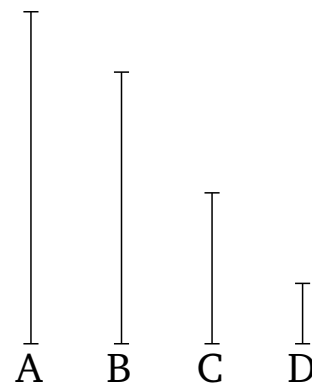


Ἐστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ Α, Β, τὸν δὲ Α μετρεῖτω τις ἀριθμὸς ὁ Γ· λέγω, ὅτι καὶ οἱ Γ, Β πρῶτοι πρὸς ἀλλήλους εἰσὶν.

Εἰ γὰρ μὴ εἰσὶν οἱ Γ, Β πρῶτοι πρὸς ἀλλήλους, μετρήσει [τις] τοὺς Γ, Β ἀριθμὸς. μετρεῖτω, καὶ ἔστω ὁ Δ. ἐπεὶ ὁ Δ τὸν Γ μετρεῖ, ὁ δὲ Γ τὸν Α μετρεῖ, καὶ ὁ Δ ἄρα τὸν Α μετρεῖ. μετρεῖ δὲ καὶ τὸν Β· ὁ Δ ἄρα τοὺς Α, Β μετρεῖ πρῶτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Γ, Β ἀριθμοὺς ἀριθμὸς τις μετρήσει. οἱ Γ, Β ἄρα πρῶτοι πρὸς ἀλλήλους εἰσὶν· ὅπερ ἔδει δεῖξαι.

Proposition 23

If two numbers are prime to one another then a number measuring one of them will be prime to the remaining (one).



Let A and B be two numbers (which are) prime to one another, and let some number C measure A . I say that C and B are also prime to one another.

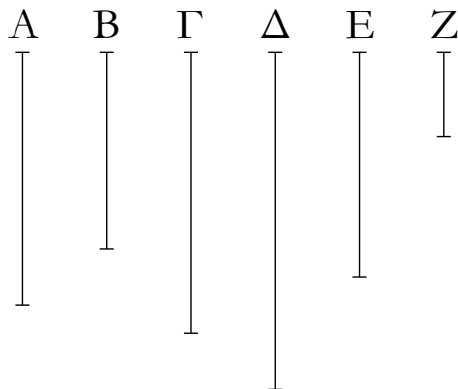
For if C and B are not prime to one another then [some] number will measure C and B . Let it (so) measure (them), and let it be D . Since D measures C , and C measures A , D thus also measures A . And (D) also measures B . Thus, D measures A and B , which are prime to one another. The very thing is impossible. Thus, some number does not measure the numbers C and B . Thus, C and B are prime to one another. (Which is) the very thing it was required to show.

κδ'.

Ἐὰν δύο ἀριθμοὶ πρὸς τινὰ ἀριθμὸν πρῶτοι ὦσιν, καὶ ὁ ἐξ αὐτῶν γενόμενος πρὸς τὸν αὐτὸν πρῶτος ἔσται.

Proposition 24

If two numbers are prime to some number then the number created from (multiplying) the former (two numbers) will also be prime to the latter (number).



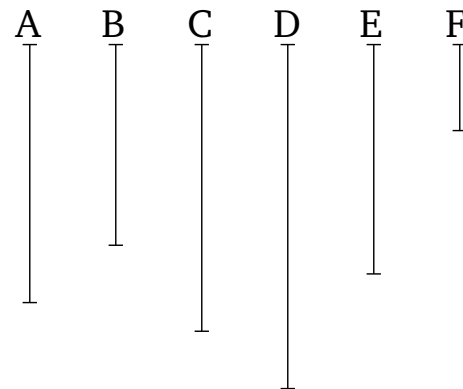
Δύο γὰρ ἀριθμοὶ οἱ A, B πρὸς τινὰ ἀριθμὸν τὸν Γ πρῶτοι ἔστωσαν, καὶ ὁ A τὸν B πολλαπλασιάσας τὸν Δ ποιείτω· λέγω, ὅτι οἱ Γ, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μὴ εἰσιν οἱ Γ, Δ πρῶτοι πρὸς ἀλλήλους, μετρήσει [τις] τοὺς Γ, Δ ἀριθμὸς· μετρεῖτω, καὶ ἔστω ὁ E . καὶ ἐπεὶ οἱ Γ, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν, τὸν δὲ Γ μετρεῖ τις ἀριθμὸς ὁ E , οἱ A, E ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ὁσάκις δὴ ὁ E τὸν Δ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Z · καὶ ὁ Z ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ E μονάδας. ὁ E ἄρα τὸν Z πολλαπλασιάσας τὸν Δ πεποίηκεν· ἀλλὰ μὴν καὶ ὁ A τὸν B πολλαπλασιάσας τὸν Δ πεποίηκεν· ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν E, Z τῷ ἐκ τῶν A, B . ἐὰν δὲ ὁ ὑπὸ τῶν ἄκρων ἴσος ᾖ τῷ ὑπὸ τῶν μέσων, οἱ τέσσαρες ἀριθμοὶ ἀνάλογόν εἰσιν· ἔστιν ἄρα ὡς ὁ E πρὸς τὸν A , οὕτως ὁ B πρὸς τὸν Z . οἱ δὲ A, E πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἔχόντων αὐτοῖς μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὃ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὃ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον· ὁ E ἄρα τὸν B μετρεῖ. μετρεῖ δὲ καὶ τὸν Γ · ὁ E ἄρα τοὺς B, Γ μετρεῖ πρῶτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς Γ, Δ ἀριθμοὺς ἀριθμὸς τις μετρήσει. οἱ Γ, Δ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

κε΄.

Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ᾦσιν, ὁ ἐκ τοῦ ἐνὸς αὐτῶν γενόμενος πρὸς τὸν λοιπὸν πρῶτος ἔσται.

Ἐστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ A, B , καὶ ὁ A ἑαυτὸν πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι



For let A and B be two numbers (which are both) prime to some number C . And let A make D (by) multiplying B . I say that C and D are prime to one another.

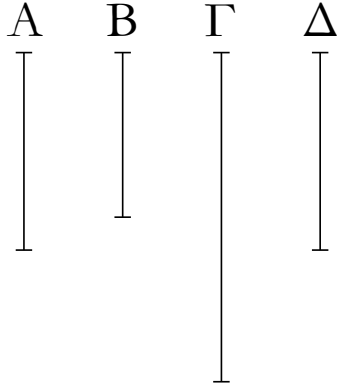
For if C and D are not prime to one another then [some] number will measure C and D . Let it (so) measure them, and let it be E . And since C and A are prime to one another, and some number E measures C , A and E are thus prime to one another [Prop. 7.23]. So as many times as E measures D , so many units let there be in F . Thus, F also measures D according to the units in E [Prop. 7.16]. Thus, E has made D (by) multiplying F [Def. 7.15]. But, in fact, A has also made D (by) multiplying B . Thus, the (number created) from (multiplying) E and F is equal to the (number created) from (multiplying) A and B . And if the (rectangle contained) by the (two) outermost is equal to the (rectangle contained) by the middle (two) then the four numbers are proportional [Prop. 6.15]. Thus, as E is to A , so B (is) to F . And A and E (are) prime (to one another). And (numbers) prime (to one another) are also the least (of those numbers having the same ratio) [Prop. 7.21]. And the least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, E measures B . And it also measures C . Thus, E measures B and C , which are prime to one another. The very thing is impossible. Thus, some number cannot measure the numbers C and D . Thus, C and D are prime to one another. (Which is) the very thing it was required to show.

Proposition 25

If two numbers are prime to one another then the number created from (squaring) one of them will be prime to the remaining (number).

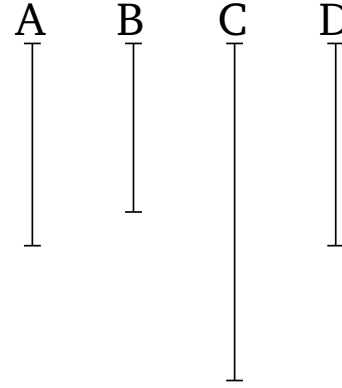
Let A and B be two numbers (which are) prime to

οἱ B, Γ πρῶτοι πρὸς ἀλλήλους εἰσίν.



Κείσθω γὰρ τῷ A ἴσος ὁ Δ. ἐπεὶ οἱ A, B πρῶτοι πρὸς ἀλλήλους εἰσίν, ἴσος δὲ ὁ A τῷ Δ, καὶ οἱ Δ, B ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ἐκάτερος ἄρα τῶν Δ, A πρὸς τὸν B πρῶτός ἐστιν· καὶ ὁ ἐκ τῶν Δ, A ἄρα γενόμενος πρὸς τὸν B πρῶτος ἔσται. ὁ δὲ ἐκ τῶν Δ, A γενόμενος ἀριθμὸς ἐστὶν ὁ Γ. οἱ Γ, B ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

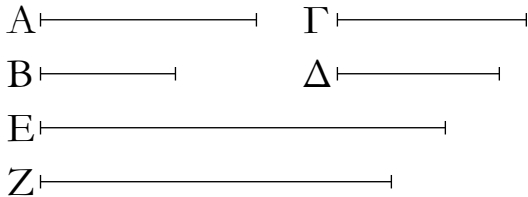
one another. And let A make C (by) multiplying itself. I say that B and C are prime to one another.



For let D be made equal to A . Since A and B are prime to one another, and A (is) equal to D , D and B are thus also prime to one another. Thus, D and A are each prime to B . Thus, the (number) created from (multilying) D and A will also be prime to B [Prop. 7.24]. And C is the number created from (multiplying) D and A . Thus, C and B are prime to one another. (Which is) the very thing it was required to show.

κζ΄.

Ἐάν δύο ἀριθμοὶ πρὸς δύο ἀριθμοὺς ἀμφοτέρωι πρὸς ἑκάτερον πρῶτοι ᾧσιν, καὶ οἱ ἐξ αὐτῶν γενόμενοι πρῶτοι πρὸς ἀλλήλους ἔσονται.

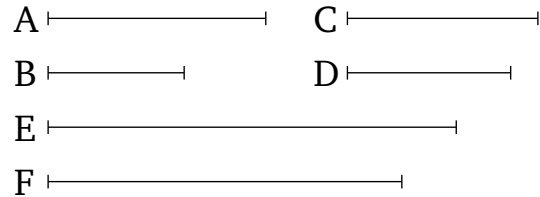


Δύο γὰρ ἀριθμοὶ οἱ A, B πρὸς δύο ἀριθμοὺς τοὺς Γ, Δ ἀμφοτέρωι πρὸς ἑκάτερον πρῶτοι ἔστωσαν, καὶ ὁ μὲν A τὸν B πολλαπλασιάσας τὸν E ποιεῖτω, ὁ δὲ Γ τὸν Δ πολλαπλασιάσας τὸν Z ποιεῖτω· λέγω, ὅτι οἱ E, Z πρῶτοι πρὸς ἀλλήλους εἰσίν.

Ἐπεὶ γὰρ ἐκάτερος τῶν A, B πρὸς τὸν Γ πρῶτός ἐστιν, καὶ ὁ ἐκ τῶν A, B ἄρα γενόμενος πρὸς τὸν Γ πρῶτος ἔσται. ὁ δὲ ἐκ τῶν A, B γενόμενός ἐστιν ὁ E· οἱ E, Γ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. διὰ τὰ αὐτὰ δὴ καὶ οἱ E, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐκάτερος ἄρα τῶν Γ, Δ πρὸς τὸν E πρῶτός ἐστιν. καὶ ὁ ἐκ τῶν Γ, Δ ἄρα γενόμενος πρὸς τὸν E πρῶτος ἔσται. ὁ δὲ ἐκ τῶν Γ, Δ γενόμενός ἐστιν ὁ Z. οἱ E, Z ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

Proposition 26

If two numbers are both prime to each of two numbers then the (numbers) created from (multiplying) them will also be prime to one another.

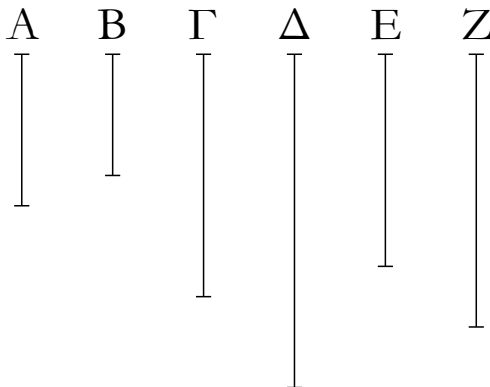


For let two numbers, A and B , both be prime to each of two numbers, C and D . And let A make E (by) multiplying B , and let C make F (by) multiplying D . I say that E and F are prime to one another.

For since A and B are each prime to C , the (number) created from (multiplying) A and B will thus also be prime to C [Prop. 7.24]. And E is the (number) created from (multiplying) A and B . Thus, E and C are prime to one another. So, for the same (reasons), E and D are also prime to one another. Thus, C and D are each prime to E . Thus, the (number) created from (multiplying) C and D will also be prime to E [Prop. 7.24]. And F is the (number) created from (multiplying) C and D . Thus, E and F are prime to one another. (Which is) the very thing it was required to show.

κζ'.

Ἐάν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ πολλαπλασιάσας ἑκάτερος ἑαυτὸν ποιῇ τινα, οἱ γενόμενοι ἐξ αὐτῶν πρῶτοι πρὸς ἀλλήλους ἔσσονται, καὶν οἱ ἐξ ἀρχῆς τοὺς γενομένους πολλαπλασιάσαντες ποιῶσι τινας, καὶ αὗτοι πρῶτοι πρὸς ἀλλήλους ἔσσονται [καὶ αἰεὶ περὶ τοὺς ἄκρους τοῦτο συμβαίνει].

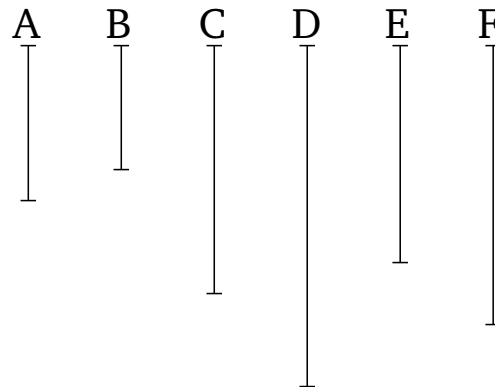


Ἐστωσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ A, B, καὶ ὁ A ἑαυτὸν μὲν πολλαπλασιάσας τὸν Γ ποιεῖτω, τὸν δὲ Γ πολλαπλασιάσας τὸν Δ ποιεῖτω, ὁ δὲ B ἑαυτὸν μὲν πολλαπλασιάσας τὸν E ποιεῖτω, τὸν δὲ E πολλαπλασιάσας τὸν Z ποιεῖτω· λέγω, ὅτι οἱ τε Γ, E καὶ οἱ Δ, Z πρῶτοι πρὸς ἀλλήλους εἰσίν.

Ἐπεὶ γὰρ οἱ A, B πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὁ A ἑαυτὸν πολλαπλασιάσας τὸν Γ πεποίηκεν, οἱ Γ, B ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐπεὶ οὖν οἱ Γ, B πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὁ B ἑαυτὸν πολλαπλασιάσας τὸν E πεποίηκεν, οἱ Γ, E ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. πάλιν, ἐπεὶ οἱ A, B πρῶτοι πρὸς ἀλλήλους εἰσίν, καὶ ὁ B ἑαυτὸν πολλαπλασιάσας τὸν E πεποίηκεν, οἱ A, E ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐπεὶ οὖν δύο ἀριθμοὶ οἱ A, Γ πρὸς δύο ἀριθμοὺς τοὺς B, E ἀμφοτέροι πρὸς ἑκάτερον πρῶτοι εἰσίν, καὶ ὁ ἐκ τῶν A, Γ ἄρα γενόμενος πρὸς τὸν ἐκ τῶν B, E πρῶτός ἐστιν. καὶ ἐστὶν ὁ μὲν ἐκ τῶν A, Γ ὁ Δ, ὁ δὲ ἐκ τῶν B, E ὁ Z. οἱ Δ, Z ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

Proposition 27[†]

If two numbers are prime to one another and each makes some (number by) multiplying itself then the numbers created from them will be prime to one another, and if the original (numbers) make some (more numbers by) multiplying the created (numbers) then these will also be prime to one another [and this always happens with the extremes].



Let A and B be two numbers prime to one another, and let A make C (by) multiplying itself, and let it make D (by) multiplying C . And let B make E (by) multiplying itself, and let it make F by multiplying E . I say that C and E , and D and F , are prime to one another.

For since A and B are prime to one another, and A has made C (by) multiplying itself, C and B are thus prime to one another [Prop. 7.25]. Therefore, since C and B are prime to one another, and B has made E (by) multiplying itself, C and E are thus prime to one another [Prop. 7.25]. Again, since A and B are prime to one another, and B has made E (by) multiplying itself, A and E are thus prime to one another [Prop. 7.25]. Therefore, since the two numbers A and C are both prime to each of the two numbers B and E , the (number) created from (multiplying) A and C is thus prime to the (number created) from (multiplying) B and E [Prop. 7.26]. And D is the (number created) from (multiplying) A and C , and F the (number created) from (multiplying) B and E . Thus, D and F are prime to one another. (Which is) the very thing it was required to show.

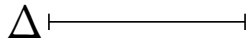
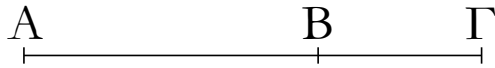
[†] In modern notation, this proposition states that if a is prime to b , then a^2 is also prime to b^2 , as well as a^3 to b^3 , etc., where all symbols denote numbers.

κη'.

Ἐάν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ συναμφοτέρος πρὸς ἑκάτερον αὐτῶν πρῶτος ἔσται· καὶ ἐὰν συναμφοτέρος πρὸς ἓνα τινὰ αὐτῶν πρῶτος ᾖ, καὶ οἱ ἐξ ἀρχῆς ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ἔσσονται.

Proposition 28

If two numbers are prime to one another then their sum will also be prime to each of them. And if the sum (of two numbers) is prime to any one of them then the original numbers will also be prime to one another.

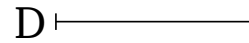
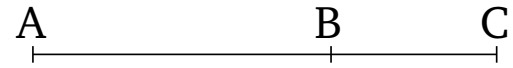


Συγκείσθωσαν γὰρ δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ AB , $BΓ$. λέγω, ὅτι καὶ συναμφότερος ὁ $ΑΓ$ πρὸς ἑκάτερον τῶν AB , $BΓ$ πρῶτός ἐστιν.

Εἰ γὰρ μὴ εἰσιν οἱ $ΓΑ$, AB πρῶτοι πρὸς ἀλλήλους, μετρήσει τις τοὺς $ΓΑ$, AB ἀριθμὸς. μετρεῖτω, καὶ ἔστω ὁ $Δ$. ἐπεὶ οὖν ὁ $Δ$ τοὺς $ΓΑ$, AB μετρεῖ, καὶ λοιπὸν ἄρα τὸν $BΓ$ μετρήσει. μετρεῖ δὲ καὶ τὸν BA . ὁ $Δ$ ἄρα τοὺς AB , $BΓ$ μετρεῖ πρῶτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς $ΓΑ$, AB ἀριθμοὺς ἀριθμὸς τις μετρήσει· οἱ $ΓΑ$, AB ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. διὰ τὰ αὐτὰ δὴ καὶ οἱ $ΑΓ$, $ΓB$ πρῶτοι πρὸς ἀλλήλους εἰσίν. ὁ $ΓΑ$ ἄρα πρὸς ἑκάτερον τῶν AB , $BΓ$ πρῶτός ἐστιν.

Ἐστωσαν δὴ πάλιν οἱ $ΓΑ$, AB πρῶτοι πρὸς ἀλλήλους· λέγω, ὅτι καὶ οἱ AB , $BΓ$ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μὴ εἰσιν οἱ AB , $BΓ$ πρῶτοι πρὸς ἀλλήλους, μετρήσει τις τοὺς AB , $BΓ$ ἀριθμὸς. μετρεῖτω, καὶ ἔστω ὁ $Δ$. καὶ ἐπεὶ ὁ $Δ$ ἑκάτερον τῶν AB , $BΓ$ μετρεῖ, καὶ ὅλον ἄρα τὸν $ΓΑ$ μετρήσει. μετρεῖ δὲ καὶ τὸν AB . ὁ $Δ$ ἄρα τοὺς $ΓΑ$, AB μετρεῖ πρῶτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς AB , $BΓ$ ἀριθμοὺς ἀριθμὸς τις μετρήσει· οἱ AB , $BΓ$ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.



For let the two numbers, AB and BC , (which are) prime to one another, be laid down together. I say that their sum AC is also prime to each of AB and BC .

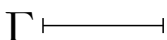
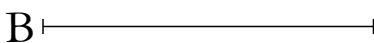
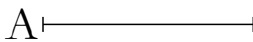
For if CA and AB are not prime to one another then some number will measure CA and AB . Let it (so) measure (them), and let it be D . Therefore, since D measures CA and AB , it will thus also measure the remainder BC . And it also measures BA . Thus, D measures AB and BC , which are prime to one another. The very thing is impossible. Thus, some number cannot measure (both) the numbers CA and AB . Thus, CA and AB are prime to one another. So, for the same (reasons), AC and CB are also prime to one another. Thus, CA is prime to each of AB and BC .

So, again, let CA and AB be prime to one another. I say that AB and BC are also prime to one another.

For if AB and BC are not prime to one another then some number will measure AB and BC . Let it (so) measure (them), and let it be D . And since D measures each of AB and BC , it will thus also measure the whole of CA . And it also measures AB . Thus, D measures CA and AB , which are prime to one another. The very thing is impossible. Thus, some number cannot measure (both) the numbers AB and BC . Thus, AB and BC are prime to one another. (Which is) the very thing it was required to show.

κθ΄.

Ἄπας πρῶτος ἀριθμὸς πρὸς ἅπαντα ἀριθμόν, ὃν μὴ μετρεῖ, πρῶτός ἐστιν.

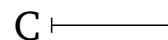
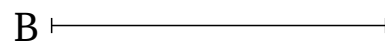


Ἐστω πρῶτος ἀριθμὸς ὁ A καὶ τὸν B μὴ μετρεῖτω· λέγω, ὅτι οἱ B , A πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰ γὰρ μὴ εἰσιν οἱ B , A πρῶτοι πρὸς ἀλλήλους, μετρήσει τις αὐτοὺς ἀριθμὸς. μετρεῖτω ὁ $Γ$. ἐπεὶ ὁ $Γ$ τὸν B μετρεῖ, ὁ δὲ A τὸν B οὐ μετρεῖ, ὁ $Γ$ ἄρα τῷ A οὐκ ἐστὶν ὁ αὐτός. καὶ ἐπεὶ ὁ $Γ$ τοὺς B , A μετρεῖ, καὶ τὸν A ἄρα μετρεῖ πρῶτον ὄντα μὴ ὦν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοὺς B , A μετρήσει τις ἀριθμὸς· οἱ A , B ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

Proposition 29

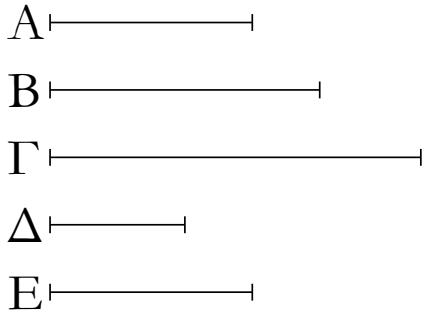
Every prime number is prime to every number which it does not measure.



Let A be a prime number, and let it not measure B . I say that B and A are prime to one another. For if B and A are not prime to one another then some number will measure them. Let C measure (them). Since C measures B , and A does not measure B , C is thus not the same as A . And since C measures B and A , it thus also measures A , which is prime, (despite) not being the same as it. The very thing is impossible. Thus, some number cannot measure (both) B and A . Thus, A and B are prime to one another. (Which is) the very thing it was required to

λ'.

Ἐάν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, τὸν δὲ γεγόμενον ἐξ αὐτῶν μετρή τις πρῶτος ἀριθμός, καὶ ἓνα τῶν ἐξ ἀρχῆς μετρήσει.



Δύο γὰρ ἀριθμοὶ οἱ A, B πολλαπλασιάσαντες ἀλλήλους τὸν Γ ποιεῖτωσαν, τὸν δὲ Γ μετρεῖτω τις πρῶτος ἀριθμός ὁ Δ· λέγω, ὅτι ὁ Δ ἓνα τῶν A, B μετρεῖ.

Τὸν γὰρ A μὴ μετρεῖτω· καὶ ἐστὶ πρῶτος ὁ Δ· οἱ A, Δ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ὅσάκις ὁ Δ τὸν Γ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ E. ἐπεὶ οὖν ὁ Δ τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ E μονάδας, ὁ Δ ἄρα τὸν E πολλαπλασιάσας τὸν Γ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ A τὸν B πολλαπλασιάσας τὸν Γ πεποίηκεν· ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Δ, E τῷ ἐκ τῶν A, B. ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν A, οὕτως ὁ B πρὸς τὸν E. οἱ δὲ Δ, A πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὃ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὃ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον· ὁ Δ ἄρα τὸν B μετρεῖ. ὁμοίως δὴ δεῖξομεν, ὅτι καὶ ἐάν τὸν B μὴ μετρή, τὸν A μετρήσει. ὁ Δ ἄρα ἓνα τῶν A, B μετρεῖ· ὅπερ εἶδει δεῖξαι.

λα'.

Ἄπας σύνθετος ἀριθμός ὑπὸ πρῶτου τινὸς ἀριθμοῦ μετρεῖται.

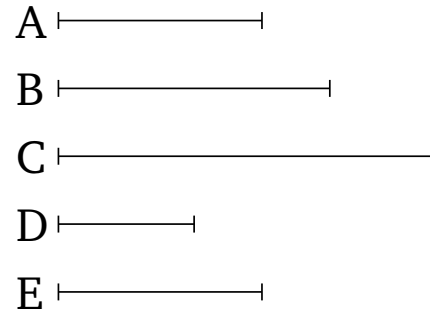
Ἐστω σύνθετος ἀριθμός ὁ A· λέγω, ὅτι ὁ A ὑπὸ πρῶτου τινὸς ἀριθμοῦ μετρεῖται.

Ἐπεὶ γὰρ σύνθετός ἐστιν ὁ A, μετρήσει τις αὐτὸν

show.

Proposition 30

If two numbers make some (number by) multiplying one another, and some prime number measures the number (so) created from them, then it will also measure one of the original (numbers).



For let two numbers A and B make C (by) multiplying one another, and let some prime number D measure C. I say that D measures one of A and B.

For let it not measure A. And since D is prime, A and D are thus prime to one another [Prop. 7.29]. And as many times as D measures C, so many units let there be in E. Therefore, since D measures C according to the units E, D has thus made C (by) multiplying E [Def. 7.15]. But, in fact, A has also made C (by) multiplying B. Thus, the (number created) from (multiplying) D and E is equal to the (number created) from (multiplying) A and B. Thus, as D is to A, so B (is) to E [Prop. 7.19]. And D and A (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, D measures B. So, similarly, we can also show that if (D) does not measure B then it will measure A. Thus, D measures one of A and B. (Which is) the very thing it was required to show.

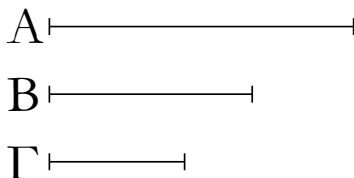
Proposition 31

Every composite number is measured by some prime number.

Let A be a composite number. I say that A is measured by some prime number.

For since A is composite, some number will measure it. Let it (so) measure (A), and let it be B. And if B

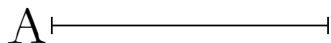
ἀριθμός. μετρεῖται, καὶ ἔστω ὁ Β. καὶ εἰ μὲν πρῶτός ἐστιν ὁ Β, γεγονός ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ σύνθετος, μετρήσει τις αὐτὸν ἀριθμός. μετρεῖται, καὶ ἔστω ὁ Γ. καὶ ἐπεὶ ὁ Γ τὸν Β μετρεῖ, ὁ δὲ Β τὸν Α μετρεῖ, καὶ ὁ Γ ἄρα τὸν Α μετρεῖ. καὶ εἰ μὲν πρῶτός ἐστιν ὁ Γ, γεγονός ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ σύνθετος, μετρήσει τις αὐτὸν ἀριθμός. τοιαύτης δὴ γινομένης ἐπισκέψεως ληφθήσεται τις πρῶτος ἀριθμός, ὃς μετρήσει. εἰ γὰρ οὐ ληφθήσεται, μετρήσουσι τὸν Α ἀριθμὸν ἄπειροι ἀριθμοί, ὧν ἕτερος ἐτέρου ἐλάσσων ἐστίν· ὅπερ ἐστὶν ἀδύνατον ἐν ἀριθμοῖς. ληφθήσεται τις ἄρα πρῶτος ἀριθμός, ὃς μετρήσει τὸν πρὸ ἑαυτοῦ, ὃς καὶ τὸν Α μετρήσει.



Ἄπας ἄρα σύνθετος ἀριθμός ὑπὸ πρῶτου τινὸς ἀριθμοῦ μετρεῖται· ὅπερ ἔδει δεῖξαι.

λβ΄.

Ἄπας ἀριθμός ἤτοι πρῶτός ἐστιν ἢ ὑπὸ πρῶτου τινὸς ἀριθμοῦ μετρεῖται.



Ἐστω ἀριθμός ὁ Α· λέγω, ὅτι ὁ Α ἤτοι πρῶτός ἐστιν ἢ ὑπὸ πρῶτου τινὸς ἀριθμοῦ μετρεῖται.

Εἰ μὲν οὖν πρῶτός ἐστιν ὁ Α, γεγονός ἂν εἴη τὸ ἐπιταχθέν. εἰ δὲ σύνθετος, μετρήσει τις αὐτὸν πρῶτος ἀριθμός.

Ἄπας ἄρα ἀριθμός ἤτοι πρῶτός ἐστιν ἢ ὑπὸ πρῶτου τινὸς ἀριθμοῦ μετρεῖται· ὅπερ ἔδει δεῖξαι.

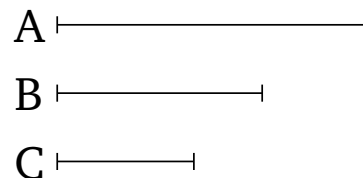
λγ΄.

Ἀριθμῶν δοθέντων ὅποσωνοῦν εὑρεῖν τοὺς ἐλάχιστους τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Ἐστωσαν οἱ δοθέντες ὅποσοιοῦν ἀριθμοὶ οἱ Α, Β, Γ· δεῖ δὴ εὑρεῖν τοὺς ἐλάχιστους τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Β, Γ.

Οἱ Α, Β, Γ γὰρ ἤτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὐ. εἰ μὲν οὖν οἱ Α, Β, Γ πρῶτοι πρὸς ἀλλήλους εἰσὶν, ἐλάχιστοι εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

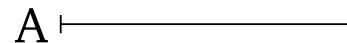
is prime then that which was prescribed has happened. And if (B is) composite then some number will measure it. Let it (so) measure (B), and let it be C . And since C measures B , and B measures A , C thus also measures A . And if C is prime then that which was prescribed has happened. And if (C is) composite then some number will measure it. So, in this manner of continued investigation, some prime number will be found which will measure (the number preceding it, which will also measure A). And if (such a number) cannot be found then an infinite (series of) numbers, each of which is less than the preceding, will measure the number A . The very thing is impossible for numbers. Thus, some prime number will (eventually) be found which will measure the (number) preceding it, which will also measure A .



Thus, every composite number is measured by some prime number. (Which is) the very thing it was required to show.

Proposition 32

Every number is either prime or is measured by some prime number.



Let A be a number. I say that A is either prime or is measured by some prime number.

In fact, if A is prime then that which was prescribed has happened. And if (it is) composite then some prime number will measure it [Prop. 7.31].

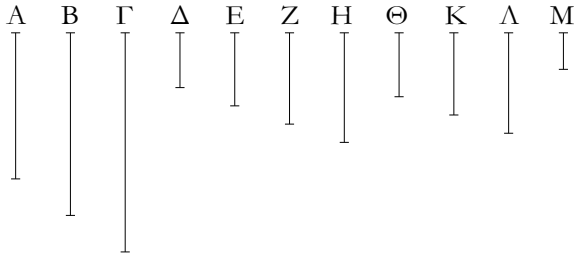
Thus, every number is either prime or is measured by some prime number. (Which is) the very thing it was required to show.

Proposition 33

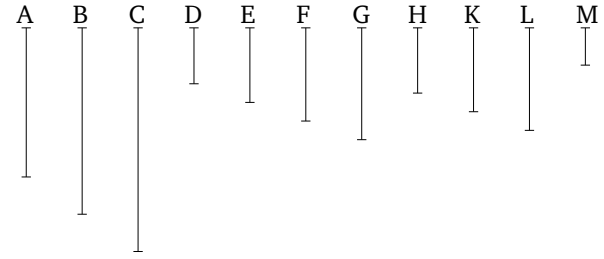
To find the least of those (numbers) having the same ratio as any given multitude of numbers.

Let A , B , and C be any given multitude of numbers. So it is required to find the least of those (numbers) having the same ratio as A , B , and C .

For A , B , and C are either prime to one another, or not. In fact, if A , B , and C are prime to one another then they are the least of those (numbers) having the same ratio as them [Prop. 7.22].



Εἰ δὲ οὐ, εἰλήφθω τῶν A, B, Γ τὸ μέγιστον κοινὸν μέτρον ὁ Δ , καὶ ὅσάκις ὁ Δ ἕκαστον τῶν A, B, Γ μετρεῖ, τοσαῦται μονάδες ἕστωσαν ἐν ἑκάστῳ τῶν E, Z, H . καὶ ἕκαστος ἄρα τῶν E, Z, H ἕκαστον τῶν A, B, Γ μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας. οἱ E, Z, H ἄρα τοὺς A, B, Γ ἰσάκις μετροῦσιν· οἱ E, Z, H ἄρα τοῖς A, B, Γ ἐν τῷ αὐτῷ λόγῳ εἰσίν. λέγω δὴ, ὅτι καὶ ἐλάχιστοι. εἰ γὰρ μὴ εἰσιν οἱ E, Z, H ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς A, B, Γ , ἔσονται [τινες] τῶν E, Z, H ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς A, B, Γ . ἕστωσαν οἱ Θ, K, Λ ἰσάκις ἄρα ὁ Θ τὸν A μετρεῖ καὶ ἑκάτερος τῶν K, Λ ἑκάτερον τῶν B, Γ . ὅσάκις δὲ ὁ Θ τὸν A μετρεῖ, τοσαῦται μονάδες ἕστωσαν ἐν τῷ M · καὶ ἑκάτερος ἄρα τῶν K, Λ ἑκάτερον τῶν B, Γ μετρεῖ κατὰ τὰς ἐν τῷ M μονάδας. καὶ ἐπεὶ ὁ Θ τὸν A μετρεῖ κατὰ τὰς ἐν τῷ M μονάδας, καὶ ὁ M ἄρα τὸν A μετρεῖ κατὰ τὰς ἐν τῷ Θ μονάδας. διὰ τὰ αὐτὰ δὴ ὁ M καὶ ἑκάτερον τῶν B, Γ μετρεῖ κατὰ τὰς ἐν ἑκατέρῳ τῶν K, Λ μονάδας· ὁ M ἄρα τοὺς A, B, Γ μετρεῖ. καὶ ἐπεὶ ὁ Θ τὸν A μετρεῖ κατὰ τὰς ἐν τῷ M μονάδας, ὁ Θ ἄρα τὸν M πολλαπλασιάσας τὸν A πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ E τὸν Δ πολλαπλασιάσας τὸν A πεποίηκεν. ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν E, Δ τῷ ἐκ τῶν Θ, M . ἔστιν ἄρα ὡς ὁ E πρὸς τὸν Θ , οὕτως ὁ M πρὸς τὸν Δ . μείζων δὲ ὁ E τοῦ Θ · μείζων ἄρα καὶ ὁ M τοῦ Δ . καὶ μετρεῖ τοὺς A, B, Γ · ὅπερ ἐστὶν ἀδύνατον· ὑπόκειται γὰρ ὁ Δ τῶν A, B, Γ τὸ μέγιστον κοινὸν μέτρον. οὐκ ἄρα ἔσονται τινες τῶν E, Z, H ἐλάσσονες ἀριθμοὶ ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς A, B, Γ . οἱ E, Z, H ἄρα ἐλάχιστοι εἰσι τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς A, B, Γ · ὅπερ εἶδει δεῖξαι.



And if not, let the greatest common measure, D , of A, B , and C have been taken [Prop. 7.3]. And as many times as D measures A, B, C , so many units let there be in E, F, G , respectively. And thus E, F, G measure A, B, C , respectively, according to the units in D [Prop. 7.15]. Thus, E, F, G measure A, B, C (respectively) an equal number of times. Thus, E, F, G are in the same ratio as A, B, C (respectively) [Def. 7.20]. So I say that (they are) also the least (of those numbers having the same ratio as A, B, C). For if E, F, G are not the least of those (numbers) having the same ratio as A, B, C (respectively), then there will be [some] numbers less than E, F, G which are in the same ratio as A, B, C (respectively). Let them be H, K, L . Thus, H measures A the same number of times that K, L also measure B, C , respectively. And as many times as H measures A , so many units let there be in M . Thus, K, L measure B, C , respectively, according to the units in M . And since H measures A according to the units in M , M thus also measures A according to the units in H [Prop. 7.15]. So, for the same (reasons), M also measures B, C according to the units in K, L , respectively. Thus, M measures A, B , and C . And since H measures A according to the units in M , H has thus made A (by) multiplying M . So, for the same (reasons), E has also made A (by) multiplying D . Thus, the (number created) from (multiplying) E and D is equal to the (number created) from (multiplying) H and M . Thus, as E (is) to H , so M (is) to D [Prop. 7.19]. And E (is) greater than H . Thus, M (is) also greater than D [Prop. 5.13]. And (M) measures A, B , and C . The very thing is impossible. For D was assumed (to be) the greatest common measure of A, B , and C . Thus, there cannot be any numbers less than E, F, G which are in the same ratio as A, B, C (respectively). Thus, E, F, G are the least of (those numbers) having the same ratio as A, B, C (respectively). (Which is) the very thing it was required to show.

λδ'.

Proposition 34

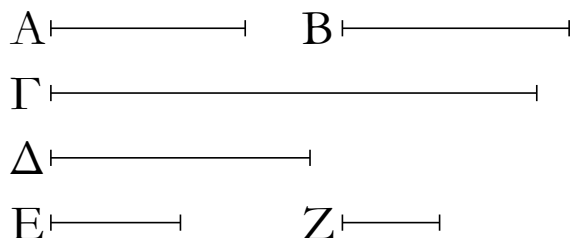
Δύο ἀριθμῶν δοθέντων εὑρεῖν, ὃν ἐλάχιστον μετροῦσιν ἀριθμὸν.

To find the least number which two given numbers (both) measure.

Ἔστωσαν οἱ δοθέντες δύο ἀριθμοὶ οἱ A, B · δεῖ δὴ εὑρεῖν,

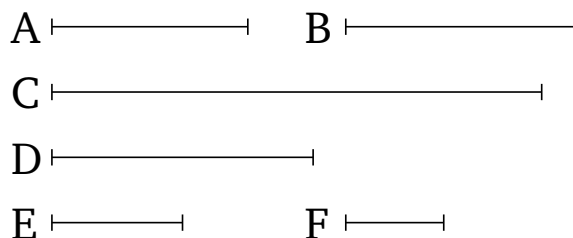
Let A and B be the two given numbers. So it is re-

ὅν ἐλάχιστον μετροῦσιν ἀριθμόν.

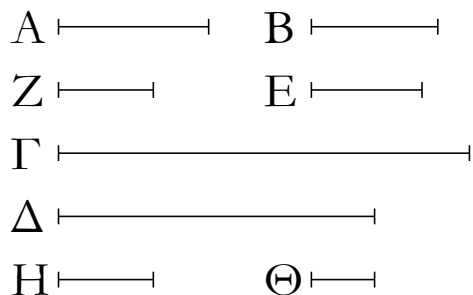


Οἱ Α, Β γάρ ἦτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὐ.
ἔστωσαν πρότερον οἱ Α, Β πρῶτοι πρὸς ἀλλήλους, καὶ ὁ Α
τὸν Β πολλαπλασιάσας τὸν Γ ποιείτω· καὶ ὁ Β ἄρα τὸν Α
πολλαπλασιάσας τὸν Γ πεποιήκεν. οἱ Α, Β ἄρα τὸν Γ με-
τροῦσιν. λέγω δὴ, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσι
τινα ἀριθμὸν οἱ Α, Β ἐλάσσονα ὄντα τοῦ Γ. μετρεῖτωσαν
τὸν Δ. καὶ ὁσάκις ὁ Α τὸν Δ μετρεῖ, τοσαῦται μονάδες
ἔστωσαν ἐν τῷ Ε, ὁσάκις δὲ ὁ Β τὸν Δ μετρεῖ, τοσαῦται
μονάδες ἔστωσαν ἐν τῷ Ζ. ὁ μὲν Α ἄρα τὸν Ε πολλα-
πλασιάσας τὸν Δ πεποιήκεν, ὁ δὲ Β τὸν Ζ πολλαπλασιάσας
τὸν Δ πεποιήκεν· ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν Α, Ε τῷ ἐκ τῶν
Β, Ζ. ἔστιν ἄρα ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Ζ πρὸς τὸν
Ε. οἱ δὲ Α, Β πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ
ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκως ὅ-
τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα· ὁ Β
ἄρα τὸν Ε μετρεῖ, ὡς ἐπόμενος ἐπόμενον. καὶ ἐπεὶ ὁ Α τοὺς
Β, Ε πολλαπλασιάσας τοὺς Γ, Δ πεποιήκεν, ἔστιν ἄρα ὡς
ὁ Β πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Δ. μετρεῖ δὲ ὁ Β τὸν
Ε· μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ ὁ μείζων τὸν ἐλάσσονα· ὅπερ
ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ Α, Β μετροῦσι τινα ἀριθμὸν
ἐλάσσονα ὄντα τοῦ Γ. ὁ Γ ἄρα ἐλάχιστος ὢν ὑπὸ τῶν Α, Β
μετρεῖται.

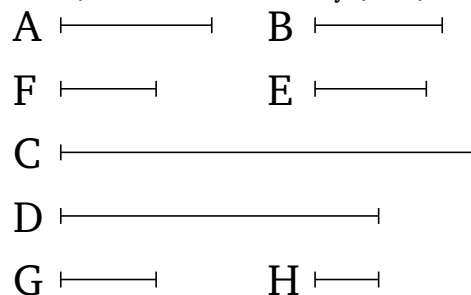
quired to find the least number which they (both) measure.



For A and B are either prime to one another, or not. Let them, first of all, be prime to one another. And let A make C (by) multiplying B . Thus, B has also made C (by) multiplying A [Prop. 7.16]. Thus, A and B (both) measure C . So I say that (C) is also the least (number which they both measure). For if not, A and B will (both) measure some (other) number which is less than C . Let them (both) measure D (which is less than C). And as many times as A measures D , so many units let there be in E . And as many times as B measures D , so many units let there be in F . Thus, A has made D (by) multiplying E , and B has made D (by) multiplying F . Thus, the (number created) from (multiplying) A and E is equal to the (number created) from (multiplying) B and F . Thus, as A (is) to B , so F (is) to E [Prop. 7.19]. And A and B are prime (to one another), and prime (numbers) are the least (of those numbers having the same ratio) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser [Prop. 7.20]. Thus, B measures E , as the following (number measuring) the following. And since A has made C and D (by) multiplying B and E (respectively), thus as B is to E , so C (is) to D [Prop. 7.17]. And B measures E . Thus, C also measures D , the greater (measuring) the lesser. The very thing is impossible. Thus, A and B do not (both) measure some number which is less than C . Thus, C is the least (number) which is measured by (both) A and B .



Μὴ ἔστωσαν δὴ οἱ A, B πρῶτοι πρὸς ἀλλήλους, καὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A, B οἱ Z, E . ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν A, E τῷ



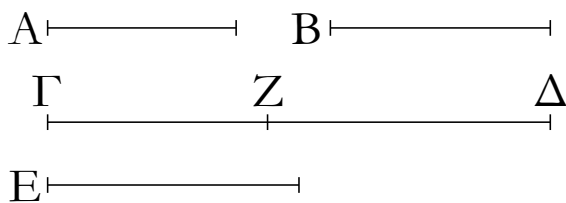
So let A and B be not prime to one another. And let the least numbers, F and E , have been taken having the same ratio as A and B (respectively) [Prop. 7.33].

ἐκ τῶν B, Z. καὶ ὁ A τὸν E πολλαπλασιάσας τὸν Γ ποιεῖται· καὶ ὁ B ἄρα τὸν Z πολλαπλασιάσας τὸν Γ πεποίηκεν· οἱ A, B ἄρα τὸν Γ μετροῦσιν. λέγω δὴ, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσι τινὰ ἀριθμὸν οἱ A, B ἐλάσσονα ὄντα τοῦ Γ. μετρεῖωσαν τὸν Δ. καὶ ὅσας μὲν ὁ A τὸν Δ μετρεῖ, τοσαῦται μονάδες ἕστωσαν ἐν τῷ H, ὅσας δὲ ὁ B τὸν Δ μετρεῖ, τοσαῦται μονάδες ἕστωσαν ἐν τῷ Θ. ὁ μὲν A ἄρα τὸν H πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ δὲ B τὸν Θ πολλαπλασιάσας τὸν Δ πεποίηκεν. ἴσος ἄρα ἐστὶν ὁ ἐκ τῶν A, H τῷ ἐκ τῶν B, Θ· ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B, οὕτως ὁ Θ πρὸς τὸν H. ὡς δὲ ὁ A πρὸς τὸν B, οὕτως ὁ Z πρὸς τὸν E· καὶ ὡς ἄρα ὁ Z πρὸς τὸν E, οὕτως ὁ Θ πρὸς τὸν H. οἱ δὲ Z, E ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκεις ὁ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα· ὁ E ἄρα τὸν H μετρεῖ. καὶ ἐπεὶ ὁ A τοὺς E, H πολλαπλασιάσας τοὺς Γ, Δ πεποίηκεν, ἔστιν ἄρα ὡς ὁ E πρὸς τὸν H, οὕτως ὁ Γ πρὸς τὸν Δ. ὁ δὲ E τὸν H μετρεῖ· καὶ ὁ Γ ἄρα τὸν Δ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ A, B μετρήσουσι τινὰ ἀριθμὸν ἐλάσσονα ὄντα τοῦ Γ. ὁ Γ ἄρα ἐλάχιστος ὢν ὑπὸ τῶν A, B μετρεῖται· ὅπερ ἔπει δεῖξαι.

Thus, the (number created) from (multiplying) A and E is equal to the (number created) from (multiplying) B and F [Prop. 7.19]. And let A make C (by) multiplying E. Thus, B has also made C (by) multiplying F. Thus, A and B (both) measure C. So I say that (C) is also the least (number which they both measure). For if not, A and B will (both) measure some number which is less than C. Let them (both) measure D (which is less than C). And as many times as A measures D, so many units let there be in G. And as many times as B measures D, so many units let there be in H. Thus, A has made D (by) multiplying G, and B has made D (by) multiplying H. Thus, the (number created) from (multiplying) A and G is equal to the (number created) from (multiplying) B and H. Thus, as A is to B, so H (is) to G [Prop. 7.19]. And as A (is) to B, so F (is) to E. Thus, also, as F (is) to E, so H (is) to G. And F and E are the least (numbers having the same ratio as A and B), and the least (numbers) measure those (numbers) having the same ratio an equal number of times, the greater (measuring) the greater, and the lesser the lesser [Prop. 7.20]. Thus, E measures G. And since A has made C and D (by) multiplying E and G (respectively), thus as E is to G, so C (is) to D [Prop. 7.17]. And E measures G. Thus, C also measures D, the greater (measuring) the lesser. The very thing is impossible. Thus, A and B do not (both) measure some (number) which is less than C. Thus, C (is) the least (number) which is measured by (both) A and B. (Which is) the very thing it was required to show.

λε΄.

Ἐὰν δύο ἀριθμοὶ ἀριθμὸν τινὰ μετρώσιν, καὶ ὁ ἐλάχιστος ὑπ' αὐτῶν μετρούμενος τὸν αὐτὸν μετρήσει.

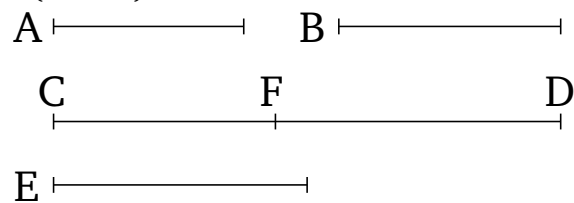


Δύο γὰρ ἀριθμοὶ οἱ A, B ἀριθμὸν τινὰ τὸν ΓΔ μετρεῖωσαν, ἐλάχιστον δὲ τὸν E· λέγω, ὅτι καὶ ὁ E τὸν ΓΔ μετρεῖ.

Εἰ γὰρ οὐ μετρεῖ ὁ E τὸν ΓΔ, ὁ E τὸν ΔZ μετρῶν λειπέτω ἑαυτοῦ ἐλάσσονα τὸν ΓZ. καὶ ἐπεὶ οἱ A, B τὸν E μετροῦσιν, ὁ δὲ E τὸν ΔZ μετρεῖ, καὶ οἱ A, B ἄρα τὸν ΔZ μετρήσουσιν. μετροῦσι δὲ καὶ ὅλον τὸν ΓΔ· καὶ λοιπὸν ἄρα τὸν ΓZ μετρήσουσιν ἐλάσσονα ὄντα τοῦ E· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οὐ μετρεῖ ὁ E τὸν ΓΔ· μετρεῖ ἄρα· ὅπερ ἔδει δεῖξαι.

Proposition 35

If two numbers (both) measure some number then the least (number) measured by them will also measure the same (number).



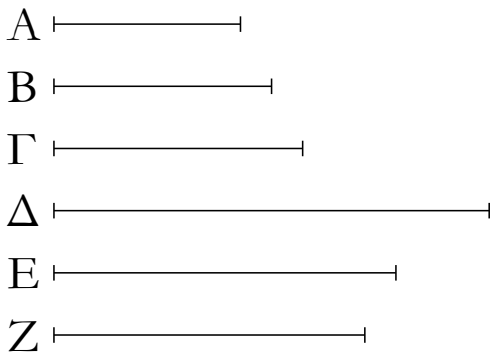
For let two numbers, A and B, (both) measure some number CD, and (let) E (be the) least (number measured by both A and B). I say that E also measures CD.

For if E does not measure CD then let E leave CF less than itself (in) measuring DF. And since A and B (both) measure E, and E measures DF, A and B will thus also measure DF. And (A and B) also measure the whole of CD. Thus, they will also measure the remainder CF, which is less than E. The very thing is impossible. Thus, E cannot not measure CD. Thus, (E) measures

λζ΄.

Τριῶν ἀριθμῶν δοθέντων εὑρεῖν, ὃν ἐλάχιστον μετροῦσιν ἀριθμόν.

Ἐστωσαν οἱ δοθέντες τρεῖς ἀριθμοὶ οἱ A , B , Γ · δεῖ δὴ εὑρεῖν, ὃν ἐλάχιστον μετροῦσιν ἀριθμόν.



Εἰλήφθω γὰρ ὑπὸ δύο τῶν A , B ἐλάχιστος μετρούμενος ὁ Δ . ὁ δὲ Γ τὸν Δ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρεῖτω πρότερον. μετροῦσι δὲ καὶ οἱ A , B τὸν Δ . οἱ A , B , Γ ἄρα τὸν Δ μετροῦσιν. λέγω δὴ, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσιν [τινα] ἀριθμόν οἱ A , B , Γ ἐλάσσονα ὄντα τοῦ Δ . μετρεῖτωσαν τὸν E . ἐπεὶ οἱ A , B , Γ τὸν E μετροῦσιν, καὶ οἱ A , B ἄρα τὸν E μετροῦσιν. καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν A , B μετρούμενος [τὸν E] μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν A , B μετρούμενός ἐστιν ὁ Δ . ὁ Δ ἄρα τὸν E μετρήσει ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ A , B , Γ μετρήσουσιν τινα ἀριθμόν ἐλάσσονα ὄντα τοῦ Δ . οἱ A , B , Γ ἄρα ἐλάχιστον τὸν Δ μετροῦσιν.

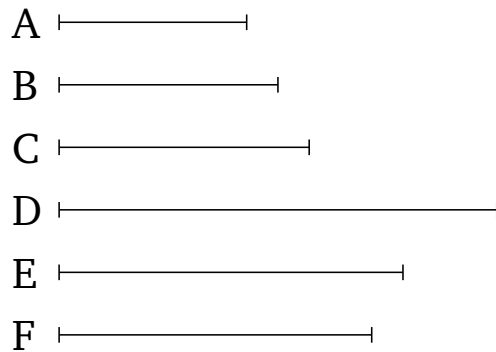
Μὴ μετρεῖτω δὴ πάλιν ὁ Γ τὸν Δ , καὶ εἰλήφθω ὑπὸ τῶν Γ , Δ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ E . ἐπεὶ οἱ A , B τὸν Δ μετροῦσιν, ὁ δὲ Δ τὸν E μετρεῖ, καὶ οἱ A , B ἄρα τὸν E μετροῦσιν. μετρεῖ δὲ καὶ ὁ Γ [τὸν E · καὶ] οἱ A , B , Γ ἄρα τὸν E μετροῦσιν. λέγω δὴ, ὅτι καὶ ἐλάχιστον. εἰ γὰρ μή, μετρήσουσιν [τινα] οἱ A , B , Γ ἐλάσσονα ὄντα τοῦ E . μετρεῖτωσαν τὸν Z . ἐπεὶ οἱ A , B , Γ τὸν Z μετροῦσιν, καὶ οἱ A , B ἄρα τὸν Z μετροῦσιν· καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν A , B μετρούμενος τὸν Z μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν A , B μετρούμενός ἐστιν ὁ Δ . ὁ Δ ἄρα τὸν Z μετρεῖ. μετρεῖ δὲ καὶ ὁ Γ τὸν Z . οἱ Δ , Γ ἄρα τὸν Z μετροῦσιν· ὥστε καὶ ὁ ἐλάχιστος ὑπὸ τῶν Δ , Γ μετρούμενος τὸν Z μετρήσει. ὁ δὲ ἐλάχιστος ὑπὸ τῶν Γ , Δ μετρούμενός ἐστιν ὁ E . ὁ E ἄρα τὸν Z μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ A , B , Γ μετρήσουσιν τινα ἀριθμόν ἐλάσσονα ὄντα τοῦ E . ὁ E ἄρα ἐλάχιστος ὢν ὑπὸ τῶν A , B , Γ μετρεῖται· ὅπερ ἔδει δεῖξαι.

(CD). (Which is) the very thing it was required to show.

Proposition 36

To find the least number which three given numbers (all) measure.

Let A , B , and C be the three given numbers. So it is required to find the least number which they (all) measure.

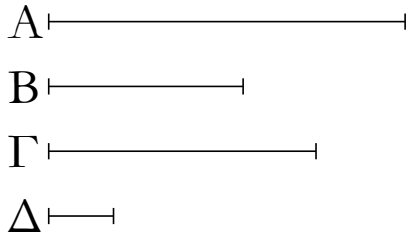


For let the least (number), D , measured by the two (numbers) A and B have been taken [Prop. 7.34]. So C either measures, or does not measure, D . Let it, first of all, measure (D). And A and B also measure D . Thus, A , B , and C (all) measure D . So I say that (D is) also the least (number measured by A , B , and C). For if not, A , B , and C will (all) measure [some] number which is less than D . Let them measure E (which is less than D). Since A , B , and C (all) measure E then A and B thus also measure E . Thus, the least (number) measured by A and B will also measure [E] [Prop. 7.35]. And D is the least (number) measured by A and B . Thus, D will measure E , the greater (measuring) the lesser. The very thing is impossible. Thus, A , B , and C cannot (all) measure some number which is less than D . Thus, A , B , and C (all) measure the least (number) D .

So, again, let C not measure D . And let the least number, E , measured by C and D have been taken [Prop. 7.34]. Since A and B measure D , and D measures E , A and B thus also measure E . And C also measures [E]. Thus, A , B , and C [also] measure E . So I say that (E is) also the least (number measured by A , B , and C). For if not, A , B , and C will (all) measure some (number) which is less than E . Let them measure F (which is less than E). Since A , B , and C (all) measure F , A and B thus also measure F . Thus, the least (number) measured by A and B will also measure F [Prop. 7.35]. And D is the least (number) measured by A and B . Thus, D measures F . And C also measures F . Thus, D and C (both) measure F . Hence, the least (number) measured by D and C will also measure F [Prop. 7.35]. And E

λζ'.

Ἐάν ἀριθμὸς ὑπὸ τινος ἀριθμοῦ μετρηῇται, ὁ μετρούμενος ὁμώνυμον μέρος ἔξει τῷ μετροῦντι.

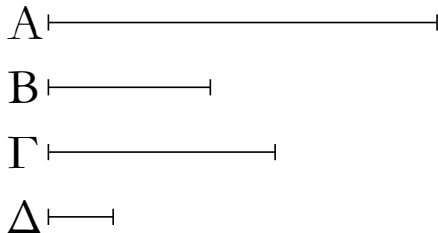


Ἀριθμὸς γάρ ὁ A ὑπὸ τινος ἀριθμοῦ τοῦ B μετρεῖσθω· λέγω, ὅτι ὁ A ὁμώνυμον μέρος ἔχει τῷ B.

Ὅσάκις γάρ ὁ B τὸν A μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Γ. ἐπεὶ ὁ B τὸν A μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας, μετρεῖ δὲ καὶ ἡ Δ μονὰς τὸν Γ ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας, ἰσάκις ἄρα ἡ Δ μονὰς τὸν Γ ἀριθμὸν μετρεῖ καὶ ὁ B τὸν A. ἐναλλάξ ἄρα ἰσάκις ἡ Δ μονὰς τὸν B ἀριθμὸν μετρεῖ καὶ ὁ Γ τὸν A· ὃ ἄρα μέρος ἐστὶν ἡ Δ μονὰς τοῦ B ἀριθμοῦ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ Γ τοῦ A. ἡ δὲ Δ μονὰς τοῦ B ἀριθμοῦ μέρος ἐστὶν ὁμώνυμον αὐτῷ· καὶ ὁ Γ ἄρα τοῦ A μέρος ἐστὶν ὁμώνυμον τῷ B. ὥστε ὁ A μέρος ἔχει τὸν Γ ὁμώνυμον ὄντα τῷ B· ὅπερ ἔδει δεῖξαι.

λη'.

Ἐάν ἀριθμὸς μέρος ἔχη ὅτιοῦν, ὑπὸ ὁμωνύμου ἀριθμοῦ μετρηθήσεται τῷ μέρει.



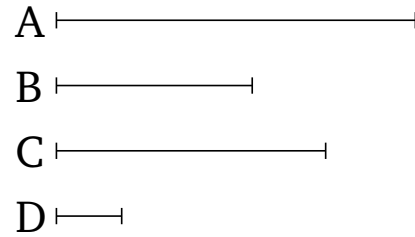
Ἀριθμὸς γάρ ὁ A μέρος ἐχέτω ὅτιοῦν τὸν B, καὶ τῷ B μέρει ὁμώνυμος ἔστω [ἀριθμὸς] ὁ Γ· λέγω, ὅτι ὁ Γ τὸν A μετρεῖ.

Ἐπεὶ γάρ ὁ B τοῦ A μέρος ἐστὶν ὁμώνυμον τῷ Γ, ἔστι δὲ καὶ ἡ Δ μονὰς τοῦ Γ μέρος ὁμώνυμον αὐτῷ, ὃ ἄρα μέρος

is the least (number) measured by C and D . Thus, E measures F , the greater (measuring) the lesser. The very thing is impossible. Thus, A , B , and C cannot measure some number which is less than E . Thus, E (is) the least (number) which is measured by A , B , and C . (Which is) the very thing it was required to show.

Proposition 37

If a number is measured by some number then the (number) measured will have a part called the same as the measuring (number).

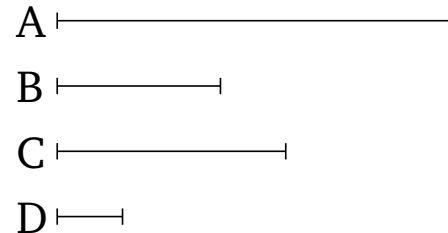


For let the number A be measured by some number B . I say that A has a part called the same as B .

For as many times as B measures A , so many units let there be in C . Since B measures A according to the units in C , and the unit D also measures C according to the units in it, the unit D thus measures the number C as many times as B (measures) A . Thus, alternately, the unit D measures the number B as many times as C (measures) A [Prop. 7.15]. Thus, which(ever) part the unit D is of the number B , C is also the same part of A . And the unit D is a part of the number B called the same as it (i.e., a B th part). Thus, C is also a part of A called the same as B (i.e., C is the B th part of A). Hence, A has a part C which is called the same as B (i.e., A has a B th part). (Which is) the very thing it was required to show.

Proposition 38

If a number has any part whatever then it will be measured by a number called the same as the part.



For let the number A have any part whatever, B . And let the [number] C be called the same as the part B (i.e., B is the C th part of A). I say that C measures A .

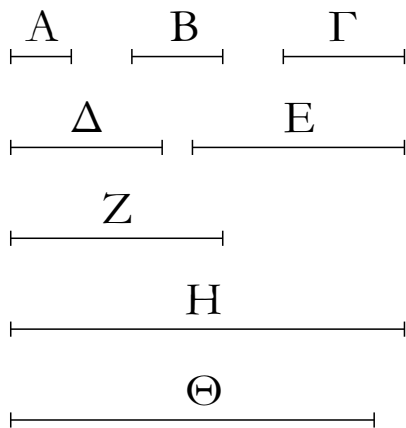
For since B is a part of A called the same as C , and the unit D is also a part of C called the same as it (i.e.,

ἐστὶν ἡ Δ μονὰς τοῦ Γ ἀριθμοῦ, τὸ αὐτὸ μέρος ἐστὶ καὶ ὁ B τοῦ A : ἰσάκεις ἄρα ἡ Δ μονὰς τὸν Γ ἀριθμὸν μετρεῖ καὶ ὁ B τὸν A . ἐναλλάξ ἄρα ἰσάκεις ἡ Δ μονὰς τὸν B ἀριθμὸν μετρεῖ καὶ ὁ Γ τὸν A . ὁ Γ ἄρα τὸν A μετρεῖ· ὅπερ ἔδει δεῖξαι.

D is the C th part of C), thus which(ever) part the unit D is of the number C , B is also the same part of A . Thus, the unit D measures the number C as many times as B (measures) A . Thus, alternately, the unit D measures the number B as many times as C (measures) A [Prop. 7.15]. Thus, C measures A . (Which is) the very thing it was required to show.

λθ΄.

Ἀριθμὸν εὐρεῖν, ὃς ἐλάχιστος ὢν ἔξει τὰ δοθέντα μέρη.



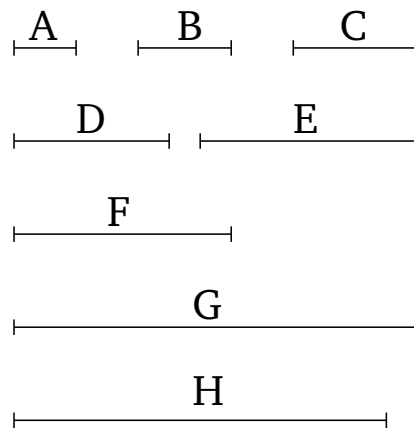
Ἐστω τὰ δοθέντα μέρη τὰ A , B , Γ . δεῖ δὴ ἀριθμὸν εὐρεῖν, ὃς ἐλάχιστος ὢν ἔξει τὰ A , B , Γ μέρη.

Ἐστωσαν γὰρ τοῖς A , B , Γ μέρεσιν ὁμώνυμοι ἀριθμοὶ οἱ Δ , E , Z , καὶ εἰλήφθω ὑπὸ τῶν Δ , E , Z ἐλάχιστος μετρούμενος ἀριθμὸς ὁ H .

Ὁ H ἄρα ὁμώνυμα μέρη ἔχει τοῖς Δ , E , Z . τοῖς δὲ Δ , E , Z ὁμώνυμα μέρη ἐστὶ τὰ A , B , Γ . ὁ H ἄρα ἔχει τὰ A , B , Γ μέρη. λέγω δὴ, ὅτι καὶ ἐλάχιστος ὢν, εἰ γὰρ μή, ἔσται τις τοῦ H ἐλάσσων ἀριθμὸς, ὃς ἔξει τὰ A , B , Γ μέρη. ἔστω ὁ Θ . ἐπεὶ ὁ Θ ἔχει τὰ A , B , Γ μέρη, ὁ Θ ἄρα ὑπὸ ὁμωνύμων ἀριθμῶν μετρηθήσεται τοῖς A , B , Γ μέρεσιν. τοῖς δὲ A , B , Γ μέρεσιν ὁμώνυμοι ἀριθμοὶ εἰσιν οἱ Δ , E , Z . ὁ Θ ἄρα ὑπὸ τῶν Δ , E , Z μετρεῖται. καὶ ἐστὶν ἐλάσσων τοῦ H . ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἔσται τις τοῦ H ἐλάσσων ἀριθμὸς, ὃς ἔξει τὰ A , B , Γ μέρη. ὅπερ ἔδει δεῖξαι.

Proposition 39

To find the least number that will have given parts.



Let A , B , and C be the given parts. So it is required to find the least number which will have the parts A , B , and C (i.e., an A th part, a B th part, and a C th part).

For let D , E , and F be numbers having the same names as the parts A , B , and C (respectively). And let the least number, G , measured by D , E , and F , have been taken [Prop. 7.36].

Thus, G has parts called the same as D , E , and F [Prop. 7.37]. And A , B , and C are parts called the same as D , E , and F (respectively). Thus, G has the parts A , B , and C . So I say that (G) is also the least (number having the parts A , B , and C). For if not, there will be some number less than G which will have the parts A , B , and C . Let it be H . Since H has the parts A , B , and C , H will thus be measured by numbers called the same as the parts A , B , and C [Prop. 7.38]. And D , E , and F are numbers called the same as the parts A , B , and C (respectively). Thus, H is measured by D , E , and F . And (H) is less than G . The very thing is impossible. Thus, there cannot be some number less than G which will have the parts A , B , and C . (Which is) the very thing it was required to show.

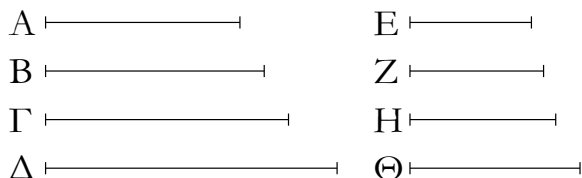
ELEMENTS BOOK 8

Continued Proportion[†]

[†]The propositions contained in Books 7–9 are generally attributed to the school of Pythagoras.

α'.

Ἐάν ὧσιν ὁσοιοποτοῦν ἀριθμοὶ ἐξῆς ἀνάλογον, οἱ δὲ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὧσιν, ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.



Ἐστωσαν ὁποιοιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον οἱ A, B, Γ, Δ, οἱ δὲ ἄκροι αὐτῶν οἱ A, Δ, πρῶτοι πρὸς ἀλλήλους ἔστωσαν· λέγω, ὅτι οἱ A, B, Γ, Δ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς.

Εἰ γὰρ μή, ἔστωσαν ἐλάττονες τῶν A, B, Γ, Δ οἱ E, Z, H, Θ ἐν τῷ αὐτῷ λόγῳ ὄντες αὐτοῖς. καὶ ἐπεὶ οἱ A, B, Γ, Δ ἐν τῷ αὐτῷ λόγῳ εἰσι τοῖς E, Z, H, Θ, καὶ ἐστὶν ἴσον τὸ πλήθος [τῶν A, B, Γ, Δ] τῷ πλήθει [τῶν E, Z, H, Θ], δι' ἴσου ἄρα ἐστὶν ὡς A πρὸς τὸν Δ, ὁ E πρὸς τὸν Θ. οἱ δὲ A, Δ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἐχοντας ἰσάκεις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον. μετρεῖ ἄρα ὁ A τὸν E ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα οἱ E, Z, H, Θ ἐλάσσονες ὄντες τῶν A, B, Γ, Δ ἐν τῷ αὐτῷ λόγῳ εἰσὶν αὐτοῖς. οἱ A, B, Γ, Δ ἄρα ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς· ὅπερ ἔδει δεῖξαι.

β'.

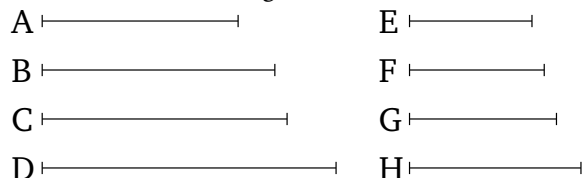
Αριθμοὺς εὑρεῖν ἐξῆς ἀνάλογον ἐλάχιστους, ὅσους ἂν ἐπιτάξῃ τις, ἐν τῷ δοθέντι λόγῳ.

Ἐστω ὁ δοθείς λόγος ἐν ἐλάχιστοις ἀριθμοῖς ὁ τοῦ A πρὸς τὸν B· δεῖ δὴ ἀριθμοὺς εὑρεῖν ἐξῆς ἀνάλογον ἐλάχιστους, ὅσους ἂν τις ἐπιτάξῃ, ἐν τῷ τοῦ A πρὸς τὸν B λόγῳ.

Ἐπιτετάχθωσαν δὴ τέσσαρες, καὶ ὁ A ἑαυτὸν πολλαπλασιάσας τὸν Γ ποιεῖτω, τὸν δὲ B πολλαπλασιάσας τὸν Δ ποιεῖτω, καὶ ἔτι ὁ B ἑαυτὸν πολλαπλασιάσας τὸν E ποιεῖτω, καὶ ἔτι ὁ A τοὺς Γ, Δ, E πολλαπλασιάσας τοὺς Z, H, Θ ποιεῖτω, ὁ δὲ B τὸν E πολλαπλασιάσας τὸν K ποιεῖτω.

Proposition 1

If there are any multitude whatsoever of continuously proportional numbers, and the outermost of them are prime to one another, then the (numbers) are the least of those (numbers) having the same ratio as them.



Let A, B, C, D be any multitude whatsoever of continuously proportional numbers. And let the outermost of them, A and D, be prime to one another. I say that A, B, C, D are the least of those (numbers) having the same ratio as them.

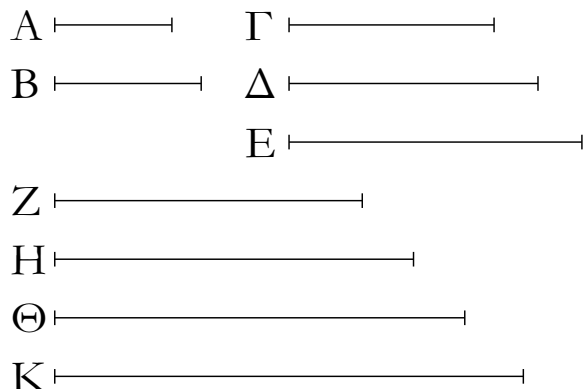
For if not, let E, F, G, H be less than A, B, C, D (respectively), being in the same ratio as them. And since A, B, C, D are in the same ratio as E, F, G, H, and the multitude [of A, B, C, D] is equal to the multitude [of E, F, G, H], thus, via equality, as A is to D, (so) E (is) to H [Prop. 7.14]. And A and D (are) prime (to one another). And prime (numbers are) also the least of those (numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures E, the greater (measuring) the lesser. The very thing is impossible. Thus, E, F, G, H, being less than A, B, C, D, are not in the same ratio as them. Thus, A, B, C, D are the least of those (numbers) having the same ratio as them. (Which is) the very thing it was required to show.

Proposition 2

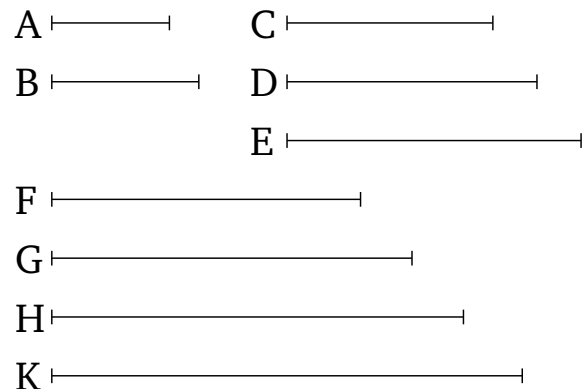
To find the least numbers, as many as may be prescribed, (which are) continuously proportional in a given ratio.

Let the given ratio, (expressed) in the least numbers, be that of A to B. So it is required to find the least numbers, as many as may be prescribed, (which are) in the ratio of A to B.

Let four (numbers) have been prescribed. And let A make C (by) multiplying itself, and let it make D (by) multiplying B. And, further, let B make E (by) multiplying itself. And, further, let A make F, G, H (by) multiplying C, D, E. And let B make K (by) multiplying E.



Καὶ ἐπεὶ ὁ A ἑαυτὸν μὲν πολλαπλασιάσας τὸν Γ πεποίηκεν, τὸν δὲ B πολλαπλασιάσας τὸν Δ πεποίηκεν, ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , [οὕτως] ὁ Γ πρὸς τὸν Δ . πάλιν, ἐπεὶ ὁ μὲν A τὸν B πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ δὲ B ἑαυτὸν πολλαπλασιάσας τὸν E πεποίηκεν, ἐκάτερος ἄρα τῶν A, B τὸν B πολλαπλασιάσας ἐκάτερον τῶν Δ, E πεποίηκεν. ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Δ πρὸς τὸν E . ἀλλ' ὡς ὁ A πρὸς τὸν B , ὁ Γ πρὸς τὸν Δ · καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ , ὁ Δ πρὸς τὸν E . καὶ ἐπεὶ ὁ A τοὺς Γ, Δ πολλαπλασιάσας τοὺς Z, H πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ , [οὕτως] ὁ Z πρὸς τὸν H . ὡς δὲ ὁ Γ πρὸς τὸν Δ , οὕτως ἦν ὁ A πρὸς τὸν B · καὶ ὡς ἄρα ὁ A πρὸς τὸν B , ὁ Z πρὸς τὸν H . πάλιν, ἐπεὶ ὁ A τοὺς Δ, E πολλαπλασιάσας τοὺς H, Θ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν E , ὁ H πρὸς τὸν Θ . ἀλλ' ὡς ὁ Δ πρὸς τὸν E , ὁ A πρὸς τὸν B . καὶ ὡς ἄρα ὁ A πρὸς τὸν B , οὕτως ὁ H πρὸς τὸν Θ . καὶ ἐπεὶ οἱ A, B τὸν E πολλαπλασιάσαντες τοὺς Θ, K πεποιήκασιν, ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Θ πρὸς τὸν K . ἀλλ' ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Z πρὸς τὸν H καὶ ὁ H πρὸς τὸν Θ . καὶ ὡς ἄρα ὁ Z πρὸς τὸν H , οὕτως ὁ Θ πρὸς τὸν K . οἱ Γ, Δ, E ἄρα καὶ οἱ Z, H, Θ, K ἀνάλογόν εἰσιν ἐν τῷ τοῦ A πρὸς τὸν B λόγῳ. λέγω δὴ, ὅτι καὶ ἐλάχιστοι. ἐπεὶ γὰρ οἱ A, B ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ δὲ ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων πρῶτοι πρὸς ἀλλήλους εἰσίν, οἱ A, B ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐκάτερος μὲν τῶν A, B ἑαυτὸν πολλαπλασιάσας ἐκάτερον τῶν Γ, E πεποίηκεν, ἐκάτερον δὲ τῶν Γ, E πολλαπλασιάσας ἐκάτερον τῶν Z, K πεποίηκεν· οἱ Γ, E ἄρα καὶ οἱ Z, K πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐὰν δὲ ὧσιν ὅποσοιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον, οἱ δὲ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὧσιν, ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς. οἱ Γ, Δ, E ἄρα καὶ οἱ Z, H, Θ, K ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A, B · ὅπερ εἶδει δεῖξαι.



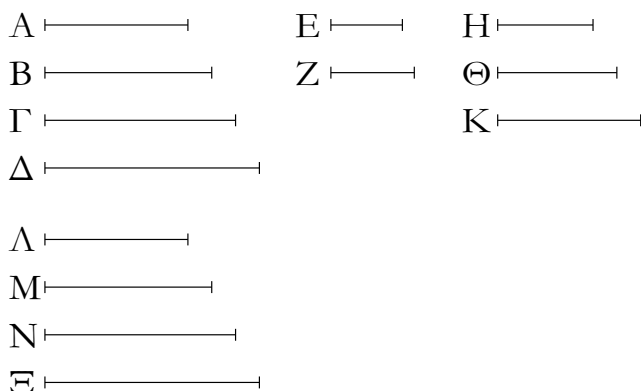
And since A has made C (by) multiplying itself, and has made D (by) multiplying B , thus as A is to B , [so] C (is) to D [Prop. 7.17]. Again, since A has made D (by) multiplying B , and B has made E (by) multiplying itself, A, B have thus made D, E , respectively, (by) multiplying B . Thus, as A is to B , so D (is) to E [Prop. 7.18]. But, as A (is) to B , (so) C (is) to D . And thus as C (is) to D , (so) D (is) to E . And since A has made F, G (by) multiplying C, D , thus as C is to D , [so] F (is) to G [Prop. 7.17]. And as C (is) to D , so A was to B . And thus as A (is) to B , (so) F (is) to G . Again, since A has made G, H (by) multiplying D, E , thus as D is to E , (so) G (is) to H [Prop. 7.17]. But, as D (is) to E , (so) A (is) to B . And thus as A (is) to B , so G (is) to H . And since A, B have made H, K (by) multiplying E , thus as A is to B , so H (is) to K . But, as A (is) to B , so F (is) to G , and G to H . And thus as F (is) to G , so G (is) to H , and H to K . Thus, C, D, E and F, G, H, K are (both continuously) proportional in the ratio of A to B . So I say that (they are) also the least (sets of numbers continuously proportional in that ratio). For since A and B are the least of those (numbers) having the same ratio as them, and the least of those (numbers) having the same ratio are prime to one another [Prop. 7.22], A and B are thus prime to one another. And A, B have made C, E , respectively, (by) multiplying themselves, and have made F, K by multiplying C, E , respectively. Thus, C, E and F, K are prime to one another [Prop. 7.27]. And if there are any multitude whatsoever of continuously proportional numbers, and the outermost of them are prime to one another, then the (numbers) are the least of those (numbers) having the same ratio as them [Prop. 8.1]. Thus, C, D, E and F, G, H, K are the least of those (continuously proportional sets of numbers) having the same ratio as A and B . (Which is) the very thing it was required to show.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν τρεῖς ἀριθμοὶ ἐξῆς ἀνάλογον ἐλάχιστοι ἀνάλογον ἐλάχιστοι ὥσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ ἄκρον αὐτῶν τετράγωνοί εἰσιν, ἐὰν δὲ τέσσαρες, κύβοι.

Υ'.

Ἐὰν ὥσιν ὅποσοι οὖν ἀριθμοὶ ἐξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν.



Ἐστῶσαν ὅποσοι οὖν ἀριθμοὶ ἐξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς οἱ A, B, Γ, Δ· λέγω, ὅτι οἱ ἄκροι αὐτῶν οἱ A, Δ πρῶτοι πρὸς ἀλλήλους εἰσίν.

Εἰλήφθωσαν γὰρ δύο μὲν ἀριθμοὶ ἐλάχιστοι ἐν τῷ τῶν A, B, Γ, Δ λόγῳ οἱ E, Z, τρεῖς δὲ οἱ H, Θ, K, καὶ ἐξῆς ἐνὶ πλείους, ἕως τὸ λαμβανόμενον πλῆθος ἴσον γέννηται τῷ πλήθει τῶν A, B, Γ, Δ. εἰλήφθωσαν καὶ ἕστωσαν οἱ Λ, Μ, Ν, Ξ.

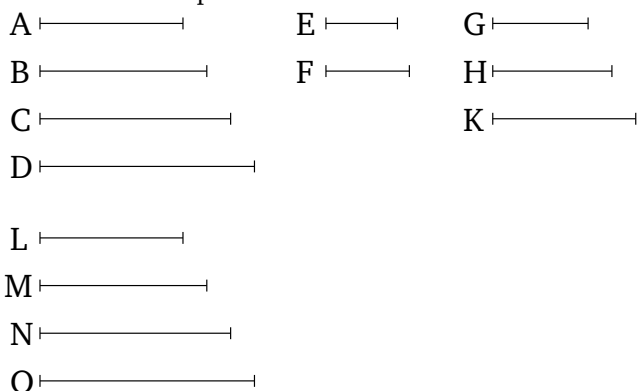
Καὶ ἐπεὶ οἱ E, Z ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ ἑκάτερος τῶν E, Z ἑαυτὸν μὲν πολλαπλασιάσας ἑκάτερον τῶν H, K πεποίηκεν, ἑκάτερον δὲ τῶν H, K πολλαπλασιάσας ἑκάτερον τῶν Λ, Ξ πεποίηκεν, καὶ οἱ H, K ἄρα καὶ οἱ Λ, Ξ πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ οἱ A, B, Γ, Δ ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, εἰσὶ δὲ καὶ οἱ Λ, Μ, Ν, Ξ ἐλάχιστοι ἐν τῷ αὐτῷ λόγῳ ὄντες τοῖς A, B, Γ, Δ, καὶ ἐστὶν ἴσον τὸ πλῆθος τῶν A, B, Γ, Δ τῷ πλήθει τῶν Λ, Μ, Ν, Ξ, ἕκαστος ἄρα τῶν A, B, Γ, Δ ἑκάστῳ τῶν Λ, Μ, Ν, Ξ ἴσος ἐστίν· ἴσος ἄρα ἐστὶν ὁ μὲν A τῷ Λ, ὁ δὲ Δ τῷ Ξ. καὶ εἰσιν οἱ Λ, Ξ πρῶτοι πρὸς ἀλλήλους. καὶ οἱ A, Δ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν· ὅπερ ἔδει δεῖξαι.

Corollary

So it is clear, from this, that if three continuously proportional numbers are the least of those (numbers) having the same ratio as them then the outermost of them are square, and, if four (numbers), cube.

Proposition 3

If there are any multitude whatsoever of continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them then the outermost of them are prime to one another.



Let A, B, C, D be any multitude whatsoever of continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them. I say that the outermost of them, A and D, are prime to one another.

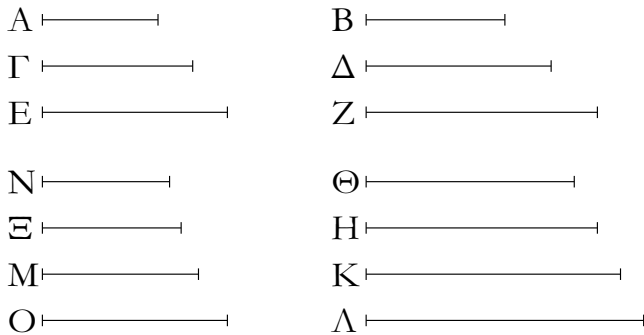
For let the two least (numbers) E, F (which are) in the same ratio as A, B, C, D have been taken [Prop. 7.33]. And the three (least numbers) G, H, K [Prop. 8.2]. And (so on), successively increasing by one, until the multitude of (numbers) taken is made equal to the multitude of A, B, C, D. Let them have been taken, and let them be L, M, N, O.

And since E and F are the least of those (numbers) having the same ratio as them they are prime to one another [Prop. 7.22]. And since E, F have made G, K, respectively, (by) multiplying themselves [Prop. 8.2 corr.], and have made L, O (by) multiplying G, K, respectively, G, K and L, O are thus also prime to one another [Prop. 7.27]. And since A, B, C, D are the least of those (numbers) having the same ratio as them, and L, M, N, O are also the least (of those numbers having the same ratio as them), being in the same ratio as A, B, C, D, and the multitude of A, B, C, D is equal to the multitude of L, M, N, O, thus A, B, C, D are equal to L, M, N, O, respectively. Thus, A is equal to L, and D to O. And L and O are prime to one another. Thus, A and D are also prime to one another. (Which is) the very thing it was

required to show.

δ'.

Λόγων δοθέντων ὁποσωνοῦν ἐν ἐλάχιστοις ἀριθμοῖς ἀριθμοὺς εὑρεῖν ἐξῆς ἀνάλογον ἐλάχιστους ἐν τοῖς δοθεῖσι λόγοις.

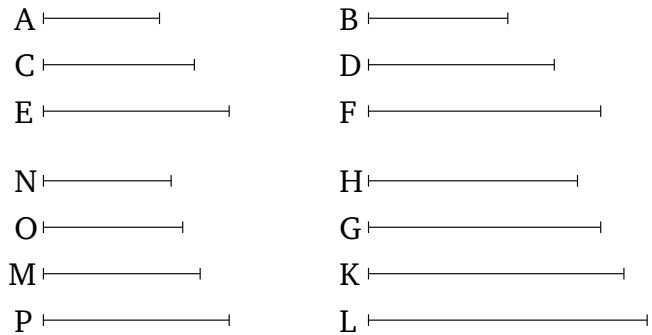


Ἐστωσαν οἱ δοθέντες λόγοι ἐν ἐλάχιστοις ἀριθμοῖς ὅ τε τοῦ A πρὸς τὸν B καὶ ὁ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι ὁ τοῦ E πρὸς τὸν Z· δεῖ δὴ ἀριθμοὺς εὑρεῖν ἐξῆς ἀνάλογον ἐλάχιστους ἐν τε τῷ τοῦ A πρὸς τὸν B λόγῳ καὶ ἐν τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τῷ τοῦ E πρὸς τὸν Z.

Εἰλήφθω γὰρ ὁ ὑπὸ τῶν B, Γ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ H. καὶ ὁσάκις μὲν ὁ B τὸν H μετρεῖ, τοσαυτάκις καὶ ὁ A τὸν Θ μετρεῖται, ὁσάκις δὲ ὁ Γ τὸν H μετρεῖ, τοσαυτάκις καὶ ὁ Δ τὸν K μετρεῖται. ὁ δὲ E τὸν K ἢ τοι μετρεῖ ἢ οὐ μετρεῖ. μετρεῖται πρότερον. καὶ ὁσάκις ὁ E τὸν K μετρεῖ, τοσαυτάκις καὶ ὁ Z τὸν Λ μετρεῖται. καὶ ἐπεὶ ἰσάκις ὁ A τὸν Θ μετρεῖ καὶ ὁ B τὸν H, ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B, οὕτως ὁ Θ πρὸς τὸν H. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ H πρὸς τὸν K, καὶ ἔτι ὡς ὁ E πρὸς τὸν Z, οὕτως ὁ K πρὸς τὸν Λ· οἱ Θ, H, K, Λ ἄρα ἐξῆς ἀνάλογόν εἰσιν ἐν τε τῷ τοῦ A πρὸς τὸν B καὶ ἐν τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι ἐν τῷ τοῦ E πρὸς τὸν Z λόγῳ. λέγω δὴ, ὅτι καὶ ἐλάχιστοι. εἰ γὰρ μὴ εἰσιν οἱ Θ, H, K, Λ ἐξῆς ἀνάλογον ἐλάχιστοι ἐν τε τοῖς τοῦ A πρὸς τὸν B καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἐν τῷ τοῦ E πρὸς τὸν Z λόγοις, ἔστωσαν οἱ N, Ξ, M, O. καὶ ἐπεὶ ἔστιν ὡς ὁ A πρὸς τὸν B, οὕτως ὁ N πρὸς τὸν Ξ, οἱ δὲ A, B ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον, ὁ B ἄρα τὸν Ξ μετρεῖ. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Ξ μετρεῖ· οἱ B, Γ ἄρα τὸν Ξ μετροῦσιν· καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν B, Γ μετρούμενος τὸν Ξ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν B, Γ μετρεῖται ὁ H· ὁ H ἄρα τὸν Ξ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἔστιν ἀδύνατον. οὐκ ἄρα ἔσσονται τινες τῶν Θ, H, K, Λ ἐλάσσονες ἀριθμοὶ ἐξῆς ἐν τε τῷ τοῦ A πρὸς τὸν B καὶ τῷ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τῷ τοῦ E πρὸς τὸν Z λόγῳ.

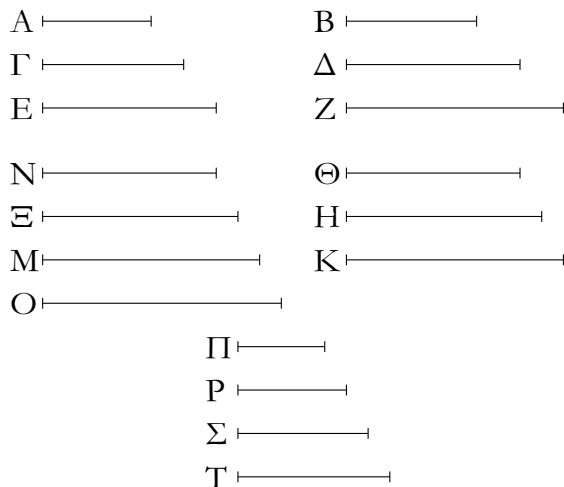
Proposition 4

For any multitude whatsoever of given ratios, (expressed) in the least numbers, to find the least numbers continuously proportional in these given ratios.



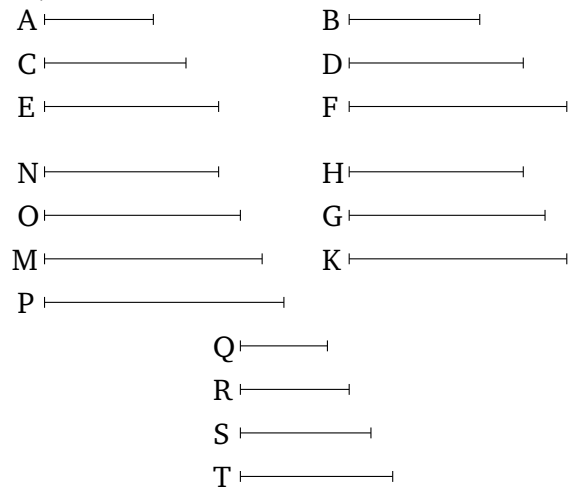
Let the given ratios, (expressed) in the least numbers, be the (ratios) of A to B , and of C to D , and, further, of E to F . So it is required to find the least numbers continuously proportional in the ratio of A to B , and of C to B , and, further, of E to F .

For let the least number, G , measured by (both) B and C have been taken [Prop. 7.34]. And as many times as B measures G , so many times let A also measure H . And as many times as C measures G , so many times let D also measure K . And E either measures, or does not measure, K . Let it, first of all, measure (K). And as many times as E measures K , so many times let F also measure L . And since A measures H the same number of times that B also (measures) G , thus as A is to B , so H (is) to G [Def. 7.20, Prop. 7.13]. And so, for the same (reasons), as C (is) to D , so G (is) to K , and, further, as E (is) to F , so K (is) to L . Thus, H, G, K, L are continuously proportional in the ratio of A to B , and of C to D , and, further, of E to F . So I say that (they are) also the least (numbers continuously proportional in these ratios). For if H, G, K, L are not the least numbers continuously proportional in the ratios of A to B , and of C to D , and of E to F , let N, O, M, P be (the least such numbers). And since as A is to B , so N (is) to O , and A and B are the least (numbers which have the same ratio as them), and the least (numbers) measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20], B thus measures O . So, for the same (reasons), C also measures O . Thus, B and C (both) measure O . Thus, the least number measured by (both) B and C will also measure O [Prop. 7.35]. And G (is) the least number measured by (both) B and C .



Μὴ μετρεῖται δὴ ὁ Ε τὸν Κ, καὶ εἰλήφθω ὑπὸ τῶν Ε, Κ ἐλάχιστος μετρούμενος ἀριθμὸς ὁ Μ. καὶ ὅσας μὲν ὁ Κ τὸν Μ μετρεῖ, τοσαυτάκις καὶ ἑκάτερος τῶν Θ, Η ἑκάτερον τῶν Ν, Ξ μετρεῖται, ὅσας δὲ ὁ Ε τὸν Μ μετρεῖ, τοσαυτάκις καὶ ὁ Ζ τὸν Ο μετρεῖται. ἐπεὶ ἰσάκις ὁ Θ τὸν Ν μετρεῖ καὶ ὁ Η τὸν Ξ, ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν Η, οὕτως ὁ Ν πρὸς τὸν Ξ. ὡς δὲ ὁ Θ πρὸς τὸν Η, οὕτως ὁ Α πρὸς τὸν Β· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Β, οὕτως ὁ Ν πρὸς τὸν Ξ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ξ πρὸς τὸν Μ. πάλιν, ἐπεὶ ἰσάκις ὁ Ε τὸν Μ μετρεῖ καὶ ὁ Ζ τὸν Ο, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Μ πρὸς τὸν Ο· οἱ Ν, Ξ, Μ, Ο ἄρα ἐξῆς ἀνάλογόν εἰσιν ἐν τοῖς τοῦ τε Α πρὸς τὸν Β καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τοῦ Ε πρὸς τὸν Ζ λόγους. λέγω δὴ, ὅτι καὶ ἐλάχιστοι ἐν τοῖς Α Β, Γ Δ, Ε Ζ λόγοις. εἰ γὰρ μὴ, ἔσονται τινες τῶν Ν, Ξ, Μ, Ο ἐλάσσονες ἀριθμοὶ ἐξῆς ἀνάλογον ἐν τοῖς Α Β, Γ Δ, Ε Ζ λόγοις. ἔστωσαν οἱ Π, Ρ, Σ, Τ. καὶ ἐπεὶ ἔστιν ὡς ὁ Π πρὸς τὸν Ρ, οὕτως ὁ Α πρὸς τὸν Β, οἱ δὲ Α, Β ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ἰσάκις ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον, ὁ Β ἄρα τὸν Ρ μετρεῖ. διὰ τὰ αὐτὰ δὴ καὶ ὁ Γ τὸν Ρ μετρεῖ· οἱ Β, Γ ἄρα τὸν Ρ μετροῦσιν. καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Β, Γ μετρούμενος τὸν Ρ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Β, Γ μετρούμενος ἔστιν ὁ Η· ὁ Η ἄρα τὸν Ρ μετρεῖ. καὶ ἔστιν ὡς ὁ Η πρὸς τὸν Ρ, οὕτως ὁ Κ πρὸς τὸν Σ· καὶ ὁ Κ ἄρα τὸν Σ μετρεῖ. μετρεῖ δὲ καὶ ὁ Ε τὸν Σ· οἱ Ε, Κ ἄρα τὸν Σ μετροῦσιν. καὶ ὁ ἐλάχιστος ἄρα ὑπὸ τῶν Ε, Κ μετρούμενος τὸν Σ μετρήσει. ἐλάχιστος δὲ ὑπὸ τῶν Ε, Κ μετρούμενός ἐστιν ὁ Μ· ὁ Μ ἄρα τὸν Σ μετρεῖ ὁ μείζων τὸν ἐλάσσονα· ὅπερ ἔστιν ἀδύνατον. οὐκ ἄρα ἔσονται τινες τῶν

Thus, G measures O , the greater (measuring) the lesser. The very thing is impossible. Thus, there cannot be any numbers less than H, G, K, L (which are) continuously (proportional) in the ratio of A to B , and of C to D , and, further, of E to F .



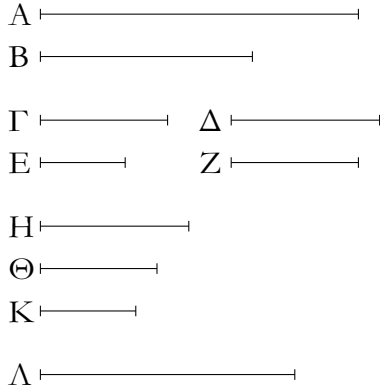
So let E not measure K . And let the least number, M , measured by (both) E and K have been taken [Prop. 7.34]. And as many times as K measures M , so many times let H, G also measure N, O , respectively. And as many times as E measures M , so many times let F also measure P . Since H measures N the same number of times as G (measures) O , thus as H is to G , so N (is) to O [Def. 7.20, Prop. 7.13]. And as H (is) to G , so A (is) to B . And thus as A (is) to B , so N (is) to O . And so, for the same (reasons), as C (is) to D , so O (is) to M . Again, since E measures M the same number of times as F (measures) P , thus as E is to F , so M (is) to P [Def. 7.20, Prop. 7.13]. Thus, N, O, M, P are continuously proportional in the ratios of A to B , and of C to D , and, further, of E to F . So I say that (they are) also the least (numbers) in the ratios of $A B, C D, E F$. For if not, then there will be some numbers less than N, O, M, P (which are) continuously proportional in the ratios of $A B, C D, E F$. Let them be Q, R, S, T . And since as Q is to R , so A (is) to B , and A and B (are) the least (numbers having the same ratio as them), and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20], B thus measures R . So, for the same (reasons), C also measures R . Thus, B and C (both) measure R . Thus, the least (number) measured by (both) B and C will also measure R [Prop. 7.35]. And G is the least number measured by (both) B and C . Thus, G measures R . And as G is to R , so K (is) to S . Thus,

N, Ξ, M, O ἐλάχιστονες ἀριθμοὶ ἐξῆς ἀνάλογον ἔν τε τοῖς τοῦ A πρὸς τὸν B καὶ τοῦ Γ πρὸς τὸν Δ καὶ ἔτι τοῦ E πρὸς τὸν Z λόγοις· οἱ N, Ξ, M, O ἄρα ἐξῆς ἀνάλογον ἐλάχιστοι εἰσιν ἐν τοῖς $A B, \Gamma \Delta, E Z$ λόγοις· ὅπερ ἔδει δεῖξαι.

K also measures S [Def. 7.20]. And E also measures S [Prop. 7.20]. Thus, E and K (both) measure S . Thus, the least (number) measured by (both) E and K will also measure S [Prop. 7.35]. And M is the least (number) measured by (both) E and K . Thus, M measures S , the greater (measuring) the lesser. The very thing is impossible. Thus there cannot be any numbers less than N, O, M, P (which are) continuously proportional in the ratios of A to B , and of C to D , and, further, of E to F . Thus, N, O, M, P are the least (numbers) continuously proportional in the ratios of $A B, C D, E F$. (Which is) the very thing it was required to show.

ε'.

Οἱ ἐπίπεδοι ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσι τὸν συγχεόμενον ἐκ τῶν πλευρῶν.



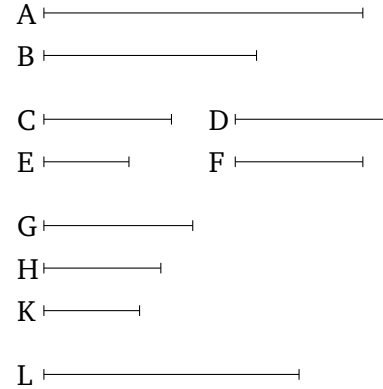
Ἐστωσαν ἐπίπεδοι ἀριθμοὶ οἱ A, B , καὶ τοῦ μὲν A πλευραὶ ἔστωσαν οἱ Γ, Δ ἀριθμοί, τοῦ δὲ B οἱ E, Z λέγω, ὅτι ὁ A πρὸς τὸν B λόγον ἔχει τὸν συγχεόμενον ἐκ τῶν πλευρῶν.

Λόγων γὰρ δοθέντων τοῦ τε ὃν ἔχει ὁ Γ πρὸς τὸν E καὶ ὁ Δ πρὸς τὸν Z εἰλήφθωσαν ἀριθμοὶ ἐξῆς ἐλάχιστοι ἐν τοῖς $\Gamma E, \Delta Z$ λόγοις, οἱ H, Θ, K , ὥστε εἶναι ὡς μὲν τὸν Γ πρὸς τὸν E , οὕτως τὸν H πρὸς τὸν Θ , ὡς δὲ τὸν Δ πρὸς τὸν Z , οὕτως τὸν Θ πρὸς τὸν K . καὶ ὁ Δ τὸν E πολλαπλασιάσας τὸν Λ ποιείτω.

Καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν A πεποίηκεν, τὸν δὲ E πολλαπλασιάσας τὸν Λ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν E , οὕτως ὁ A πρὸς τὸν Λ . ὡς δὲ ὁ Γ πρὸς τὸν E , οὕτως ὁ H πρὸς τὸν Θ . καὶ ὡς ἄρα ὁ H πρὸς τὸν Θ , οὕτως ὁ A πρὸς τὸν Λ . πάλιν, ἐπεὶ ὁ E τὸν Δ πολλαπλασιάσας τὸν Λ πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Z πολλαπλασιάσας τὸν B πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Z , οὕτως ὁ Λ πρὸς τὸν B . ἀλλ' ὡς ὁ Δ πρὸς τὸν Z , οὕτως ὁ Θ πρὸς τὸν K . καὶ ὡς ἄρα ὁ Θ πρὸς τὸν K , οὕτως ὁ Λ πρὸς τὸν B . ἐδείχθη δὲ καὶ ὡς ὁ H πρὸς τὸν Θ , οὕτως ὁ A πρὸς τὸν Λ . δι' ἴσου ἄρα ἔστιν ὡς ὁ H πρὸς τὸν K , [οὕτως] ὁ A πρὸς τὸν B . ὁ δὲ H πρὸς τὸν K λόγον ἔχει

Proposition 5

Plane numbers have to one another the ratio compounded† out of (the ratios of) their sides.



Let A and B be plane numbers, and let the numbers C, D be the sides of A , and (the numbers) E, F (the sides) of B . I say that A has to B the ratio compounded out of (the ratios of) their sides.

For given the ratios which C has to E , and D (has) to F , let the least numbers, G, H, K , continuously proportional in the ratios $C E, D F$ have been taken [Prop. 8.4], so that as C is to E , so G (is) to H , and as D (is) to F , so H (is) to K . And let D make L (by) multiplying E .

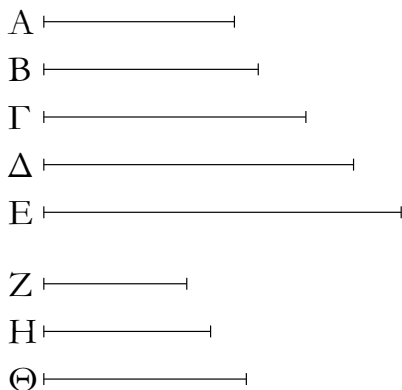
And since D has made A (by) multiplying C , and has made L (by) multiplying E , thus as C is to E , so A (is) to L [Prop. 7.17]. And as C (is) to E , so G (is) to H . And thus as G (is) to H , so A (is) to L . Again, since E has made L (by) multiplying D [Prop. 7.16], but, in fact, has also made B (by) multiplying F , thus as D is to F , so L (is) to B [Prop. 7.17]. But, as D (is) to F , so H (is) to K . And thus as H (is) to K , so L (is) to B . And it was also shown that as G (is) to H , so A (is) to L . Thus, via equality, as G is to K , [so] A (is) to B [Prop. 7.14]. And G has to K the ratio compounded out of (the ratios of) the sides (of A and B). Thus, A also has to B the ratio compounded out of (the ratios of) the sides (of A and B).

τὸν συγκείμενον ἐκ τῶν πλευρῶν· καὶ ὁ A ἄρα πρὸς τὸν B λόγον ἔχει τὸν συγκείμενον ἐκ τῶν πλευρῶν· ὅπερ ἔδει δεῖξαι.

† i.e., multiplied.

ε'.

Ἐάν ὧσιν ὁποσοιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον, ὁ δὲ πρῶτος τὸν δεύτερον μὴ μετρήῃ, οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει.



Ἐστωσαν ὁποσοιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον οἱ A, B, Γ, Δ, E , ὁ δὲ A τὸν B μὴ μετρεῖται· λέγω, ὅτι οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει.

Ὅτι μὲν οὖν οἱ A, B, Γ, Δ, E ἐξῆς ἀλλήλους οὐ μετροῦσιν, φανερόν· οὐδὲ γὰρ ὁ A τὸν B μετρεῖ. λέγω δὴ, ὅτι οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει. εἰ γὰρ δυνατόν, μετρεῖται ὁ A τὸν Γ . καὶ ὅσοι εἰσὶν οἱ A, B, Γ , τοσοῦτοι εἰληφθῶσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A, B, Γ οἱ Z, H, Θ . καὶ ἐπεὶ οἱ Z, H, Θ ἐν τῷ αὐτῷ λόγῳ εἰσὶ τοῖς A, B, Γ , καὶ ἐστὶν ἴσον τὸ πλῆθος τῶν A, B, Γ τῷ πλῆθει τῶν Z, H, Θ , δι' ἴσου ἄρα ἐστὶν ὡς ὁ A πρὸς τὸν Γ , οὕτως ὁ Z πρὸς τὸν Θ . καὶ ἐπεὶ ἐστὶν ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Z πρὸς τὸν H , οὐ μετρεῖ δὲ ὁ A τὸν B , οὐ μετρεῖ ἄρα οὐδὲ ὁ Z τὸν H · οὐκ ἄρα μονὰς ἐστὶν ὁ Z · ἢ γὰρ μονὰς πάντα ἀριθμὸν μετρεῖ. καὶ εἰσὶν οἱ Z, Θ πρῶτοι πρὸς ἀλλήλους [οὐδὲ ὁ Z ἄρα τὸν Θ μετρεῖ]. καὶ ἐστὶν ὡς ὁ Z πρὸς τὸν Θ , οὕτως ὁ A πρὸς τὸν Γ · οὐδὲ ὁ A ἄρα τὸν Γ μετρεῖ. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει· ὅπερ ἔδει δεῖξαι.

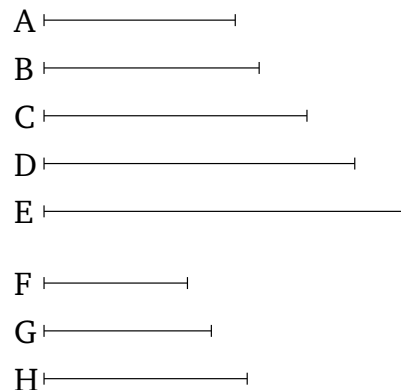
ζ'.

Ἐάν ὧσιν ὁποσοιοῦν ἀριθμοὶ [ἐξῆς] ἀνάλογον, ὁ δὲ πρῶτος τὸν ἑσχατον μετρήῃ, καὶ τὸν δεύτερον μετρήσει.

(Which is) the very thing it was required to show.

Proposition 6

If there are any multitude whatsoever of continuously proportional numbers, and the first does not measure the second, then no other (number) will measure any other (number) either.

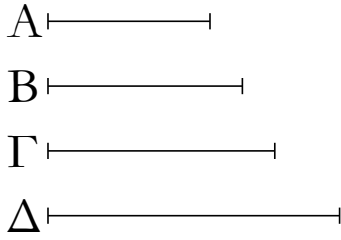


Let A, B, C, D, E be any multitude whatsoever of continuously proportional numbers, and let A not measure B . I say that no other (number) will measure any other (number) either.

Now, (it is) clear that A, B, C, D, E do not successively measure one another. For A does not even measure B . So I say that no other (number) will measure any other (number) either. For, if possible, let A measure C . And as many (numbers) as are A, B, C , let so many of the least numbers, F, G, H , have been taken of those (numbers) having the same ratio as A, B, C [Prop. 7.33]. And since F, G, H are in the same ratio as A, B, C , and the multitude of A, B, C is equal to the multitude of F, G, H , thus, via equality, as A is to C , so F (is) to H [Prop. 7.14]. And since as A is to B , so F (is) to G , and A does not measure B , F does not measure G either [Def. 7.20]. Thus, F is not a unit. For a unit measures all numbers. And F and H are prime to one another [Prop. 8.3] [and thus F does not measure H]. And as F is to H , so A (is) to C . And thus A does not measure C either [Def. 7.20]. So, similarly, we can show that no other (number) can measure any other (number) either. (Which is) the very thing it was required to show.

Proposition 7

If there are any multitude whatsoever of [continuously] proportional numbers, and the first measures the

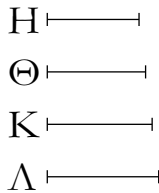
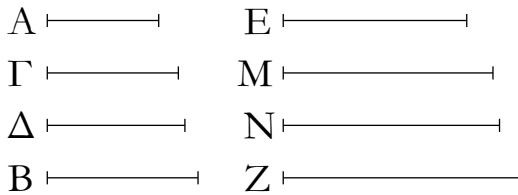


Ἐστωσαν ὁποσοιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον οἱ A, B, Γ, Δ , ὁ δὲ A τὸν Δ μετρεῖτω· λέγω, ὅτι καὶ ὁ A τὸν B μετρεῖ.

Εἰ γὰρ οὐ μετρεῖ ὁ A τὸν B , οὐδὲ ἄλλος οὐδεὶς οὐδένα μετρήσει· μετρεῖ δὲ ὁ A τὸν Δ . μετρεῖ ἄρα καὶ ὁ A τὸν B · ὅπερ ἔδει δεῖξαι.

η'.

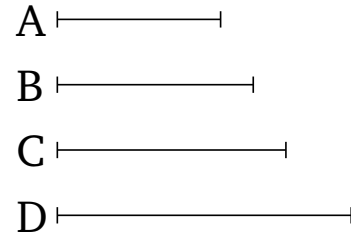
Ἐάν δύο ἀριθμῶν μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας [αὐτοῖς] μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται



Δύο γὰρ ἀριθμῶν τῶν A, B μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτέτωσαν ἀριθμοὶ οἱ Γ, Δ , καὶ πεποιήσθω ὡς ὁ A πρὸς τὸν B , οὕτως ὁ E πρὸς τὸν Z · λέγω, ὅτι ὅσοι εἰς τοὺς A, B μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς E, Z μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Ὅσοι γὰρ εἰσι τῷ πλήθει οἱ A, B, Γ, Δ , τοσοῦτοι εἰληφθῶσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς A, Γ, Δ, B οἱ H, Θ, K, Λ · οἱ ἄρα ἄκροι αὐτῶν οἱ H, Λ πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ οἱ A, Γ, Δ, B τοῖς H, Θ, K, Λ ἐν τῷ αὐτῷ λόγῳ εἰσίν, καὶ ἐστὶν ἴσον τὸ πλήθος τῶν A, Γ, Δ, B τῷ πλήθει τῶν H, Θ, K, Λ , δι' ἴσου ἄρα ἐστὶν ὡς ὁ A πρὸς τὸν B , οὕτως ὁ H πρὸς τὸν Λ . ὡς δὲ ὁ A πρὸς τὸν B , οὕτως ὁ E πρὸς τὸν Z · καὶ

last, then (the first) will also measure the second.

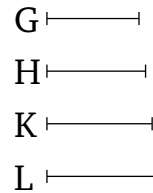
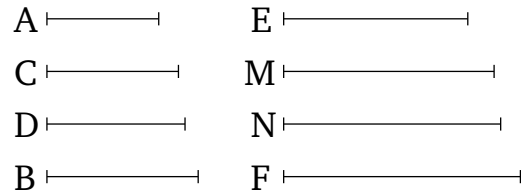


Let A, B, C, D be any number whatsoever of continuously proportional numbers. And let A measure D . I say that A also measures B .

For if A does not measure B then no other (number) will measure any other (number) either [Prop. 8.6]. But A measures D . Thus, A also measures B . (Which is) the very thing it was required to show.

Proposition 8

If between two numbers there fall (some) numbers in continued proportion then, as many numbers as fall in between them in continued proportion, so many (numbers) will also fall in between (any two numbers) having the same ratio [as them] in continued proportion.



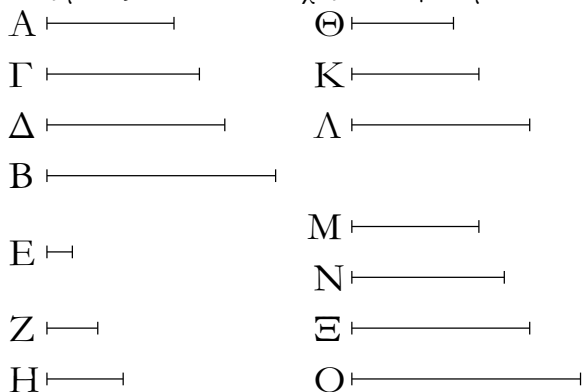
For let the numbers, C and D , fall between two numbers, A and B , in continued proportion, and let it have been contrived (that) as A (is) to B , so E (is) to F . I say that, as many numbers as have fallen in between A and B in continued proportion, so many (numbers) will also fall in between E and F in continued proportion.

For as many as A, B, C, D are in multitude, let so many of the least numbers, G, H, K, L , having the same ratio as A, B, C, D , have been taken [Prop. 7.33]. Thus, the outermost of them, G and L , are prime to one another [Prop. 8.3]. And since A, B, C, D are in the same ratio as G, H, K, L , and the multitude of A, B, C, D is equal to the multitude of G, H, K, L , thus, via equality, as A is to B , so G (is) to L [Prop. 7.14]. And as A (is) to B , so

ὥς ἄρα ὁ H πρὸς τὸν Λ , οὕτως ὁ E πρὸς τὸν Z . οἱ δὲ H , Λ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκεις ὃ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὃ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον. ἰσάκεις ἄρα ὁ H τὸν E μετρεῖ καὶ ὁ Λ τὸν Z . ὁσάκεις δὴ ὁ H τὸν E μετρεῖ, τοσαυτάκεις καὶ ἑκάτερος τῶν Θ , K ἑκάτερον τῶν M , N μετρεῖται· οἱ H , Θ , K , Λ ἄρα τοὺς E , M , N , Z ἰσάκεις μετροῦσιν. οἱ H , Θ , K , Λ ἄρα τοῖς E , M , N , Z ἐν τῷ αὐτῷ λόγῳ εἰσίν. ἀλλὰ οἱ H , Θ , K , Λ τοῖς A , Γ , Δ , B ἐν τῷ αὐτῷ λόγῳ εἰσίν· καὶ οἱ A , Γ , Δ , B ἄρα τοῖς E , M , N , Z ἐν τῷ αὐτῷ λόγῳ εἰσίν. οἱ δὲ A , Γ , Δ , B ἐξῆς ἀνάλογόν εἰσιν· καὶ οἱ E , M , N , Z ἄρα ἐξῆς ἀνάλογόν εἰσιν. ὅσοι ἄρα εἰς τοὺς A , B μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς E , Z μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί· ὅπερ ἔδει δεῖξαι.

θ'.

Ἐάν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ ἑκατέρου αὐτῶν καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.



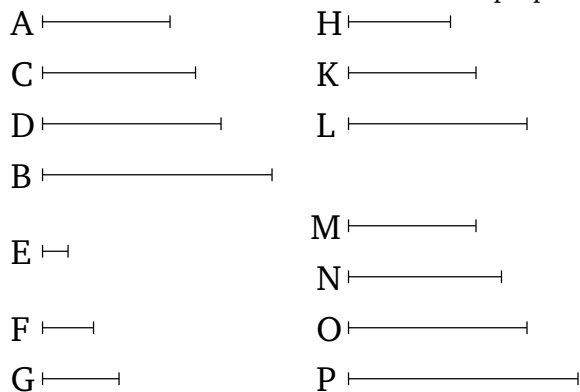
Ἐστῶσαν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους οἱ A , B , καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτέωσαν οἱ Γ , Δ , καὶ ἐκχείσθω ἡ E μονάς· λέγω, ὅτι ὅσοι εἰς τοὺς A , B μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ ἑκατέρου τῶν A , B καὶ τῆς μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

Εἰλήφθωσαν γὰρ δύο μὲν ἀριθμοὶ ἐλάχιστοι ἐν τῷ τῶν A , Γ , Δ , B λόγῳ ὄντες οἱ Z , H , τρεῖς δὲ οἱ Θ , K , Λ , καὶ αἰ

E (is) to F . And thus as G (is) to L , so E (is) to F . And G and L (are) prime (to one another). And (numbers) prime (to one another are) also the least (numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, G measures E the same number of times as L (measures) F . So as many times as G measures E , so many times let H , K also measure M , N , respectively. Thus, G , H , K , L measure E , M , N , F (respectively) an equal number of times. Thus, G , H , K , L are in the same ratio as E , M , N , F [Def. 7.20]. But, G , H , K , L are in the same ratio as A , C , D , B . Thus, A , C , D , B are also in the same ratio as E , M , N , F . And A , C , D , B are continuously proportional. Thus, E , M , N , F are also continuously proportional. Thus, as many numbers as have fallen in between A and B in continued proportion, so many numbers have also fallen in between E and F in continued proportion. (Which is) the very thing it was required to show.

Proposition 9

If two numbers are prime to one another and there fall in between them (some) numbers in continued proportion then, as many numbers as fall in between them in continued proportion, so many (numbers) will also fall between each of them and a unit in continued proportion.



Let A and B be two numbers (which are) prime to one another, and let the (numbers) C and D fall in between them in continued proportion. And let the unit E be set out. I say that, as many numbers as have fallen in between A and B in continued proportion, so many (numbers) will also fall between each of A and B and the unit in continued proportion.

For let the least two numbers, F and G , which are in the ratio of A , C , D , B , have been taken [Prop. 7.33].

ἐξῆς ἐνὶ πλείους, ἕως ἂν ἴσον γένηται τὸ πλῆθος αὐτῶν τῷ πλήθει τῶν A, Γ, Δ, B . εἰλήφθωσαν, καὶ ἔστωσαν οἱ M, N, Ξ, O . φανερόν δὲ, ὅτι ὁ μὲν Z ἑαυτὸν πολλαπλασιάσας τὸν Θ πεποίηκεν, τὸν δὲ Θ πολλαπλασιάσας τὸν M πεποίηκεν, καὶ ὁ H ἑαυτὸν μὲν πολλαπλασιάσας τὸν Λ πεποίηκεν, τὸν δὲ Λ πολλαπλασιάσας τὸν O πεποίηκεν. καὶ ἐπεὶ οἱ M, N, Ξ, O ἐλάχιστοι εἰσι τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς Z, H , εἰσὶ δὲ καὶ οἱ A, Γ, Δ, B ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς Z, H , καὶ ἔστιν ἴσον τὸ πλῆθος τῶν M, N, Ξ, O τῷ πλήθει τῶν A, Γ, Δ, B , ἕκαστος ἄρα τῶν M, N, Ξ, O ἐκάστῳ τῶν A, Γ, Δ, B ἴσος ἐστίν· ἴσος ἄρα ἐστὶν ὁ μὲν M τῷ A , ὁ δὲ O τῷ B . καὶ ἐπεὶ ὁ Z ἑαυτὸν πολλαπλασιάσας τὸν Θ πεποίηκεν, ὁ Z ἄρα τὸν Θ μετρεῖ κατὰ τὰς ἐν τῷ Z μονάδας. μετρεῖ δὲ καὶ ἡ E μονὰς τὸν Z κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκεις ἄρα ἡ E μονὰς τὸν Z ἀριθμὸν μετρεῖ καὶ ὁ Z τὸν Θ . ἔστιν ἄρα ὡς ἡ E μονὰς πρὸς τὸν Z ἀριθμὸν, οὕτως ὁ Z πρὸς τὸν Θ . πάλιν, ἐπεὶ ὁ Z τὸν Θ πολλαπλασιάσας τὸν M πεποίηκεν, ὁ Θ ἄρα τὸν M μετρεῖ κατὰ τὰς ἐν τῷ Z μονάδας. μετρεῖ δὲ καὶ ἡ E μονὰς τὸν Z ἀριθμὸν κατὰ τὰς ἐν αὐτῷ μονάδας· ἰσάκεις ἄρα ἡ E μονὰς τὸν Z ἀριθμὸν μετρεῖ καὶ ὁ Θ τὸν M . ἔστιν ἄρα ὡς ἡ E μονὰς πρὸς τὸν Z ἀριθμὸν, οὕτως ὁ Θ πρὸς τὸν M . ἐδείχθη δὲ καὶ ὡς ἡ E μονὰς πρὸς τὸν Z ἀριθμὸν, οὕτως ὁ Z πρὸς τὸν Θ . καὶ ὡς ἄρα ἡ E μονὰς πρὸς τὸν Z ἀριθμὸν, οὕτως ὁ Z πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν M . ἴσος δὲ ὁ M τῷ A · ἔστιν ἄρα ὡς ἡ E μονὰς πρὸς τὸν Z ἀριθμὸν, οὕτως ὁ Z πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν A . διὰ τὰ αὐτὰ δὴ καὶ ὡς ἡ E μονὰς πρὸς τὸν H ἀριθμὸν, οὕτως ὁ H πρὸς τὸν Λ καὶ ὁ Λ πρὸς τὸν B . ὅσοι ἄρα εἰς τοὺς A, B μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ ἐκατέρου τῶν A, B καὶ μονάδος τῆς E μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί· ὅπερ ἔδει δεῖξαι.

ι'.

Ἐάν δύο ἀριθμῶν ἑκατέρου καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι ἑκατέρου αὐτῶν καὶ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς αὐτοὺς μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

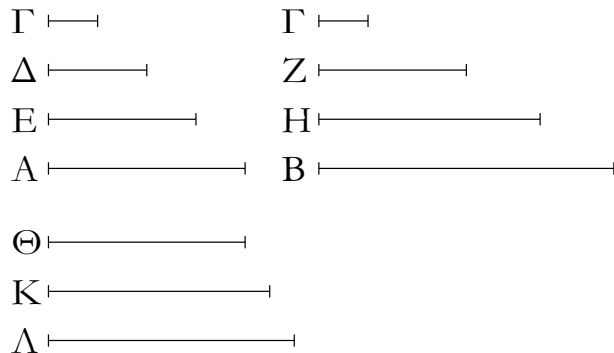
Δύο γὰρ ἀριθμῶν τῶν A, B καὶ μονάδος τῆς Γ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτέωσαν ἀριθμοί οἱ Δ, E καὶ οἱ Z, H . λέγω, ὅτι ὅσοι ἑκατέρου τῶν A, B καὶ μονάδος τῆς Γ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεπτώκασιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς A, B μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπεσοῦνται.

And the (least) three (numbers), H, K, L . And so on, successively increasing by one, until the multitude of the (least numbers taken) is made equal to the multitude of A, C, D, B [Prop. 8.2]. Let them have been taken, and let them be M, N, O, P . So (it is) clear that F has made H (by) multiplying itself, and has made M (by) multiplying H . And G has made L (by) multiplying itself, and has made P (by) multiplying L [Prop. 8.2 corr.]. And since M, N, O, P are the least of those (numbers) having the same ratio as F, G , and A, C, D, B are also the least of those (numbers) having the same ratio as F, G [Prop. 8.2], and the multitude of M, N, O, P is equal to the multitude of A, C, D, B , thus M, N, O, P are equal to A, C, D, B , respectively. Thus, M is equal to A , and P to B . And since F has made H (by) multiplying itself, F thus measures H according to the units in F [Def. 7.15]. And the unit E also measures F according to the units in it. Thus, the unit E measures the number F as many times as F (measures) H . Thus, as the unit E is to the number F , so F (is) to H [Def. 7.20]. Again, since F has made M (by) multiplying H , H thus measures M according to the units in F [Def. 7.15]. And the unit E also measures the number F according to the units in it. Thus, the unit E measures the number F as many times as H (measures) M . Thus, as the unit E is to the number F , so H (is) to M [Prop. 7.20]. And it was shown that as the unit E (is) to the number F , so F (is) to H . And thus as the unit E (is) to the number F , so F (is) to H , and H (is) to M . And M (is) equal to A . Thus, as the unit E is to the number F , so F (is) to H , and H to A . And so, for the same (reasons), as the unit E (is) to the number G , so G (is) to L , and L to B . Thus, as many (numbers) as have fallen in between A and B in continued proportion, so many numbers have also fallen between each of A and B and the unit E in continued proportion. (Which is) the very thing it was required to show.

Proposition 10

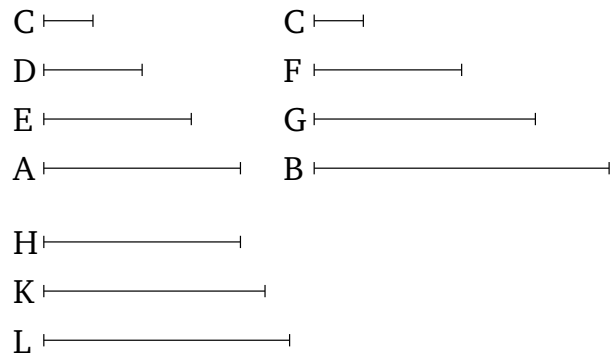
If (some) numbers fall between each of two numbers and a unit in continued proportion then, as many (numbers) as fall between each of the (two numbers) and the unit in continued proportion, so many (numbers) will also fall in between the (two numbers) themselves in continued proportion.

For let the numbers D, E and F, G fall between the numbers A and B (respectively) and the unit C in continued proportion. I say that, as many numbers as have fallen between each of A and B and the unit C in continued proportion, so many will also fall in between A and B in continued proportion.



Ὁ Δ γὰρ τὸν Ζ πολλαπλασιάσας τὸν Θ ποιείτω, ἑκάτερος δὲ τῶν Δ, Ζ τὸν Θ πολλαπλασιάσας ἑκάτερον τῶν Κ, Λ ποιείτω.

Καὶ ἐπεὶ ἔστιν ὡς ἡ Γ μονὰς πρὸς τὸν Δ ἀριθμὸν, οὕτως ὁ Δ πρὸς τὸν Ε, ἰσάκεις ἄρα ἡ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ Δ τὸν Ε. ἡ δὲ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· καὶ ὁ Δ ἄρα ἀριθμὸς τὸν Ε μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὁ Δ ἄρα ἑαυτὸν πολλαπλασιάσας τὸν Ε πεποίηκεν. πάλιν, ἐπεὶ ἔστιν ὡς ἡ Γ [μονὰς] πρὸς τὸν Δ ἀριθμὸν, οὕτως ὁ Ε πρὸς τὸν Α, ἰσάκεις ἄρα ἡ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ καὶ ὁ Ε τὸν Α. ἡ δὲ Γ μονὰς τὸν Δ ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· καὶ ὁ Ε ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὁ Δ ἄρα τὸν Ε πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ μὲν Ζ ἑαυτὸν πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Η πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Δ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Θ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Ε πρὸς τὸν Θ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Θ πρὸς τὸν Η. καὶ ὡς ἄρα ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Θ πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Δ ἑκάτερον τῶν Ε, Θ πολλαπλασιάσας ἑκάτερον τῶν Α, Κ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Κ. ἀλλ' ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Δ πρὸς τὸν Ζ· καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Κ. πάλιν, ἐπεὶ ἑκάτερος τῶν Δ, Ζ τὸν Θ πολλαπλασιάσας ἑκάτερον τῶν Κ, Λ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Κ πρὸς τὸν Λ. ἀλλ' ὡς ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Κ· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Κ, οὕτως ὁ Κ πρὸς τὸν Λ. ἔτι ἐπεὶ ὁ Ζ ἑκάτερον τῶν Θ, Η πολλαπλασιάσας ἑκάτερον τῶν Α, Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν Η, οὕτως ὁ Α πρὸς τὸν Β. ὡς δὲ ὁ Θ πρὸς τὸν Η, οὕτως ὁ Δ πρὸς τὸν Ζ· καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Κ καὶ ὁ Κ πρὸς τὸν Λ· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Κ, οὕτως ὁ Κ πρὸς τὸν Λ καὶ ὁ Λ πρὸς τὸν Β. οἱ Α, Κ, Λ, Β ἄρα κατὰ τὸ συνεχὲς ἐξῆς εἰσιν ἀνάλογον. ὅσοι ἄρα ἑκατέρου τῶν Α, Β καὶ τῆς Γ μονάδος μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν ἀριθμοί, τοσοῦτοι καὶ εἰς τοὺς Α, Β μεταξὺ κατὰ τὸ συνεχὲς ἐμπεσοῦνται· ὅπερ ἔδει



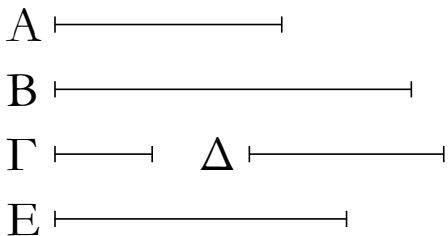
For let D make H (by) multiplying F . And let D, F make K, L , respectively, by multiplying H .

As since as the unit C is to the number D , so D (is) to E , the unit C thus measures the number D as many times as D (measures) E [Def. 7.20]. And the unit C measures the number D according to the units in D . Thus, the number D also measures E according to the units in D . Thus, D has made E (by) multiplying itself. Again, since as the [unit] C is to the number D , so E (is) to A , the unit C thus measures the number D as many times as E (measures) A [Def. 7.20]. And the unit C measures the number D according to the units in D . Thus, E also measures A according to the units in D . Thus, D has made A (by) multiplying E . And so, for the same (reasons), F has made G (by) multiplying itself, and has made B (by) multiplying G . And since D has made E (by) multiplying itself, and has made H (by) multiplying F , thus as D is to F , so E (is) to H [Prop 7.17]. And so, for the same reasons, as D (is) to F , so H (is) to G [Prop. 7.18]. And thus as E (is) to H , so H (is) to G . Again, since D has made A, K (by) multiplying E, H , respectively, thus as E is to H , so A (is) to K [Prop 7.17]. But, as E (is) to H , so D (is) to F . And thus as D (is) to F , so A (is) to K . Again, since D, F have made K, L , respectively, (by) multiplying H , thus as D is to F , so K (is) to L [Prop. 7.18]. But, as D (is) to F , so A (is) to K . And thus as A (is) to K , so K (is) to L . Further, since F has made L, B (by) multiplying H, G , respectively, thus as H is to G , so L (is) to B [Prop 7.17]. And as H (is) to G , so D (is) to F . And thus as D (is) to F , so L (is) to B . And it was also shown that as D (is) to F , so A (is) to K , and K to L . And thus as A (is) to K , so K (is) to L , and L to B . Thus, A, K, L, B are successively in continued proportion. Thus, as many numbers as fall between each of A and B and the unit C in continued proportion, so many will also fall in between A and B in continued proportion. (Which is) the very thing it was required to show.

δείξαι.

ια'.

Δύο τετραγώνων ἀριθμῶν εἰς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ τετράγωνος πρὸς τὸν τετράγωνον διπλασίονα λόγον ἔχει ἥπερ ἡ πλευρὰ πρὸς τὴν πλευράν.



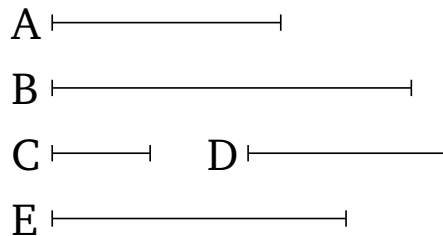
Ἐστωσαν τετράγωνοι ἀριθμοὶ οἱ A, B , καὶ τοῦ μὲν A πλευρὰ ἔστω ὁ Γ , τοῦ δὲ B ὁ Δ . λέγω, ὅτι τῶν A, B εἰς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ A πρὸς τὸν B διπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ .

Ὁ Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν E ποιεῖται. καὶ ἐπεὶ τετράγωνός ἐστιν ὁ A , πλευρὰ δὲ αὐτοῦ ἐστιν ὁ Γ , ὁ Γ ἄρα ἑαυτὸν πολλαπλασιάσας τὸν A πεποιήκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Δ ἑαυτὸν πολλαπλασιάσας τὸν B πεποιήκεν. ἐπεὶ οὖν ὁ Γ ἑκάτερον τῶν Γ, Δ πολλαπλασιάσας ἑκάτερον τῶν A, E πεποιήκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ , οὕτως ὁ A πρὸς τὸν E . διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ , οὕτως ὁ E πρὸς τὸν B . καὶ ὡς ἄρα ὁ A πρὸς τὸν E , οὕτως ὁ E πρὸς τὸν B . τῶν A, B ἄρα εἰς μέσος ἀνάλογόν ἐστιν ἀριθμός.

Λέγω δὴ, ὅτι καὶ ὁ A πρὸς τὸν B διπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ . ἐπεὶ γὰρ τρεῖς ἀριθμοὶ ἀνάλογόν εἰσιν οἱ A, E, B , ὁ A ἄρα πρὸς τὸν B διπλασίονα λόγον ἔχει ἥπερ ὁ A πρὸς τὸν E . ὡς δὲ ὁ A πρὸς τὸν E , οὕτως ὁ Γ πρὸς τὸν Δ . ὁ A ἄρα πρὸς τὸν B διπλασίονα λόγον ἔχει ἥπερ ἡ Γ πλευρὰ πρὸς τὴν Δ . ὅπερ ἔδει δείξαι.

Proposition 11

There exists one number in mean proportion to two (given) square numbers.[†] And (one) square (number) has to the (other) square (number) a squared[‡] ratio with respect to (that) the side (of the former has) to the side (of the latter).



Let A and B be square numbers, and let C be the side of A , and D (the side) of B . I say that there exists one number in mean proportion to A and B , and that A has to B a squared ratio with respect to (that) C (has) to D .

For let C make E (by) multiplying D . And since A is square, and C is its side, C has thus made A (by) multiplying itself. And so, for the same (reasons), D has made B (by) multiplying itself. Therefore, since C has made A , E (by) multiplying C , D , respectively, thus as C is to D , so A (is) to E [Prop. 7.17]. And so, for the same (reasons), as C (is) to D , so E (is) to B [Prop. 7.18]. And thus as A (is) to E , so E (is) to B . Thus, one number (namely, E) is in mean proportion to A and B .

So I say that A also has to B a squared ratio with respect to (that) C (has) to D . For since A, E, B are three (continuously) proportional numbers, A thus has to B a squared ratio with respect to (that) A (has) to E [Def. 5.9]. And as A (is) to E , so C (is) to D . Thus, A has to B a squared ratio with respect to (that) side C (has) to (side) D . (Which is) the very thing it was required to show.

[†] In other words, between two given square numbers there exists a number in continued proportion.

[‡] Literally, "double".

ιβ'.

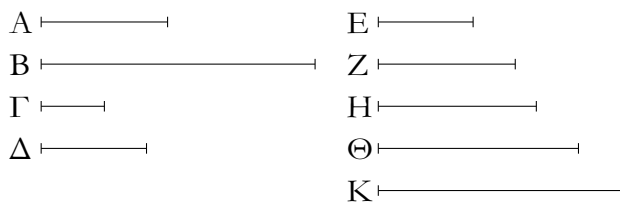
Δύο κύβων ἀριθμῶν δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί, καὶ ὁ κύβος πρὸς τὸν κύβον τριπλασίονα λόγον ἔχει ἥπερ ἡ πλευρὰ πρὸς τὴν πλευράν.

Ἐστωσαν κύβοι ἀριθμοὶ οἱ A, B καὶ τοῦ μὲν A πλευρὰ ἔστω ὁ Γ , τοῦ δὲ B ὁ Δ . λέγω, ὅτι τῶν A, B δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί, καὶ ὁ A πρὸς τὸν B τριπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ .

Proposition 12

There exist two numbers in mean proportion to two (given) cube numbers.[†] And (one) cube (number) has to the (other) cube (number) a cubed[‡] ratio with respect to (that) the side (of the former has) to the side (of the latter).

Let A and B be cube numbers, and let C be the side of A , and D (the side) of B . I say that there exist two numbers in mean proportion to A and B , and that A has



Ὁ γὰρ Γ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε ποιεῖτω, τὸν δὲ Δ πολλαπλασιάσας τὸν Ζ ποιεῖτω, ὁ δὲ Δ ἑαυτὸν πολλαπλασιάσας τὸν Η ποιεῖτω, ἐκάτερος δὲ τῶν Γ, Δ τὸν Ζ πολλαπλασιάσας ἐκάτερον τῶν Θ, Κ ποιεῖτω.

Καὶ ἐπεὶ κύβος ἐστὶν ὁ Α, πλευρὰ δὲ αὐτοῦ ὁ Γ, καὶ ὁ Γ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηκεν, ὁ Γ ἄρα ἑαυτὸν μὲν πολλαπλασιάσας τὸν Ε πεποίηκεν, τὸν δὲ Ε πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Δ ἑαυτὸν μὲν πολλαπλασιάσας τὸν Η πεποίηκεν, τὸν δὲ Η πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Γ ἐκάτερον τῶν Γ, Δ πολλαπλασιάσας ἐκάτερον τῶν Ε, Ζ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ε πρὸς τὸν Ζ. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν Η. πάλιν, ἐπεὶ ὁ Γ ἐκάτερον τῶν Ε, Ζ πολλαπλασιάσας ἐκάτερον τῶν Α, Θ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Α πρὸς τὸν Θ. ὡς δὲ ὁ Ε πρὸς τὸν Ζ, οὕτως ὁ Γ πρὸς τὸν Δ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Θ. πάλιν, ἐπεὶ ἐκάτερος τῶν Γ, Δ τὸν Ζ πολλαπλασιάσας ἐκάτερον τῶν Θ, Κ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Θ πρὸς τὸν Κ. πάλιν, ἐπεὶ ὁ Δ ἐκάτερον τῶν Ζ, Η πολλαπλασιάσας ἐκάτερον τῶν Κ, Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ζ πρὸς τὸν Η, οὕτως ὁ Κ πρὸς τὸν Β. ὡς δὲ ὁ Ζ πρὸς τὸν Η, οὕτως ὁ Γ πρὸς τὸν Δ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Θ καὶ ὁ Θ πρὸς τὸν Κ καὶ ὁ Κ πρὸς τὸν Β. τῶν Α, Β ἄρα δύο μέσοι ἀνάλογόν εἰσιν οἱ Θ, Κ.

Λέγω δὴ, ὅτι καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ. ἐπεὶ γὰρ τέσσαρες ἀριθμοὶ ἀνάλογόν εἰσιν οἱ Α, Θ, Κ, Β, ὁ Α ἄρα πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Α πρὸς τὸν Θ. ὡς δὲ ὁ Α πρὸς τὸν Θ, οὕτως ὁ Γ πρὸς τὸν Δ· καὶ ὁ Α [ἄρα] πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Δ· ὅπερ ἔδει δεῖξαι.

† In other words, between two given cube numbers there exist two numbers in continued proportion.

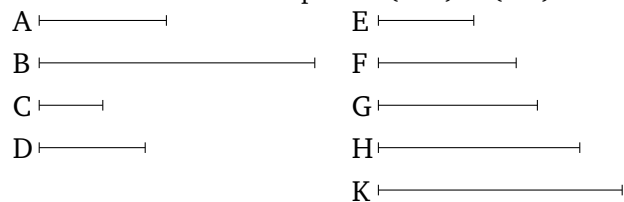
‡ Literally, "triple".

ιγ'.

Ἐὰν ὧσιν ὁποιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον, καὶ πολλαπλασιάσας ἕκαστος ἑαυτὸν ποιῇ τινα, οἱ γεγόμενοι ἐξ αὐτῶν ἀνάλογον ἔσονται· καὶ ἐὰν οἱ ἐξ ἀρχῆς τοὺς γενομένους πολλαπλασιάσαντες ποιῶσιν τινας, καὶ αὐτοὶ ἀνάλογον ἔσονται [καὶ αἰεὶ περὶ τοὺς ἄχρους τοῦτο συμβαίνει].

Ἔστωσαν ὁποιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον, οἱ Α, Β,

to *B* a cubed ratio with respect to (that) *C* (has) to *D*.



For let *C* make *E* (by) multiplying itself, and let it make *F* (by) multiplying *D*. And let *D* make *G* (by) multiplying itself, and let *C*, *D* make *H*, *K*, respectively, (by) multiplying *F*.

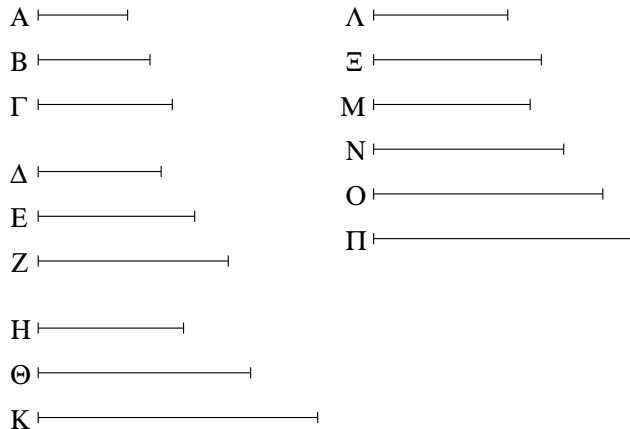
And since *A* is cube, and *C* (is) its side, and *C* has made *E* (by) multiplying itself, *C* has thus made *E* (by) multiplying itself, and has made *A* (by) multiplying *E*. And so, for the same (reasons), *D* has made *G* (by) multiplying itself, and has made *B* (by) multiplying *G*. And since *C* has made *E*, *F* (by) multiplying *C*, *D*, respectively, thus as *C* is to *D*, so *E* (is) to *F* [Prop. 7.17]. And so, for the same (reasons), as *C* (is) to *D*, so *F* (is) to *G* [Prop. 7.18]. Again, since *C* has made *A*, *H* (by) multiplying *E*, *F*, respectively, thus as *E* is to *F*, so *A* (is) to *H* [Prop. 7.17]. And as *E* (is) to *F*, so *C* (is) to *D*. And thus as *C* (is) to *D*, so *A* (is) to *H*. Again, since *C*, *D* have made *H*, *K*, respectively, (by) multiplying *F*, thus as *C* is to *D*, so *H* (is) to *K* [Prop. 7.18]. Again, since *D* has made *K*, *B* (by) multiplying *F*, *G*, respectively, thus as *F* is to *G*, so *K* (is) to *B* [Prop. 7.17]. And as *F* (is) to *G*, so *C* (is) to *D*. And thus as *C* (is) to *D*, so *A* (is) to *H*, and *H* to *K*, and *K* to *B*. Thus, *H* and *K* are two (numbers) in mean proportion to *A* and *B*.

So I say that *A* also has to *B* a cubed ratio with respect to (that) *C* (has) to *D*. For since *A*, *H*, *K*, *B* are four (continuously) proportional numbers, *A* thus has to *B* a cubed ratio with respect to (that) *A* (has) to *H* [Def. 5.10]. And as *A* (is) to *H*, so *C* (is) to *D*. And [thus] *A* has to *B* a cubed ratio with respect to (that) *C* (has) to *D*. (Which is) the very thing it was required to show.

Proposition 13

If there are any multitude whatsoever of continuously proportional numbers, and each makes some (number by) multiplying itself, then the (numbers) created from them will (also) be (continuously) proportional. And if the original (numbers) make some (more numbers by) multiplying the created (numbers) then these will also

Γ, ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ, καὶ οἱ Α, Β, Γ ἑαυτοὺς μὲν πολλαπλασιάσαντες τοὺς Δ, Ε, Ζ ποιείωσαν, τοὺς δὲ Δ, Ε, Ζ πολλαπλασιάσαντες τοὺς Η, Θ, Κ ποιείωσαν· λέγω, ὅτι οἱ τε Δ, Ε, Ζ καὶ οἱ Η, Θ, Κ ἐξῆς ἀνάλογον εἰσιν.



Ὁ μὲν γὰρ Α τὸν Β πολλαπλασιάσας τὸν Λ ποιείτω, ἑκάτερος δὲ τῶν Α, Β τὸν Λ πολλαπλασιάσας ἑκάτερον τῶν Μ, Ν ποιείτω. καὶ πάλιν ὁ μὲν Β τὸν Γ πολλαπλασιάσας τὸν Ξ ποιείτω, ἑκάτερος δὲ τῶν Β, Γ τὸν Ξ πολλαπλασιάσας ἑκάτερον τῶν Ο, Π ποιείτω.

Ὁμοίως δὴ τοῖς ἐπάνω δεῖξομεν, ὅτι οἱ Δ, Α, Ε καὶ οἱ Η, Μ, Ν, Θ ἐξῆς εἰσιν ἀνάλογον ἐν τῷ τοῦ Α πρὸς τὸν Β λόγῳ, καὶ ἔτι οἱ Ε, Ξ, Ζ καὶ οἱ Θ, Ο, Π, Κ ἐξῆς εἰσιν ἀνάλογον ἐν τῷ τοῦ Β πρὸς τὸν Γ λόγῳ. καὶ ἐστὶν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ· καὶ οἱ Δ, Α, Ε ἄρα τοῖς Ε, Ξ, Ζ ἐν τῷ αὐτῷ λόγῳ εἰσὶ καὶ ἔτι οἱ Η, Μ, Ν, Θ τοῖς Θ, Ο, Π, Κ. καὶ ἐστὶν ἴσον τὸ μὲν τῶν Δ, Α, Ε πλήθος τῷ τῶν Ε, Ξ, Ζ πλήθει, τὸ δὲ τῶν Η, Μ, Ν, Θ τῷ τῶν Θ, Ο, Π, Κ δι' ἴσου ἄρα ἐστὶν ὡς μὲν ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Ε πρὸς τὸν Ζ, ὡς δὲ ὁ Η πρὸς τὸν Θ, οὕτως ὁ Θ πρὸς τὸν Κ· ὅπερ ἔδει δεῖξαι.

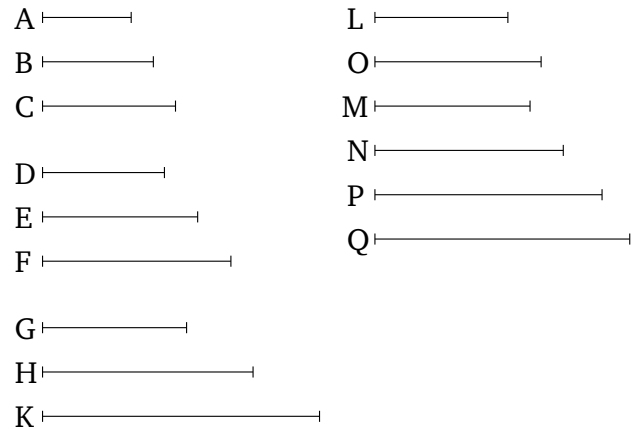
ιδ'.

Ἐὰν τετράγωνος τετράγωνον μετρήῃ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσῃ· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρήῃ, καὶ ὁ τετράγωνος τὸν τετράγωνον μετρήσῃ.

Ἐστωσαν τετράγωνοι ἀριθμοὶ οἱ Α, Β, πλευραὶ δὲ αὐτῶν ἔστωσαν οἱ Γ, Δ, ὁ δὲ Α τὸν Β μετρεῖτω· λέγω, ὅτι καὶ ὁ Γ τὸν Δ μετρεῖ.

be (continuously) proportional [and this always happens with the extremes].

Let A, B, C be any multitude whatsoever of continuously proportional numbers, (such that) as A (is) to B , so B (is) to C . And let A, B, C make D, E, F (by) multiplying themselves, and let them make G, H, K (by) multiplying D, E, F . I say that D, E, F and G, H, K are continuously proportional.



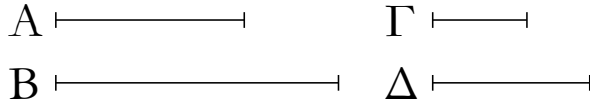
For let A make L (by) multiplying B . And let A, B make M, N , respectively, (by) multiplying L . And, again, let B make O (by) multiplying C . And let B, C make P, Q , respectively, (by) multiplying O .

So, similarly to the above, we can show that D, L, E and G, M, N, H are continuously proportional in the ratio of A to B , and, further, (that) E, O, F and H, P, Q, K are continuously proportional in the ratio of B to C . And as A is to B , so B (is) to C . And thus D, L, E are in the same ratio as E, O, F , and, further, G, M, N, H (are in the same ratio) as H, P, Q, K . And the multitude of D, L, E is equal to the multitude of E, O, F , and that of G, M, N, H to that of H, P, Q, K . Thus, via equality, as D is to E , so E (is) to F , and as G (is) to H , so H (is) to K [Prop. 7.14]. (Which is) the very thing it was required to show.

Proposition 14

If a square (number) measures a(nother) square (number) then the side (of the former) will also measure the side (of the latter). And if the side (of a square number) measures the side (of another square number) then the (former) square (number) will also measure the (latter) square (number).

Let A and B be square numbers, and let C and D be their sides (respectively). And let A measure B . I say that C also measures D .



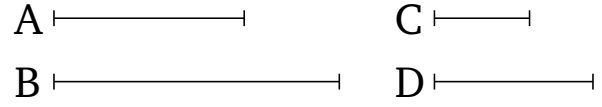
E —————

Ὁ Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν Ε ποιεῖτω· οἱ Α, Ε, Β ἄρα ἐξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ οἱ Α, Ε, Β ἐξῆς ἀνάλογόν εἰσιν, καὶ μετρεῖ ὁ Α τὸν Β, μετρεῖ ἄρα καὶ ὁ Α τὸν Ε. καὶ ἐστὶν ὡς ὁ Α πρὸς τὸν Ε, οὕτως ὁ Γ πρὸς τὸν Δ· μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ.

Πάλιν δὴ ὁ Γ τὸν Δ μετρεῖται· λέγω, ὅτι καὶ ὁ Α τὸν Β μετρεῖ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι οἱ Α, Ε, Β ἐξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ ἐστὶν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Ε, μετρεῖ δὲ ὁ Γ τὸν Δ, μετρεῖ ἄρα καὶ ὁ Α τὸν Ε. καὶ εἰσιν οἱ Α, Ε, Β ἐξῆς ἀνάλογον· μετρεῖ ἄρα καὶ ὁ Α τὸν Β.

Ἐὰν ἄρα τετράγωνος τετράγωνον μετρήῃ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσῃ· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρήῃ, καὶ ὁ τετράγωνος τὸν τετράγωνον μετρήσῃ· ὅπερ ἔδει δεῖξαι.



E —————

For let C make E (by) multiplying D . Thus, A , E , B are continuously proportional in the ratio of C to D [Prop. 8.11]. And since A , E , B are continuously proportional, and A measures B , A thus also measures E [Prop. 8.7]. And as A is to E , so C (is) to D . Thus, C also measures D [Def. 7.20].

So, again, let C measure D . I say that A also measures B .

For similarly, with the same construction, we can show that A , E , B are continuously proportional in the ratio of C to D . And since as C is to D , so A (is) to E , and C measures D , A thus also measures E [Def. 7.20]. And A , E , B are continuously proportional. Thus, A also measures B .

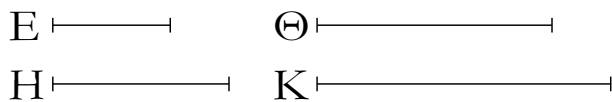
Thus, if a square (number) measures a(nother) square (number) then the side (of the former) will also measure the side (of the latter). And if the side (of a square number) measures the side (of another square number) then the (former) square (number) will also measure the (latter) square (number). (Which is) the very thing it was required to show.

ιε'.

Proposition 15

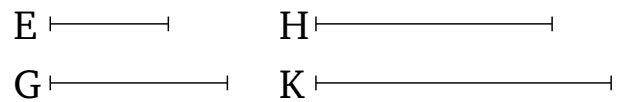
Ἐὰν κύβος ἀριθμὸς κύβον ἀριθμὸν μετρήῃ, καὶ ἡ πλευρὰ τὴν πλευρὰν μετρήσῃ· καὶ ἐὰν ἡ πλευρὰ τὴν πλευρὰν μετρήῃ, καὶ ὁ κύβος τὸν κύβον μετρήσῃ.

Κύβος γὰρ ἀριθμὸς ὁ Α κύβον τὸν Β μετρεῖται, καὶ τοῦ μὲν Α πλευρὰ ἔστω ὁ Γ, τοῦ δὲ Β ὁ Δ· λέγω, ὅτι ὁ Γ τὸν Δ μετρεῖ.



Z —————

Ὁ Γ γὰρ ἑαυτὸν πολλαπλασιάσας τὸν Ε ποιεῖται, ὁ δὲ Δ



F —————

For let C make E (by) multiplying itself. And let

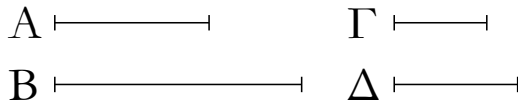
ἑαυτὸν πολλαπλασιάσας τὸν Η ποιεῖτω, καὶ ἔτι ὁ Γ τὸν Δ πολλαπλασιάσας τὸν Ζ [ποιεῖτω], ἐκάτερος δὲ τῶν Γ, Δ τὸν Ζ πολλαπλασιάσας ἐκάτερον τῶν Θ, Κ ποιεῖτω. φανερόν δὴ, ὅτι οἱ Ε, Ζ, Η καὶ οἱ Α, Θ, Κ, Β ἐξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ οἱ Α, Θ, Κ, Β ἐξῆς ἀνάλογόν εἰσιν, καὶ μετρεῖ ὁ Α τὸν Β, μετρεῖ ἄρα καὶ τὸν Θ. καὶ ἐστὶν ὡς ὁ Α πρὸς τὸν Θ, οὕτως ὁ Γ πρὸς τὸν Δ· μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ.

Ἀλλὰ δὴ μετρεῖτω ὁ Γ τὸν Δ· λέγω, ὅτι καὶ ὁ Α τὸν Β μετρήσει.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δὴ δείξομεν, ὅτι οἱ Α, Θ, Κ, Β ἐξῆς ἀνάλογόν εἰσιν ἐν τῷ τοῦ Γ πρὸς τὸν Δ λόγῳ. καὶ ἐπεὶ ὁ Γ τὸν Δ μετρεῖ, καὶ ἐστὶν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Α πρὸς τὸν Θ, καὶ ὁ Α ἄρα τὸν Θ μετρεῖ· ὥστε καὶ τὸν Β μετρεῖ ὁ Α· ὅπερ ἔδει δεῖξαι.

ιϛ'.

Ἐὰν τετράγωνος ἀριθμὸς τετράγωνον ἀριθμὸν μὴ μετρήῃ, οὐδὲ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· καὶ ἡ πλευρὰ τὴν πλευρὰν μὴ μετρήῃ, οὐδὲ ὁ τετράγωνος τὸν τετράγωνον μετρήσει.



Ἐστωσαν τετράγωνοι ἀριθμοὶ οἱ Α, Β, πλευραὶ δὲ αὐτῶν ἔστωσαν οἱ Γ, Δ, καὶ μὴ μετρεῖτω ὁ Α τὸν Β· λέγω, ὅτι οὐδὲ ὁ Γ τὸν Δ μετρεῖ.

Εἰ γὰρ μετρεῖ ὁ Γ τὸν Δ, μετρήσει καὶ ὁ Α τὸν Β. οὐ μετρεῖ δὲ ὁ Α τὸν Β· οὐδὲ ἄρα ὁ Γ τὸν Δ μετρήσει.

Μὴ μετρεῖτω [δὴ] πάλιν ὁ Γ τὸν Δ· λέγω, ὅτι οὐδὲ ὁ Α τὸν Β μετρήσει.

Εἰ γὰρ μετρεῖ ὁ Α τὸν Β, μετρήσει καὶ ὁ Γ τὸν Δ. οὐ μετρεῖ δὲ ὁ Γ τὸν Δ· οὐδ' ἄρα ὁ Α τὸν Β μετρήσει· ὅπερ ἔδει δεῖξαι.

ιζ'.

Ἐὰν κύβος ἀριθμὸς κύβον ἀριθμὸν μὴ μετρήῃ, οὐδὲ ἡ πλευρὰ τὴν πλευρὰν μετρήσει· καὶ ἡ πλευρὰ τὴν πλευρὰν μὴ μετρήῃ, οὐδὲ ὁ κύβος τὸν κύβον μετρήσει.

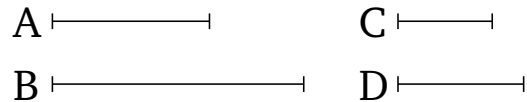
D make G (by) multiplying itself. And, further, [let] C [make] F (by) multiplying D , and let C, D make H, K , respectively, (by) multiplying F . So it is clear that E, F, G and A, H, K, B are continuously proportional in the ratio of C to D [Prop. 8.12]. And since A, H, K, B are continuously proportional, and A measures B , (A) thus also measures H [Prop. 8.7]. And as A is to H , so C (is) to D . Thus, C also measures D [Def. 7.20].

And so let C measure D . I say that A will also measure B .

For similarly, with the same construction, we can show that A, H, K, B are continuously proportional in the ratio of C to D . And since C measures D , and as C is to D , so A (is) to H , A thus also measures H [Def. 7.20]. Hence, A also measures B . (Which is) the very thing it was required to show.

Proposition 16

If a square number does not measure a(nother) square number then the side (of the former) will not measure the side (of the latter) either. And if the side (of a square number) does not measure the side (of another square number) then the (former) square (number) will not measure the (latter) square (number) either.



Let A and B be square numbers, and let C and D be their sides (respectively). And let A not measure B . I say that C does not measure D either.

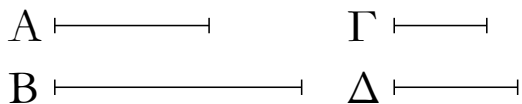
For if C measures D then A will also measure B [Prop. 8.14]. And A does not measure B . Thus, C will not measure D either.

[So], again, let C not measure D . I say that A will not measure B either.

For if A measures B then C will also measure D [Prop. 8.14]. And C does not measure D . Thus, A will not measure B either. (Which is) the very thing it was required to show.

Proposition 17

If a cube number does not measure a(nother) cube number then the side (of the former) will not measure the side (of the latter) either. And if the side (of a cube number) does not measure the side (of another cube number) then the (former) cube (number) will not measure the (latter) cube (number) either.

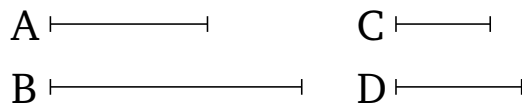


Κύβος γὰρ ἀριθμὸς ὁ A κύβον ἀριθμὸν τὸν B μὴ μετρεῖτω, καὶ τοῦ μὲν A πλευρὰ ἔστω ὁ Γ , τοῦ δὲ B ὁ Δ . λέγω, ὅτι ὁ Γ τὸν Δ οὐ μετρήσει.

Εἰ γὰρ μετρεῖ ὁ Γ τὸν Δ , καὶ ὁ A τὸν B μετρήσει. οὐ μετρεῖ δὲ ὁ A τὸν B : οὐδ' ἄρα ὁ Γ τὸν Δ μετρεῖ.

Ἀλλὰ δὴ μὴ μετρεῖτω ὁ Γ τὸν Δ . λέγω, ὅτι οὐδὲ ὁ A τὸν B μετρήσει.

Εἰ γὰρ ὁ A τὸν B μετρεῖ, καὶ ὁ Γ τὸν Δ μετρήσει. οὐ μετρεῖ δὲ ὁ Γ τὸν Δ : οὐδ' ἄρα ὁ A τὸν B μετρήσει· ὅπερ ἔδει δεῖξαι.



For let the cube number A not measure the cube number B . And let C be the side of A , and D (the side) of B . I say that C will not measure D .

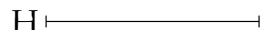
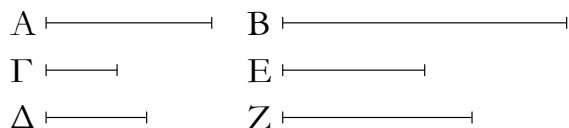
For if C measures D then A will also measure B [Prop. 8.15]. And A does not measure B . Thus, C does not measure D either.

And so let C not measure D . I say that A will not measure B either.

For if A measures B then C will also measure D [Prop. 8.15]. And C does not measure D . Thus, A will not measure B either. (Which is) the very thing it was required to show.

ιη'.

Δύο ὁμοίων ἐπιπέδων ἀριθμῶν εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός· καὶ ὁ ἐπίπεδος πρὸς τὸν ἐπίπεδον διπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν.

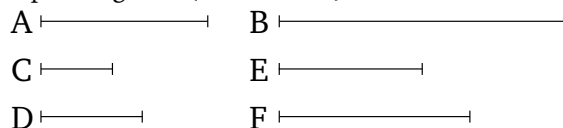


Ἐστωσαν δύο ὅμοιοι ἐπίπεδοι ἀριθμοὶ οἱ A , B , καὶ τοῦ μὲν A πλευραὶ ἔστωσαν οἱ Γ , Δ ἀριθμοί, τοῦ δὲ B οἱ E , Z . καὶ ἐπεὶ ὅμοιοι ἐπίπεδοι εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Δ , οὕτως ὁ E πρὸς τὸν Z . λέγω οὖν, ὅτι τῶν A , B εἷς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ A πρὸς τὸν B διπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν E ἢ ὁ Δ πρὸς τὸν Z , τουτέστιν ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον [πλευράν].

Καὶ ἐπεὶ ἔστιν ὡς ὁ Γ πρὸς τὸν Δ , οὕτως ὁ E πρὸς τὸν Z , ἐναλλάξ ἄρα ἐστὶν ὡς ὁ Γ πρὸς τὸν E , ὁ Δ πρὸς τὸν Z . καὶ ἐπεὶ ἐπίπεδός ἐστιν ὁ A , πλευραὶ δὲ αὐτοῦ οἱ Γ , Δ , ὁ Δ ἄρα τὸν Γ πολλαπλασιάσας τὸν A πεποιήκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ E τὸν Z πολλαπλασιάσας τὸν B πεποιήκεν. ὁ Δ δὴ τὸν E πολλαπλασιάσας τὸν H ποιεῖτω. καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν A πεποιήκεν, τὸν δὲ E πολλαπλασιάσας τὸν H πεποιήκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν E , οὕτως ὁ A πρὸς τὸν H . ἀλλ' ὡς ὁ Γ πρὸς τὸν E , [οὕτως] ὁ Δ πρὸς τὸν Z : καὶ ὡς ἄρα ὁ Δ πρὸς τὸν Z , οὕτως ὁ A πρὸς τὸν H . πάλιν, ἐπεὶ ὁ E τὸν μὲν Δ πολλαπλασιάσας τὸν H πεποιήκεν, τὸν δὲ Z πολλαπλασιάσας τὸν B πεποιήκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν Z , οὕτως ὁ H πρὸς τὸν B . ἐδείχθη δὲ καὶ ὡς ὁ Δ πρὸς τὸν Z , οὕτως ὁ A πρὸς τὸν

Proposition 18

There exists one number in mean proportion to two similar plane numbers. And (one) plane (number) has to the (other) plane (number) a squared[†] ratio with respect to (that) a corresponding side (of the former has) to a corresponding side (of the latter).



Let A and B be two similar plane numbers. And let the numbers C , D be the sides of A , and E , F (the sides) of B . And since similar numbers are those having proportional sides [Def. 7.21], thus as C is to D , so E (is) to F . Therefore, I say that there exists one number in mean proportion to A and B , and that A has to B a squared ratio with respect to that C (has) to E , or D to F —that is to say, with respect to (that) a corresponding side (has) to a corresponding [side].

For since as C is to D , so E (is) to F , thus, alternately, as C is to E , so D (is) to F [Prop. 7.13]. And since A is plane, and C , D its sides, D has thus made A (by) multiplying C . And so, for the same (reasons), E has made B (by) multiplying F . So let D make G (by) multiplying E . And since D has made A (by) multiplying C , and has made G (by) multiplying E , thus as C is to E , so A (is) to G [Prop. 7.17]. But as C (is) to E , [so] D (is) to F . And thus as D (is) to F , so A (is) to G . Again, since E has made G (by) multiplying D , and has made B (by) multiplying F , thus as D is to F , so G (is) to B [Prop. 7.17]. And it was also shown that as D (is) to F , so A (is) to G . And thus as A (is) to G , so G (is) to B . Thus, A , G , B are

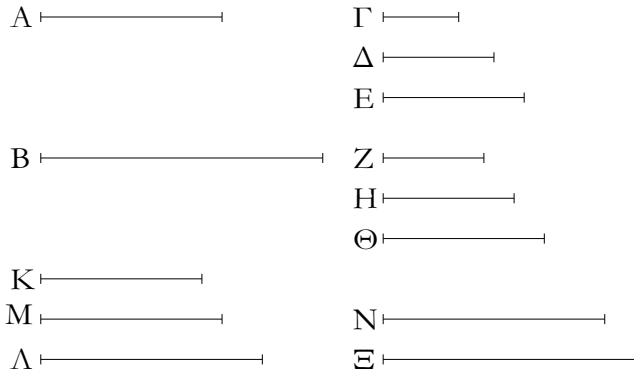
Η· καὶ ὡς ἄρα ὁ Α πρὸς τὸν Η, οὕτως ὁ Η πρὸς τὸν Β. οἱ Α, Η, Β ἄρα ἐξῆς ἀνάλογόν εἰσιν. τῶν Α, Β ἄρα εἷς μέσος ἀνάλογόν ἐστὶν ἀριθμός.

Λέγω δὴ, ὅτι καὶ ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἥπερ ὁ Γ πρὸς τὸν Ε ἢ ὁ Δ πρὸς τὸν Ζ. ἐπεὶ γὰρ οἱ Α, Η, Β ἐξῆς ἀνάλογόν εἰσιν, ὁ Α πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ πρὸς τὸν Η. καὶ ἐστὶν ὡς ὁ Α πρὸς τὸν Η, οὕτως ὁ τε Γ πρὸς τὸν Ε καὶ ὁ Δ πρὸς τὸν Ζ. καὶ ὁ Α ἄρα πρὸς τὸν Β διπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Ε ἢ ὁ Δ πρὸς τὸν Ζ· ὅπερ ἔδει δεῖξαι.

† Literally, “double”.

ιθ'.

Δύο ὁμοίων στερεῶν ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί· καὶ ὁ στερεὸς πρὸς τὸν ὅμοιον στερεὸν τριπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν.



Ἐστῶσαν δύο ὅμοιοι στερεοὶ οἱ Α, Β, καὶ τοῦ μὲν Α πλευραὶ ἔστωσαν οἱ Γ, Δ, Ε, τοῦ δὲ Β οἱ Ζ, Η, Θ. καὶ ἐπεὶ ὅμοιοι στερεοὶ εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς, ἔστιν ἄρα ὡς μὲν ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν Η, ὡς δὲ ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Η πρὸς τὸν Θ. λέγω, ὅτι τῶν Α, Β δύο μέσοι ἀνάλογόν ἐμπίπτουσιν ἀριθμοί, καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ.

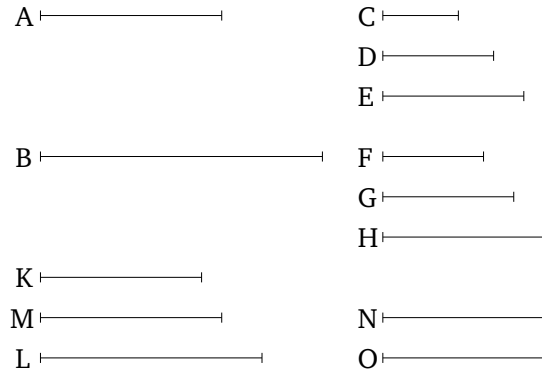
Ὁ Γ γὰρ τὸν Δ πολλαπλασιάσας τὸν Κ ποιεῖτω, ὁ δὲ Ζ τὸν Η πολλαπλασιάσας τὸν Λ ποιεῖτω. καὶ ἐπεὶ οἱ Γ, Δ τοῖς Ζ, Η ἐν τῷ αὐτῷ λόγῳ εἰσιν, καὶ ἐκ μὲν τῶν Γ, Δ ἐστὶν ὁ Κ, ἐκ δὲ τῶν Ζ, Η ὁ Λ, οἱ Κ, Λ [ἄρα] ὅμοιοι ἐπίπεδοι εἰσιν ἀριθμοί· τῶν Κ, Λ ἄρα εἷς μέσος ἀνάλογόν ἐστὶν ἀριθμός. ἔστω ὁ Μ. ὁ Μ ἄρα ἐστὶν ὁ ἐκ τῶν Δ, Ζ, ὡς ἐν τῷ πρὸ τούτου θεωρήματι ἐδείχθη. καὶ ἐπεὶ ὁ Δ τὸν μὲν Γ πολλαπλασιάσας τὸν Κ πεποίηκεν, τὸν δὲ Ζ πολλαπλασιάσας τὸν Μ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Γ πρὸς τὸν Ζ, οὕτως ὁ Κ πρὸς τὸν Μ. ἀλλ' ὡς ὁ Κ πρὸς τὸν Μ, ὁ Μ πρὸς τὸν Λ. οἱ Κ, Μ, Λ ἄρα ἐξῆς εἰσιν ἀνάλογον ἐν

continuously proportional. Thus, there exists one number (namely, G) in mean proportion to A and B .

So I say that A also has to B a squared ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) C (has) to E , or D to F . For since A, G, B are continuously proportional, A has to B a squared ratio with respect to (that A has) to G [Prop. 5.9]. And as A is to G , so C (is) to E , and D to F . And thus A has to B a squared ratio with respect to (that) C (has) to E , or D to F . (Which is) the very thing it was required to show.

Proposition 19

Two numbers fall (between) two similar solid numbers in mean proportion. And a solid (number) has to a similar solid (number) a cubed† ratio with respect to (that) a corresponding side (has) to a corresponding side.



Let A and B be two similar solid numbers, and let C, D, E be the sides of A , and F, G, H (the sides) of B . And since similar solid (numbers) are those having proportional sides [Def. 7.21], thus as C is to D , so F (is) to G , and as D (is) to E , so G (is) to H . I say that two numbers fall (between) A and B in mean proportion, and (that) A has to B a cubed ratio with respect to (that) C (has) to F , and D to G , and, further, E to H .

For let C make K (by) multiplying D , and let F make L (by) multiplying G . And since C, D are in the same ratio as F, G , and K is the (number created) from (multiplying) C, D , and L the (number created) from (multiplying) F, G , [thus] K and L are similar plane numbers [Def. 7.21]. Thus, there exists one number in mean proportion to K and L [Prop. 8.18]. Let it be M . Thus, M is the (number created) from (multiplying) D, F , as shown in the theorem before this (one). And since D has made K (by) multiplying C , and has made M (by) multiplying F , thus as C is to F , so K (is) to M [Prop. 7.17]. But, as

τῷ τοῦ Γ πρὸς τὸν Ζ λόγῳ. καὶ ἐπεὶ ἐστὶν ὡς ὁ Γ πρὸς τὸν Δ, οὕτως ὁ Ζ πρὸς τὸν Η, ἐναλλάξ ἄρα ἐστὶν ὡς ὁ Γ πρὸς τὸν Ζ, οὕτως ὁ Δ πρὸς τὸν Η. διὰ τὰ αὐτὰ δὴ καὶ ὡς ὁ Δ πρὸς τὸν Η, οὕτως ὁ Ε πρὸς τὸν Θ. οἱ Κ, Μ, Λ ἄρα ἐξῆς εἰσὶν ἀνάλογον ἐν τε τῷ τοῦ Γ πρὸς τὸν Ζ λόγῳ καὶ τῷ τοῦ Δ πρὸς τὸν Η καὶ ἔτι τῷ τοῦ Ε πρὸς τὸν Θ. ἑκάτερος δὴ τῶν Ε, Θ τὸν Μ πολλαπλασιάσας ἑκάτερον τῶν Ν, Ξ ποιεῖτω. καὶ ἐπεὶ στερεὸς ἐστὶν ὁ Α, πλευραὶ δὲ αὐτοῦ εἰσὶν οἱ Γ, Δ, Ε, ὁ Ε ἄρα τὸν ἐκ τῶν Γ, Δ πολλαπλασιάσας τὸν Α πεποίηκεν. ὁ δὲ ἐκ τῶν Γ, Δ ἐστὶν ὁ Κ· ὁ Ε ἄρα τὸν Κ πολλαπλασιάσας τὸν Α πεποίηκεν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Θ τὸν Λ πολλαπλασιάσας τὸν Β πεποίηκεν. καὶ ἐπεὶ ὁ Ε τὸν Κ πολλαπλασιάσας τὸν Α πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Μ πολλαπλασιάσας τὸν Ν πεποίηκεν, ἔστιν ἄρα ὡς ὁ Κ πρὸς τὸν Μ, οὕτως ὁ Α πρὸς τὸν Ν. ὡς δὲ ὁ Κ πρὸς τὸν Μ, οὕτως ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Ν. πάλιν, ἐπεὶ ἑκάτερος τῶν Ε, Θ τὸν Μ πολλαπλασιάσας ἑκάτερον τῶν Ν, Ξ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Ν πρὸς τὸν Ξ. ἀλλ' ὡς ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η· καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ, οὕτως ὁ Α πρὸς τὸν Ν καὶ ὁ Ν πρὸς τὸν Ξ. πάλιν, ἐπεὶ ὁ Θ τὸν Μ πολλαπλασιάσας τὸν Ξ πεποίηκεν, ἀλλὰ μὴν καὶ τὸν Λ πολλαπλασιάσας τὸν Β πεποίηκεν, ἔστιν ἄρα ὡς ὁ Μ πρὸς τὸν Λ, οὕτως ὁ Ξ πρὸς τὸν Β. ἀλλ' ὡς ὁ Μ πρὸς τὸν Λ, οὕτως ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ. καὶ ὡς ἄρα ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ὁ Ε πρὸς τὸν Θ, οὕτως οὐ μόνον ὁ Ξ πρὸς τὸν Β, ἀλλὰ καὶ ὁ Α πρὸς τὸν Ν καὶ ὁ Ν πρὸς τὸν Ξ. οἱ Α, Ν, Ξ, Β ἄρα ἐξῆς εἰσὶν ἀνάλογον ἐν τοῖς εἰρημένους τῶν πλευρῶν λόγοις.

Λέγω, ὅτι καὶ ὁ Α πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἥπερ ὁ Γ ἀριθμὸς πρὸς τὸν Ζ ἢ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ. ἐπεὶ γὰρ τέσσαρες ἀριθμοὶ ἐξῆς ἀνάλογον εἰσὶν οἱ Α, Ν, Ξ, Β, ὁ Α ἄρα πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ὁ Α πρὸς τὸν Ν. ἀλλ' ὡς ὁ Α πρὸς τὸν Ν, οὕτως ἐδείχθη ὁ Γ πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ. καὶ ὁ Α ἄρα πρὸς τὸν Β τριπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἥπερ ὁ Γ ἀριθμὸς πρὸς τὸν Ζ καὶ ὁ Δ πρὸς τὸν Η καὶ ἔτι ὁ Ε πρὸς τὸν Θ· ὅπερ εἶδει δεῖξαι.

† Literally, "triple".

κ'.

Ἐὰν δύο ἀριθμῶν εἷς μέσος ἀνάλογον ἐμπίπτῃ ἀριθμός, ὁμοιοὶ ἐπίπεδοι ἔσονται οἱ ἀριθμοί.

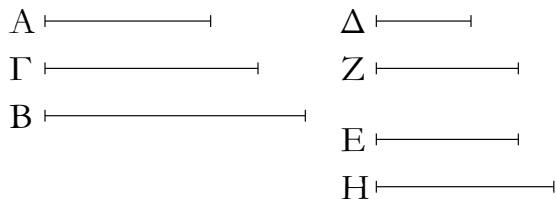
K (is) to *M*, (so) *M* (is) to *L*. Thus, *K*, *M*, *L* are continuously proportional in the ratio of *C* to *F*. And since as *C* is to *D*, so *F* (is) to *G*, thus, alternately, as *C* is to *F*, so *D* (is) to *G* [Prop. 7.13]. And so, for the same (reasons), as *D* (is) to *G*, so *E* (is) to *H*. Thus, *K*, *M*, *L* are continuously proportional in the ratio of *C* to *F*, and of *D* to *G*, and, further, of *E* to *H*. So let *E*, *H* make *N*, *O*, respectively, (by) multiplying *M*. And since *A* is solid, and *C*, *D*, *E* are its sides, *E* has thus made *A* (by) multiplying the (number created) from (multiplying) *C*, *D*. And *K* is the (number created) from (multiplying) *C*, *D*. Thus, *E* has made *A* (by) multiplying *K*. And so, for the same (reasons), *H* has made *B* (by) multiplying *L*. And since *E* has made *A* (by) multiplying *K*, but has, in fact, also made *N* (by) multiplying *M*, thus as *K* is to *M*, so *A* (is) to *N* [Prop. 7.17]. And as *K* (is) to *M*, so *C* (is) to *F*, and *D* to *G*, and, further, *E* to *H*. And thus as *C* (is) to *F*, and *D* to *G*, and *E* to *H*, so *A* (is) to *N*. Again, since *E*, *H* have made *N*, *O*, respectively, (by) multiplying *M*, thus as *E* is to *H*, so *N* (is) to *O* [Prop. 7.18]. But, as *E* (is) to *H*, so *C* (is) to *F*, and *D* to *G*. And thus as *C* (is) to *F*, and *D* to *G*, and *E* to *H*, so (is) *A* to *N*, and *N* to *O*. Again, since *H* has made *O* (by) multiplying *M*, but has, in fact, also made *B* (by) multiplying *L*, thus as *M* (is) to *L*, so *O* (is) to *B* [Prop. 7.17]. But, as *M* (is) to *L*, so *C* (is) to *F*, and *D* to *G*, and *E* to *H*. And thus as *C* (is) to *F*, and *D* to *G*, and *E* to *H*, so not only (is) *O* to *B*, but also *A* to *N*, and *N* to *O*. Thus, *A*, *N*, *O*, *B* are continuously proportional in the aforementioned ratios of the sides.

So I say that *A* also has to *B* a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) the number *C* (has) to *F*, or *D* to *G*, and, further, *E* to *H*. For since *A*, *N*, *O*, *B* are four continuously proportional numbers, *A* thus has to *B* a cubed ratio with respect to (that) *A* (has) to *N* [Def. 5.10]. But, as *A* (is) to *N*, so it was shown (is) *C* to *F*, and *D* to *G*, and, further, *E* to *H*. And thus *A* has to *B* a cubed ratio with respect to (that) a corresponding side (has) to a corresponding side—that is to say, with respect to (that) the number *C* (has) to *F*, and *D* to *G*, and, further, *E* to *H*. (Which is) the very thing it was required to show.

Proposition 20

If one number falls between two numbers in mean proportion then the numbers will be similar plane (num-

Δύο γὰρ ἀριθμῶν τῶν A, B εἷς μέσος ἀνάλογον ἐμπίπττω ἀριθμὸς ὁ Γ . λέγω, ὅτι οἱ A, B ὅμοιοι ἐπίπεδοι εἰσιν ἀριθμοί.



Εἰλήφθωσαν [γὰρ] ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A, Γ οἱ Δ, E . ἰσάκεις ἄρα ὁ Δ τὸν A μετρεῖ καὶ ὁ E τὸν Γ . ὁσάκεις δὴ ὁ Δ τὸν A μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Z . ὁ Z ἄρα τὸν Δ πολλαπλασιάσας τὸν A πεποίηκεν. ὥστε ὁ A ἐπίπεδός ἐστιν, πλευραὶ δὲ αὐτοῦ οἱ Δ, Z . πάλιν, ἐπεὶ οἱ Δ, E ἐλάχιστοι εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Γ, B , ἰσάκεις ἄρα ὁ Δ τὸν Γ μετρεῖ καὶ ὁ E τὸν B . ὁσάκεις δὴ ὁ E τὸν B μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ H . ὁ H ἄρα τὸν E πολλαπλασιάσας τὸν B πεποίηκεν. ὁ B ἄρα ἐπίπεδος ἐστι, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ E, H . οἱ A, B ἄρα ἐπίπεδοι εἰσιν ἀριθμοί. λέγω δὴ, ὅτι καὶ ὅμοιοι. ἐπεὶ γὰρ ὁ Z τὸν μὲν Δ πολλαπλασιάσας τὸν A πεποίηκεν, τὸν δὲ E πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ Δ πρὸς τὸν E , οὕτως ὁ A πρὸς τὸν Γ , τουτέστιν ὁ Γ πρὸς τὸν B . πάλιν, ἐπεὶ ὁ E ἐκάτερον τῶν Z, H πολλαπλασιάσας τοὺς Γ, B πεποίηκεν, ἔστιν ἄρα ὡς ὁ Z πρὸς τὸν H , οὕτως ὁ Γ πρὸς τὸν B . ὡς δὲ ὁ Γ πρὸς τὸν B , οὕτως ὁ Δ πρὸς τὸν E . καὶ ὡς ἄρα ὁ Δ πρὸς τὸν E , οὕτως ὁ Z πρὸς τὸν H . καὶ ἐναλλάξ ὡς ὁ Δ πρὸς τὸν Z , οὕτως ὁ E πρὸς τὸν H . οἱ A, B ἄρα ὅμοιοι ἐπίπεδοι ἀριθμοὶ εἰσιν· αἱ γὰρ πλευραὶ αὐτῶν ἀνάλογόν εἰσιν· ὅπερ ἔδει δεῖξαι.

bers).

For let one number C fall between the two numbers A and B in mean proportion. I say that A and B are similar plane numbers.



[For] let the least numbers, D and E , having the same ratio as A and C have been taken [Prop. 7.33]. Thus, D measures A as many times as E (measures) C [Prop. 7.20]. So as many times as D measures A , so many units let there be in F . Thus, F has made A (by) multiplying D [Def. 7.15]. Hence, A is plane, and D, F (are) its sides. Again, since D and E are the least of those (numbers) having the same ratio as C and B , D thus measures C as many times as E (measures) B [Prop. 7.20]. So as many times as E measures B , so many units let there be in G . Thus, G has made B (by) multiplying E [Def. 7.15]. Thus, B is plane, and E, G are its sides. Thus, A and B are (both) plane numbers. So I say that (they are) also similar. For since F has made A (by) multiplying D , and has made C (by) multiplying E , thus as D is to E , so A (is) to C —that is to say, C to B [Prop. 7.17].[†] Again, since E has made C, B (by) multiplying F, G , respectively, thus as F is to G , so C (is) to B [Prop. 7.17]. And as C (is) to B , so D (is) to E . And thus as D (is) to E , so F (is) to G . And, alternately, as D (is) to F , so E (is) to G [Prop. 7.13]. Thus, A and B are similar plane numbers. For their sides are proportional [Def. 7.21]. (Which is) the very thing it was required to show.

[†] This part of the proof is defective, since it is not demonstrated that $F \times E = C$. Furthermore, it is not necessary to show that $D : E :: A : C$, because this is true by hypothesis.

κα'.

Proposition 21

Ἐὰν δύο ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅμοιοι στερεοὶ εἰσιν οἱ ἀριθμοί.

Δύο γὰρ ἀριθμῶν τῶν A, B δύο μέσοι ἀνάλογον ἐμπίπτέωσαν ἀριθμοὶ οἱ Γ, Δ . λέγω, ὅτι οἱ A, B ὅμοιοι στερεοὶ εἰσιν.

Εἰλήφθωσαν γὰρ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A, Γ , Δ τρεῖς οἱ E, Z, H . οἱ ἄρα ἄκροι αὐτῶν οἱ E, H πρῶτοι πρὸς ἀλλήλους εἰσιν. καὶ ἐπεὶ τῶν E, H εἷς μέσος ἀνάλογον ἐμπίπτωκεν ἀριθμὸς ὁ Z , οἱ E, H ἄρα ἀριθμοὶ ὅμοιοι ἐπίπεδοι εἰσιν. ἔστωσαν οὖν τοῦ μὲν

If two numbers fall between two numbers in mean proportion then the (latter) are similar solid (numbers).

For let the two numbers C and D fall between the two numbers A and B in mean proportion. I say that A and B are similar solid (numbers).

For let the three least numbers E, F, G having the same ratio as A, C, D have been taken [Prop. 8.2]. Thus, the outermost of them, E and G , are prime to one another [Prop. 8.3]. And since one number, F , has fallen (between) E and G in mean proportion, E and G are

Ε πλευραὶ οἱ Θ, Κ, τοῦ δὲ Η οἱ Λ, Μ. φανερόν ἄρα ἐστὶν ἐκ τοῦ πρὸ τούτου, ὅτι οἱ Ε, Ζ, Η ἐξῆς εἰσὶν ἀνάλογον ἐν τε τῷ τοῦ Θ πρὸς τὸν Λ λόγῳ καὶ τῷ τοῦ Κ πρὸς τὸν Μ. καὶ ἐπεὶ οἱ Ε, Ζ, Η ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Α, Γ, Δ, καὶ ἐστὶν ἴσον τὸ πλῆθος τῶν Ε, Ζ, Η τῷ πλῆθει τῶν Α, Γ, Δ, δι' ἴσου ἄρα ἐστὶν ὡς ὁ Ε πρὸς τὸν Η, οὕτως ὁ Α πρὸς τὸν Δ. οἱ δὲ Ε, Η πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ἰσάκεις ὅ τε μείζων τὸν μείζονα καὶ ὁ ἐλάσσων τὸν ἐλάσσονα, τουτέστιν ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον· ἰσάκεις ἄρα ὁ Ε τὸν Α μετρεῖ καὶ ὁ Η τὸν Δ. ὁσάκεις δὴ ὁ Ε τὸν Α μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ν. ὁ Ν ἄρα τὸν Ε πολλαπλασιάσας τὸν Α πεποίηκεν. ὁ δὲ Ε ἐστὶν ὁ ἐκ τῶν Θ, Κ· ὁ Ν ἄρα τὸν ἐκ τῶν Θ, Κ πολλαπλασιάσας τὸν Α πεποίηκεν. στερεὸς ἄρα ἐστὶν ὁ Α, πλευραὶ δὲ αὐτοῦ εἰσὶν οἱ Θ, Κ, Ν. πάλιν, ἐπεὶ οἱ Ε, Ζ, Η ἐλάχιστοί εἰσι τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς Γ, Δ, Β, ἰσάκεις ἄρα ὁ Ε τὸν Γ μετρεῖ καὶ ὁ Η τὸν Β. ὁσάκεις δὴ ὁ Ε τὸν Γ μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Ξ. ὁ Ξ ἄρα τὸν Η πολλαπλασιάσας τὸν Β πεποίηκεν. ὁ δὲ Η ἐστὶν ὁ ἐκ τῶν Λ, Μ· ὁ Ξ ἄρα τὸν ἐκ τῶν Λ, Μ πολλαπλασιάσας τὸν Β πεποίηκεν. στερεὸς ἄρα ἐστὶν ὁ Β, πλευραὶ δὲ αὐτοῦ εἰσὶν οἱ Λ, Μ, Ξ· οἱ Α, Β ἄρα στερεοὶ εἰσιν.

Α	—————	Θ	—————
Γ	—————	Κ	—————
Δ	—————	Ν	—————
Β	—————		

Ε	—————	Λ	—————
Ζ	—————	Μ	—————
Η	—————	Ξ	—————

Λέγω [δὴ], ὅτι καὶ ὁμοιοί. ἐπεὶ γὰρ οἱ Ν, Ξ τὸν Ε πολλαπλασιάσαντες τοὺς Α, Γ πεποίηκασιν, ἐστὶν ἄρα ὡς ὁ Ν πρὸς τὸν Ξ, ὁ Α πρὸς τὸν Γ, τουτέστιν ὁ Ε πρὸς τὸν Ζ. ἀλλ' ὡς ὁ Ε πρὸς τὸν Ζ, ὁ Θ πρὸς τὸν Λ καὶ ὁ Κ πρὸς τὸν Μ· καὶ ὡς ἄρα ὁ Θ πρὸς τὸν Λ, οὕτως ὁ Κ πρὸς τὸν Μ καὶ ὁ Ν πρὸς τὸν Ξ. καὶ εἰσὶν οἱ μὲν Θ, Κ, Ν πλευραὶ τοῦ Α,

thus similar plane numbers [Prop. 8.20]. Therefore, let H, K be the sides of E , and L, M (the sides) of G . Thus, it is clear from the (proposition) before this (one) that E, F, G are continuously proportional in the ratio of H to L , and of K to M . And since E, F, G are the least (numbers) having the same ratio as A, C, D , and the multitude of E, F, G is equal to the multitude of A, C, D , thus, via equality, as E is to G , so A (is) to D [Prop. 7.14]. And E and G (are) prime (to one another), and prime (numbers) are also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser—that is to say, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, E measures A the same number of times as G (measures) D . So as many times as E measures A , so many units let there be in N . Thus, N has made A (by) multiplying E [Def. 7.15]. And E is the (number created) from (multiplying) H and K . Thus, N has made A (by) multiplying the (number created) from (multiplying) H and K . Thus, A is solid, and its sides are H, K, N . Again, since E, F, G are the least (numbers) having the same ratio as C, D, B , thus E measures C the same number of times as G (measures) B [Prop. 7.20]. So as many times as E measures C , so many units let there be in O . Thus, G measures B according to the units in O . Thus, O has made B (by) multiplying G . And G is the (number created) from (multiplying) L and M . Thus, O has made B (by) multiplying the (number created) from (multiplying) L and M . Thus, B is solid, and its sides are L, M, O . Thus, A and B are (both) solid.

Α	—————	Η	—————
Κ	—————	Κ	—————
Δ	—————	Ν	—————
Β	—————		

Ε	—————	Λ	—————
Ζ	—————	Μ	—————
Η	—————	Ο	—————

[So] I say that (they are) also similar. For since N, O have made A, C (by) multiplying E , thus as N is to O , so A (is) to C —that is to say, E to F [Prop. 7.18]. But, as E (is) to F , so H (is) to L , and K to M . And thus as H (is) to L , so K (is) to M , and N to O . And H, K, N are the sides of A , and L, M, O the sides of B . Thus, A and

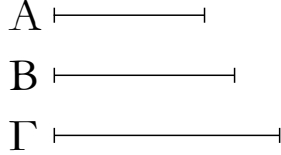
οἱ δὲ Ξ , Λ , M πλευραὶ τοῦ B . οἱ A , B ἄρα ἀριθμοὶ ὅμοιοι στερεοὶ εἰσιν· ὅπερ ἔδει δεῖξαι.

B are similar solid numbers [Def. 7.21]. (Which is) the very thing it was required to show.

† The Greek text has “ O , L , M ”, which is obviously a mistake.

κβ'.

Ἐὰν τρεῖς ἀριθμοὶ ἐξῆς ἀνάλογον ὦσιν, ὁ δὲ πρῶτος τετράγωνος ᾗ, καὶ ὁ τρίτος τετράγωνος ἔσται.

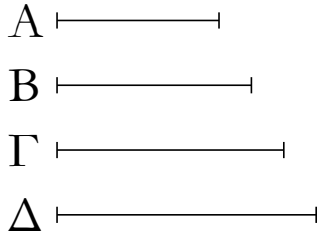


Ἐστῶσαν τρεῖς ἀριθμοὶ ἐξῆς ἀνάλογον οἱ A , B , Γ , ὁ δὲ πρῶτος ὁ A τετράγωνος ἔστω· λέγω, ὅτι καὶ ὁ τρίτος ὁ Γ τετράγωνός ἐστιν.

Ἐπεὶ γὰρ τῶν A , Γ εἷς μέσος ἀναλογόν ἐστιν ἀριθμὸς ὁ B , οἱ A , Γ ἄρα ὅμοιοι ἐπίπεδοι εἰσιν. τετράγωνος δὲ ὁ A · τετράγωνος ἄρα καὶ ὁ Γ · ὅπερ ἔδει δεῖξαι.

κγ'.

Ἐὰν τέσσαρες ἀριθμοὶ ἐξῆς ἀνάλογον ὦσιν, ὁ δὲ πρῶτος κύβος ᾗ, καὶ ὁ τέταρτος κύβος ἔσται.



Ἐστῶσαν τέσσαρες ἀριθμοὶ ἐξῆς ἀνάλογον οἱ A , B , Γ , Δ , ὁ δὲ A κύβος ἔστω· λέγω, ὅτι καὶ ὁ Δ κύβος ἐστίν.

Ἐπεὶ γὰρ τῶν A , Δ δύο μέσοι ἀναλογόν εἰσιν ἀριθμοὶ οἱ B , Γ , οἱ A , Δ ἄρα ὅμοιοι εἰσι στερεοὶ ἀριθμοί. κύβος δὲ ὁ A · κύβος ἄρα καὶ ὁ Δ · ὅπερ ἔδει δεῖξαι.

κδ'.

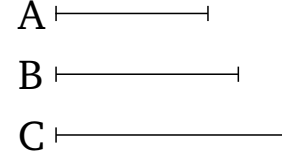
Ἐὰν δύο ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχωσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, ὁ δὲ πρῶτος τετράγωνος ᾗ, καὶ ὁ δεύτερος τετράγωνος ἔσται.



Δύο γὰρ ἀριθμοὶ οἱ A , B πρὸς ἀλλήλους λόγον

Proposition 22

If three numbers are continuously proportional, and the first is square, then the third will also be square.

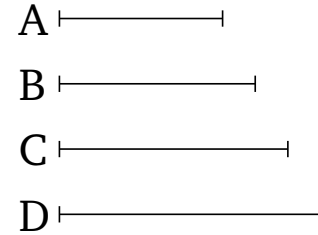


Let A , B , C be three continuously proportional numbers, and let the first A be square. I say that the third C is also square.

For since one number, B , is in mean proportion to A and C , A and C are thus similar plane (numbers) [Prop. 8.20]. And A is square. Thus, C is also square [Def. 7.21]. (Which is) the very thing it was required to show.

Proposition 23

If four numbers are continuously proportional, and the first is cube, then the fourth will also be cube.

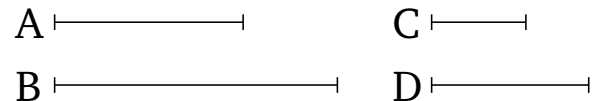


Let A , B , C , D be four continuously proportional numbers, and let A be cube. I say that D is also cube.

For since two numbers, B and C , are in mean proportion to A and D , A and D are thus similar solid numbers [Prop. 8.21]. And A (is) cube. Thus, D (is) also cube [Def. 7.21]. (Which is) the very thing it was required to show.

Proposition 24

If two numbers have to one another the ratio which a square number (has) to a(nother) square number, and the first is square, then the second will also be square.



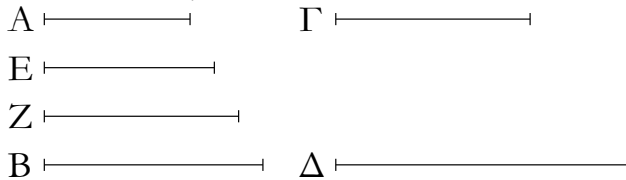
For let two numbers, A and B , have to one another

ἐχέτωσαν, ὃν τετράγωνος ἀριθμὸς ὁ Γ πρὸς τετράγωνον ἀριθμὸν τὸν Δ, ὁ δὲ Α τετράγωνος ἔστω· λέγω, ὅτι καὶ ὁ Β τετράγωνός ἐστιν.

Ἐπεὶ γὰρ οἱ Γ, Δ τετράγωνοι εἰσιν, οἱ Γ, Δ ἄρα ὅμοιοι ἐπίπεδοι εἰσιν. τῶν Γ, Δ ἄρα εἰς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. καὶ ἐστὶν ὡς ὁ Γ πρὸς τὸν Δ, ὁ Α πρὸς τὸν Β· καὶ τῶν Α, Β ἄρα εἰς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. καὶ ἐστὶν ὁ Α τετράγωνος· καὶ ὁ Β ἄρα τετράγωνός ἐστιν· ὅπερ ἔδει δεῖξαι.

κε'.

Ἐὰν δύο ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχωσιν, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμὸν, ὁ δὲ πρῶτος κύβος ᾗ, καὶ ὁ δεύτερος κύβος ἔσται.



Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρὸς ἀλλήλους λόγον ἐχέτωσαν, ὃν κύβος ἀριθμὸς ὁ Γ πρὸς κύβον ἀριθμὸν τὸν Δ, κύβος δὲ ἔστω ὁ Α· λέγω [δὴ], ὅτι καὶ ὁ Β κύβος ἐστίν.

Ἐπεὶ γὰρ οἱ Γ, Δ κύβοι εἰσίν, οἱ Γ, Δ ὅμοιοι στερεοὶ εἰσιν· τῶν Γ, Δ ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ὅσοι δὲ εἰς τοὺς Γ, Δ μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτουσιν, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς· ὥστε καὶ τῶν Α, Β δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ἐμπίπτέτωσαν οἱ Ε, Ζ. ἐπεὶ οὖν τέσσαρες ἀριθμοὶ οἱ Α, Ε, Ζ, Β ἐξῆς ἀνάλογόν εἰσιν, καὶ ἐστὶ κύβος ὁ Α, κύβος ἄρα καὶ ὁ Β· ὅπερ ἔδει δεῖξαι.

κε'.

Οἱ ὅμοιοι ἐπίπεδοι ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν.



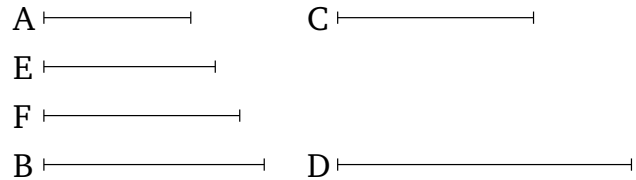
Ἐστωσαν ὅμοιοι ἐπίπεδοι ἀριθμοὶ οἱ Α, Β· λέγω, ὅτι ὁ Α πρὸς τὸν Β λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς

the ratio which the square number C (has) to the square number D . And let A be square. I say that B is also square.

For since C and D are square, C and D are thus similar plane (numbers). Thus, one number falls (between) C and D in mean proportion [Prop. 8.18]. And as C is to D , (so) A (is) to B . Thus, one number also falls (between) A and B in mean proportion [Prop. 8.8]. And A is square. Thus, B is also square [Prop. 8.22]. (Which is) the very thing it was required to show.

Proposition 25

If two numbers have to one another the ratio which a cube number (has) to a(nother) cube number, and the first is cube, then the second will also be cube.

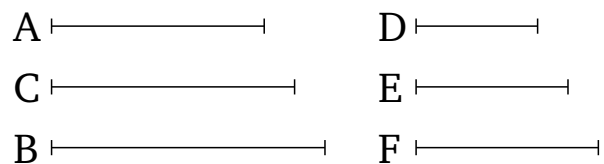


For let two numbers, A and B , have to one another the ratio which the cube number C (has) to the cube number D . And let A be cube. [So] I say that B is also cube.

For since C and D are cube (numbers), C and D are (thus) similar solid (numbers). Thus, two numbers fall (between) C and D in mean proportion [Prop. 8.19]. And as many (numbers) as fall in between C and D in continued proportion, so many also (fall) in (between) those (numbers) having the same ratio as them (in continued proportion) [Prop. 8.8]. And hence two numbers fall (between) A and B in mean proportion. Let E and F (so) fall. Therefore, since the four numbers A, E, F, B are continuously proportional, and A is cube, B (is) thus also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

Proposition 26

Similar plane numbers have to one another the ratio which (some) square number (has) to a(nother) square number.



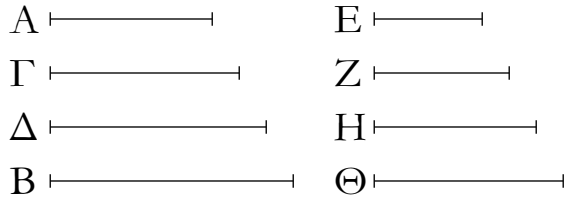
Let A and B be similar plane numbers. I say that A has to B the ratio which (some) square number (has) to

τετράγωνον ἀριθμόν.

Ἐπεὶ γὰρ οἱ A, B ὅμοιοι ἐπίπεδοι εἰσιν, τῶν A, B ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. ἐμπιπτέτω καὶ ἔστω ὁ Γ , καὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς A, Γ, B οἱ Δ, E, Z · οἱ ἄρα ἄκροι αὐτῶν οἱ Δ, Z τετράγωνοι εἰσιν. καὶ ἐπεὶ ἔστιν ὡς ὁ Δ πρὸς τὸν Z , οὕτως ὁ A πρὸς τὸν B , καὶ εἰσιν οἱ Δ, Z τετράγωνοι, ὁ A ἄρα πρὸς τὸν B λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ὅπερ ἔδει δεῖξαι.

κζ'.

Οἱ ὅμοιοι στερεοὶ ἀριθμοὶ πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν.



Ἐστωσαν ὅμοιοι στερεοὶ ἀριθμοὶ οἱ A, B · λέγω, ὅτι ὁ A πρὸς τὸν B λόγον ἔχει, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν.

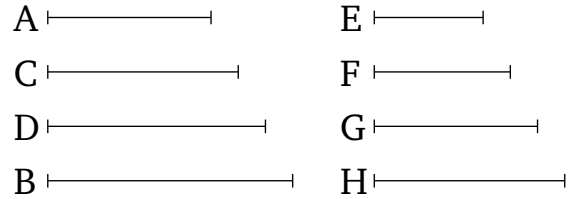
Ἐπεὶ γὰρ οἱ A, B ὅμοιοι στερεοὶ εἰσιν, τῶν A, B ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. ἐμπιπτέτωσαν οἱ Γ, Δ , καὶ εἰλήφθωσαν ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἔχόντων τοῖς A, Γ, Δ, B ἴσοι αὐτοῖς τὸ πλῆθος οἱ E, Z, H, Θ · οἱ ἄρα ἄκροι αὐτῶν οἱ E, Θ κύβοι εἰσίν. καὶ ἔστιν ὡς ὁ E πρὸς τὸν Θ , οὕτως ὁ A πρὸς τὸν B · καὶ ὁ A ἄρα πρὸς τὸν B λόγον ἔχει, ὃν κύβος ἀριθμὸς πρὸς κύβον ἀριθμόν· ὅπερ ἔδει δεῖξαι.

a(nother) square number.

For since A and B are similar plane numbers, one number thus falls (between) A and B in mean proportion [Prop. 8.18]. Let it (so) fall, and let it be C . And let the least numbers, D, E, F , having the same ratio as A, C, B have been taken [Prop. 8.2]. The outermost of them, D and F , are thus square [Prop. 8.2 corr.]. And since as D is to F , so A (is) to B , and D and F are square, A thus has to B the ratio which (some) square number (has) to a(nother) square number. (Which is) the very thing it was required to show.

Proposition 27

Similar solid numbers have to one another the ratio which (some) cube number (has) to a(nother) cube number.



Let A and B be similar solid numbers. I say that A has to B the ratio which (some) cube number (has) to a(nother) cube number.

For since A and B are similar solid (numbers), two numbers thus fall (between) A and B in mean proportion [Prop. 8.19]. Let C and D have (so) fallen. And let the least numbers, E, F, G, H , having the same ratio as A, C, D, B , (and) equal in multitude to them, have been taken [Prop. 8.2]. Thus, the outermost of them, E and H , are cube [Prop. 8.2 corr.]. And as E is to H , so A (is) to B . And thus A has to B the ratio which (some) cube number (has) to a(nother) cube number. (Which is) the very thing it was required to show.

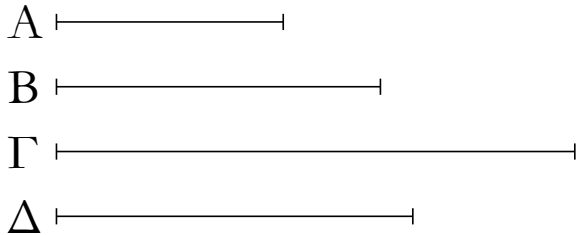
ELEMENTS BOOK 9

Applications of Number Theory[†]

[†]The propositions contained in Books 7–9 are generally attributed to the school of Pythagoras.

α'.

Ἐάν δύο ὅμοιοι ἐπίπεδοι ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσιν τινά, ὁ γενόμενος τετράγωνος ἔσται.

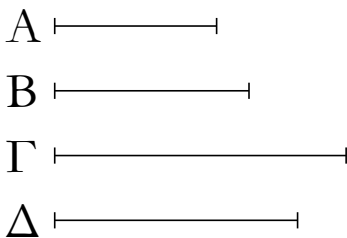


Ἐστωσαν δύο ὅμοιοι ἐπίπεδοι ἀριθμοὶ οἱ A , B , καὶ ὁ A τὸν B πολλαπλασιάσας τὸν Γ ποιείτω· λέγω, ὅτι ὁ Γ τετράγωνός ἐστιν.

Ὁ γὰρ A ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω. ὁ Δ ἄρα τετράγωνός ἐστιν. ἐπεὶ οὖν ὁ A ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ B πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Δ πρὸς τὸν Γ . καὶ ἐπεὶ οἱ A , B ὅμοιοι ἐπίπεδοι εἰσιν ἀριθμοί, τῶν A , B ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. ἐὰν δὲ δύο ἀριθμῶν μεταξὺ κατὰ τὸ συνεχὲς ἀνάλογον ἐμπίπτωσιν ἀριθμοί, ὅσοι εἰς αὐτοὺς ἐμπίπτουσι, τοσοῦτοι καὶ εἰς τοὺς τὸν αὐτὸν λόγον ἔχοντας· ὥστε καὶ τῶν Δ , Γ εἷς μέσος ἀνάλογον ἐμπίπτει ἀριθμός. καὶ ἐστὶ τετράγωνος ὁ Δ · τετράγωνος ἄρα καὶ ὁ Γ · ὅπερ εἶδει δεῖξαι.

β'.

Ἐάν δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσιν τετράγωνον, ὅμοιοι ἐπίπεδοι εἰσιν ἀριθμοί.

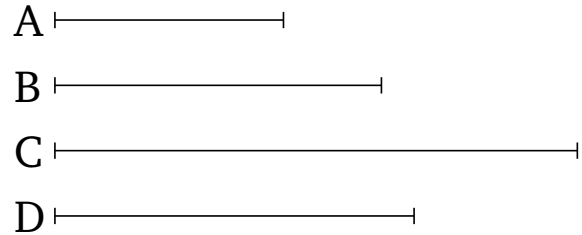


Ἐστωσαν δύο ἀριθμοὶ οἱ A , B , καὶ ὁ A τὸν B πολλαπλασιάσας τετράγωνον τὸν Γ ποιείτω· λέγω, ὅτι οἱ A , B ὅμοιοι ἐπίπεδοι εἰσιν ἀριθμοί.

Ὁ γὰρ A ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιείτω· ὁ Δ ἄρα τετράγωνός ἐστιν. καὶ ἐπεὶ ὁ A ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ B πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , ὁ Δ πρὸς τὸν Γ . καὶ ἐπεὶ ὁ Δ τετράγωνός ἐστιν, ἀλλὰ καὶ ὁ Γ , οἱ Δ , Γ ἄρα ὅμοιοι ἐπίπεδοι εἰσιν. τῶν Δ , Γ ἄρα εἷς μέσος ἀνάλογον

Proposition 1

If two similar plane numbers make some (number by) multiplying one another then the created (number) will be square.

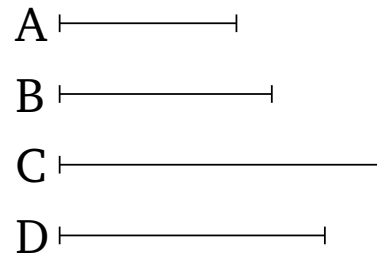


Let A and B be two similar plane numbers, and let A make C (by) multiplying B . I say that C is square.

For let A make D (by) multiplying itself. D is thus square. Therefore, since A has made D (by) multiplying itself, and has made C (by) multiplying B , thus as A is to B , so D (is) to C [Prop. 7.17]. And since A and B are similar plane numbers, one number thus falls (between) A and B in mean proportion [Prop. 8.18]. And if (some) numbers fall between two numbers in continued proportion then, as many (numbers) as fall in (between) them (in continued proportion), so many also (fall) in (between numbers) having the same ratio (as them in continued proportion) [Prop. 8.8]. And hence one number falls (between) D and C in mean proportion. And D is square. Thus, C (is) also square [Prop. 8.22]. (Which is) the very thing it was required to show.

Proposition 2

If two numbers make a square (number by) multiplying one another then they are similar plane numbers.



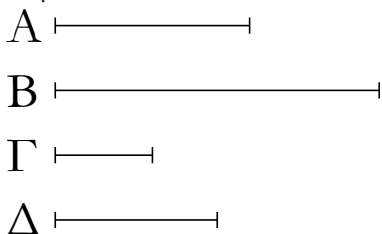
Let A and B be two numbers, and let A make the square (number) C (by) multiplying B . I say that A and B are similar plane numbers.

For let A make D (by) multiplying itself. Thus, D is square. And since A has made D (by) multiplying itself, and has made C (by) multiplying B , thus as A is to B , so D (is) to C [Prop. 7.17]. And since D is square, and C (is) also, D and C are thus similar plane numbers. Thus, one (number) falls (between) D and C in mean propor-

ἐμπίπτει. καὶ ἐστὶν ὡς ὁ Δ πρὸς τὸν Γ, οὕτως ὁ Α πρὸς τὸν Β· καὶ τῶν Α, Β ἄρα εἷς μέσος ἀνάλογον ἐμπίπτει. ἐὰν δὲ δύο ἀριθμῶν εἷς μέσος ἀνάλογον ἐμπίπτῃ, ὅμοιοι ἐπίπεδοι εἰσιν [οἱ] ἀριθμοί· οἱ ἄρα Α, Β ὅμοιοι εἰσιν ἐπίπεδοι· ὅπερ ἔδει δεῖξαι.

Υ'.

Ἐὰν κύβος ἀριθμὸς ἑαυτὸν πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος κύβος ἔσται.



Κύβος γὰρ ἀριθμὸς ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β ποιεῖτω· λέγω, ὅτι ὁ Β κύβος ἐστίν.

Εἰλήφθω γὰρ τοῦ Α πλευρὰ ὁ Γ, καὶ ὁ Γ ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιεῖτω. φανερόν δὴ ἐστίν, ὅτι ὁ Γ τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν. καὶ ἐπεὶ ὁ Γ ἑαυτὸν πολλαπλασιάσας τὸν Δ πεποίηκεν, ὁ Γ ἄρα τὸν Δ μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· ἀλλὰ μὴν καὶ ἡ μονὰς τὸν Γ μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Γ, ὁ Γ πρὸς τὸν Δ. πάλιν, ἐπεὶ ὁ Γ τὸν Δ πολλαπλασιάσας τὸν Α πεποίηκεν, ὁ Δ ἄρα τὸν Α μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας· μετρεῖ δὲ καὶ ἡ μονὰς τὸν Γ κατὰ τὰς ἐν αὐτῷ μονάδας· ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Γ, ὁ Δ πρὸς τὸν Α. ἀλλ' ὡς ἡ μονὰς πρὸς τὸν Γ, ὁ Γ πρὸς τὸν Δ· καὶ ὡς ἄρα ἡ μονὰς πρὸς τὸν Γ, οὕτως ὁ Γ πρὸς τὸν Δ καὶ ὁ Δ πρὸς τὸν Α. τῆς ἄρα μονάδος καὶ τοῦ Α ἀριθμοῦ δύο μέσοι ἀνάλογον κατὰ τὸ συνεχὲς ἐμπεπτώκασιν ἀριθμοί· οἱ Γ, Δ. πάλιν, ἐπεὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν, ὁ Α ἄρα τὸν Β μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· μετρεῖ δὲ καὶ ἡ μονὰς τὸν Α κατὰ τὰς ἐν αὐτῷ μονάδας· ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν Α, ὁ Α πρὸς τὸν Β. τῆς δὲ μονάδος καὶ τοῦ Α δύο μέσοι ἀνάλογον ἐμπεπτώκασιν ἀριθμοί· καὶ τῶν Α, Β ἄρα δύο μέσοι ἀνάλογον ἐμπεσοῦνται ἀριθμοί. ἐὰν δὲ δύο ἀριθμῶν δύο μέσοι ἀνάλογον ἐμπίπτωσιν, ὁ δὲ πρῶτος κύβος ᾗ, καὶ ὁ δεῦτερος κύβος ἔσται. καὶ ἐστὶν ὁ Α κύβος· καὶ ὁ Β ἄρα κύβος ἐστίν· ὅπερ ἔδει δεῖξαι.

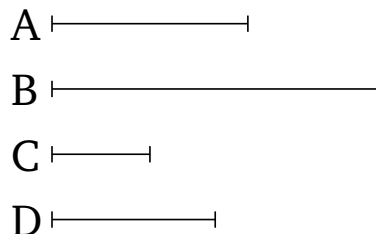
δ'.

Ἐὰν κύβος ἀριθμὸς κύβον ἀριθμὸν πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος κύβος ἔσται.

tion [Prop. 8.18]. And as D is to C , so A (is) to B . Thus, one (number) also falls (between) A and B in mean proportion [Prop. 8.8]. And if one (number) falls (between) two numbers in mean proportion then [the] numbers are similar plane (numbers) [Prop. 8.20]. Thus, A and B are similar plane (numbers). (Which is) the very thing it was required to show.

Proposition 3

If a cube number makes some (number by) multiplying itself then the created (number) will be cube.

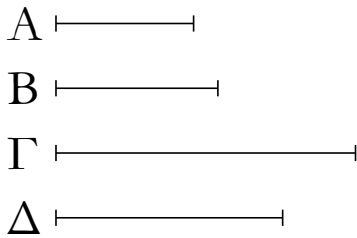


For let the cube number A make B (by) multiplying itself. I say that B is cube.

For let the side C of A have been taken. And let C make D by multiplying itself. So it is clear that C has made A (by) multiplying D . And since C has made D (by) multiplying itself, C thus measures D according to the units in it [Def. 7.15]. But, in fact, a unit also measures C according to the units in it [Def. 7.20]. Thus, as a unit is to C , so C (is) to D . Again, since C has made A (by) multiplying D , D thus measures A according to the units in C . And a unit also measures C according to the units in it. Thus, as a unit is to C , so D (is) to A . But, as a unit (is) to C , so C (is) to D . And thus as a unit (is) to C , so C (is) to D , and D to A . Thus, two numbers, C and D , have fallen (between) a unit and the number A in continued mean proportion. Again, since A has made B (by) multiplying itself, A thus measures B according to the units in it. And a unit also measures A according to the units in it. Thus, as a unit is to A , so A (is) to B . And two numbers have fallen (between) a unit and A in mean proportion. Thus two numbers will also fall (between) A and B in mean proportion [Prop. 8.8]. And if two (numbers) fall (between) two numbers in mean proportion, and the first (number) is cube, then the second will also be cube [Prop. 8.23]. And A is cube. Thus, B is also cube. (Which is) the very thing it was required to show.

Proposition 4

If a cube number makes some (number by) multiplying a(nother) cube number then the created (number)

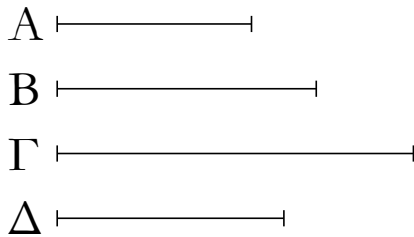


Κύβος γὰρ ἀριθμὸς ὁ A κύβον ἀριθμὸν τὸν B πολλαπλασιάσας τὸν Γ ποιεῖτω· λέγω, ὅτι ὁ Γ κύβος ἐστίν.

Ὁ γὰρ A ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιεῖτω· ὁ Δ ἄρα κύβος ἐστίν. καὶ ἐπεὶ ὁ A ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ B πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , οὕτως ὁ Δ πρὸς τὸν Γ . καὶ ἐπεὶ οἱ A, B κύβοι εἰσίν, ὅμοιοι στερεοὶ εἰσιν οἱ A, B . τῶν A, B ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί· ὥστε καὶ τῶν Δ, Γ δύο μέσοι ἀνάλογον ἐμπεσοῦνται ἀριθμοί. καὶ ἐστὶ κύβος ὁ Δ · κύβος ἄρα καὶ ὁ Γ · ὅπερ ἔδει δεῖξαι.

ε'.

Ἐὰν κύβος ἀριθμὸς ἀριθμὸν τινα πολλαπλασιάσας κύβον ποιῇ, καὶ ὁ πολλαπλασιασθεὶς κύβος ἔσται.



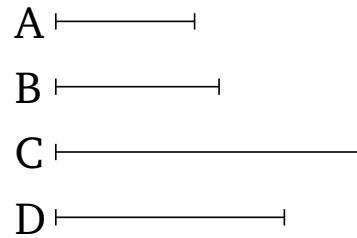
Κύβος γὰρ ἀριθμὸς ὁ A ἀριθμὸν τινα τὸν B πολλαπλασιάσας κύβον τὸν Γ ποιεῖτω· λέγω, ὅτι ὁ B κύβος ἐστίν.

Ὁ γὰρ A ἑαυτὸν πολλαπλασιάσας τὸν Δ ποιεῖτω· κύβος ἄρα ἐστὶν ὁ Δ . καὶ ἐπεὶ ὁ A ἑαυτὸν μὲν πολλαπλασιάσας τὸν Δ πεποίηκεν, τὸν δὲ B πολλαπλασιάσας τὸν Γ πεποίηκεν, ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , ὁ Δ πρὸς τὸν Γ . καὶ ἐπεὶ οἱ Δ, Γ κύβοι εἰσίν, ὅμοιοι στερεοὶ εἰσιν. τῶν Δ, Γ ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. καὶ ἐστὶν ὡς ὁ Δ πρὸς τὸν Γ , οὕτως ὁ A πρὸς τὸν B · καὶ τῶν A, B ἄρα δύο μέσοι ἀνάλογον ἐμπίπτουσιν ἀριθμοί. καὶ ἐστὶ κύβος ὁ A · κύβος ἄρα ἐστὶ καὶ ὁ B · ὅπερ ἔδει δεῖξαι.

ς'.

Ἐὰν ἀριθμὸς ἑαυτὸν πολλαπλασιάσας κύβον ποιῇ, καὶ

will be cube.

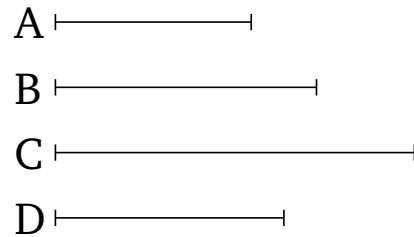


For let the cube number A make C (by) multiplying the cube number B . I say that C is cube.

For let A make D (by) multiplying itself. Thus, D is cube [Prop. 9.3]. And since A has made D (by) multiplying itself, and has made C (by) multiplying B , thus as A is to B , so D (is) to C [Prop. 7.17]. And since A and B are cube, A and B are similar solid (numbers). Thus, two numbers fall (between) A and B in mean proportion [Prop. 8.19]. Hence, two numbers will also fall (between) D and C in mean proportion [Prop. 8.8]. And D is cube. Thus, C (is) also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

Proposition 5

If a cube number makes a(nother) cube number (by) multiplying some (number) then the (number) multiplied will also be cube.



For let the cube number A make the cube (number) C (by) multiplying some number B . I say that B is cube.

For let A make D (by) multiplying itself. D is thus cube [Prop. 9.3]. And since A has made D (by) multiplying itself, and has made C (by) multiplying B , thus as A is to B , so D (is) to C [Prop. 7.17]. And since D and C are (both) cube, they are similar solid (numbers). Thus, two numbers fall (between) D and C in mean proportion [Prop. 8.19]. And as D is to C , so A (is) to B . Thus, two numbers also fall (between) A and B in mean proportion [Prop. 8.8]. And A is cube. Thus, B is also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

Proposition 6

If a number makes a cube (number by) multiplying

αὐτὸς κύβος ἔσται.

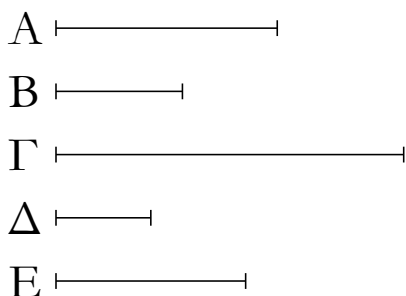


Ἀριθμὸς γὰρ ὁ A ἑαυτὸν πολλαπλασιάσας κύβον τὸν B ποιῶν· λέγω, ὅτι καὶ ὁ A κύβος ἐστίν.

Ὁ γὰρ A τὸν B πολλαπλασιάσας τὸν Γ ποιῶν. ἐπεὶ οὖν ὁ A ἑαυτὸν μὲν πολλαπλασιάσας τὸν B πεποίηκεν, τὸν δὲ B πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Γ ἄρα κύβος ἐστίν. καὶ ἐπεὶ ὁ A ἑαυτὸν πολλαπλασιάσας τὸν B πεποίηκεν, ὁ A ἄρα τὸν B μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας. μετρεῖ δὲ καὶ ἡ μονὰς τὸν A κατὰ τὰς ἐν αὐτῇ μονάδας. ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν A , οὕτως ὁ A πρὸς τὸν B . καὶ ἐπεὶ ὁ A τὸν B πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ B ἄρα τὸν Γ μετρεῖ κατὰ τὰς ἐν τῷ A μονάδας. μετρεῖ δὲ καὶ ἡ μονὰς τὸν A κατὰ τὰς ἐν αὐτῇ μονάδας. ἔστιν ἄρα ὡς ἡ μονὰς πρὸς τὸν A , οὕτως ὁ B πρὸς τὸν Γ . ἀλλ' ὡς ἡ μονὰς πρὸς τὸν A , οὕτως ὁ A πρὸς τὸν B . καὶ ὡς ἄρα ὁ A πρὸς τὸν B , ὁ B πρὸς τὸν Γ . καὶ ἐπεὶ οἱ B , Γ κύβοι εἰσίν, ὅμοιοι στερεοὶ εἰσιν. τῶν B , Γ ἄρα δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί. καὶ ἐστὶν ὡς ὁ B πρὸς τὸν Γ , ὁ A πρὸς τὸν B . καὶ τῶν A , B ἄρα δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί. καὶ ἐστὶν κύβος ὁ B . κύβος ἄρα ἐστὶ καὶ ὁ A . ὅπερ ἔδει δεῖξαι.

ζ'.

Ἐὰν σύνθετος ἀριθμὸς ἀριθμὸν τινα πολλαπλασιάσας ποιῇ τινα, ὁ γεγόμενος στερεὸς ἔσται.



Σύνθετος γὰρ ἀριθμὸς ὁ A ἀριθμὸν τινα τὸν B πολλαπλασιάσας τὸν Γ ποιῶν· λέγω, ὅτι ὁ Γ στερεὸς ἐστίν.

Ἐπεὶ γὰρ ὁ A σύνθετός ἐστιν, ὑπὸ ἀριθμοῦ τινος μετρηθήσεται. μετρεῖσθω ὑπὸ τοῦ Δ , καὶ ὅσας ὁ Δ τὸν A μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ E . ἐπεὶ οὖν ὁ Δ τὸν A μετρεῖ κατὰ τὰς ἐν τῷ E μονάδας, ὁ E ἄρα τὸν Δ πολλαπλασιάσας τὸν A πεποίηκεν. καὶ ἐπεὶ ὁ A τὸν B πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ δὲ A ἐστὶν ὁ ἐκ τῶν Δ , E , ὁ ἄρα ἐκ τῶν Δ , E τὸν B πολλαπλασιάσας τὸν Γ πεποίηκεν. ὁ Γ ἄρα στερεὸς ἐστίν, πλευραὶ δὲ αὐτοῦ εἰσιν οἱ Δ , E , B .

itself then it itself will also be cube.

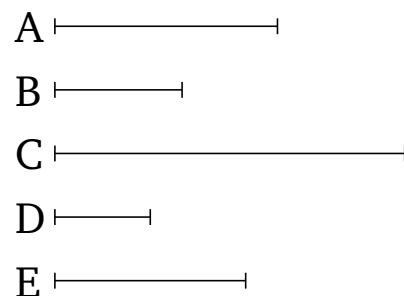


For let the number A make the cube (number) B (by) multiplying itself. I say that A is also cube.

For let A make C (by) multiplying B . Therefore, since A has made B (by) multiplying itself, and has made C (by) multiplying B , C is thus cube. And since A has made B (by) multiplying itself, A thus measures B according to the units in (A). And a unit also measures A according to the units in it. Thus, as a unit is to A , so A (is) to B . And since A has made C (by) multiplying B , B thus measures C according to the units in A . And a unit also measures A according to the units in it. Thus, as a unit is to A , so B (is) to C . But, as a unit (is) to A , so A (is) to B . And thus as A (is) to B , (so) B (is) to C . And since B and C are cube, they are similar solid (numbers). Thus, there exist two numbers in mean proportion (between) B and C [Prop. 8.19]. And as B is to C , (so) A (is) to B . Thus, there also exist two numbers in mean proportion (between) A and B [Prop. 8.8]. And B is cube. Thus, A is also cube [Prop. 8.23]. (Which is) the very thing it was required to show.

Proposition 7

If a composite number makes some (number by) multiplying some (other) number then the created (number) will be solid.



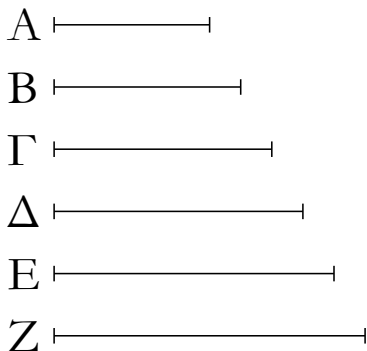
For let the composite number A make C (by) multiplying some number B . I say that C is solid.

For since A is a composite (number), it will be measured by some number. Let it be measured by D . And, as many times as D measures A , so many units let there be in E . Therefore, since D measures A according to the units in E , E has thus made A (by) multiplying D [Def. 7.15]. And since A has made C (by) multiplying B , and A is the (number created) from (multiplying) D , E , the (number created) from (multiplying) D , E has thus

ὅπερ ἔδει δεῖξαι.

η'.

Ἐάν ἀπὸ μονάδος ὅποσοιῦν ἀριθμοὶ ἐξῆς ἀνάλογον ὦσιν, ὁ μὲν τρίτος ἀπὸ τῆς μονάδος τετράγωνος ἔσται καὶ οἱ ἕνα διαλείποντες, ὁ δὲ τέταρτος κύβος καὶ οἱ δύο διαλείποντες πάντες, ὁ δὲ ἕβδομος κύβος ἅμα καὶ τετράγωνος καὶ οἱ πέντε διαλείποντες.



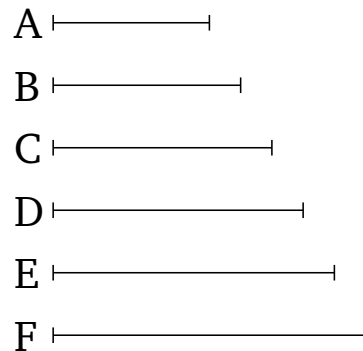
Ἐστωσαν ἀπὸ μονάδος ὅποσοιῦν ἀριθμοὶ ἐξῆς ἀνάλογον οἱ A, B, Γ, Δ, E, Z· λέγω, ὅτι ὁ μὲν τρίτος ἀπὸ τῆς μονάδος ὁ B τετράγωνος ἔστι καὶ οἱ ἕνα διαλείποντες πάντες, ὁ δὲ τέταρτος ὁ Γ κύβος καὶ οἱ δύο διαλείποντες πάντες, ὁ δὲ ἕβδομος ὁ Z κύβος ἅμα καὶ τετράγωνος καὶ οἱ πέντε διαλείποντες πάντες.

Ἐπεὶ γάρ ἐστιν ὡς ἡ μονὰς πρὸς τὸν A, οὕτως ὁ A πρὸς τὸν B, ἰσάκεις ἄρα ἡ μονὰς τὸν A ἀριθμὸν μετρεῖ καὶ ὁ A τὸν B. ἡ δὲ μονὰς τὸν A ἀριθμὸν μετρεῖ κατὰ τὰς ἐν αὐτῇ μονάδας· καὶ ὁ A ἄρα τὸν B μετρεῖ κατὰ τὰς ἐν τῇ A μονάδας. ὁ A ἄρα ἑαυτὸν πολλαπλασιάσας τὸν B πεποίηκεν· τετράγωνος ἄρα ἐστὶν ὁ B. καὶ ἐπεὶ οἱ B, Γ, Δ ἐξῆς ἀνάλογόν εἰσιν, ὁ δὲ B τετράγωνος ἐστίν, καὶ ὁ Δ ἄρα τετράγωνος ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ὁ Z τετράγωνος ἐστίν. ὁμοίως δὲ δείξομεν, ὅτι καὶ οἱ ἕνα διαλείποντες πάντες τετράγωνοί εἰσιν. λέγω δὴ, ὅτι καὶ ὁ τέταρτος ἀπὸ τῆς μονάδος ὁ Γ κύβος ἐστὶ καὶ οἱ δύο διαλείποντες πάντες. ἐπεὶ γάρ ἐστιν ὡς ἡ μονὰς πρὸς τὸν A, οὕτως ὁ B πρὸς τὸν Γ, ἰσάκεις ἄρα ἡ μονὰς τὸν A ἀριθμὸν μετρεῖ καὶ ὁ B τὸν Γ. ἡ δὲ μονὰς τὸν A ἀριθμὸν μετρεῖ κατὰ τὰς ἐν τῇ A μονάδας· καὶ ὁ B ἄρα τὸν Γ μετρεῖ κατὰ τὰς ἐν τῇ A μονάδας· ὁ A ἄρα τὸν B πολλαπλασιάσας τὸν Γ πεποίηκεν. ἐπεὶ οὖν ὁ A ἑαυτὸν μὲν πολλαπλασιάσας τὸν B πεποίηκεν, τὸν δὲ B πολλαπλασιάσας τὸν Γ πεποίηκεν, κύβος ἄρα ἐστὶν ὁ Γ. καὶ ἐπεὶ οἱ Γ, Δ, E, Z ἐξῆς ἀνάλογόν εἰσιν, ὁ δὲ Γ κύβος ἐστίν,

made *C* (by) multiplying *B*. Thus, *C* is solid, and its sides are *D*, *E*, *B*. (Which is) the very thing it was required to show.

Proposition 8

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then the third from the unit will be square, and (all) those (numbers after that) which leave an interval of one (number), and the fourth (will be) cube, and all those (numbers after that) which leave an interval of two (numbers), and the seventh (will be) both cube and square, and (all) those (numbers after that) which leave an interval of five (numbers).



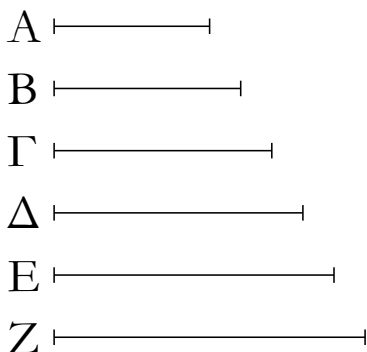
Let any multitude whatsoever of numbers, *A*, *B*, *C*, *D*, *E*, *F*, be continuously proportional, (starting) from a unit. I say that the third from the unit, *B*, is square, and all those (numbers after that) which leave an interval of one (number). And the fourth (from the unit), *C*, (is) cube, and all those (numbers after that) which leave an interval of two (numbers). And the seventh (from the unit), *F*, (is) both cube and square, and all those (numbers after that) which leave an interval of five (numbers).

For since as the unit is to *A*, so *A* (is) to *B*, the unit thus measures the number *A* the same number of times as *A* (measures) *B* [Def. 7.20]. And the unit measures the number *A* according to the units in it. Thus, *A* also measures *B* according to the units in *A*. *A* has thus made *B* (by) multiplying itself [Def. 7.15]. Thus, *B* is square. And since *B*, *C*, *D* are continuously proportional, and *B* is square, *D* is thus also square [Prop. 8.22]. So, for the same (reasons), *F* is also square. So, similarly, we can also show that all those (numbers after that) which leave an interval of one (number) are square. So I also say that the fourth (number) from the unit, *C*, is cube, and all those (numbers after that) which leave an interval of two (numbers). For since as the unit is to *A*, so *B* (is) to *C*, the unit thus measures the number *A* the same number of times that *B* (measures) *C*. And the unit measures the

καὶ ὁ Z ἄρα κύβος ἐστίν. ἐδείχθη δὲ καὶ τετράγωνος· ὁ ἄρα ἑβδομος ἀπὸ τῆς μονάδος κύβος τέ ἐστι καὶ τετράγωνος. ὁμοίως δὴ δείξομεν, ὅτι καὶ οἱ πέντε διαλείποντες πάντες κύβοι τέ εἰσι καὶ τετράγωνοι· ὅπερ ἔδει δείξαι.

Θ'.

Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἐξῆς κατὰ τὸ συνεχὲς ἀριθμοὶ ἀνάλογον ὦσιν, ὁ δὲ μετὰ τὴν μονάδα τετράγωνος ᾗ, καὶ οἱ λοιποὶ πάντες τετράγωνοι ἔσονται. καὶ ἐὰν ὁ μετὰ τὴν μονάδα κύβος ᾗ, καὶ οἱ λοιποὶ πάντες κύβοι ἔσονται.



Ἐστωσαν ἀπὸ μονάδος ἐξῆς ἀνάλογον ὁσοιδηποῦν ἀριθμοὶ οἱ $A, B, \Gamma, \Delta, E, Z$, ὁ δὲ μετὰ τὴν μονάδα ὁ A τετράγωνος ἔστω· λέγω, ὅτι καὶ οἱ λοιποὶ πάντες τετράγωνοι ἔσονται.

Ὅτι μὲν οὖν ὁ τρίτος ἀπὸ τῆς μονάδος ὁ B τετράγωνός ἐστι καὶ οἱ ἕνα διαλείποντες πάντες, δέδεικται· λέγω [δή], ὅτι καὶ οἱ λοιποὶ πάντες τετράγωνοί εἰσιν. ἐπεὶ γὰρ οἱ A, B, Γ ἐξῆς ἀνάλογόν εἰσιν, καὶ ἐστὶν ὁ A τετράγωνος, καὶ ὁ Γ [ἄρα] τετράγωνος ἐστίν. πάλιν, ἐπεὶ [καὶ] οἱ B, Γ, Δ ἐξῆς ἀνάλογόν εἰσιν, καὶ ἐστὶν ὁ B τετράγωνος, καὶ ὁ Δ [ἄρα] τετράγωνός ἐστιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ οἱ λοιποὶ πάντες τετράγωνοί εἰσιν.

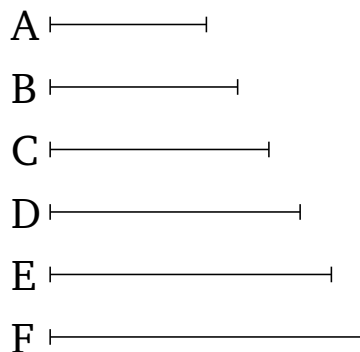
Ἀλλὰ δὴ ἔστω ὁ A κύβος· λέγω, ὅτι καὶ οἱ λοιποὶ πάντες κύβοι εἰσιν.

Ὅτι μὲν οὖν ὁ τέταρτος ἀπὸ τῆς μονάδος ὁ Γ κύβος ἐστὶ καὶ οἱ δύο διαλείποντες πάντες, δέδεικται· λέγω [δή], ὅτι καὶ οἱ λοιποὶ πάντες κύβοι εἰσίν. ἐπεὶ γὰρ ἐστὶν ὡς ἡ μονὰς πρὸς τὸν A , οὕτως ὁ A πρὸς τὸν B , ἰσάκως ἄρα ἡ μονὰς πρὸς τὸν A μετρεῖ καὶ ὁ A τὸν B . ἡ δὲ μονὰς τὸν A μετρεῖ κατὰ τὰς ἐν

number A according to the units in A . And thus B measures C according to the units in A . A has thus made C (by) multiplying B . Therefore, since A has made B (by) multiplying itself, and has made C (by) multiplying B , C is thus cube. And since C, D, E, F are continuously proportional, and C is cube, F is thus also cube [Prop. 8.23]. And it was also shown (to be) square. Thus, the seventh (number) from the unit is (both) cube and square. So, similarly, we can show that all those (numbers after that) which leave an interval of five (numbers) are (both) cube and square. (Which is) the very thing it was required to show.

Proposition 9

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, and the (number) after the unit is square, then all the remaining (numbers) will also be square. And if the (number) after the unit is cube, then all the remaining (numbers) will also be cube.



Let any multitude whatsoever of numbers, A, B, C, D, E, F , be continuously proportional, (starting) from a unit. And let the (number) after the unit, A , be square. I say that all the remaining (numbers) will also be square.

In fact, it has (already) been shown that the third (number) from the unit, B , is square, and all those (numbers after that) which leave an interval of one (number) [Prop. 9.8]. [So] I say that all the remaining (numbers) are also square. For since A, B, C are continuously proportional, and A (is) square, C is [thus] also square [Prop. 8.22]. Again, since B, C, D are [also] continuously proportional, and B is square, D is [thus] also square [Prop. 8.22]. So, similarly, we can show that all the remaining (numbers) are also square.

And so let A be cube. I say that all the remaining (numbers) are also cube.

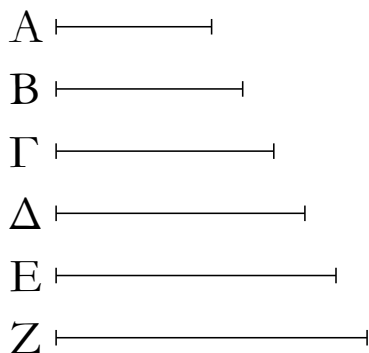
In fact, it has (already) been shown that the fourth (number) from the unit, C , is cube, and all those (numbers after that) which leave an interval of two (numbers)

αὐτῶ μονάδας· καὶ ὁ A ἄρα τὸν B μετρεῖ κατὰ τὰς ἐν αὐτῶ μονάδας· ὁ A ἄρα ἑαυτὸν πολλαπλασιάσας τὸν B πεποίηκεν· καὶ ἐστὶν ὁ A κύβος. ἐὰν δὲ κύβος ἀριθμὸς ἑαυτὸν πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος κύβος ἐστίν· καὶ ὁ B ἄρα κύβος ἐστίν. καὶ ἐπεὶ τέσσαρες ἀριθμοὶ οἱ A, B, Γ, Δ ἐξῆς ἀνάλογόν εἰσιν, καὶ ἐστὶν ὁ A κύβος, καὶ ὁ Δ ἄρα κύβος ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ὁ E κύβος ἐστίν, καὶ ὁμοίως οἱ λοιποὶ πάντες κύβοι εἰσίν· ὅπερ ἔδει δεῖξαι.

[Prop. 9.8]. [So] I say that all the remaining (numbers) are also cube. For since as the unit is to A , so A (is) to B , the unit thus measures A the same number of times as A (measures) B . And the unit measures A according to the units in it. Thus, A also measures B according to the units in (A). A has thus made B (by) multiplying itself. And A is cube. And if a cube number makes some (number by) multiplying itself then the created (number) is cube [Prop. 9.3]. Thus, B is also cube. And since the four numbers A, B, C, D are continuously proportional, and A is cube, D is thus also cube [Prop. 8.23]. So, for the same (reasons), E is also cube, and, similarly, all the remaining (numbers) are cube. (Which is) the very thing it was required to show.

ι'.

Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ [ἐξῆς] ἀνάλογον ὦσιν, ὁ δὲ μετὰ τὴν μονάδα μὴ ᾗ τετράγωνος, οὐδ' ἄλλος οὐδεὶς τετράγωνος ἔσται χωρὶς τοῦ τρίτου ἀπὸ τῆς μονάδος καὶ τῶν ἑνα διαλειπόντων πάντων. καὶ ἐὰν ὁ μετὰ τὴν μονάδα κύβος μὴ ᾗ, οὐδὲ ἄλλος οὐδεὶς κύβος ἔσται χωρὶς τοῦ τετάρτου ἀπὸ τῆς μονάδος καὶ τῶν δύο διαλειπόντων πάντων.

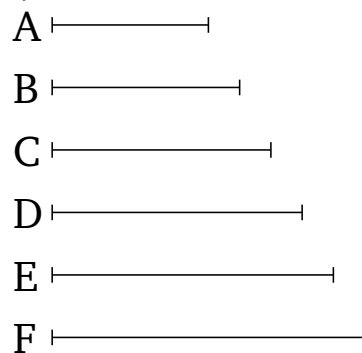


Ἐστωσαν ἀπὸ μονάδος ἐξῆς ἀνάλογον ὁσοιδηποῦν ἀριθμοὶ οἱ $A, B, \Gamma, \Delta, E, Z$, ὁ μετὰ τὴν μονάδα ὁ A μὴ ἔστω τετράγωνος· λέγω, ὅτι οὐδὲ ἄλλος οὐδεὶς τετράγωνος ἔσται χωρὶς τοῦ τρίτου ἀπὸ τῆς μονάδος [καὶ τῶν ἑνα διαλειπόντων].

Εἰ γὰρ δυνατόν, ἔστω ὁ Γ τετράγωνος. ἔστι δὲ καὶ ὁ B τετράγωνος· οἱ B, Γ ἄρα πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ ἐστὶν ὡς ὁ B πρὸς τὸν Γ , ὁ A πρὸς τὸν B · οἱ A, B ἄρα πρὸς ἀλλήλους λόγον ἔχουσιν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ὥστε οἱ A, B ὅμοιοι ἐπίπεδοι εἰσιν. καὶ ἐστὶ τετράγωνος ὁ B · τετράγωνος ἄρα ἐστὶ καὶ ὁ A · ὅπερ οὐχ ὑπέκειτο. οὐκ ἄρα ὁ Γ τετράγωνός ἐστιν. ὁμοίως δὴ δείξομεν, ὅτι οὐδ' ἄλλος οὐδεὶς τετράγωνός ἐστι χωρὶς

Proposition 10

If any multitude whatsoever of numbers is [continuously] proportional, (starting) from a unit, and the (number) after the unit is not square, then no other (number) will be square either, apart from the third from the unit, and all those (numbers after that) which leave an interval of one (number). And if the (number) after the unit is not cube, then no other (number) will be cube either, apart from the fourth from the unit, and all those (numbers after that) which leave an interval of two (numbers).



Let any multitude whatsoever of numbers, A, B, C, D, E, F , be continuously proportional, (starting) from a unit. And let the (number) after the unit, A , not be square. I say that no other (number) will be square either, apart from the third from the unit [and (all) those (numbers after that) which leave an interval of one (number)].

For, if possible, let C be square. And B is also square [Prop. 9.8]. Thus, B and C have to one another (the) ratio which (some) square number (has) to (some other) square number. And as B is to C , (so) A (is) to B . Thus, A and B have to one another (the) ratio which (some) square number has to (some other) square number. Hence, A and B are similar plane (numbers)

τοῦ τρίτου ἀπὸ τῆς μονάδος καὶ τῶν ἑνα διαλειπόντων.

Ἀλλὰ δὴ μὴ ἔστω ὁ A κύβος. λέγω, ὅτι οὐδ' ἄλλος οὐδεὶς κύβος ἔσται χωρὶς τοῦ τετάρτου ἀπὸ τῆς μονάδος καὶ τῶν δύο διαλειπόντων.

Εἰ γὰρ δυνατόν, ἔστω ὁ Δ κύβος. ἔστι δὲ καὶ ὁ Γ κύβος· τέταρτος γάρ ἐστιν ἀπὸ τῆς μονάδος. καὶ ἐστὶν ὡς ὁ Γ πρὸς τὸν Δ , ὁ B πρὸς τὸν Γ · καὶ ὁ B ἄρα πρὸς τὸν Γ λόγον ἔχει, ὃν κύβος πρὸς κύβον. καὶ ἐστὶν ὁ Γ κύβος· καὶ ὁ B ἄρα κύβος ἐστίν. καὶ ἐπεὶ ἐστὶν ὡς ἡ μονὰς πρὸς τὸν A , ὁ A πρὸς τὸν B , ἡ δὲ μονὰς τὸν A μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας, καὶ ὁ A ἄρα τὸν B μετρεῖ κατὰ τὰς ἐν αὐτῷ μονάδας· ὁ A ἄρα ἑαυτὸν πολλαπλασιάσας κύβον τὸν B πεποίηκεν. ἐὰν δὲ ἀριθμὸς ἑαυτὸν πολλαπλασιάσας κύβον ποιῇ, καὶ αὐτὸς κύβος ἔσται. κύβος ἄρα καὶ ὁ A · ὅπερ οὐχ ὑπόκειται. οὐκ ἄρα ὁ Δ κύβος ἐστίν. ὁμοίως δὴ δείξομεν, ὅτι οὐδ' ἄλλος οὐδεὶς κύβος ἐστὶ χωρὶς τοῦ τετάρτου ἀπὸ τῆς μονάδος καὶ τῶν δύο διαλειπόντων· ὅπερ εἶδει δείξαι.

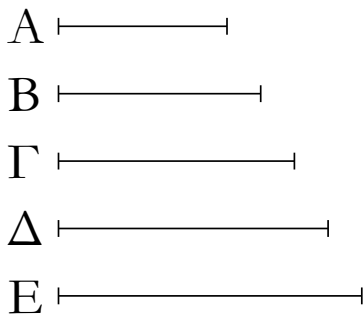
[Prop. 8.26]. And B is square. Thus, A is also square. The very opposite thing was assumed. C is thus not square. So, similarly, we can show that no other (number is) square either, apart from the third from the unit, and (all) those (numbers after that) which leave an interval of one (number).

And so let A not be cube. I say that no other (number) will be cube either, apart from the fourth from the unit, and (all) those (numbers after that) which leave an interval of two (numbers).

For, if possible, let D be cube. And C is also cube [Prop. 9.8]. For it is the fourth (number) from the unit. And as C is to D , (so) B (is) to C . And B thus has to C the ratio which (some) cube (number has) to (some other) cube (number). And C is cube. Thus, B is also cube [Props. 7.13, 8.25]. And since as the unit is to A , (so) A (is) to B , and the unit measures A according to the units in it, A thus also measures B according to the units in (A). Thus, A has made the cube (number) B (by) multiplying itself. And if a number makes a cube (number by) multiplying itself then it itself will be cube [Prop. 9.6]. Thus, A (is) also cube. The very opposite thing was assumed. Thus, D is not cube. So, similarly, we can show that no other (number) is cube either, apart from the fourth from the unit, and (all) those (numbers after that) which leave an interval of two (numbers). (Which is) the very thing it was required to show.

ια'.

Ἐὰν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον ὦσιν, ὁ ἐλάττων τὸν μείζονα μετρεῖ κατὰ τινὰ τῶν ὑπαρχόντων ἐν τοῖς ἀνάλογον ἀριθμοῖς.

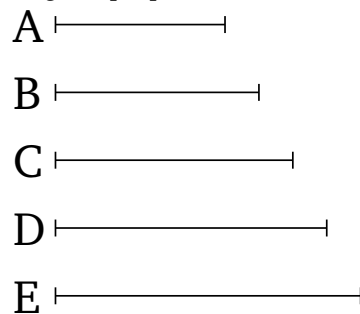


Ἐστωσαν ἀπὸ μονάδος τῆς A ὁποσοιοῦν ἀριθμοὶ ἐξῆς ἀνάλογον οἱ B , Γ , Δ , E · λέγω, ὅτι τῶν B , Γ , Δ , E ὁ ἐλάχιστος ὁ B τὸν E μετρεῖ κατὰ τινὰ τῶν Γ , Δ .

Ἐπεὶ γάρ ἐστιν ὡς ἡ A μονὰς πρὸς τὸν B , οὕτως ὁ Δ πρὸς τὸν E , ἰσάκεις ἄρα ἡ A μονὰς τὸν B ἀριθμὸν μετρεῖ καὶ ὁ Δ τὸν E · ἐναλλάξ ἄρα ἰσάκεις ἡ A μονὰς τὸν Δ μετρεῖ καὶ ὁ B τὸν E . ἡ δὲ A μονὰς τὸν Δ μετρεῖ κατὰ τὰς ἐν

Proposition 11

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then a lesser (number) measures a greater according to some existing (number) among the proportional numbers.



Let any multitude whatsoever of numbers, B , C , D , E , be continuously proportional, (starting) from the unit A . I say that, for B , C , D , E , the least (number), B , measures E according to some (one) of C , D .

For since as the unit A is to B , so D (is) to E , the unit A thus measures the number B the same number of times as D (measures) E . Thus, alternately, the unit A

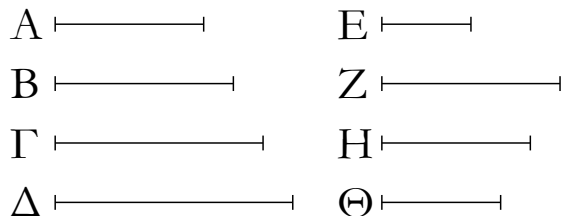
αὐτῷ μονάδας· καὶ ὁ B ἄρα τὸν E μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας· ὥστε ὁ ἐλάχιστος ὁ B τὸν μείζονα τὸν E μετρεῖ κατὰ τινὰ ἀριθμὸν τῶν ὑπαρχόντων ἐν τοῖς ἀνάλογον ἀριθμοῖς.

Πόρισμα.

Καὶ φανερόν, ὅτι ἣν ἔχει τάξιν ὁ μετρῶν ἀπὸ μονάδος, τὴν αὐτὴν ἔχει καὶ ὁ καθ' ὃν μετρεῖ ἀπὸ τοῦ μετρομένου ἐπὶ τὸ πρὸ αὐτοῦ. ὅπερ εἶδει δεῖξαι.

ιβ'.

Ἐάν ἀπὸ μονάδος ὅποσοιῦν ἀριθμοὶ ἐξῆς ἀνάλογον ὦσιν, ὑφ' ὧσων ἂν ὁ ἔσχατος πρῶτων ἀριθμῶν μετρηῖται, ὑπὸ τῶν αὐτῶν καὶ ὁ παρὰ τὴν μονάδα μετρηθήσεται.



Ἐστωσαν ἀπὸ μονάδος ὅποσοιδηποτοῦν ἀριθμοὶ ἀνάλογον οἱ A, B, Γ, Δ · λέγω, ὅτι ὑφ' ὧσων ἂν ὁ Δ πρῶτων ἀριθμῶν μετρηῖται, ὑπὸ τῶν αὐτῶν καὶ ὁ A μετρηθήσεται.

Μετρεῖσθω γὰρ ὁ Δ ὑπὸ τινος πρῶτου ἀριθμοῦ τοῦ E · λέγω, ὅτι ὁ E τὸν A μετρεῖ. μὴ γάρ· καὶ ἐστὶν ὁ E πρῶτος, ἅπας δὲ πρῶτος ἀριθμὸς πρὸς ἅπαντα, ὃν μὴ μετρεῖ, πρῶτός ἐστιν· οἱ E, A ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. καὶ ἐπεὶ ὁ E τὸν Δ μετρεῖ, μετρεῖτω αὐτὸν κατὰ τὸν Z · ὁ E ἄρα τὸν Z πολλαπλασιάσας τὸν Δ πεποίηκεν. πάλιν, ἐπεὶ ὁ A τὸν Δ μετρεῖ κατὰ τὰς ἐν τῷ Γ μονάδας, ὁ A ἄρα τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ E τὸν Z πολλαπλασιάσας τὸν Δ πεποίηκεν· ὁ ἄρα ἐκ τῶν A, Γ ἴσος ἐστὶ τῷ ἐκ τῶν E, Z . ἐστὶν ἄρα ὡς ὁ A πρὸς τὸν E , ὁ Z πρὸς τὸν Γ . οἱ δὲ A, E πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκως ὃ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον· μετρεῖ ἄρα ὁ E τὸν Γ . μετρεῖτω αὐτὸν κατὰ τὸν H · ὁ E ἄρα τὸν H πολλαπλασιάσας τὸν Γ πεποίηκεν. ἀλλὰ μὴν διὰ τὸ πρὸ τούτου καὶ ὁ A τὸν B πολλαπλασιάσας τὸν Γ πεποίηκεν. ὁ ἄρα ἐκ τῶν A, B ἴσος ἐστὶ τῷ ἐκ τῶν E, H . ἐστὶν ἄρα ὡς ὁ A πρὸς τὸν E , ὁ H πρὸς τὸν B . οἱ δὲ A, E πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ

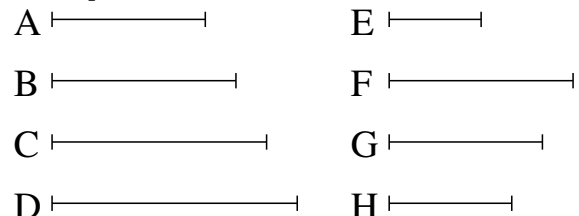
measures D the same number of times as B (measures) E [Prop. 7.15]. And the unit A measures D according to the units in it. Thus, B also measures E according to the units in D . Hence, the lesser (number) B measures the greater E according to some existing number among the proportional numbers (namely, D).

Corollary

And (it is) clear that what(ever relative) place the measuring (number) has from the unit, the (number) according to which it measures has the same (relative) place from the measured (number), in (the direction of the number) before it. (Which is) the very thing it was required to show.

Proposition 12

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, then however many prime numbers the last (number) is measured by, the (number) next to the unit will also be measured by the same (prime numbers).



Let any multitude whatsoever of numbers, A, B, C, D , be (continuously) proportional, (starting) from a unit. I say that however many prime numbers D is measured by, A will also be measured by the same (prime numbers).

For let D be measured by some prime number E . I say that E measures A . For (suppose it does) not. E is prime, and every prime number is prime to every number which it does not measure [Prop. 7.29]. Thus, E and A are prime to one another. And since E measures D , let it measure it according to F . Thus, E has made D (by) multiplying F . Again, since A measures D according to the units in C [Prop. 9.11 corr.], A has thus made D (by) multiplying C . But, in fact, E has also made D (by) multiplying F . Thus, the (number created) from (multiplying) A, C is equal to the (number created) from (multiplying) E, F . Thus, as A is to E , (so) F (is) to C [Prop. 7.19]. And A and E (are) prime (to one another), and (numbers) prime (to one another) are also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the lead-

μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας αὐτοῖς ἰσάκεις ὃ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον· μετρεῖ ἄρα ὁ E τὸν B . μετρεῖτω αὐτὸν κατὰ τὸν Θ · ὁ E ἄρα τὸν Θ πολλαπλασιάσας τὸν B πεποίηκεν. ἀλλὰ μὴν καὶ ὁ A ἑαυτὸν πολλαπλασιάσας τὸν B πεποίηκεν· ὁ ἄρα ἐκ τῶν E , Θ ἴσος ἐστὶ τῷ ἀπὸ τοῦ A . ἔστιν ἄρα ὡς ὁ E πρὸς τὸν A , ὁ A πρὸς τὸν Θ . οἱ δὲ A , E πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκεις ὃ ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον· μετρεῖ ἄρα ὁ E τὸν A ὡς ἡγούμενος ἡγούμενον. ἀλλὰ μὴν καὶ οὐ μετρεῖ· ὅπερ ἀδύνατον. οὐκ ἄρα οἱ E , A πρῶτοι πρὸς ἀλλήλους εἰσίν. σύνθετοι ἄρα. οἱ δὲ σύνθετοι ὑπὸ [πρώτου] ἀριθμοῦ τινος μετροῦνται. καὶ ἐπεὶ ὁ E πρῶτος ὑπόκειται, ὁ δὲ πρῶτος ὑπὸ ἐτέρου ἀριθμοῦ οὐ μετρεῖται ἢ ὑφ' ἑαυτοῦ, ὁ E ἄρα τοὺς A , E μετρεῖ· ὥστε ὁ E τὸν A μετρεῖ. μετρεῖ δὲ καὶ τὸν Δ · ὁ E ἄρα τοὺς A , Δ μετρεῖ. ὁμοίως δὲ δεῖξομεν, ὅτι ὑφ' ὧσων ἂν ὁ Δ πρώτων ἀριθμῶν μετρηται, ὑπὸ τῶν αὐτῶν καὶ ὁ A μετρηθήσεται· ὅπερ ἔδει δεῖξαι.

ing, and the following the following [Prop. 7.20]. Thus, E measures C . Let it measure it according to G . Thus, E has made C (by) multiplying G . But, in fact, via the (proposition) before this, A has also made C (by) multiplying B [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) A , B is equal to the (number created) from (multiplying) E , G . Thus, as A is to E , (so) G (is) to B [Prop. 7.19]. And A and E (are) prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, E measures B . Let it measure it according to H . Thus, E has made B (by) multiplying H . But, in fact, A has also made B (by) multiplying itself [Prop. 9.8]. Thus, the (number created) from (multiplying) E , H is equal to the (square) on A . Thus, as E is to A , (so) A (is) to H [Prop. 7.19]. And A and E are prime (to one another), and (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, E measures A , as the leading (measuring the) leading. But, in fact, (E) also does not measure (A). The very thing (is) impossible. Thus, E and A are not prime to one another. Thus, (they are) composite (to one another). And (numbers) composite (to one another) are (both) measured by some [prime] number [Def. 7.14]. And since E is assumed (to be) prime, and a prime (number) is not measured by another number (other) than itself [Def. 7.11], E thus measures (both) A and E . Hence, E measures A . And it also measures D . Thus, E measures (both) A and D . So, similarly, we can show that however many prime numbers D is measured by, A will also be measured by the same (prime numbers). (Which is) the very thing it was required to show.

ιγ'.

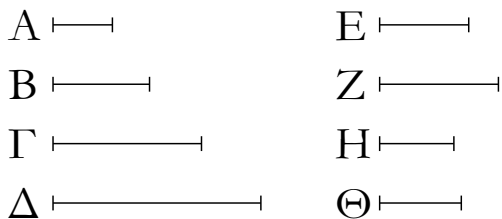
Proposition 13

Ἐὰν ἀπὸ μονάδος ὅποσοιῶν ἀριθμοὶ ἐξῆς ἀνάλογον ὦσιν, ὁ δὲ μετὰ τὴν μονάδα πρῶτος ἦ, ὁ μέγιστος ὑπ' οὐδενὸς [ἄλλου] μετρηθήσεται παρὲς τῶν ὑπαρχόντων ἐν τοῖς ἀνάλογον ἀριθμοῖς.

If any multitude whatsoever of numbers is continuously proportional, (starting) from a unit, and the (number) after the unit is prime, then the greatest (number) will be measured by no [other] (numbers) except (numbers) existing among the proportional numbers.

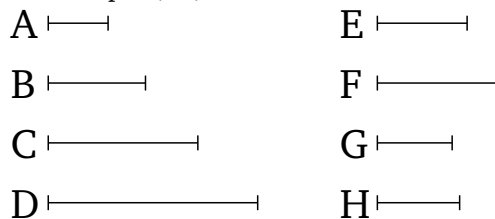
Ἐστῶσαν ἀπὸ μονάδος ὅποσοιῶν ἀριθμοὶ ἐξῆς ἀνάλογον οἱ A , B , Γ , Δ , ὁ δὲ μετὰ τὴν μονάδα ὁ A πρῶτος ἔστω· λέγω, ὅτι ὁ μέγιστος αὐτῶν ὁ Δ ὑπ' οὐδενὸς ἄλλου μετρηθήσεται παρὲς τῶν A , B , Γ .

Let any multitude whatsoever of numbers, A , B , C , D , be continuously proportional, (starting) from a unit. And let the (number) after the unit, A , be prime. I say



Εἰ γὰρ δυνατόν, μετρείσθω ὑπὸ τοῦ E , καὶ ὁ E μηδενὶ τῶν A, B, Γ ἕστω ὁ αὐτός. φανερόν δὴ, ὅτι ὁ E πρῶτος οὐκ ἔστιν. εἰ γὰρ ὁ E πρῶτός ἐστι καὶ μετρεῖ τὸν Δ , καὶ τὸν A μετρήσει πρῶτον ὄντα μὴ ὦν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ E πρῶτός ἐστιν. σύνθετος ἄρα. πᾶς δὲ σύνθετος ἀριθμὸς ὑπὸ πρῶτου τινὸς ἀριθμοῦ μετρεῖται· ὁ E ἄρα ὑπὸ πρῶτου τινὸς ἀριθμοῦ μετρεῖται. λέγω δὴ, ὅτι ὑπ' οὐδενὸς ἄλλου πρῶτου μετρηθήσεται πλὴν τοῦ A . εἰ γὰρ ὑφ' ἐτέρου μετρεῖται ὁ E , ὁ δὲ E τὸν Δ μετρεῖ, χάκεινος ἄρα τὸν Δ μετρήσει· ὥστε καὶ τὸν A μετρήσει πρῶτον ὄντα μὴ ὦν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. ὁ A ἄρα τὸν E μετρεῖ. καὶ ἐπεὶ ὁ E τὸν Δ μετρεῖ, μετρεῖται αὐτὸν κατὰ τὸν Z . λέγω, ὅτι ὁ Z οὐδενὶ τῶν A, B, Γ ἔστιν ὁ αὐτός. εἰ γὰρ ὁ Z ἐνὶ τῶν A, B, Γ ἔστιν ὁ αὐτός καὶ μετρεῖ τὸν Δ κατὰ τὸν E , καὶ εἷς ἄρα τῶν A, B, Γ τὸν Δ μετρεῖ κατὰ τὸν E . ἀλλὰ εἷς τῶν A, B, Γ τὸν Δ μετρεῖ κατὰ τινὰ τῶν A, B, Γ · καὶ ὁ E ἄρα ἐνὶ τῶν A, B, Γ ἔστιν ὁ αὐτός· ὅπερ οὐχ ὑπόκειται. οὐκ ἄρα ὁ Z ἐνὶ τῶν A, B, Γ ἔστιν ὁ αὐτός. ὁμοίως δὴ δείξομεν, ὅτι μετρεῖται ὁ Z ὑπὸ τοῦ A , δεικνύντες πάλιν, ὅτι ὁ Z οὐκ ἔστι πρῶτος. εἰ γὰρ, καὶ μετρεῖ τὸν Δ , καὶ τὸν A μετρήσει πρῶτον ὄντα μὴ ὦν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον· οὐκ ἄρα πρῶτός ἐστιν ὁ Z · σύνθετος ἄρα. ἅπας δὲ σύνθετος ἀριθμὸς ὑπὸ πρῶτου τινὸς ἀριθμοῦ μετρεῖται· ὁ Z ἄρα ὑπὸ πρῶτου τινὸς ἀριθμοῦ μετρεῖται. λέγω δὴ, ὅτι ὑφ' ἐτέρου πρῶτου οὐ μετρηθήσεται πλὴν τοῦ A . εἰ γὰρ ἕτερός τις πρῶτος τὸν Z μετρεῖ, ὁ δὲ Z τὸν Δ μετρεῖ, χάκεινος ἄρα τὸν Δ μετρήσει· ὥστε καὶ τὸν A μετρήσει πρῶτον ὄντα μὴ ὦν αὐτῷ ὁ αὐτός· ὅπερ ἐστὶν ἀδύνατον. ὁ A ἄρα τὸν Z μετρεῖ. καὶ ἐπεὶ ὁ E τὸν Δ μετρεῖ κατὰ τὸν Z , ὁ E ἄρα τὸν Z πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ A τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν· ὁ ἄρα ἐκ τῶν A, Γ ἴσος ἐστὶ τῷ ἐκ τῶν E, Z . ἀνάλογον ἄρα ἐστὶν ὡς ὁ A πρὸς τὸν E , οὕτως ὁ Z πρὸς τὸν Γ . ὁ δὲ A τὸν E μετρεῖ· καὶ ὁ Z ἄρα τὸν Γ μετρεῖ. μετρεῖται αὐτὸν κατὰ τὸν H . ὁμοίως δὴ δείξομεν, ὅτι ὁ H οὐδενὶ τῶν A, B ἔστιν ὁ αὐτός, καὶ ὅτι μετρεῖται ὑπὸ τοῦ A . καὶ ἐπεὶ ὁ Z τὸν Γ μετρεῖ κατὰ τὸν H , ὁ Z ἄρα τὸν H πολλαπλασιάσας τὸν Γ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ A τὸν B πολλαπλασιάσας τὸν Γ πεποίηκεν· ὁ ἄρα ἐκ τῶν A, B ἴσος ἐστὶ τῷ ἐκ τῶν Z, H . ἀνάλογον ἄρα ὡς ὁ A πρὸς τὸν Z , ὁ H πρὸς τὸν B . μετρεῖ δὲ ὁ A τὸν Z · μετρεῖ ἄρα καὶ ὁ H τὸν B . μετρεῖται αὐτὸν κατὰ τὸν Θ . ὁμοίως δὴ δείξομεν, ὅτι ὁ Θ τῷ A οὐκ ἔστιν ὁ αὐτός. καὶ ἐπεὶ ὁ H τὸν

that the greatest of them, D , will be measured by no other (numbers) except A, B, C .



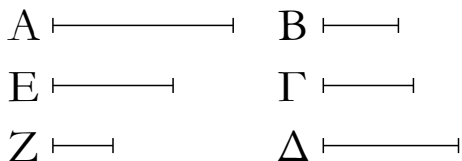
For, if possible, let it be measured by E , and let E not be the same as one of A, B, C . So it is clear that E is not prime. For if E is prime, and measures D , then it will also measure A , (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, E is not prime. Thus, (it is) composite. And every composite number is measured by some prime number [Prop. 7.31]. Thus, E is measured by some prime number. So I say that it will be measured by no other prime number than A . For if E is measured by another (prime number), and E measures D , then this (prime number) will thus also measure D . Hence, it will also measure A , (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, A measures E . And since E measures D , let it measure it according to F . I say that F is not the same as one of A, B, C . For if F is the same as one of A, B, C , and measures D according to E , then one of A, B, C thus also measures D according to E . But one of A, B, C (only) measures D according to some (one) of A, B, C [Prop. 9.11]. And thus E is the same as one of A, B, C . The very opposite thing was assumed. Thus, F is not the same as one of A, B, C . Similarly, we can show that F is measured by A , (by) again showing that F is not prime. For if (F is prime), and measures D , then it will also measure A , (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, F is not prime. Thus, (it is) composite. And every composite number is measured by some prime number [Prop. 7.31]. Thus, F is measured by some prime number. So I say that it will be measured by no other prime number than A . For if some other prime (number) measures F , and F measures D , then this (prime number) will thus also measure D . Hence, it will also measure A , (despite A) being prime (and) not being the same as it [Prop. 9.12]. The very thing is impossible. Thus, A measures F . And since E measures D according to F , E has thus made D (by) multiplying F . But, in fact, A has also made D (by) multiplying C [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) A, C is equal to the (number created) from (multiplying) E, F . Thus, proportionally, as A is to E , so F (is) to C [Prop. 7.19]. And A measures

Β μετρεῖ κατὰ τὸν Θ, ὁ Η ἄρα τὸν Θ πολλαπλασιάσας τὸν Β πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Α ἑαυτὸν πολλαπλασιάσας τὸν Β πεποίηκεν· ὁ ἄρα ὑπὸ Θ, Η ἴσος ἐστὶ τῷ ἀπὸ τοῦ Α τετραγώνῳ· ἔστιν ἄρα ὡς ὁ Θ πρὸς τὸν Α, ὁ Α πρὸς τὸν Η. μετρεῖ δὲ ὁ Α τὸν Η· μετρεῖ ἄρα καὶ ὁ Θ τὸν Α πρῶτον ὄντα μὴ ὦν αὐτῷ ὁ αὐτός· ὅπερ ἄτοπον. οὐκ ἄρα ὁ μέγιστος ὁ Δ ὑπὸ ἐτέρου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν Α, Β, Γ· ὅπερ ἔδει δεῖξαι.

E. Thus, *F* also measures *C*. Let it measure it according to *G*. So, similarly, we can show that *G* is not the same as one of *A*, *B*, and that it is measured by *A*. And since *F* measures *C* according to *G*, *F* has thus made *C* (by) multiplying *G*. But, in fact, *A* has also made *C* (by) multiplying *B* [Prop. 9.11 corr.]. Thus, the (number created) from (multiplying) *A*, *B* is equal to the (number created) from (multiplying) *F*, *G*. Thus, proportionally, as *A* (is) to *F*, so *G* (is) to *B* [Prop. 7.19]. And *A* measures *F*. Thus, *G* also measures *B*. Let it measure it according to *H*. So, similarly, we can show that *H* is not the same as *A*. And since *G* measures *B* according to *H*, *G* has thus made *B* (by) multiplying *H*. But, in fact, *A* has also made *B* (by) multiplying itself [Prop. 9.8]. Thus, the (number created) from (multiplying) *H*, *G* is equal to the square on *A*. Thus, as *H* is to *A*, (so) *A* (is) to *G* [Prop. 7.19]. And *A* measures *G*. Thus, *H* also measures *A*, (despite *A*) being prime (and) not being the same as it. The very thing (is) absurd. Thus, the greatest (number) *D* cannot be measured by another (number) except (one of) *A*, *B*, *C*. (Which is) the very thing it was required to show.

ιδ'.

Ἐὰν ἐλάχιστος ἀριθμὸς ὑπὸ πρώτων ἀριθμῶν μετρήται, ὑπ' οὐδενὸς ἄλλου πρώτου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν ἐξ ἀρχῆς μετρούντων.

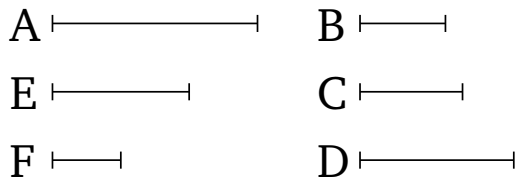


Ἐλάχιστος γὰρ ἀριθμὸς ὁ Α ὑπὸ πρώτων ἀριθμῶν τῶν Β, Γ, Δ μετρεῖσθω· λέγω, ὅτι ὁ Α ὑπ' οὐδενὸς ἄλλου πρώτου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν Β, Γ, Δ.

Εἰ γὰρ δυνατόν, μετρεῖσθω ὑπὸ πρώτου τοῦ Ε, καὶ ὁ Ε μηδενὶ τῶν Β, Γ, Δ ἔστω ὁ αὐτός. καὶ ἐπεὶ ὁ Ε τὸν Α μετρεῖ, μετρεῖτω αὐτὸν κατὰ τὸν Ζ· ὁ Ε ἄρα τὸν Ζ πολλαπλασιάσας τὸν Α πεποίηκεν. καὶ μετρεῖται ὁ Α ὑπὸ πρώτων ἀριθμῶν τῶν Β, Γ, Δ. ἐὰν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσιν τινα, τὸν δὲ γεγόμενον ἐξ αὐτῶν μετρή τις πρῶτος ἀριθμὸς, καὶ ἓνα τῶν ἐξ ἀρχῆς μετρήσει· οἱ Β, Γ, Δ ἄρα ἓνα τῶν Ε, Ζ μετρήσουσιν. τὸν μὲν οὖν Ε οὐ μετρήσουσιν· ὁ γὰρ Ε πρῶτός ἐστι καὶ οὐδενὶ τῶν Β, Γ, Δ ὁ αὐτός. τὸν Ζ ἄρα μετροῦσιν ἐλάσσονα ὄντα τοῦ Α· ὅπερ ἀδύνατον. ὁ γὰρ Α ὑπόκειται ἐλάχιστος ὑπὸ τῶν Β, Γ, Δ μετρούμενος. οὐκ ἄρα τὸν Α μετρήσει πρῶτος ἀριθμὸς παρὲξ τῶν Β, Γ, Δ· ὅπερ ἔδει δεῖξαι.

Proposition 14

If a least number is measured by (some) prime numbers then it will not be measured by any other prime number except (one of) the original measuring (numbers).

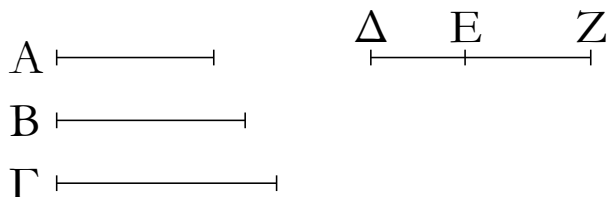


For let *A* be the least number measured by the prime numbers *B*, *C*, *D*. I say that *A* will not be measured by any other prime number except (one of) *B*, *C*, *D*.

For, if possible, let it be measured by the prime (number) *E*. And let *E* not be the same as one of *B*, *C*, *D*. And since *E* measures *A*, let it measure it according to *F*. Thus, *E* has made *A* (by) multiplying *F*. And *A* is measured by the prime numbers *B*, *C*, *D*. And if two numbers make some (number by) multiplying one another, and some prime number measures the number created from them, then (the prime number) will also measure one of the original (numbers) [Prop. 7.30]. Thus, *B*, *C*, *D* will measure one of *E*, *F*. In fact, they do not measure *E*. For *E* is prime, and not the same as one of *B*, *C*, *D*. Thus, they (all) measure *F*, which is less than *A*. The very thing (is) impossible. For *A* was assumed (to be) the least (number) measured by *B*, *C*, *D*. Thus, no prime

ιε'.

Ἐάν τρεῖς ἀριθμοὶ ἐξῆς ἀνάλογον ὦσιν ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς, δύο ὅποιοι οὖν συντεθέντες πρὸς τὸν λοιπὸν πρῶτοι εἰσιν.



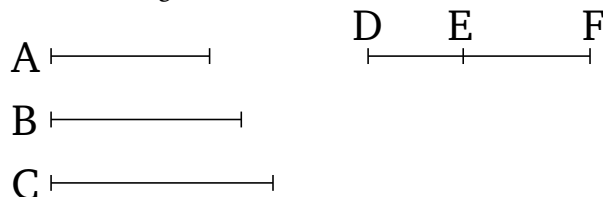
Ἐστωσαν τρεῖς ἀριθμοὶ ἐξῆς ἀνάλογον ἐλάχιστοι τῶν τὸν αὐτὸν λόγον ἐχόντων αὐτοῖς οἱ A, B, Γ · λέγω, ὅτι τῶν A, B, Γ δύο ὅποιοι οὖν συντεθέντες πρὸς τὸν λοιπὸν πρῶτοι εἰσιν, οἱ μὲν A, B πρὸς τὸν Γ , οἱ δὲ B, Γ πρὸς τὸν A καὶ ἔτι οἱ A, Γ πρὸς τὸν B .

Εἰλήφθωσαν γὰρ ἐλάχιστοι ἀριθμοὶ τῶν τὸν αὐτὸν λόγον ἐχόντων τοῖς A, B, Γ δύο οἱ $\Delta E, EZ$. φανερὸν δὴ, ὅτι ὁ μὲν ΔE ἑαυτὸν πολλαπλασιάσας τὸν A πεποίηκεν, τὸν δὲ EZ πολλαπλασιάσας τὸν B πεποίηκεν, καὶ ἔτι ὁ EZ ἑαυτὸν πολλαπλασιάσας τὸν Γ πεποίηκεν. καὶ ἐπεὶ οἱ $\Delta E, EZ$ ἐλάχιστοι εἰσιν, πρῶτοι πρὸς ἀλλήλους εἰσίν. ἐὰν δὲ δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, καὶ συναμφοτέρος πρὸς ἐκάτερον πρῶτός ἐστιν· καὶ ὁ ΔZ ἄρα πρὸς ἐκάτερον τῶν $\Delta E, EZ$ πρῶτός ἐστιν. ἀλλὰ μὴν καὶ ὁ ΔE πρὸς τὸν EZ πρῶτός ἐστιν· οἱ $\Delta Z, \Delta E$ ἄρα πρὸς τὸν EZ πρῶτοι εἰσιν. ἐὰν δὲ δύο ἀριθμοὶ πρὸς τινὰ ἀριθμὸν πρῶτοι ὦσιν, καὶ ὁ ἐξ αὐτῶν γενόμενος πρὸς τὸν λοιπὸν πρῶτός ἐστιν· ὥστε ὁ ἐκ τῶν $Z\Delta, \Delta E$ πρὸς τὸν EZ πρῶτός ἐστιν· ὥστε καὶ ὁ ἐκ τῶν $Z\Delta, \Delta E$ πρὸς τὸν ἀπὸ τοῦ EZ πρῶτός ἐστιν. [ἐὰν γὰρ δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, ὁ ἐκ τοῦ ἐνὸς αὐτῶν γενόμενος πρὸς τὸν λοιπὸν πρῶτός ἐστιν]. ἀλλ' ὁ ἐκ τῶν $Z\Delta, \Delta E$ ὁ ἀπὸ τοῦ ΔE ἐστὶ μετὰ τοῦ ἐκ τῶν $\Delta E, EZ$ · ὁ ἄρα ἀπὸ τοῦ ΔE μετὰ τοῦ ἐκ τῶν $\Delta E, EZ$ πρὸς τὸν ἀπὸ τοῦ EZ πρῶτός ἐστιν. καὶ ἐστὶν ὁ μὲν ἀπὸ τοῦ ΔE ὁ A , ὁ δὲ ἐκ τῶν $\Delta E, EZ$ ὁ B , ὁ δὲ ἀπὸ τοῦ EZ ὁ Γ · οἱ A, B ἄρα συντεθέντες πρὸς τὸν Γ πρῶτοι εἰσιν. ὁμοίως δὲ δείξομεν, ὅτι καὶ οἱ B, Γ πρὸς τὸν A πρῶτοι εἰσιν. λέγω δὴ, ὅτι καὶ οἱ A, Γ πρὸς τὸν B πρῶτοι εἰσιν. ἐπεὶ γὰρ ὁ ΔZ πρὸς ἐκάτερον τῶν $\Delta E, EZ$ πρῶτός ἐστιν, καὶ ὁ ἀπὸ τοῦ ΔZ πρὸς τὸν ἐκ τῶν $\Delta E, EZ$ πρῶτός ἐστιν. ἀλλὰ τῷ ἀπὸ τοῦ ΔZ ἴσοι εἰσὶν οἱ ἀπὸ τῶν $\Delta E, EZ$ μετὰ τοῦ δις ἐκ τῶν $\Delta E, EZ$ · καὶ οἱ ἀπὸ τῶν $\Delta E, EZ$ ἄρα μετὰ τοῦ δις ὑπὸ τῶν $\Delta E, EZ$ πρὸς τὸν ὑπὸ τῶν $\Delta E, EZ$ πρῶτοι [εἰσι]. διελόντι οἱ ἀπὸ τῶν $\Delta E, EZ$ μετὰ τοῦ ἁπαξ ὑπὸ $\Delta E, EZ$ πρὸς τὸν ὑπὸ $\Delta E, EZ$ πρῶτοι εἰσιν. ἔτι διελόντι οἱ ἀπὸ τῶν $\Delta E, EZ$ ἄρα πρὸς τὸν ὑπὸ $\Delta E, EZ$ πρῶτοι εἰσιν. καὶ ἐστὶν ὁ μὲν

number can measure A except (one of) B, C, D . (Which is) the very thing it was required to show.

Proposition 15

If three continuously proportional numbers are the least of those (numbers) having the same ratio as them then two (of them) added together in any way are prime to the remaining (one).



Let A, B, C be three continuously proportional numbers (which are) the least of those (numbers) having the same ratio as them. I say that two of A, B, C added together in any way are prime to the remaining (one), (that is) A and B (prime) to C , B and C to A , and, further, A and C to B .

Let the two least numbers, DE and EF , having the same ratio as A, B, C , have been taken [Prop. 8.2]. So it is clear that DE has made A (by) multiplying itself, and has made B (by) multiplying EF , and, further, EF has made C (by) multiplying itself [Prop. 8.2]. And since DE, EF are the least (of those numbers having the same ratio as them), they are prime to one another [Prop. 7.22]. And if two numbers are prime to one another then the sum (of them) is also prime to each [Prop. 7.28]. Thus, DF is also prime to each of DE, EF . But, in fact, DE is also prime to EF . Thus, DF, DE are (both) prime to EF . And if two numbers are (both) prime to some number then the (number) created from (multiplying) them is also prime to the remaining (number) [Prop. 7.24]. Hence, the (number created) from (multiplying) FD, DE is prime to EF . Hence, the (number created) from (multiplying) FD, DE is also prime to the (square) on EF [Prop. 7.25]. [For if two numbers are prime to one another then the (number) created from (squaring) one of them is prime to the remaining (number).] But the (number created) from (multiplying) FD, DE is the (square) on DE plus the (number created) from (multiplying) DE, EF [Prop. 2.3]. Thus, the (square) on DE plus the (number created) from (multiplying) DE, EF is prime to the (square) on EF . And the (square) on DE is A , and the (number created) from (multiplying) DE, EF (is) B , and the (square) on EF (is) C . Thus, A, B summed is prime to C . So, similarly, we can show that B, C (summed) is also prime to A . So I say that A, C (summed) is also prime to B . For since

ἀπὸ τοῦ ΔΕ ὁ Α, ὁ δὲ ὑπὸ τῶν ΔΕ, ΕΖ ὁ Β, ὁ δὲ ἀπὸ τοῦ ΕΖ ὁ Γ. οἱ Α, Γ ἄρα συντεθέντες πρὸς τὸν Β πρῶτοί εἰσιν· ὅπερ ἔδει δεῖξαι.

DF is prime to each of DE , EF then the (square) on DF is also prime to the (number created) from (multiplying) DE , EF [Prop. 7.25]. But, the (sum of the squares) on DE , EF plus twice the (number created) from (multiplying) DE , EF is equal to the (square) on DF [Prop. 2.4]. And thus the (sum of the squares) on DE , EF plus twice the (rectangle contained) by DE , EF [is] prime to the (rectangle contained) by DE , EF . By separation, the (sum of the squares) on DE , EF plus once the (rectangle contained) by DE , EF is prime to the (rectangle contained) by DE , EF .[†] Again, by separation, the (sum of the squares) on DE , EF is prime to the (rectangle contained) by DE , EF . And the (square) on DE is A , and the (rectangle contained) by DE , EF (is) B , and the (square) on EF (is) C . Thus, A , C summed is prime to B . (Which is) the very thing it was required to show.

[†] Since if $\alpha\beta$ measures $\alpha^2 + \beta^2 + 2\alpha\beta$ then it also measures $\alpha^2 + \beta^2 + \alpha\beta$, and vice versa.

ιϛ'.

Ἐὰν δύο ἀριθμοὶ πρῶτοι πρὸς ἀλλήλους ὦσιν, οὐκ ἔσται ὡς ὁ πρῶτος πρὸς τὸν δεύτερον, οὕτως ὁ δεύτερος πρὸς ἄλλον τινά.



Δύο γὰρ ἀριθμοὶ οἱ Α, Β πρῶτοι πρὸς ἀλλήλους ἔστωσαν· λέγω, ὅτι οὐκ ἔστιν ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς ἄλλον τινά.

Εἰ γὰρ δυνατόν, ἔστω ὡς ὁ Α πρὸς τὸν Β, ὁ Β πρὸς τὸν Γ. οἱ δὲ Α, Β πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκως ὅ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον· μετρεῖ ἄρα ὁ Α τὸν Β ὡς ἡγούμενος ἡγούμενον. μετρεῖ δὲ καὶ ἑαυτόν· ὁ Α ἄρα τοὺς Α, Β μετρεῖ πρῶτους ὄντας πρὸς ἀλλήλους· ὅπερ ἄτοπον. οὐκ ἄρα ἔσται ὡς ὁ Α πρὸς τὸν Β, οὕτως ὁ Β πρὸς τὸν Γ· ὅπερ ἔδει δεῖξαι.

ιζ'.

Ἐὰν ὦσιν ὅσοιδηποτοῦν ἀριθμοὶ ἐξῆς ἀνάλογον, οἱ δὲ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους ὦσιν, οὐκ ἔσται ὡς ὁ πρῶτος πρὸς τὸν δεύτερον, οὕτως ὁ ἔσχατος πρὸς ἄλλον

Proposition 16

If two numbers are prime to one another then as the first is to the second, so the second (will) not (be) to some other (number).



For let the two numbers A and B be prime to one another. I say that as A is to B , so B is not to some other (number).

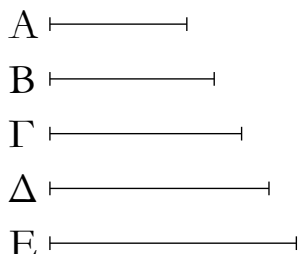
For, if possible, let it be that as A (is) to B , (so) B (is) to C . And A and B (are) prime (to one another). And (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures B , as the leading (measuring) the leading. And (A) also measures itself. Thus, A measures A and B , which are prime to one another. The very thing (is) absurd. Thus, as A (is) to B , so B cannot be to C . (Which is) the very thing it was required to show.

Proposition 17

If any multitude whatsoever of numbers is continuously proportional, and the outermost of them are prime to one another, then as the first (is) to the second, so the

τινά.

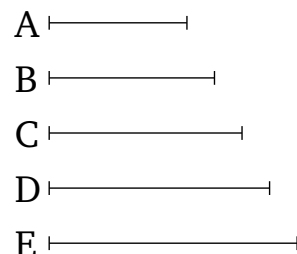
Ἐστωσαν ὅσοιδηποτοῦν ἀριθμοὶ ἐξῆς ἀνάλογον οἱ A, B, Γ, Δ , οἱ δὲ ἄκροι αὐτῶν οἱ A, Δ πρῶτοι πρὸς ἀλλήλους ἔστωσαν· λέγω, ὅτι οὐκ ἔστιν ὥς ὁ A πρὸς τὸν B , οὕτως ὁ Δ πρὸς ἄλλον τινά.



Εἰ γὰρ δυνατόν, ἔστω ὥς ὁ A πρὸς τὸν B , οὕτως ὁ Δ πρὸς τὸν E · ἐναλλάξ ἄρα ἔστιν ὥς ὁ A πρὸς τὸν Δ , ὁ B πρὸς τὸν E . οἱ δὲ A, Δ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι ἀριθμοὶ μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκως ὃ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον. μετρεῖ ἄρα ὁ A τὸν B . καὶ ἔστιν ὥς ὁ A πρὸς τὸν B , ὁ B πρὸς τὸν Γ . καὶ ὁ B ἄρα τὸν Γ μετρεῖ· ὥστε καὶ ὁ A τὸν Γ μετρεῖ. καὶ ἐπεὶ ἔστιν ὥς ὁ B πρὸς τὸν Γ , ὁ Γ πρὸς τὸν Δ , μετρεῖ δὲ ὁ B τὸν Γ , μετρεῖ ἄρα καὶ ὁ Γ τὸν Δ . ἀλλ' ὁ A τὸν Γ ἐμέτρει· ὥστε ὁ A καὶ τὸν Δ μετρεῖ. μετρεῖ δὲ καὶ ἑαυτὸν. ὁ A ἄρα τοὺς A, Δ μετρεῖ πρῶτους ὄντας πρὸς ἀλλήλους· ὅπερ ἔστιν ἀδύνατον. οὐκ ἄρα ἔσται ὥς ὁ A πρὸς τὸν B , οὕτως ὁ Δ πρὸς ἄλλον τινά· ὅπερ ἔδει δεῖξαι.

last will not be to some other (number).

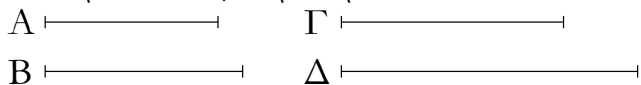
Let A, B, C, D be any multitude whatsoever of continuously proportional numbers. And let the outermost of them, A and D , be prime to one another. I say that as A is to B , so D (is) not to some other (number).



For, if possible, let it be that as A (is) to B , so D (is) to E . Thus, alternately, as A is to D , (so) B (is) to E [Prop. 7.13]. And A and D are prime (to one another). And (numbers) prime (to one another) are also the least (of those numbers having the same ratio as them) [Prop. 7.21]. And the least numbers measure those (numbers) having the same ratio (as them) an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures B . And as A is to B , (so) B (is) to C . Thus, B also measures C . And hence A measures C [Def. 7.20]. And since as B is to C , (so) C (is) to D , and B measures C , C thus also measures D [Def. 7.20]. But, A was (found to be) measuring C . And hence A also measures D . And (A) also measures itself. Thus, A measures A and D , which are prime to one another. The very thing is impossible. Thus, as A (is) to B , so D cannot be to some other (number). (Which is) the very thing it was required to show.

ιη'.

Δύο ἀριθμῶν δοθέντων ἐπισκέψασθαι, εἰ δυνατόν ἔστιν αὐτοῖς τρίτον ἀνάλογον προσερεῖν.



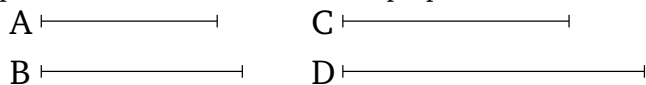
Ἐστωσαν οἱ δοθέντες δύο ἀριθμοὶ οἱ A, B , καὶ δέον ἔστω ἐπισκέψασθαι, εἰ δυνατόν ἔστιν αὐτοῖς τρίτον ἀνάλογον προσερεῖν.

Οἱ δὲ A, B ἥτοι πρῶτοι πρὸς ἀλλήλους εἰσὶν ἢ οὐ. καὶ εἰ πρῶτοι πρὸς ἀλλήλους εἰσὶν, δέδεικται, ὅτι ἀδύνατον ἔστιν αὐτοῖς τρίτον ἀνάλογον προσερεῖν.

Ἀλλὰ δὲ μὴ ἔστωσαν οἱ A, B πρῶτοι πρὸς ἀλλήλους, καὶ ὁ B ἑαυτὸν πολλαπλασιάσας τὸν Γ ποιεῖτω. ὁ A δὲ τὸν Γ ἥτοι μετρεῖ ἢ οὐ μετρεῖ. μετρεῖται πρότερον κατὰ τὸν Δ · ὁ A ἄρα τὸν Δ πολλαπλασιάσας τὸν Γ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ B ἑαυτὸν πολλαπλασιάσας τὸν Γ πεποίηκεν· ὁ ἄρα

Proposition 18

For two given numbers, to investigate whether it is possible to find a third (number) proportional to them.



Let A and B be the two given numbers. And let it be required to investigate whether it is possible to find a third (number) proportional to them.

So A and B are either prime to one another, or not. And if they are prime to one another then it has (already) been show that it is impossible to find a third (number) proportional to them [Prop. 9.16].

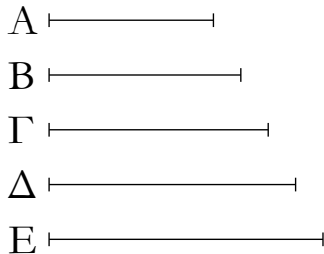
And so let A and B not be prime to one another. And let B make C (by) multiplying itself. So A either measures, or does not measure, C . Let it first of all measure (C) according to D . Thus, A has made C (by) multiply-

ἐκ τῶν A, Δ ἴσος ἐστὶ τῷ ἀπὸ τοῦ B . ἔστιν ἄρα ὡς ὁ A πρὸς τὸν B , ὁ B πρὸς τὸν Δ . τοῖς A, B ἄρα τρίτος ἀριθμὸς ἀνάλογον προσηύρηται ὁ Δ .

Ἀλλὰ δὴ μὴ μετρεῖτω ὁ A τὸν Γ . λέγω, ὅτι τοῖς A, B ἀδύνατόν ἐστι τρίτον ἀνάλογον προσεμερεῖν ἀριθμόν. εἰ γὰρ δυνατόν, προσηυρήσθω ὁ Δ . ὁ ἄρα ἐκ τῶν A, Δ ἴσος ἐστὶ τῷ ἀπὸ τοῦ B . ὁ δὲ ἀπὸ τοῦ B ἐστὶν ὁ Γ . ὁ ἄρα ἐκ τῶν A, Δ ἴσος ἐστὶ τῷ Γ . ὥστε ὁ A τὸν Δ πολλαπλασιάσας τὸν Γ πεποίηκεν. ὁ A ἄρα τὸν Γ μετρεῖ κατὰ τὸν Δ . ἀλλὰ μὴν ὑπόκειται καὶ μὴ μετρῶν. ὅπερ ἄτοπον. οὐκ ἄρα δυνατόν ἐστι τοῖς A, B τρίτον ἀνάλογον προσεμερεῖν ἀριθμόν, ὅταν ὁ A τὸν Γ μὴ μετρή. ὅπερ ἔδει δεῖξαι.

ιθ'.

Τριῶν ἀριθμῶν δοθέντων ἐπισκέψασθαι, πότε δυνατόν ἐστὶν αὐτοῖς τέταρτον ἀνάλογον προσεμερεῖν.



Ἐστωσαν οἱ δοθέντες τρεῖς ἀριθμοὶ οἱ A, B, Γ , καὶ δέον ἔστω ἐπισκέψασθαι, πότε δυνατόν ἐστὶν αὐτοῖς τέταρτον ἀνάλογον προσεμερεῖν.

Ἦτοι οὖν οὐκ εἰσιν ἐξῆς ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν, ἢ ἐξῆς εἰσιν ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν οὐκ εἰσὶ πρῶτοι πρὸς ἀλλήλους, ἢ οὔτε ἐξῆς εἰσιν ἀνάλογον, οὔτε οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν, ἢ καὶ ἐξῆς εἰσιν ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν πρῶτοι πρὸς ἀλλήλους εἰσίν.

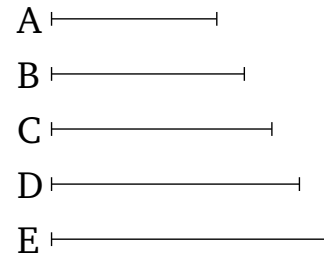
Εἰ μὲν οὖν οἱ A, B, Γ ἐξῆς εἰσιν ἀνάλογον, καὶ οἱ ἄκροι αὐτῶν οἱ A, Γ πρῶτοι πρὸς ἀλλήλους εἰσίν, δέδεικται, ὅτι ἀδύνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσεμερεῖν ἀριθμόν. μὴ ἔστωσαν δὴ οἱ A, B, Γ ἐξῆς ἀνάλογον τῶν ἀκρῶν πάλιν ὄντων πρῶτων πρὸς ἀλλήλους. λέγω, ὅτι καὶ οὕτως ἀδύνατόν ἐστιν αὐτοῖς τέταρτον ἀνάλογον προσεμερεῖν. εἰ γὰρ δυνατόν, προσεμερήσθω ὁ Δ , ὥστε εἶναι ὡς τὸν A πρὸς τὸν B , τὸν Γ πρὸς τὸν Δ , καὶ γεγονέτω ὡς ὁ B πρὸς τὸν Γ , ὁ Δ πρὸς τὸν E . καὶ ἐπεὶ ἐστὶν ὡς μὲν ὁ A πρὸς τὸν B , ὁ Γ πρὸς τὸν Δ , ὡς δὲ ὁ B πρὸς τὸν Γ , ὁ Δ πρὸς τὸν E , δι' ἴσου ἄρα ὡς ὁ A πρὸς τὸν Γ , ὁ Γ πρὸς τὸν E . οἱ δὲ A, Γ πρῶτοι, οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι

ἰν D . But, in fact, B has also made C (by) multiplying itself. Thus, the (number created) from (multiplying) A, D is equal to the (square) on B . Thus, as A is to B , (so) B (is) to D [Prop. 7.19]. Thus, a third number has been found proportional to A, B , (namely) D .

And so let A not measure C . I say that it is impossible to find a third number proportional to A, B . For, if possible, let it have been found, (and let it be) D . Thus, the (number created) from (multiplying) A, D is equal to the (square) on B [Prop. 7.19]. And the (square) on B is C . Thus, the (number created) from (multiplying) A, D is equal to C . Hence, A has made C (by) multiplying D . Thus, A measures C according to D . But (A) was, in fact, also assumed (to be) not measuring (C). The very thing (is) absurd. Thus, it is not possible to find a third number proportional to A, B when A does not measure C . (Which is) the very thing it was required to show.

Proposition 19[†]

For three given numbers, to investigate when it is possible to find a fourth (number) proportional to them.



Let A, B, C be the three given numbers. And let it be required to investigate when it is possible to find a fourth (number) proportional to them.

In fact, (A, B, C) are either not continuously proportional and the outermost of them are prime to one another, or are continuously proportional and the outermost of them are not prime to one another, or are neither continuously proportional nor are the outermost of them prime to one another, or are continuously proportional and the outermost of them are prime to one another.

In fact, if A, B, C are continuously proportional, and the outermost of them, A and C , are prime to one another, (then) it has (already) been shown that it is impossible to find a fourth number proportional to them [Prop. 9.17]. So let A, B, C not be continuously proportional, (with) the outermost of them again being prime to one another. I say that, in this case, it is also impossible to find a fourth (number) proportional to them. For, if possible, let it have been found, (and let it be) D . Hence, it will be that as A (is) to B , (so) C (is) to D . And let it be contrived that as B (is) to C , (so) D (is) to E . And since

μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ὃ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον. μετρεῖ ἄρα ὁ A τὸν Γ ὡς ἡγούμενος ἡγούμενον. μετρεῖ δὲ καὶ ἑαυτὸν· ὁ A ἄρα τοὺς A , Γ μετρεῖ πρώτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τοῖς A , B , Γ δυνατόν ἐστι τέταρτον ἀνάλογον προσσευρεῖν.

Ἀλλὰ δὴ πάλιν ἔστωσαν οἱ A , B , Γ ἐξῆς ἀνάλογον, οἱ δὲ A , Γ μὴ ἔστωσαν πρῶτοι πρὸς ἀλλήλους. λέγω, ὅτι δυνατόν ἐστι αὐτοῖς τέταρτον ἀνάλογον προσσευρεῖν. ὁ γὰρ B τὸν Γ πολλαπλασιάσας τὸν Δ ποιεῖτω· ὁ A ἄρα τὸν Δ ἤτοι μετρεῖ ἢ οὐ μετρεῖ. μετρεῖτω αὐτὸν πρότερον κατὰ τὸν E · ὁ A ἄρα τὸν E πολλαπλασιάσας τὸν Δ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ B τὸν Γ πολλαπλασιάσας τὸν Δ πεποίηκεν· ὁ ἄρα ἐκ τῶν A , E ἴσος ἐστὶ τῷ ἐκ τῶν B , Γ . ἀνάλογον ἄρα [ἐστὶν] ὡς ὁ A πρὸς τὸν B , ὁ Γ πρὸς τὸν E · τοῖς A , B , Γ ἄρα τέταρτος ἀνάλογον προσηήρηται ὁ E .

Ἀλλὰ δὴ μὴ μετρεῖτω ὁ A τὸν Δ · λέγω, ὅτι ἀδύνατόν ἐστι τοῖς A , B , Γ τέταρτον ἀνάλογον προσσευρεῖν ἀριθμόν. εἰ γὰρ δυνατόν, προσευρήσθω ὁ E · ὁ ἄρα ἐκ τῶν A , E ἴσος ἐστὶ τῷ ἐκ τῶν B , Γ . ἀλλὰ ὁ ἐκ τῶν B , Γ ἐστὶν ὁ Δ · καὶ ὁ ἐκ τῶν A , E ἄρα ἴσος ἐστὶ τῷ Δ . ὁ A ἄρα τὸν E πολλαπλασιάσας τὸν Δ πεποίηκεν· ὁ A ἄρα τὸν Δ μετρεῖ κατὰ τὸν E · ὥστε μετρεῖ ὁ A τὸν Δ . ἀλλὰ καὶ οὐ μετρεῖ· ὅπερ ἄτοπον. οὐκ ἄρα δυνατόν ἐστι τοῖς A , B , Γ τέταρτον ἀνάλογον προσσευρεῖν ἀριθμόν, ὅταν ὁ A τὸν Δ μὴ μετρή. ἀλλὰ δὴ οἱ A , B , Γ μήτε ἐξῆς ἔστωσαν ἀνάλογον μήτε οἱ ἄκροι πρῶτοι πρὸς ἀλλήλους. καὶ ὁ B τὸν Γ πολλαπλασιάσας τὸν Δ ποιεῖτω. ὁμοίως δὴ δειχθήσεται, ὅτι εἰ μὲν μετρεῖ ὁ A τὸν Δ , δυνατόν ἐστὶν αὐτοῖς ἀνάλογον προσσευρεῖν, εἰ δὲ οὐ μετρεῖ, ἀδύνατον· ὅπερ ἔδει δεῖξαι.

as A is to B , (so) C (is) to D , and as B (is) to C , (so) D (is) to E , thus, via equality, as A (is) to C , (so) C (is) to E [Prop. 7.14]. And A and C (are) prime (to one another). And (numbers) prime (to one another are) also the least (numbers having the same ratio as them) [Prop. 7.21]. And the least (numbers) measure those numbers having the same ratio as them (the same number of times), the leading (measuring) the leading, and the following the following [Prop. 7.20]. Thus, A measures C , (as) the leading (measuring) the leading. And it also measures itself. Thus, A measures A and C , which are prime to one another. The very thing is impossible. Thus, it is not possible to find a fourth (number) proportional to A , B , C .

And so let A , B , C again be continuously proportional, and let A and C not be prime to one another. I say that it is possible to find a fourth (number) proportional to them. For let B make D (by) multiplying C . Thus, A either measures or does not measure D . Let it, first of all, measure (D) according to E . Thus, A has made D (by) multiplying E . But, in fact, B has also made D (by) multiplying C . Thus, the (number created) from (multiplying) A , E is equal to the (number created) from (multiplying) B , C . Thus, proportionally, as A [is] to B , (so) C (is) to E [Prop. 7.19]. Thus, a fourth (number) proportional to A , B , C has been found, (namely) E .

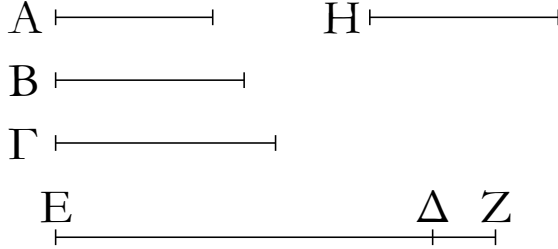
And so let A not measure D . I say that it is impossible to find a fourth number proportional to A , B , C . For, if possible, let it have been found, (and let it be) E . Thus, the (number created) from (multiplying) A , E is equal to the (number created) from (multiplying) B , C . But, the (number created) from (multiplying) B , C is D . And thus the (number created) from (multiplying) A , E is equal to D . Thus, A has made D (by) multiplying E . Thus, A measures D according to E . Hence, A measures D . But, it also does not measure (D). The very thing (is) absurd. Thus, it is not possible to find a fourth number proportional to A , B , C when A does not measure D . And so (let) A , B , C (be) neither continuously proportional, nor (let) the outermost of them (be) prime to one another. And let B make D (by) multiplying C . So, similarly, it can be show that if A measures D then it is possible to find a fourth (number) proportional to (A , B , C), and impossible if (A) does not measure (D). (Which is) the very thing it was required to show.

† The proof of this proposition is incorrect. There are, in fact, only two cases. Either A , B , C are continuously proportional, with A and C prime to one another, or not. In the first case, it is impossible to find a fourth proportional number. In the second case, it is possible to find a fourth proportional number provided that A measures B times C . Of the four cases considered by Euclid, the proof given in the second case is incorrect, since it only demonstrates that if $A : B :: C : D$ then a number E cannot be found such that $B : C :: D : E$. The proofs given in the other three

cases are correct.

κ'.

Οἱ πρῶτοι ἀριθμοὶ πλείους εἰσὶ παντὸς τοῦ προτεθέντος πλῆθους πρῶτων ἀριθμῶν.



Ἐστωσαν οἱ προτεθέντες πρῶτοι ἀριθμοὶ οἱ A, B, Γ . λέγω, ὅτι τῶν A, B, Γ πλείους εἰσὶ πρῶτοι ἀριθμοί.

Εἰλήφθω γὰρ ὁ ὑπὸ τῶν A, B, Γ ἐλάχιστος μετρούμενος καὶ ἔστω ΔE , καὶ προσκείσθω τῷ ΔE μονὰς ἡ ΔZ . ὁ δὲ EZ ἤτοι πρῶτός ἐστιν ἢ οὐ. ἔστω πρότερον πρῶτος· εὐρημένοι ἄρα εἰσὶ πρῶτοι ἀριθμοὶ οἱ A, B, Γ, EZ πλείους τῶν A, B, Γ .

Ἀλλὰ δὴ μὴ ἔστω ὁ EZ πρῶτος· ὑπὸ πρώτου ἄρα τινὸς ἀριθμοῦ μετρεῖται. μετρεῖσθω ὑπὸ πρώτου τοῦ H . λέγω, ὅτι ὁ H οὐδενὶ τῶν A, B, Γ ἐστὶν ὁ αὐτός. εἰ γὰρ δυνατόν, ἔστω. οἱ δὲ A, B, Γ τὸν ΔE μετροῦσιν· καὶ ὁ H ἄρα τὸν ΔE μετρήσει. μετρεῖ δὲ καὶ τὸν EZ · καὶ λοιπὴν τὴν ΔZ μονάδα μετρήσει ὁ H ἀριθμὸς ὧν· ὅπερ ἄτοπον. οὐκ ἄρα ὁ H ἐνὶ τῶν A, B, Γ ἐστὶν ὁ αὐτός. καὶ ὑπόκειται πρῶτος. εὐρημένοι ἄρα εἰσὶ πρῶτοι ἀριθμοὶ πλείους τοῦ προτεθέντος πλῆθους τῶν A, B, Γ οἱ A, B, Γ, H · ὅπερ ἔδει δεῖξαι.

κα'.

Ἐὰν ἄρτιοι ἀριθμοὶ ὅποσοιῦν συντεθῶσιν, ὁ ὅλος ἄρτιός ἐστιν.

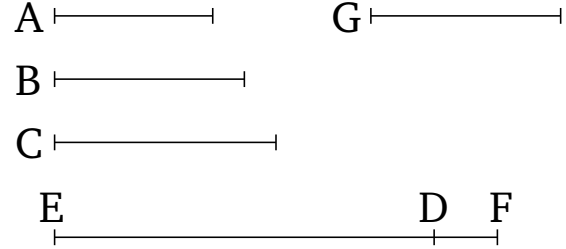


Συγκείσθωσαν γὰρ ἄρτιοι ἀριθμοὶ ὅποσοιῦν οἱ $AB, B\Gamma, \Gamma\Delta, \Delta E$. λέγω, ὅτι ὅλος ὁ AE ἄρτιός ἐστιν.

Ἐπεὶ γὰρ ἕκαστος τῶν $AB, B\Gamma, \Gamma\Delta, \Delta E$ ἄρτιός ἐστιν, ἔχει μέρος ἡμισυ· ὥστε καὶ ὅλος ὁ AE ἔχει μέρος ἡμισυ. ἄρτιος δὲ ἀριθμὸς ἐστὶν ὁ δίχα διαιρούμενος· ἄρτιος ἄρα ἐστὶν ὁ AE · ὅπερ ἔδει δεῖξαι.

Proposition 20

The (set of all) prime numbers is more numerous than any assigned multitude of prime numbers.



Let A, B, C be the assigned prime numbers. I say that the (set of all) primes numbers is more numerous than A, B, C .

For let the least number measured by A, B, C have been taken, and let it be DE [Prop. 7.36]. And let the unit DF have been added to DE . So EF is either prime, or not. Let it, first of all, be prime. Thus, the (set of) prime numbers A, B, C, EF , (which is) more numerous than A, B, C , has been found.

And so let EF not be prime. Thus, it is measured by some prime number [Prop. 7.31]. Let it be measured by the prime (number) G . I say that G is not the same as any of A, B, C . For, if possible, let it be (the same). And A, B, C (all) measure DE . Thus, G will also measure DE . And it also measures EF . (So) G will also measure the remainder, unit DF , (despite) being a number [Prop. 7.28]. The very thing (is) absurd. Thus, G is not the same as one of A, B, C . And it was assumed (to be) prime. Thus, the (set of) prime numbers A, B, C, G , (which is) more numerous than the assigned multitude (of prime numbers), A, B, C , has been found. (Which is) the very thing it was required to show.

Proposition 21

If any multitude whatsoever of even numbers is added together then the whole is even.



For let any multitude whatsoever of even numbers, AB, BC, CD, DE , lie together. I say that the whole, AE , is even.

For since everyone of AB, BC, CD, DE is even, it has a half part [Def. 7.6]. And hence the whole AE has a half part. And an even number is one (which can be) divided in half [Def. 7.6]. Thus, AE is even. (Which is)

κβ'.

Ἐάν περισσοὶ ἀριθμοὶ ὅποιοι οὖν συντεθῶσιν, τὸ δὲ πλῆθος αὐτῶν ἄρτιον ἦ, ὁ ὅλος ἄρτιος ἔσται.

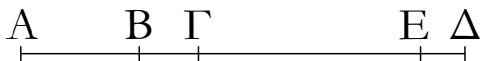


Συγκείσθωσαν γὰρ περισσοὶ ἀριθμοὶ ὅσοιδηποῦν ἄρτιοι τὸ πλῆθος οἱ AB, BΓ, ΓΔ, ΔΕ· λέγω, ὅτι ὁλος ὁ AE ἄρτιός ἐστιν.

Ἐπεὶ γὰρ ἕκαστος τῶν AB, BΓ, ΓΔ, ΔΕ περιττός ἐστιν, ἀφαιρεθείσης μονάδος ἀφ' ἑκάστου ἕκαστος τῶν λοιπῶν ἄρτιος ἔσται· ὥστε καὶ ὁ συγκείμενος ἐξ αὐτῶν ἄρτιος ἔσται. ἔστι δὲ καὶ τὸ πλῆθος τῶν μονάδων ἄρτιον. καὶ ὁλος ἄρα ὁ AE ἄρτιός ἐστιν· ὅπερ ἔδει δεῖξαι.

κγ'.

Ἐάν περισσοὶ ἀριθμοὶ ὅποιοι οὖν συντεθῶσιν, τὸ δὲ πλῆθος αὐτῶν περισσὸν ἦ, καὶ ὁ ὅλος περισσοὺς ἔσται.



Συγκείσθωσαν γὰρ ὅποιοι οὖν περισσοὶ ἀριθμοί, ὧν τὸ πλῆθος περισσὸν ἔστω, οἱ AB, BΓ, ΓΔ· λέγω, ὅτι καὶ ὁλος ὁ AΔ περισσοὺς ἐστιν.

Ἀφηρήσθω ἀπὸ τοῦ ΓΔ μονὰς ἡ ΔΕ· λοιπὸς ἄρα ὁ ΓΕ ἄρτιός ἐστιν. ἔστι δὲ καὶ ὁ ΓΑ ἄρτιος· καὶ ὁλος ἄρα ὁ AE ἄρτιός ἐστιν. καὶ ἐστὶ μονὰς ἡ ΔΕ. περισσοὺς ἄρα ἐστὶν ὁ AΔ· ὅπερ ἔδει δεῖξαι.

κδ'.

Ἐάν ἀπὸ ἀρτίου ἀριθμοῦ ἄρτιος ἀφαιρεθῇ, ὁ λοιπὸς ἄρτιος ἔσται.



Ἀπὸ γὰρ ἀρτίου τοῦ AB ἄρτιος ἀφηρήσθω ὁ BΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ ἄρτιός ἐστιν.

Ἐπεὶ γὰρ ὁ AB ἄρτιός ἐστιν, ἔχει μέρος ἡμισυ. διὰ τὰ αὐτὰ δὲ καὶ ὁ BΓ ἔχει μέρος ἡμισυ· ὥστε καὶ λοιπὸς [ὁ ΓΑ ἔχει μέρος ἡμισυ] ἄρτιος [ἄρα] ἐστὶν ὁ ΑΓ· ὅπερ ἔδει δεῖξαι.

the very thing it was required to show.

Proposition 22

If any multitude whatsoever of odd numbers is added together, and the multitude of them is even, then the whole will be even.



For let any even multitude whatsoever of odd numbers, AB , BC , CD , DE , lie together. I say that the whole, AE , is even.

For since everyone of AB , BC , CD , DE is odd then, a unit being subtracted from each, everyone of the remainders will be (made) even [Def. 7.7]. And hence the sum of them will be even [Prop. 9.21]. And the multitude of the units is even. Thus, the whole AE is also even [Prop. 9.21]. (Which is) the very thing it was required to show.

Proposition 23

If any multitude whatsoever of odd numbers is added together, and the multitude of them is odd, then the whole will also be odd.



For let any multitude whatsoever of odd numbers, AB , BC , CD , lie together, and let the multitude of them be odd. I say that the whole, AD , is also odd.

For let the unit DE have been subtracted from CD . The remainder CE is thus even [Def. 7.7]. And CA is also even [Prop. 9.22]. Thus, the whole AE is also even [Prop. 9.21]. And DE is a unit. Thus, AD is odd [Def. 7.7]. (Which is) the very thing it was required to show.

Proposition 24

If an even (number) is subtracted from an (other) even number then the remainder will be even.

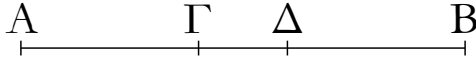


For let the even (number) BC have been subtracted from the even number AB . I say that the remainder CA is even.

For since AB is even, it has a half part [Def. 7.6]. So, for the same (reasons), BC also has a half part. And hence the remainder [CA has a half part]. [Thus,] AC is even. (Which is) the very thing it was required to show.

κε'.

Ἐάν ἀπὸ ἄρτιου ἀριθμοῦ περισσὸς ἀφαιρεθῇ, ὁ λοιπὸς περισσὸς ἔσται.

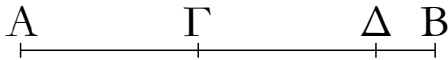


Ἀπὸ γὰρ ἄρτιου τοῦ AB περισσὸς ἀφηρήσθω ὁ BΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ περισσὸς ἔστιν.

Ἀφηρήσθω γὰρ ἀπὸ τοῦ BΓ μονὰς ἡ ΓΔ· ὁ ΔB ἄρα ἄρτιός ἐστιν. ἔστι δὲ καὶ ὁ AB ἄρτιος· καὶ λοιπὸς ἄρα ὁ ΑΔ ἄρτιός ἐστιν. καὶ ἔστι μονὰς ἡ ΓΔ· ὁ ΓΑ ἄρα περισσὸς ἔστιν· ὅπερ ἔδει δεῖξαι.

κς'.

Ἐάν ἀπὸ περισσοῦ ἀριθμοῦ περισσὸς ἀφαιρεθῇ, ὁ λοιπὸς ἄρτιος ἔσται.



Ἀπὸ γὰρ περισσοῦ τοῦ AB περισσὸς ἀφηρήσθω ὁ BΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ ἄρτιός ἐστιν.

Ἐπεὶ γὰρ ὁ AB περισσὸς ἔστιν, ἀφηρήσθω μονὰς ἡ BΔ· λοιπὸς ἄρα ὁ ΑΔ ἄρτιός ἐστιν. διὰ τὰ αὐτὰ δὲ καὶ ὁ ΓΔ ἄρτιός ἐστιν· ὥστε καὶ λοιπὸς ὁ ΓΑ ἄρτιός ἐστιν· ὅπερ ἔδει δεῖξαι.

κζ'.

Ἐάν ἀπὸ περισσοῦ ἀριθμοῦ ἄρτιος ἀφαιρεθῇ, ὁ λοιπὸς περισσὸς ἔσται.



Ἀπὸ γὰρ περισσοῦ τοῦ AB ἄρτιος ἀφηρήσθω ὁ BΓ· λέγω, ὅτι ὁ λοιπὸς ὁ ΓΑ περισσὸς ἔστιν.

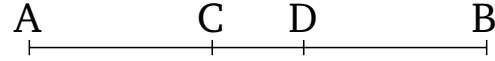
Ἀφηρήσθω [γὰρ] μονὰς ἡ ΑΔ· ὁ ΔB ἄρα ἄρτιός ἐστιν. ἔστι δὲ καὶ ὁ BΓ ἄρτιος· καὶ λοιπὸς ἄρα ὁ ΓΔ ἄρτιός ἐστιν. περισσὸς ἄρα ὁ ΓΑ· ὅπερ ἔδει δεῖξαι.

κη'.

Ἐάν περισσὸς ἀριθμὸς ἄρτιον πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος ἄρτιος ἔσται.

Proposition 25

If an odd (number) is subtracted from an even number then the remainder will be odd.

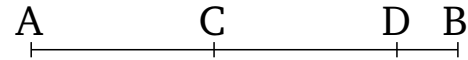


For let the odd (number) BC have been subtracted from the even number AB. I say that the remainder CA is odd.

For let the unit CD have been subtracted from BC. DB is thus even [Def. 7.7]. And AB is also even. And thus the remainder AD is even [Prop. 9.24]. And CD is a unit. Thus, CA is odd [Def. 7.7]. (Which is) the very thing it was required to show.

Proposition 26

If an odd (number) is subtracted from an odd number then the remainder will be even.



For let the odd (number) BC have been subtracted from the odd (number) AB. I say that the remainder CA is even.

For since AB is odd, let the unit BD have been subtracted (from it). Thus, the remainder AD is even [Def. 7.7]. So, for the same (reasons), CD is also even. And hence the remainder CA is even [Prop. 9.24]. (Which is) the very thing it was required to show.

Proposition 27

If an even (number) is subtracted from an odd number then the remainder will be odd.



For let the even (number) BC have been subtracted from the odd (number) AB. I say that the remainder CA is odd.

[For] let the unit AD have been subtracted (from AB). DB is thus even [Def. 7.7]. And BC is also even. Thus, the remainder CD is also even [Prop. 9.24]. CA (is) thus odd [Def. 7.7]. (Which is) the very thing it was required to show.

Proposition 28

If an odd number makes some (number by) multiplying an even (number) then the created (number) will be even.



Περισσός γάρ ἀριθμὸς ὁ A ἄρτιον τὸν B πολλαπλασιάσας τὸν Γ ποιεῖτω· λέγω, ὅτι ὁ Γ ἄρτιός ἐστιν.

Ἐπεὶ γὰρ ὁ A τὸν B πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Γ ἄρα σύγκειται ἐκ τοσοῦτων ἴσων τῷ B , ὅσαι εἰσὶν ἐν τῷ A μονάδες. καὶ ἐστὶν ὁ B ἄρτιος· ὁ Γ ἄρα σύγκειται ἐξ ἄρτίων. ἐὰν δὲ ἄρτιοι ἀριθμοὶ ὅποιοι οὖν συντεθῶσιν, ὁ ὅλος ἄρτιός ἐστιν. ἄρτιος ἄρα ἐστὶν ὁ Γ · ὅπερ ἔδει δεῖξαι.

κθ'.

Ἐὰν περισσὸς ἀριθμὸς περισσὸν ἀριθμὸν πολλαπλασιάσας ποιῇ τινα, ὁ γενόμενος περισσὸς ἔσται.

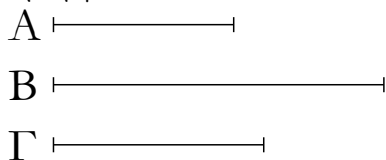


Περισσὸς γάρ ἀριθμὸς ὁ A περισσὸν τὸν B πολλαπλασιάσας τὸν Γ ποιεῖτω· λέγω, ὅτι ὁ Γ περισσός ἐστιν.

Ἐπεὶ γὰρ ὁ A τὸν B πολλαπλασιάσας τὸν Γ πεποίηκεν, ὁ Γ ἄρα σύγκειται ἐκ τοσοῦτων ἴσων τῷ B , ὅσαι εἰσὶν ἐν τῷ A μονάδες. καὶ ἐστὶν ἐκάτερος τῶν A , B περισσός· ὁ Γ ἄρα σύγκειται ἐκ περισσῶν ἀριθμῶν, ὧν τὸ πλῆθος περισσόν ἐστιν. ὥστε ὁ Γ περισσός ἐστιν· ὅπερ ἔδει δεῖξαι.

λ'.

Ἐὰν περισσὸς ἀριθμὸς ἄρτιον ἀριθμὸν μετρήῃ, καὶ τὸν ἡμισυν αὐτοῦ μετρήσῃ.



Περισσὸς γάρ ἀριθμὸς ὁ A ἄρτιον τὸν B μετρεῖτω· λέγω, ὅτι καὶ τὸν ἡμισυν αὐτοῦ μετρήσῃ.

Ἐπεὶ γὰρ ὁ A τὸν B μετρεῖ, μετρεῖτω αὐτὸν κατὰ τὸν Γ · λέγω, ὅτι ὁ Γ οὐκ ἔστι περισσός. εἰ γὰρ δυνατόν, ἔστω. καὶ ἐπεὶ ὁ A τὸν B μετρεῖ κατὰ τὸν Γ , ὁ A ἄρα τὸν Γ πολλαπλασιάσας τὸν B πεποίηκεν. ὁ B ἄρα σύγκειται ἐκ περισσῶν ἀριθμῶν, ὧν τὸ πλῆθος περισσόν ἐστιν. ὁ B ἄρα



For let the odd number A make C (by) multiplying the even (number) B . I say that C is even.

For since A has made C (by) multiplying B , C is thus composed out of so many (magnitudes) equal to B , as many as (there) are units in A [Def. 7.15]. And B is even. Thus, C is composed out of even (numbers). And if any multitude whatsoever of even numbers is added together then the whole is even [Prop. 9.21]. Thus, C is even. (Which is) the very thing it was required to show.

Proposition 29

If an odd number makes some (number by) multiplying an odd (number) then the created (number) will be odd.

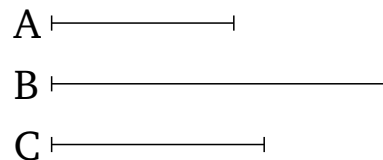


For let the odd number A make C (by) multiplying the odd (number) B . I say that C is odd.

For since A has made C (by) multiplying B , C is thus composed out of so many (magnitudes) equal to B , as many as (there) are units in A [Def. 7.15]. And each of A , B is odd. Thus, C is composed out of odd (numbers), (and) the multitude of them is odd. Hence C is odd [Prop. 9.23]. (Which is) the very thing it was required to show.

Proposition 30

If an odd number measures an even number then it will also measure (one) half of it.



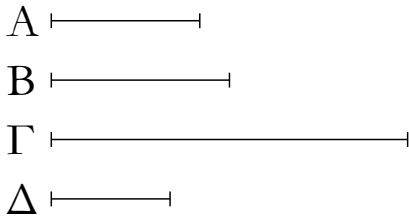
For let the odd number A measure the even (number) B . I say that (A) will also measure (one) half of (B).

For since A measures B , let it measure it according to C . I say that C is not odd. For, if possible, let it be (odd). And since A measures B according to C , A has thus made B (by) multiplying C . Thus, B is composed out of odd numbers, (and) the multitude of them is odd. B is thus

περισσός ἐστιν· ὅπερ ἄτοπον· ὑπόκειται γὰρ ἄρτιος. οὐκ ἄρα ὁ Γ περισσός ἐστιν· ἄρτιος ἄρα ἐστὶν ὁ Γ. ὥστε ὁ Α τὸν Β μετρεῖ ἀρτιάκις. διὰ δὴ τοῦτο καὶ τὸν ἥμισυν αὐτοῦ μετρήσει· ὅπερ ἔδει δεῖξαι.

λα'.

Ἐὰν περισσὸς ἀριθμὸς πρὸς τινὰ ἀριθμὸν πρῶτος ᾖ, καὶ πρὸς τὸν διπλασίονα αὐτοῦ πρῶτος ἔσται.

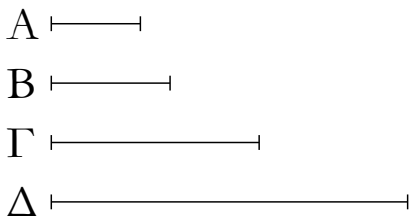


Περισσὸς γὰρ ἀριθμὸς ὁ Α πρὸς τινὰ ἀριθμὸν τὸν Β πρῶτος ἔστω, τοῦ δὲ Β διπλασίον ἔστω ὁ Γ· λέγω, ὅτι ὁ Α [καὶ] πρὸς τὸν Γ πρῶτός ἐστιν.

Εἰ γὰρ μὴ εἰσιν [οἱ Α, Γ] πρῶτοι, μετρήσει τις αὐτοὺς ἀριθμός. μετρεῖτω, καὶ ἔστω ὁ Δ. καὶ ἐστὶν ὁ Α περισσός· περισσὸς ἄρα καὶ ὁ Δ. καὶ ἐπεὶ ὁ Δ περισσὸς ὢν τὸν Γ μετρεῖ, καὶ ἐστὶν ὁ Γ ἄρτιος, καὶ τὸν ἥμισυν ἄρα τοῦ Γ μετρήσει [ὁ Δ]. τοῦ δὲ Γ ἥμισύ ἐστὶν ὁ Β· ὁ Δ ἄρα τὸν Β μετρεῖ. μετρεῖ δὲ καὶ τὸν Α. ὁ Δ ἄρα τοὺς Α, Β μετρεῖ πρῶτους ὄντας πρὸς ἀλλήλους· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ Α πρὸς τὸν Γ πρῶτος οὐκ ἐστὶν. οἱ Α, Γ ἄρα πρῶτοι πρὸς ἀλλήλους εἰσιν· ὅπερ ἔδει δεῖξαι.

λβ'.

Τῶν ἀπὸ δυάδος διπλασιαζομένων ἀριθμῶν ἕκαστος ἀρτιάκις ἀρτιός ἐστι μόνον.



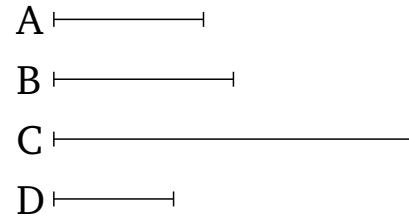
Ἀπὸ γὰρ δυάδος τῆς Α δεδιπλασιάσθωσαν ὅσοιδη-ποτοῦν ἀριθμοὶ οἱ Β, Γ, Δ· λέγω, ὅτι οἱ Β, Γ, Δ ἀρτιάκις ἀρτιοὶ εἰσι μόνον.

Ὅτι μὲν οὖν ἕκαστος [τῶν Β, Γ, Δ] ἀρτιάκις ἀρτιός ἐστιν, φανερόν· ἀπὸ γὰρ δυάδος ἐστὶ διπλασιασθείς. λέγω, ὅτι καὶ μόνον. ἐκχείσθω γὰρ μονάς. ἐπεὶ οὖν ἀπὸ μονάδος ὁποσοιοῦν ἀριθμοὶ ἐξῆς ἀνάλογόν εἰσιν, ὁ δὲ μετὰ τὴν μονάδα ὁ Α πρῶτός ἐστιν, ὁ μέγιστος τῶν Α, Β, Γ, Δ ὁ

odd [Prop. 9.23]. The very thing (is) absurd. For (B) was assumed (to be) even. Thus, C is not odd. Thus, C is even. Hence, A measures B an even number of times. So, on account of this, (A) will also measure (one) half of (B). (Which is) the very thing it was required to show.

Proposition 31

If an odd number is prime to some number then it will also be prime to its double.

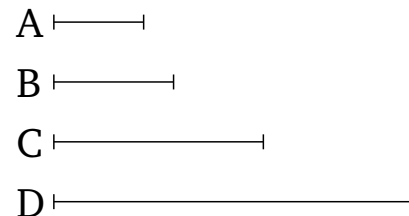


For let the odd number A be prime to some number B. And let C be double B. I say that A is [also] prime to C.

For if [A and C] are not prime (to one another) then some number will measure them. Let it measure (them), and let it be D. And A is odd. Thus, D (is) also odd. And since D, which is odd, measures C, and C is even, [D] will thus also measure half of C [Prop. 9.30]. And B is half of C. Thus, D measures B. And it also measures A. Thus, D measures (both) A and B, (despite) them being prime to one another. The very thing is impossible. Thus, A is not unprime to C. Thus, A and C are prime to one another. (Which is) the very thing it was required to show.

Proposition 32

Each of the numbers (which is continually) doubled, (starting) from a dyad, is an even-times-even (number) only.



For let any multitude of numbers whatsoever, B, C, D, have been (continually) doubled, (starting) from the dyad A. I say that B, C, D are even-times-even (numbers) only.

In fact, (it is) clear that each [of B, C, D] is an even-times-even (number). For it is doubled from a dyad [Def. 7.8]. I also say that (they are even-times-even numbers) only. For let a unit be laid down. Therefore, since

Δ ὑπ' οὐδενὸς ἄλλου μετρηθήσεται παρὲξ τῶν Α, Β, Γ. καὶ ἐστὶν ἕκαστος τῶν Α, Β, Γ ἄρτιος· ὁ Δ ἄρα ἀρτιάκις ἄρτιός ἐστι μόνον. ὁμοίως δὴ δείξομεν, ὅτι [καὶ] ἑκάτερος τῶν Β, Γ ἀρτιάκις ἄρτιός ἐστι μόνον· ὅπερ ἔδει δεῖξαι.

λγ'.

Ἐὰν ἀριθμὸς τὸν ἥμισυν ἔχη περισσόν, ἀρτιάκις περισσὸς ἐστὶ μόνον.

A —————

Ἀριθμὸς γὰρ ὁ Α τὸν ἥμισυν ἐχέτω περισσόν· λέγω, ὅτι ὁ Α ἀρτιάκις περισσὸς ἐστὶ μόνον.

Ὅτι μὲν οὖν ἀρτιάκις περισσὸς ἐστίν, φανερόν· ὁ γὰρ ἥμισυς αὐτοῦ περισσὸς ὢν μετρεῖ αὐτὸν ἀρτιάκις, λέγω δὴ, ὅτι καὶ μόνον. εἰ γὰρ ἔσται ὁ Α καὶ ἀρτιάκις ἄρτιος, μετρηθήσεται ὑπὸ ἀρτίου κατὰ ἄρτιον ἀριθμόν· ὥστε καὶ ὁ ἥμισυς αὐτοῦ μετρηθήσεται ὑπὸ ἀρτίου ἀριθμοῦ περισσὸς ὢν· ὅπερ ἐστὶν ἄτοπον. ὁ Α ἄρα ἀρτιάκις περισσὸς ἐστὶ μόνον· ὅπερ ἔδει δεῖξαι.

λδ'.

Ἐὰν ἀριθμὸς μήτε τῶν ἀπὸ δυάδος διπλασιαζομένων ἢ, μήτε τὸν ἥμισυν ἔχη περισσόν, ἀρτιάκις τε ἄρτιός ἐστι καὶ ἀρτιάκις περισσός.

A —————

Ἀριθμὸς γὰρ ὁ Α μήτε τῶν ἀπὸ δυάδος διπλασιαζομένων ἔστω μήτε τὸν ἥμισυν ἐχέτω περισσόν· λέγω, ὅτι ὁ Α ἀρτιάκις τέ ἐστὶν ἄρτιος καὶ ἀρτιάκις περισσός.

Ὅτι μὲν οὖν ὁ Α ἀρτιάκις ἐστὶν ἄρτιος, φανερόν· τὸν γὰρ ἥμισυν οὐκ ἔχει περισσόν. λέγω δὴ, ὅτι καὶ ἀρτιάκις περισσὸς ἐστίν. ἐὰν γὰρ τὸν Α τέμνωμεν δίχα καὶ τὸν ἥμισυν αὐτοῦ δίχα καὶ τοῦτο αἰ ποιοῦμεν, καταστήσομεν εἰς τινὰ ἀριθμὸν περισσόν, ὃς μετρήσει τὸν Α κατὰ ἄρτιον ἀριθμόν. εἰ γὰρ οὐ, καταστήσομεν εἰς δυάδα, καὶ ἔσται ὁ Α τῶν ἀπὸ δυάδος διπλασιαζομένων· ὅπερ οὐχ ὑπόκειται. ὥστε ὁ Α ἀρτιάκις περισσόν ἐστίν. ἐδείχθη δὲ καὶ ἀρτιάκις ἄρτιος. ὁ Α ἄρα ἀρτιάκις τε ἄρτιός ἐστι καὶ ἀρτιάκις περισσός· ὅπερ ἔδει δεῖξαι.

any multitude of numbers whatsoever are continuously proportional, starting from a unit, and the (number) A after the unit is prime, the greatest of A, B, C, D , (namely) D , will not be measured by any other (numbers) except A, B, C [Prop. 9.13]. And each of A, B, C is even. Thus, D is an even-time-even (number) only [Def. 7.8]. So, similarly, we can show that each of B, C is [also] an even-time-even (number) only. (Which is) the very thing it was required to show.

Proposition 33

If a number has an odd half then it is an even-time-odd (number) only.

A —————

For let the number A have an odd half. I say that A is an even-times-odd (number) only.

In fact, (it is) clear that (A) is an even-times-odd (number). For its half, being odd, measures it an even number of times [Def. 7.9]. So I also say that (it is an even-times-odd number) only. For if A is also an even-times-even (number) then it will be measured by an even (number) according to an even number [Def. 7.8]. Hence, its half will also be measured by an even number, (despite) being odd. The very thing is absurd. Thus, A is an even-times-odd (number) only. (Which is) the very thing it was required to show.

Proposition 34

If a number is neither (one) of the (numbers) doubled from a dyad, nor has an odd half, then it is (both) an even-times-even and an even-times-odd (number).

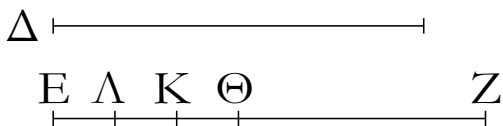
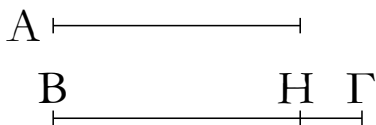
A —————

For let the number A neither be (one) of the (numbers) doubled from a dyad, nor let it have an odd half. I say that A is (both) an even-times-even and an even-times-odd (number).

In fact, (it is) clear that A is an even-times-even (number) [Def. 7.8]. For it does not have an odd half. So I say that it is also an even-times-odd (number). For if we cut A in half, and (then cut) its half in half, and we do this continually, then we will arrive at some odd number which will measure A according to an even number. For if not, we will arrive at a dyad, and A will be (one) of the (numbers) doubled from a dyad. The very opposite thing (was) assumed. Hence, A is an even-times-odd (number) [Def. 7.9]. And it was also shown (to be) an even-times-even (number). Thus, A is (both) an even-times-even and an even-times-odd (number). (Which is)

λε'.

Ἐάν ὧσιν ὁποιοιηποτοῦν ἀριθμοὶ ἐξῆς ἀνάλογον, ἀφαιρεθῶσι δὲ ἀπὸ τε τοῦ δευτέρου καὶ τοῦ ἐσχάτου ἴσοι τῷ πρώτῳ, ἔσται ὡς ἡ τοῦ δευτέρου ὑπεροχὴ πρὸς τὸν πρώτον, οὕτως ἡ τοῦ ἐσχάτου ὑπεροχὴ πρὸς τοὺς πρὸ ἑαυτοῦ πάντας.



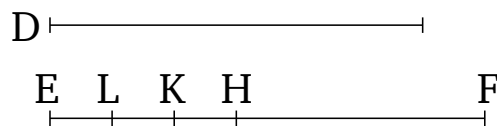
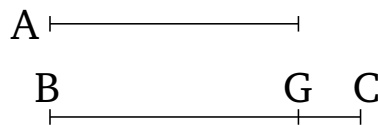
Ἐστωσαν ὁποιοιηποτοῦν ἀριθμοὶ ἐξῆς ἀνάλογον οἱ Α, ΒΓ, Δ, ΕΖ ἀφχόμενοι ἀπὸ ἐλαχίστου τοῦ Α, καὶ ἀφηρήσθω ἀπὸ τοῦ ΒΓ καὶ τοῦ ΕΖ τῷ Α ἴσος ἑκάτερος τῶν ΒΗ, ΖΘ· λέγω, ὅτι ἔστιν ὡς ὁ ΗΓ πρὸς τὸν Α, οὕτως ὁ ΕΘ πρὸς τοὺς Α, ΒΓ, Δ.

Κείσθω γὰρ τῷ μὲν ΒΓ ἴσος ὁ ΖΚ, τῷ δὲ Δ ἴσος ὁ ΖΛ, καὶ ἐπεὶ ὁ ΖΚ τῷ ΒΓ ἴσος ἐστίν, ὧν ὁ ΖΘ τῷ ΒΗ ἴσος ἐστίν, λοιπὸς ἄρα ὁ ΘΚ λοιπὸν τῷ ΗΓ ἐστὶν ἴσος. καὶ ἐπεὶ ἐστὶν ὡς ὁ ΕΖ πρὸς τὸν Δ, οὕτως ὁ Δ πρὸς τὸν ΒΓ καὶ ὁ ΒΓ πρὸς τὸν Α, ἴσος δὲ ὁ μὲν Δ τῷ ΖΛ, ὁ δὲ ΒΓ τῷ ΖΚ, ὁ δὲ Α τῷ ΖΘ, ἔστιν ἄρα ὡς ὁ ΕΖ πρὸς τὸν ΖΛ, οὕτως ὁ ΑΖ πρὸς τὸν ΖΚ καὶ ὁ ΖΚ πρὸς τὸν ΖΘ. διελόντι, ὡς ὁ ΕΛ πρὸς τὸν ΑΖ, οὕτως ὁ ΑΚ πρὸς τὸν ΖΚ καὶ ὁ ΚΘ πρὸς τὸν ΖΘ. ἔστιν ἄρα καὶ ὡς εἷς τῶν ἡγούμενων πρὸς ἓνα τῶν ἐπομένων, οὕτως ἅπαντες οἱ ἡγούμενοι πρὸς ἅπαντας τοὺς ἐπομένους· ἔστιν ἄρα ὡς ὁ ΚΘ πρὸς τὸν ΖΘ, οὕτως οἱ ΕΛ, ΑΚ, ΚΘ πρὸς τοὺς ΑΖ, ΖΚ, ΘΖ. ἴσος δὲ ὁ μὲν ΚΘ τῷ ΓΗ, ὁ δὲ ΖΘ τῷ Α, οἱ δὲ ΑΖ, ΖΚ, ΘΖ τοῖς Δ, ΒΓ, Α· ἔστιν ἄρα ὡς ὁ ΓΗ πρὸς τὸν Α, οὕτως ὁ ΕΘ πρὸς τοὺς Δ, ΒΓ, Α. ἔστιν ἄρα ὡς ἡ τοῦ δευτέρου ὑπεροχὴ πρὸς τὸν πρώτον, οὕτως ἡ τοῦ ἐσχάτου ὑπεροχὴ πρὸς τοὺς πρὸ ἑαυτοῦ πάντας· ὅπερ ἔδει δεῖξαι.

the very thing it was required to show.

Proposition 35[†]

If there is any multitude whatsoever of continually proportional numbers, and (numbers) equal to the first are subtracted from (both) the second and the last, then as the excess of the second (number is) to the first, so the excess of the last will be to (the sum of) all those (numbers) before it.



Let A, BC, D, EF be any multitude whatsoever of continuously proportional numbers, beginning from the least A . And let BG and FH , each equal to A , have been subtracted from BC and EF (respectively). I say that as GC is to A , so EH is to A, BC, D .

For let FK be made equal to BC , and FL to D . And since FK is equal to BC , of which FH is equal to BG , the remainder HK is thus equal to the remainder GC . And since as EF is to D , so D (is) to BC , and BC to A [Prop. 7.13], and D (is) equal to FL , and BC to FK , and A to FH , thus as EF is to FL , so LF (is) to FK , and FK to FH . By separation, as EL (is) to LF , so LK (is) to FK , and KH to FH [Props. 7.11, 7.13]. And thus as one of the leading (numbers) is to one of the following, so (the sum of) all of the leading (numbers is) to (the sum of) all of the following [Prop. 7.12]. Thus, as KH is to FH , so EL, LK, KH (are) to LF, FK, HF . And KH (is) equal to CG , and FH to A , and LF, FK, HF to D, BC, A . Thus, as CG is to A , so EH (is) to D, BC, A . Thus, as the excess of the second (number) is to the first, so the excess of the last (is) to (the sum of) all those (numbers) before it. (Which is) the very thing it was required to show.

[†] This proposition allows us to sum a geometric series of the form $a, ar, ar^2, ar^3, \dots, ar^{n-1}$. According to Euclid, the sum S_n satisfies $(ar - a)/a = (ar^n - a)/S_n$. Hence, $S_n = a(r^n - 1)/(r - 1)$.

λζ'.

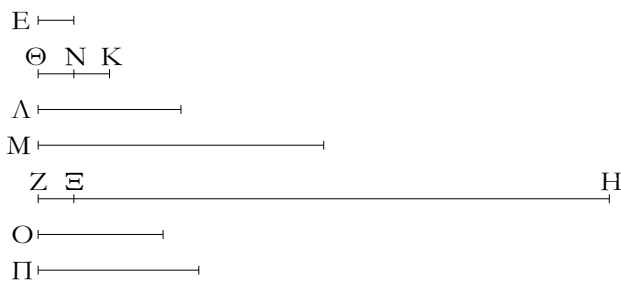
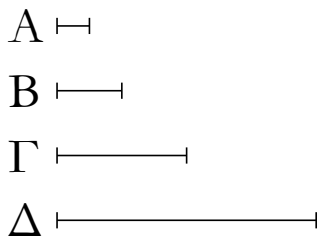
Ἐάν ἀπὸ μονάδος ὁποιοιοῦν ἀριθμοὶ ἐξῆς ἐκτεθῶσιν ἐν τῇ διπλασίονι ἀναλογίᾳ, ἕως οὗ ὁ σύμπαρ συντεθεῖς πρῶτος γένηται, καὶ ὁ σύμπαρ ἐπὶ τὸν ἐσχάτον πολλαπλασιασθεὶς

Proposition 36[†]

If any multitude whatsoever of numbers is set out continuously in a double proportion, (starting) from a unit, until the whole sum added together becomes prime, and

ποιῇ τινα, ὁ γενόμενος τέλειος ἔσται.

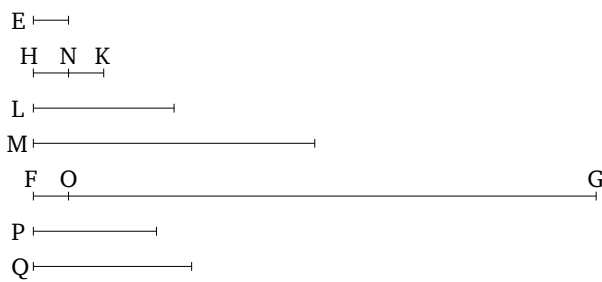
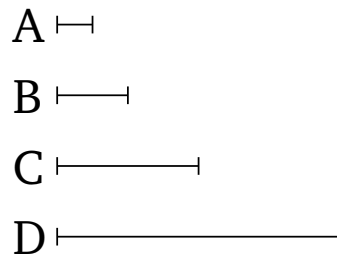
Ἀπὸ γὰρ μονάδος ἐκκείσθωσαν ὁσοιδηποτοῦν ἀριθμοὶ ἐν τῇ διπλασίονι ἀναλογίᾳ, ἕως οὗ ὁ σύμπαρ συντεθεὶς πρῶτος γένηται, οἱ A, B, Γ, Δ , καὶ τῷ σύμπαντι ἴσος ἔστω ὁ E , καὶ ὁ E τὸν Δ πολλαπλασιάσας τὸν ZH ποιεῖτω. λέγω, ὅτι ὁ ZH τέλειός ἐστιν.



Ὅσοι γὰρ εἰσιν οἱ A, B, Γ, Δ τῷ πλήθει, τοσοῦτοι ἀπὸ τοῦ E εἰλήφθωσαν ἐν τῇ διπλασίονι ἀναλογίᾳ οἱ $E, \Theta K, \Lambda, M$. δι' ἴσου ἄρα ἐστὶν ὡς ὁ A πρὸς τὸν Δ , οὕτως ὁ E πρὸς τὸν M . ὁ ἄρα ἐκ τῶν E, Δ ἴσος ἐστὶ τῷ ἐκ τῶν A, M . καὶ ἐστὶν ὁ ἐκ τῶν E, Δ ὁ ZH . καὶ ὁ ἐκ τῶν A, M ἄρα ἐστὶν ὁ ZH . ὁ A ἄρα τὸν M πολλαπλασιάσας τὸν ZH πεποίηκεν. ὁ M ἄρα τὸν ZH μετρεῖ κατὰ τὰς ἐν τῷ A μονάδας. καὶ ἐστὶ δυὰς ὁ A . διπλάσιος ἄρα ἐστὶν ὁ ZH τοῦ M . εἰσὶ δὲ καὶ οἱ $M, \Lambda, \Theta K, E$ ἐξῆς διπλάσιοι ἀλλήλων. οἱ $E, \Theta K, \Lambda, M, ZH$ ἄρα ἐξῆς ἀνάλογόν εἰσιν ἐν τῇ διπλασίονι ἀναλογίᾳ. ἀφηρήσθω δὴ ἀπὸ τοῦ δευτέρου τοῦ ΘK καὶ τοῦ ἐσχάτου τοῦ ZH τῷ πρώτῳ τῷ E ἴσος ἐκάτερος τῶν $\Theta N, Z\Xi$. ἔστιν ἄρα ὡς ἡ τοῦ δευτέρου ἀριθμοῦ ὑπεροχὴ πρὸς τὸν πρῶτον, οὕτως ἡ τοῦ ἐσχάτου ὑπεροχὴ πρὸς τοὺς πρὸ ἑαυτοῦ πάντας. ἔστιν ἄρα ὡς ὁ NK πρὸς τὸν E , οὕτως ὁ ΞH πρὸς τοὺς $M, \Lambda, K\Theta, E$. καὶ ἐστὶν ὁ NK ἴσος τῷ E . καὶ ὁ ΞH ἄρα ἴσος ἐστὶ τοῖς $M, \Lambda, \Theta K, E$. ἔστι δὲ καὶ ὁ $Z\Xi$ τῷ E ἴσος, ὁ δὲ E τοῖς A, B, Γ, Δ καὶ τῇ μονάδι. ὅλος ἄρα ὁ ZH ἴσος ἐστὶ τοῖς τε $E, \Theta K, \Lambda, M$ καὶ τοῖς A, B, Γ, Δ καὶ τῇ μονάδι. καὶ μετρεῖται ὑπ' αὐτῶν. λέγω, ὅτι καὶ ὁ ZH ὑπ' οὐδενὸς ἄλλου μετρηθήσεται παρὲξ τῶν $A, B, \Gamma, \Delta, E, \Theta K, \Lambda, M$ καὶ τῆς μονάδος. εἰ γὰρ δυνατόν, μετρεῖται τις τὸν ZH ὁ O , καὶ ὁ O μηδενὶ τῶν $A, B, \Gamma, \Delta, E, \Theta K, \Lambda, M$ ἔστω ὁ αὐτός. καὶ ὁσάκις ὁ O τὸν ZH μετρεῖ, τοσαῦται μονάδες

the sum multiplied into the last (number) makes some (number), then the (number so) created will be perfect.

For let any multitude of numbers, A, B, C, D , be set out (continuously) in a double proportion, until the whole sum added together is made prime. And let E be equal to the sum. And let E make FG (by) multiplying D . I say that FG is a perfect (number).



For as many as is the multitude of A, B, C, D , let so many (numbers), E, HK, L, M , have been taken in a double proportion, (starting) from E . Thus, via equality, as A is to D , so E (is) to M [Prop. 7.14]. Thus, the (number created) from (multiplying) E, D is equal to the (number created) from (multiplying) A, M . And FG is the (number created) from (multiplying) E, D . Thus, FG is also the (number created) from (multiplying) A, M [Prop. 7.19]. Thus, A has made FG (by) multiplying M . Thus, M measures FG according to the units in A . And A is a dyad. Thus, FG is double M . And M, L, HK, E are also continuously double one another. Thus, E, HK, L, M, FG are continuously proportional in a double proportion. So let HN and FO , each equal to the first (number) E , have been subtracted from the second (number) HK and the last FG (respectively). Thus, as the excess of the second number is to the first, so the excess of the last (is) to (the sum of) all those (numbers) before it [Prop. 9.35]. Thus, as NK is to E , so OG (is) to M, L, KH, E . And NK is equal to E . And thus OG is equal to M, L, HK, E . And FO is also equal to E , and E to A, B, C, D , and a unit. Thus, the whole of FG is equal to E, HK, L, M , and A, B, C, D , and a unit. And it is measured by them. I also say that FG will be

ἔστωσαν ἐν τῷ Π· ὁ Π ἄρα τὸν Ο πολλαπλασιάσας τὸν ΖΗ πεποίηκεν. ἀλλὰ μὴν καὶ ὁ Ε τὸν Δ πολλαπλασιάσας τὸν ΖΗ πεποίηκεν· ἔστιν ἄρα ὡς ὁ Ε πρὸς τὸν Π, ὁ Ο πρὸς τὸν Δ. καὶ ἐπεὶ ἀπὸ μονάδος ἐξῆς ἀνάλογόν εἰσιν οἱ Α, Β, Γ, Δ, ὁ Δ ἄρα ὑπ' οὐδενὸς ἄλλου ἀριθμοῦ μετρηθήσεται παρὲξ τῶν Α, Β, Γ. καὶ ὑπόκειται ὁ Ο οὐδενὶ τῶν Α, Β, Γ ὁ αὐτός· οὐκ ἄρα μετρήσει ὁ Ο τὸν Δ. ἀλλ' ὡς ὁ Ο πρὸς τὸν Δ, ὁ Ε πρὸς τὸν Π· οὐδὲ ὁ Ε ἄρα τὸν Π μετρεῖ. καὶ ἔστιν ὁ Ε πρῶτος· πᾶς δὲ πρῶτος ἀριθμὸς πρὸς ἅπαντα, ὃν μὴ μετρεῖ, πρῶτός [ἔστιν]. οἱ Ε, Π ἄρα πρῶτοι πρὸς ἀλλήλους εἰσίν. οἱ δὲ πρῶτοι καὶ ἐλάχιστοι, οἱ δὲ ἐλάχιστοι μετροῦσι τοὺς τὸν αὐτὸν λόγον ἔχοντας ἰσάκεις ὃ τε ἡγούμενος τὸν ἡγούμενον καὶ ὁ ἐπόμενος τὸν ἐπόμενον· καὶ ἔστιν ὡς ὁ Ε πρὸς τὸν Π, ὁ Ο πρὸς τὸν Δ. ἰσάκεις ἄρα ὁ Ε τὸν Ο μετρεῖ καὶ ὁ Π τὸν Δ. ὁ δὲ Δ ὑπ' οὐδενὸς ἄλλου μετρεῖται παρὲξ τῶν Α, Β, Γ· ὁ Π ἄρα ἐνὶ τῶν Α, Β, Γ ἔστιν ὁ αὐτός. ἔστω τῷ Β ὁ αὐτός. καὶ ὅσοι εἰσὶν οἱ Β, Γ, Δ τῷ πλήθει τοσοῦτοι εἰλήφθωσαν ἀπὸ τοῦ Ε οἱ Ε, ΘΚ, Λ. καὶ εἰσὶν οἱ Ε, ΘΚ, Λ τοῖς Β, Γ, Δ ἐν τῷ αὐτῷ λόγῳ· δι' ἴσου ἄρα ἔστιν ὡς ὁ Β πρὸς τὸν Δ, ὁ Ε πρὸς τὸν Λ. ὁ ἄρα ἐκ τῶν Β, Λ ἴσος ἐστὶ τῷ ἐκ τῶν Δ, Ε· ἀλλ' ὁ ἐκ τῶν Δ, Ε ἴσος ἐστὶ τῷ ἐκ τῶν Π, Ο· καὶ ὁ ἐκ τῶν Π, Ο ἄρα ἴσος ἐστὶ τῷ ἐκ τῶν Β, Λ. ἔστιν ἄρα ὡς ὁ Π πρὸς τὸν Β, ὁ Λ πρὸς τὸν Ο. καὶ ἔστιν ὁ Π τῷ Β ὁ αὐτός· καὶ ὁ Λ ἄρα τῷ Ο ἔστιν ὁ αὐτός· ὅπερ ἀδύνατον· ὁ γὰρ Ο ὑπόκειται μηδενὶ τῶν ἐκχειμένων ὁ αὐτός· οὐκ ἄρα τὸν ΖΗ μετρήσει τις ἀριθμὸς παρὲξ τῶν Α, Β, Γ, Δ, Ε, ΘΚ, Λ, Μ καὶ τῆς μονάδος. καὶ ἐδείχθη ὁ ΖΗ τοῖς Α, Β, Γ, Δ, Ε, ΘΚ, Λ, Μ καὶ τῇ μονάδι ἴσος. τέλειος δὲ ἀριθμὸς ἔστιν ὁ τοῖς ἑαυτοῦ μέρεσιν ἴσος ὢν· τέλειος ἄρα ἔστιν ὁ ΖΗ· ὅπερ ἔδει δεῖξαι.

measured by no other (numbers) except A, B, C, D, E, HK, L, M , and a unit. For, if possible, let some (number) P measure FG , and let P not be the same as any of A, B, C, D, E, HK, L, M . And as many times as P measures FG , so many units let there be in Q . Thus, Q has made FG (by) multiplying P . But, in fact, E has also made FG (by) multiplying D . Thus, as E is to Q , so P (is) to D [Prop. 7.19]. And since A, B, C, D are continually proportional, (starting) from a unit, D will thus not be measured by any other numbers except A, B, C [Prop. 9.13]. And P was assumed not (to be) the same as any of A, B, C . Thus, P does not measure D . But, as P (is) to D , so E (is) to Q . Thus, E does not measure Q either [Def. 7.20]. And E is a prime (number). And every prime number [is] prime to every (number) which it does not measure [Prop. 7.29]. Thus, E and Q are prime to one another. And (numbers) prime (to one another are) also the least (of those numbers having the same ratio as them) [Prop. 7.21], and the least (numbers) measure those (numbers) having the same ratio as them an equal number of times, the leading (measuring) the leading, and the following the following [Prop. 7.20]. And as E is to Q , (so) P (is) to D . Thus, E measures P the same number of times as Q (measures) D . And D is not measured by any other (numbers) except A, B, C . Thus, Q is the same as one of A, B, C . Let it be the same as B . And as many as is the multitude of B, C, D , let so many (of the set out numbers) have been taken, (starting) from E , (namely) E, HK, L . And E, HK, L are in the same ratio as B, C, D . Thus, via equality, as B (is) to D , (so) E (is) to L [Prop. 7.14]. Thus, the (number created) from (multiplying) B, L is equal to the (number created) from multiplying D, E [Prop. 7.19]. But, the (number created) from (multiplying) D, E is equal to the (number created) from (multiplying) Q, P . Thus, the (number created) from (multiplying) Q, P is equal to the (number created) from (multiplying) B, L . Thus, as Q is to B , (so) L (is) to P [Prop. 7.19]. And Q is the same as B . Thus, L is also the same as P . The very thing (is) impossible. For P was assumed not (to be) the same as any of the (numbers) set out. Thus, FG cannot be measured by any number except A, B, C, D, E, HK, L, M , and a unit. And FG was shown (to be) equal to (the sum of) A, B, C, D, E, HK, L, M , and a unit. And a perfect number is one which is equal to (the sum of) its own parts [Def. 7.22]. Thus, FG is a perfect (number). (Which is) the very thing it was required to show.

† This proposition demonstrates that perfect numbers take the form $2^{n-1}(2^n - 1)$ provided that $2^n - 1$ is a prime number. The ancient Greeks knew of four perfect numbers: 6, 28, 496, and 8128, which correspond to $n = 2, 3, 5$, and 7, respectively.

ELEMENTS BOOK 10

Incommensurable Magnitudes[†]

[†]The theory of incommensurable magnitudes set out in this book is generally attributed to Theaetetus of Athens. In the footnotes throughout this book, k , k' , etc. stand for distinct ratios of positive integers.

Ὅροι.

α'. Σύμμετρα μεγέθη λέγεται τὰ τῷ αὐτῷ μετρῷ μετρούμενα, ἀσύμμετρα δέ, ὧν μηδὲν ἐνδέχεται κοινὸν μέτρον γενέσθαι.

β'. Εὐθεῖαι δυνάμει σύμμετροί εἰσιν, ὅταν τὰ ἀπ' αὐτῶν τετράγωνα τῷ αὐτῷ χωρίῳ μετρηῇται, ἀσύμμετροι δέ, ὅταν τοῖς ἀπ' αὐτῶν τετραγώνοις μηδὲν ἐνδέχεται χωρίον κοινὸν μέτρον γενέσθαι.

γ'. Τούτων ὑποκειμένων δείκνυται, ὅτι τῇ προτεθείσῃ εὐθείᾳ ὑπάρχουσιν εὐθεῖαι πλῆθει ἄπειροι σύμμετροί τε καὶ ἀσύμμετροι αἱ μὲν μήκει μόνον, αἱ δὲ καὶ δυνάμει. καλεῖσθω οὖν ἡ μὲν προτεθείσα εὐθεῖα ῥητή, καὶ αἱ ταύτῃ σύμμετροι εἴτε μήκει καὶ δυνάμει εἴτε δυνάμει μόνον ῥηταί, αἱ δὲ ταύτῃ ἀσύμμετροι ἄλλοι καλεῖσθωσαν.

δ'. Καὶ τὸ μὲν ἀπὸ τῆς προτεθείσης εὐθείας τετράγωνον ῥητόν, καὶ τὰ τούτῳ σύμμετρα ῥητά, τὰ δὲ τούτῳ ἀσύμμετρα ἄλλα καλεῖσθω, καὶ αἱ δυνάμεναι αὐτὰ ἄλλοι, εἰ μὲν τετράγωνα εἴη, αὐταὶ αἱ πλευραί, εἰ δὲ ἕτερα τινὰ εὐθύγραμμα, αἱ ἴσα αὐτοῖς τετράγωνα ἀναγράφουσαι.

Definitions

1. Those magnitudes measured by the same measure are said (to be) commensurable, but (those) of which no (magnitude) admits to be a common measure (are said to be) incommensurable.[†]

2. (Two) straight-lines are commensurable in square[‡] when the squares on them are measured by the same area, but (are) incommensurable (in square) when no area admits to be a common measure of the squares on them.[§]

3. These things being assumed, it is proved that there exist an infinite multitude of straight-lines commensurable and incommensurable with an assigned straight-line—those (incommensurable) in length only, and those also (commensurable or incommensurable) in square.[¶] Therefore, let the assigned straight-line be called rational. And (let) the (straight-lines) commensurable with it, either in length and square, or in square only, (also be called) rational. But let the (straight-lines) incommensurable with it be called irrational.*

4. And let the square on the assigned straight-line be called rational. And (let areas) commensurable with it (also be called) rational. But (let areas) incommensurable with it (be called) irrational, and (let) their square-roots[§] (also be called) irrational—the sides themselves, if the (areas) are squares, and the (straight-lines) describing squares equal to them, if the (areas) are some other rectilinear (figure).^{||}

[†] In other words, two magnitudes α and β are commensurable if $\alpha : \beta :: 1 : k$, and incommensurable otherwise.

[‡] Literally, “in power”.

[§] In other words, two straight-lines of length α and β are commensurable in square if $\alpha : \beta :: 1 : k^{1/2}$, and incommensurable in square otherwise. Likewise, the straight-lines are commensurable in length if $\alpha : \beta :: 1 : k$, and incommensurable in length otherwise.

[¶] To be more exact, straight-lines can either be commensurable in square only, incommensurable in length only, or commensurable/incommensurable in both length and square, with an assigned straight-line.

* Let the length of the assigned straight-line be unity. Then rational straight-lines have lengths expressible as k or $k^{1/2}$, depending on whether the lengths are commensurable in length, or in square only, respectively, with unity. All other straight-lines are irrational.

[§] The square-root of an area is the length of the side of an equal area square.

^{||} The area of the square on the assigned straight-line is unity. Rational areas are expressible as k . All other areas are irrational. Thus, squares whose sides are of rational length have rational areas, and *vice versa*.

α'.

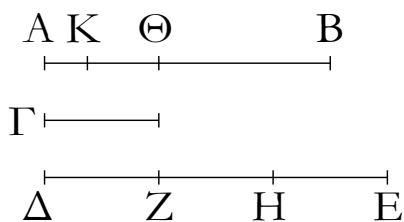
Proposition 1[†]

Δύο μεγεθῶν ἀνίσων ἐκκειμένων, ἐὰν ἀπὸ τοῦ μείζονος ἀφαιρεθῇ μείζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μείζον ἢ τὸ ἥμισυ, καὶ τοῦτο αἰεὶ γίγνηται, λειψθήσεται τι μέγεθος, ὃ ἔσται ἕλασσον τοῦ ἐκκειμένου ἐλάσσονος μεγέθους.

Ἐστω δύο μεγέθη ἄνισα τὰ AB, Γ, ὧν μείζον τὸ AB.

If, from the greater of two unequal magnitudes (which are) laid out, (a part) greater than half is subtracted, and (if from) the remainder (a part) greater than half (is subtracted), and (if) this happens continually, then some magnitude will (eventually) be left which will

λέγω, ὅτι, ἐὰν ἀπὸ τοῦ AB ἀφαιρεθῇ μείζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μείζον ἢ τὸ ἥμισυ, καὶ τοῦτο ἀεὶ γίγνηται, λειφθήσεται τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ Γ μεγέθους.



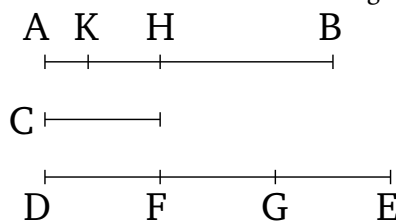
Τὸ Γ γὰρ πολλαπλασιαζόμενον ἔσται ποτὲ τοῦ AB μείζον. πεπολλαπλασιάσθω, καὶ ἔστω τὸ ΔE τοῦ μὲν Γ πολλαπλάσιον, τοῦ δὲ AB μείζον, καὶ διηρήσθω τὸ ΔE εἰς τὰ τῷ Γ ἴσα τὰ ΔZ , ZH , HE , καὶ ἀφηρήσθω ἀπὸ μὲν τοῦ AB μείζον ἢ τὸ ἥμισυ τὸ $B\Theta$, ἀπὸ δὲ τοῦ $A\Theta$ μείζον ἢ τὸ ἥμισυ τὸ ΘK , καὶ τοῦτο ἀεὶ γιγνέσθω, ἕως ἂν αἱ ἐν τῷ AB διαιρέσεις ἰσοπληθεῖς γένωνται ταῖς ἐν τῷ ΔE διαιρέσεσιν.

Ἐστῶσαν οὖν αἱ AK , $K\Theta$, ΘB διαιρέσεις ἰσοπληθεῖς οὔσαι ταῖς ΔZ , ZH , HE · καὶ ἐπεὶ μείζον ἔστι τὸ ΔE τοῦ AB , καὶ ἀφῆρηται ἀπὸ μὲν τοῦ ΔE ἔλασσον τοῦ ἡμίσεως τὸ EH , ἀπὸ δὲ τοῦ AB μείζον ἢ τὸ ἥμισυ τὸ $B\Theta$, λοιπὸν ἄρα τὸ $H\Delta$ λοιποῦ τοῦ ΘA μείζον ἔστιν. καὶ ἐπεὶ μείζον ἔστι τὸ $H\Delta$ τοῦ ΘA , καὶ ἀφῆρηται τοῦ μὲν $H\Delta$ ἥμισυ τὸ HZ , τοῦ δὲ ΘA μείζον ἢ τὸ ἥμισυ τὸ ΘK , λοιπὸν ἄρα τὸ ΔZ λοιποῦ τοῦ AK μείζον ἔστιν. ἴσον δὲ τὸ ΔZ τῷ Γ · καὶ τὸ Γ ἄρα τοῦ AK μείζον ἔστιν. ἔλασσον ἄρα τὸ AK τοῦ Γ .

Καταλείπεται ἄρα ἀπὸ τοῦ AB μεγέθους τὸ AK μέγεθος ἔλασσον ὅν τοῦ ἐκκειμένου ἐλάσσονος μεγέθους τοῦ Γ · ὅπερ ἔδει δεῖξαι. — ὁμοίως δὲ δειχθήσεται, καὶ ἡμίση ἢ τὰ ἀφαιρούμενα.

be less than the lesser laid out magnitude.

Let AB and C be two unequal magnitudes, of which (let) AB (be) the greater. I say that if (a part) greater than half is subtracted from AB , and (if a part) greater than half (is subtracted) from the remainder, and (if) this happens continually, then some magnitude will (eventually) be left which will be less than the magnitude C .



For C , when multiplied (by some number), will sometimes be greater than AB [Def. 5.4]. Let it have been (so) multiplied. And let DE be (both) a multiple of C , and greater than AB . And let DE have been divided into the (divisions) DF , FG , GE , equal to C . And let BH , (which is) greater than half, have been subtracted from AB . And (let) HK , (which is) greater than half, (have been subtracted) from AH . And let this happen continually, until the divisions in AB become equal in number to the divisions in DE .

Therefore, let the divisions (in AB) be AK , KH , HB , being equal in number to DF , FG , GE . And since DE is greater than AB , and EG , (which is) less than half, has been subtracted from DE , and BH , (which is) greater than half, from AB , the remainder GD is thus greater than the remainder HA . And since GD is greater than HA , and the half GF has been subtracted from GD , and HK , (which is) greater than half, from HA , the remainder DF is thus greater than the remainder AK . And DF (is) equal to C . C is thus also greater than AK . Thus, AK (is) less than C .

Thus, the magnitude AK , which is less than the lesser laid out magnitude C , is left over from the magnitude AB . (Which is) the very thing it was required to show. — (The theorem) can similarly be proved even if the (parts) subtracted are halves.

† This theorem is the basis of the so-called *method of exhaustion*, and is generally attributed to Eudoxus of Cnidus.

β'.

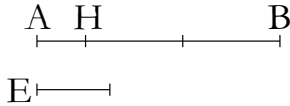
Ἐὰν δύο μεγεθῶν [ἐκκειμένων] ἀνίσων ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος τὸ καταλειπόμενον μηδέποτε καταμετρήῃ τὸ πρὸ ἑαυτοῦ, ἀσύμμετρα ἔσται τὰ μεγέθη.

Δύο γὰρ μεγεθῶν ὄντων ἀνίσων τῶν AB , $\Gamma\Delta$ καὶ ἐλάσσονος τοῦ AB ἀνθυφαιρουμένου ἀεὶ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος τὸ περιλειπόμενον μηδέποτε καταμε-

If the remainder of two unequal magnitudes (which are) [laid out] never measures the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater, then the (original) magnitudes will be incommensurable.

For, AB and CD being two unequal magnitudes, and AB (being) the lesser, let the remainder never measure

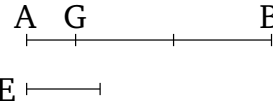
τρίτω τὸ πρὸ ἑαυτοῦ· λέγω, ὅτι ἀσύμμετρά ἐστι τὰ AB , $\Gamma\Delta$ μεγέθη.



Εἰ γὰρ ἐστι σύμμετρα, μετρήσει τι αὐτὰ μέγεθος. μετρίτω, εἰ δυνατόν, καὶ ἔστω τὸ E · καὶ τὸ μὲν AB τὸ ΔZ καταμετροῦν λειπέτω ἑαυτοῦ ἔλασσον τὸ ΓZ , τὸ δὲ ΓZ τὸ BH καταμετροῦν λειπέτω ἑαυτοῦ ἔλασσον τὸ AH , καὶ τοῦτο αἰ γινέσθω, ἕως οὗ λειφθῇ τι μέγεθος, ὃ ἐστὶν ἔλασσον τοῦ E . γεγονέντω, καὶ λελειφθῶ τὸ AH ἔλασσον τοῦ E . ἐπεὶ οὖν τὸ E τὸ AB μετρεῖ, ἀλλὰ τὸ AB τὸ ΔZ μετρεῖ, καὶ τὸ E ἄρα τὸ ΔZ μετρήσει. μετρεῖ δὲ καὶ ὅλον τὸ $\Gamma\Delta$ · καὶ λοιπὸν ἄρα τὸ ΓZ μετρήσει. ἀλλὰ τὸ ΓZ τὸ BH μετρεῖ· καὶ τὸ E ἄρα τὸ BH μετρεῖ. μετρεῖ δὲ καὶ ὅλον τὸ AB · καὶ λοιπὸν ἄρα τὸ AH μετρήσει, τὸ μείζον τὸ ἔλασσον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ AB , $\Gamma\Delta$ μεγέθη μετρήσει τι μέγεθος· ἀσύμμετρα ἄρα ἐστὶ τὰ AB , $\Gamma\Delta$ μεγέθη.

Ἐὰν ἄρα δύο μεγεθῶν ἀνίσων, καὶ τὰ ἐξῆς.

the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater. I say that the magnitudes AB and CD are incommensurable.



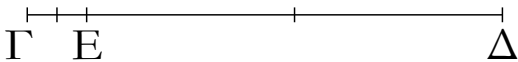
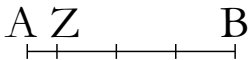
For if they are commensurable then some magnitude will measure them (both). If possible, let it (so) measure (them), and let it be E . And let AB leave CF less than itself (in) measuring FD , and let CF leave AG less than itself (in) measuring BG , and let this happen continually, until some magnitude which is less than E is left. Let (this) have occurred,[†] and let AG , (which is) less than E , have been left. Therefore, since E measures AB , but AB measures DF , E will thus also measure FD . And it also measures the whole (of) CD . Thus, it will also measure the remainder CF . But, CF measures BG . Thus, E also measures BG . And it also measures the whole (of) AB . Thus, it will also measure the remainder AG , the greater (measuring) the lesser. The very thing is impossible. Thus, some magnitude cannot measure (both) the magnitudes AB and CD . Thus, the magnitudes AB and CD are incommensurable [Def. 10.1].

Thus, if . . . of two unequal magnitudes, and so on . . .

[†] The fact that this will eventually occur is guaranteed by Prop. 10.1.

γ'.

Δύο μεγεθῶν συμμέτρων δοθέντων τὸ μέγιστον αὐτῶν κοινὸν μέτρον εὑρεῖν.



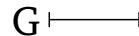
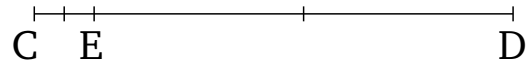
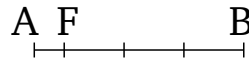
Ἐστω τὰ δοθέντα δύο μεγέθη σύμμετρα τὰ AB , $\Gamma\Delta$, ὧν ἔλασσον τὸ AB · δεῖ δὴ τῶν AB , $\Gamma\Delta$ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

Τὸ AB γὰρ μέγεθος ἦτοι μετρεῖ τὸ $\Gamma\Delta$ ἢ οὐ. εἰ μὲν οὖν μετρεῖ, μετρεῖ δὲ καὶ ἑαυτό, τὸ AB ἄρα τῶν AB , $\Gamma\Delta$ κοινὸν μέτρον ἐστίν· καὶ φανερόν, ὅτι καὶ μέγιστον. μείζον γὰρ τοῦ AB μεγέθους τὸ AB οὐ μετρήσει.

Μὴ μετρεῖτω δὴ τὸ AB τὸ $\Gamma\Delta$. καὶ ἀνθυφαιρουμένου αἰ τοῦ ἐλάσσονος ἀπὸ τοῦ μείζονος, τὸ περιλειπόμενον μετρήσει ποτὲ τὸ πρὸ ἑαυτοῦ διὰ τὸ μὴ εἶναι ἀσύμμετρα τὰ AB , $\Gamma\Delta$ · καὶ τὸ μὲν AB τὸ $E\Delta$ καταμετροῦν λειπέτω ἑαυτοῦ

Proposition 3

To find the greatest common measure of two given commensurable magnitudes.



Let AB and CD be the two given magnitudes, of which (let) AB (be) the lesser. So, it is required to find the greatest common measure of AB and CD .

For the magnitude AB either measures, or (does) not (measure), CD . Therefore, if it measures (CD), and (since) it also measures itself, AB is thus a common measure of AB and CD . And (it is) clear that (it is) also (the) greatest. For a (magnitude) greater than magnitude AB cannot measure AB .

So let AB not measure CD . And continually subtracting in turn the lesser (magnitude) from the greater, the

ἔλασσον τὸ ΕΓ, τὸ δὲ ΕΓ τὸ ΖΒ καταμετροῦν λειπέτω
ἑαυτοῦ ἔλασσον τὸ ΑΖ, τὸ δὲ ΑΖ τὸ ΓΕ μετρεῖτω.

Ἐπεὶ οὖν τὸ ΑΖ τὸ ΓΕ μετρεῖ, ἀλλὰ τὸ ΓΕ τὸ ΖΒ μετρεῖ,
καὶ τὸ ΑΖ ἄρα τὸ ΖΒ μετρήσει. μετρεῖ δὲ καὶ ἑαυτό· καὶ
ὅλον ἄρα τὸ ΑΒ μετρήσει τὸ ΑΖ. ἀλλὰ τὸ ΑΒ τὸ ΔΕ μετρεῖ·
καὶ τὸ ΑΖ ἄρα τὸ ΕΔ μετρήσει. μετρεῖ δὲ καὶ τὸ ΓΕ· καὶ
ὅλον ἄρα τὸ ΓΔ μετρεῖ· τὸ ΑΖ ἄρα τῶν ΑΒ, ΓΔ κοινὸν
μέτρον ἐστίν. λέγω δὴ, ὅτι καὶ μέγιστον. εἰ γὰρ μή, ἔσται
τι μέγεθος μείζον τοῦ ΑΖ, ὃ μετρήσει τὰ ΑΒ, ΓΔ. ἔστω τὸ
Η. ἐπεὶ οὖν τὸ Η τὸ ΑΒ μετρεῖ, ἀλλὰ τὸ ΑΒ τὸ ΕΔ μετρεῖ,
καὶ τὸ Η ἄρα τὸ ΕΔ μετρήσει. μετρεῖ δὲ καὶ ὅλον τὸ ΓΔ·
καὶ λοιπὸν ἄρα τὸ ΓΕ μετρήσει τὸ Η. ἀλλὰ τὸ ΓΕ τὸ ΖΒ
μετρεῖ· καὶ τὸ Η ἄρα τὸ ΖΒ μετρήσει. μετρεῖ δὲ καὶ ὅλον
τὸ ΑΒ, καὶ λοιπὸν τὸ ΑΖ μετρήσει, τὸ μείζον τὸ ἔλασσον·
ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα μείζον τι μέγεθος τοῦ ΑΖ
τὰ ΑΒ, ΓΔ μετρήσει· τὸ ΑΖ ἄρα τῶν ΑΒ, ΓΔ τὸ μέγιστον
κοινὸν μέτρον ἐστίν.

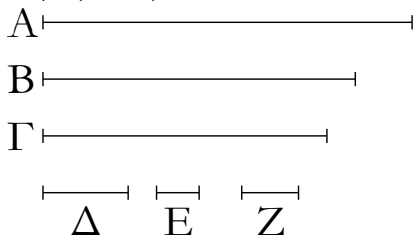
Δύο ἄρα μεγεθῶν συμμετρῶν δοθέντων τῶν ΑΒ, ΓΔ
τὸ μέγιστον κοινὸν μέτρον ἡύρηται· ὅπερ ἔδει δεῖξαι.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν μέγεθος δύο μεγέθη
μετρῇ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει.

δ'.

Τριῶν μεγεθῶν συμμετρῶν δοθέντων τὸ μέγιστον
αὐτῶν κοινὸν μέτρον εὑρεῖν.



Ἐστω τὰ δοθέντα τρία μεγέθη σύμμετρα τὰ Α, Β, Γ·
δεῖ δὴ τῶν Α, Β, Γ τὸ μέγιστον κοινὸν μέτρον εὑρεῖν.

Εἰλήφθω γὰρ δύο τῶν Α, Β τὸ μέγιστον κοινὸν μέτρον,
καὶ ἔστω τὸ Δ· τὸ δὴ Δ τὸ Γ ἤτοι μετρεῖ ἢ οὐ [μετρεῖ].
μετρεῖτω πρότερον. ἐπεὶ οὖν τὸ Δ τὸ Γ μετρεῖ, μετρεῖ δὲ

remaining (magnitude) will (at) some time measure the
(magnitude) before it, on account of AB and CD not be-
ing incommensurable [Prop. 10.2]. And let AB leave EC
less than itself (in) measuring ED , and let EC leave AF
less than itself (in) measuring FB , and let AF measure
 CE .

Therefore, since AF measures CE , but CE measures
 FB , AF will thus also measure FB . And it also mea-
sures itself. Thus, AF will also measure the whole (of)
 AB . But, AB measures DE . Thus, AF will also mea-
sure ED . And it also measures CE . Thus, it also mea-
sures the whole of CD . Thus, AF is a common measure
of AB and CD . So I say that (it is) also (the) greatest
(common measure). For, if not, there will be some mag-
nitude, greater than AF , which will measure (both) AB
and CD . Let it be G . Therefore, since G measures AB ,
but AB measures ED , G will thus also measure ED . And
it also measures the whole of CD . Thus, G will also mea-
sure the remainder CE . But CE measures FB . Thus, G
will also measure FB . And it also measures the whole
(of) AB . And (so) it will measure the remainder AF ,
the greater (measuring) the lesser. The very thing is im-
possible. Thus, some magnitude greater than AF cannot
measure (both) AB and CD . Thus, AF is the greatest
common measure of AB and CD .

Thus, the greatest common measure of two given
commensurable magnitudes, AB and CD , has been
found. (Which is) the very thing it was required to show.

Corollary

So (it is) clear, from this, that if a magnitude measures
two magnitudes then it will also measure their greatest
common measure.

Proposition 4

To find the greatest common measure of three given
commensurable magnitudes.



Let A , B , C be the three given commensurable mag-
nitudes. So it is required to find the greatest common
measure of A , B , C .

For let the greatest common measure of the two (mag-
nitudes) A and B have been taken [Prop. 10.3], and let it

καὶ τὰ A, B , τὸ Δ ἄρα τὰ A, B, Γ μετρεῖ· τὸ Δ ἄρα τῶν A, B, Γ κοινὸν μέτρον ἐστίν. καὶ φανερόν, ὅτι καὶ μέγιστον· μεῖζον γὰρ τοῦ Δ μεγέθους τὰ A, B οὐ μετρεῖ.

Μὴ μετρεῖται δὴ τὸ Δ τὸ Γ . λέγω πρῶτον, ὅτι σύμμετρά ἐστι τὰ Γ, Δ . ἐπεὶ γὰρ σύμμετρά ἐστι τὰ A, B, Γ , μετρήσει τι αὐτὰ μέγεθος, ὃ δηλαδὴ καὶ τὰ A, B μετρήσει· ὥστε καὶ τὸ τῶν A, B μέγιστον κοινὸν μέτρον τὸ Δ μετρήσει. μετρεῖ δὲ καὶ τὸ Γ · ὥστε τὸ εἰρημένον μέγεθος μετρήσει τὰ Γ, Δ · σύμμετρα ἄρα ἐστὶ τὰ Γ, Δ . εἰλήφθω οὖν αὐτῶν τὸ μέγιστον κοινὸν μέτρον, καὶ ἔστω τὸ E . ἐπεὶ οὖν τὸ E τὸ Δ μετρεῖ, ἀλλὰ τὸ Δ τὰ A, B μετρεῖ, καὶ τὸ E ἄρα τὰ A, B μετρήσει. μετρεῖ δὲ καὶ τὸ Γ . τὸ E ἄρα τὰ A, B, Γ μετρεῖ· τὸ E ἄρα τῶν A, B, Γ κοινὸν ἐστὶ μέτρον. λέγω δὴ, ὅτι καὶ μέγιστον. εἰ γὰρ δυνατόν, ἔστω τι τοῦ E μεῖζον μέγεθος τὸ Z , καὶ μετρεῖται τὰ A, B, Γ . καὶ ἐπεὶ τὸ Z τὰ A, B, Γ μετρεῖ, καὶ τὰ A, B ἄρα μετρήσει καὶ τὸ τῶν A, B μέγιστον κοινὸν μέτρον μετρήσει. τὸ δὲ τῶν A, B μέγιστον κοινὸν μέτρον ἐστὶ τὸ Δ · τὸ Z ἄρα τὸ Δ μετρεῖ. μετρεῖ δὲ καὶ τὸ Γ · τὸ Z ἄρα τὰ Γ, Δ μετρεῖ· καὶ τὸ τῶν Γ, Δ ἄρα μέγιστον κοινὸν μέτρον μετρήσει τὸ Z . ἔστι δὲ τὸ E · τὸ Z ἄρα τὸ E μετρήσει, τὸ μεῖζον τὸ ἔλασσον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα μεῖζόν τι τοῦ E μεγέθους [μέγεθος] τὰ A, B, Γ μετρεῖ· τὸ E ἄρα τῶν A, B, Γ τὸ μέγιστον κοινὸν μέτρον ἐστίν, ἐὰν μὴ μετρήῃ τὸ Δ τὸ Γ , ἐὰν δὲ μετρήῃ, αὐτὸ τὸ Δ .

Τριῶν ἄρα μεγεθῶν συμμέτρων δοθέντων τὸ μέγιστον κοινὸν μέτρον ἡύρηται [ὅπερ ἔδει δεῖξαι].

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν μέγεθος τρία μεγέθη μετρήῃ, καὶ τὸ μέγιστον αὐτῶν κοινὸν μέτρον μετρήσει.

Ὅμοίως δὴ καὶ ἐπὶ πλειόνων τὸ μέγιστον κοινὸν μέτρον ληφθήσεται, καὶ τὸ πόρισμα προχωρήσει. ὅπερ ἔδει δεῖξαι.

be D . So D either measures, or [does] not [measure], C . Let it, first of all, measure (C). Therefore, since D measures C , and it also measures A and B , D thus measures A, B, C . Thus, D is a common measure of A, B, C . And (it is) clear that (it is) also (the) greatest (common measure). For no magnitude larger than D measures (both) A and B .

So let D not measure C . I say, first, that C and D are commensurable. For if A, B, C are commensurable then some magnitude will measure them which will clearly also measure A and B . Hence, it will also measure D , the greatest common measure of A and B [Prop. 10.3 corr.]. And it also measures C . Hence, the aforementioned magnitude will measure (both) C and D . Thus, C and D are commensurable [Def. 10.1]. Therefore, let their greatest common measure have been taken [Prop. 10.3], and let it be E . Therefore, since E measures D , but D measures (both) A and B , E will thus also measure A and B . And it also measures C . Thus, E measures A, B, C . Thus, E is a common measure of A, B, C . So I say that (it is) also (the) greatest (common measure). For, if possible, let F be some magnitude greater than E , and let it measure A, B, C . And since F measures A, B, C , it will thus also measure A and B , and will (thus) measure the greatest common measure of A and B [Prop. 10.3 corr.]. And D is the greatest common measure of A and B . Thus, F measures D . And it also measures C . Thus, F measures (both) C and D . Thus, F will also measure the greatest common measure of C and D [Prop. 10.3 corr.]. And it is E . Thus, F will measure E , the greater (measuring) the lesser. The very thing is impossible. Thus, some [magnitude] greater than the magnitude E cannot measure A, B, C . Thus, if D does not measure C then E is the greatest common measure of A, B, C . And if it does measure (C) then D itself (is the greatest common measure).

Thus, the greatest common measure of three given commensurable magnitudes has been found. [(Which is) the very thing it was required to show.]

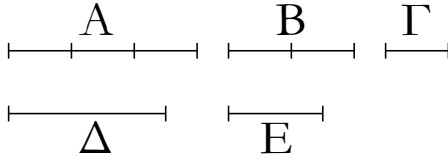
Corollary

So (it is) clear, from this, that if a magnitude measures three magnitudes then it will also measure their greatest common measure.

So, similarly, the greatest common measure of more (magnitudes) can also be taken, and the (above) corollary will go forward. (Which is) the very thing it was required to show.

ε'.

Τὰ σύμμετρα μεγέθη πρὸς ἄλληλα λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμὸν.



Ἐστω σύμμετρα μεγέθη τὰ A , B · λέγω, ὅτι τὸ A πρὸς τὸ B λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμὸν.

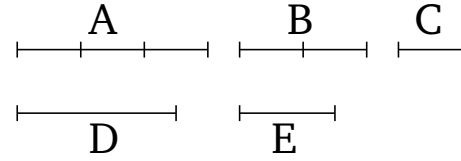
Ἐπεὶ γὰρ σύμμετρά ἐστι τὰ A , B , μετρήσει τι αὐτὰ μέγεθος· μετρεῖτω, καὶ ἔστω τὸ Γ . καὶ ὅσάκις τὸ Γ τὸ A μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ Δ , ὅσάκις δὲ τὸ Γ τὸ B μετρεῖ, τοσαῦται μονάδες ἔστωσαν ἐν τῷ E .

Ἐπεὶ οὖν τὸ Γ τὸ A μετρεῖ κατὰ τὰς ἐν τῷ Δ μονάδας, μετρεῖ δὲ καὶ ἡ μονὰς τὸν Δ κατὰ τὰς ἐν αὐτῷ μονάδας, ἰσάκις ἄρα ἡ μονὰς τὸν Δ μετρεῖ ἀριθμὸν καὶ τὸ Γ μέγεθος τὸ A · ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ A , οὕτως ἡ μονὰς πρὸς τὸν Δ · ἀνάπαλιν ἄρα, ὡς τὸ A πρὸς τὸ Γ , οὕτως ὁ Δ πρὸς τὴν μονάδα. πάλιν ἐπεὶ τὸ Γ τὸ B μετρεῖ κατὰ τὰς ἐν τῷ E μονάδας, μετρεῖ δὲ καὶ ἡ μονὰς τὸν E κατὰ τὰς ἐν αὐτῷ μονάδας, ἰσάκις ἄρα ἡ μονὰς τὸν E μετρεῖ καὶ τὸ Γ τὸ B · ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ B , οὕτως ἡ μονὰς πρὸς τὸν E . ἐδείχθη δὲ καὶ ὡς τὸ A πρὸς τὸ Γ , ὁ Δ πρὸς τὴν μονάδα· δι' ἴσου ἄρα ἐστὶν ὡς τὸ A πρὸς τὸ B , οὕτως ὁ Δ ἀριθμὸς πρὸς τὸν E .

Τὰ ἄρα σύμμετρα μεγέθη τὰ A , B πρὸς ἄλληλα λόγον ἔχει, ὃν ἀριθμὸς ὁ Δ πρὸς ἀριθμὸν τὸν E · ὅπερ ἔδει δεῖξαι.

Proposition 5

Commensurable magnitudes have to one another the ratio which (some) number (has) to (some) number.



Let A and B be commensurable magnitudes. I say that A has to B the ratio which (some) number (has) to (some) number.

For if A and B are commensurable (magnitudes) then some magnitude will measure them. Let it (so) measure (them), and let it be C . And as many times as C measures A , so many units let there be in D . And as many times as C measures B , so many units let there be in E .

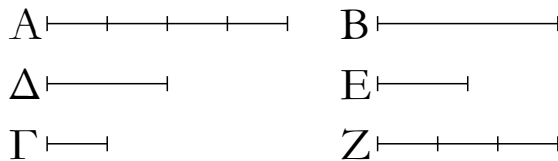
Therefore, since C measures A according to the units in D , and a unit also measures D according to the units in it, a unit thus measures the number D as many times as the magnitude C (measures) A . Thus, as C is to A , so a unit (is) to D [Def. 7.20].[†] Thus, inversely, as A (is) to C , so D (is) to a unit [Prop. 5.7 corr.]. Again, since C measures B according to the units in E , and a unit also measures E according to the units in it, a unit thus measures E the same number of times that C (measures) B . Thus, as C is to B , so a unit (is) to E [Def. 7.20]. And it was also shown that as A (is) to C , so D (is) to a unit. Thus, via equality, as A is to B , so the number D (is) to the (number) E [Prop. 5.22].

Thus, the commensurable magnitudes A and B have to one another the ratio which the number D (has) to the number E . (Which is) the very thing it was required to show.

[†] There is a slight logical gap here, since Def. 7.20 applies to four numbers, rather than two number and two magnitudes.

ζ'.

Ἐὰν δύο μεγέθη πρὸς ἄλληλα λόγον ἔχῃ, ὃν ἀριθμὸς πρὸς ἀριθμὸν, σύμμετρα ἔσται τὰ μεγέθη.

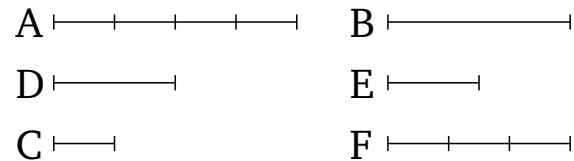


Δύο γὰρ μεγέθη τὰ A , B πρὸς ἄλληλα λόγον ἔχέτω, ὃν ἀριθμὸς ὁ Δ πρὸς ἀριθμὸν τὸν E · λέγω, ὅτι σύμμετρά ἐστι τὰ A , B μεγέθη.

Ὅσαι γὰρ εἰσιν ἐν τῷ Δ μονάδες, εἰς τοσαῦτα ἴσα

Proposition 6

If two magnitudes have to one another the ratio which (some) number (has) to (some) number then the magnitudes will be commensurable.



For let the two magnitudes A and B have to one another the ratio which the number D (has) to the number E . I say that the magnitudes A and B are commensurable.

διηρήσθω τὸ A , καὶ ἐνὶ αὐτῶν ἴσον ἔστω τὸ Γ . ὅσαι δὲ εἰσιν ἐν τῷ E μονάδες, ἐκ τοσούτων μεγεθῶν ἴσων τῷ Γ συγχεῖσθω τὸ Z .

Ἐπεὶ οὖν, ὅσαι εἰσιν ἐν τῷ Δ μονάδες, τοσαῦτά εἰσι καὶ ἐν τῷ A μεγέθει ἴσα τῷ Γ , ὃ ἄρα μέρος ἐστὶν ἢ μονὰς τοῦ Δ , τὸ αὐτὸ μέρος ἐστὶ καὶ τὸ Γ τοῦ A . ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ A , οὕτως ἢ μονὰς πρὸς τὸν Δ . μετρεῖ δὲ ἢ μονὰς τὸν Δ ἀριθμὸν· μετρεῖ ἄρα καὶ τὸ Γ τὸ A . καὶ ἐπεὶ ἐστὶν ὡς τὸ Γ πρὸς τὸ A , οὕτως ἢ μονὰς πρὸς τὸν Δ [ἀριθμὸν], ἀνάπαλιν ἄρα ὡς τὸ A πρὸς τὸ Γ , οὕτως ὁ Δ ἀριθμὸς πρὸς τὴν μονάδα. πάλιν ἐπεὶ, ὅσαι εἰσιν ἐν τῷ E μονάδες, τοσαῦτά εἰσι καὶ ἐν τῷ Z ἴσα τῷ Γ , ἔστιν ἄρα ὡς τὸ Γ πρὸς τὸ Z , οὕτως ἢ μονὰς πρὸς τὸν E [ἀριθμὸν]. ἐδείχθη δὲ καὶ ὡς τὸ A πρὸς τὸ Γ , οὕτως ὁ Δ πρὸς τὴν μονάδα· δι' ἴσου ἄρα ἐστὶν ὡς τὸ A πρὸς τὸ Z , οὕτως ὁ Δ πρὸς τὸν E . ἀλλ' ὡς ὁ Δ πρὸς τὸν E , οὕτως ἐστὶ τὸ A πρὸς τὸ B · καὶ ὡς ἄρα τὸ A πρὸς τὸ B , οὕτως καὶ πρὸς τὸ Z . τὸ A ἄρα πρὸς ἐκάτερον τῶν B , Z τὸν αὐτὸν ἔχει λόγον· ἴσον ἄρα ἐστὶ τὸ B τῷ Z . μετρεῖ δὲ τὸ Γ τὸ Z · μετρεῖ ἄρα καὶ τὸ B . ἀλλὰ μὴν καὶ τὸ A · τὸ Γ ἄρα τὰ A , B μετρεῖ. σύμμετρον ἄρα ἐστὶ τὸ A τῷ B .

Ἐὰν ἄρα δύο μεγέθη πρὸς ἄλληλα, καὶ τὰ ἐξῆς.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι, ἐὰν ὦσι δύο ἀριθμοί, ὡς οἱ Δ , E , καὶ εὐθεΐα, ὡς ἡ A , δυνατόν ἐστι ποιῆσαι ὡς ὁ Δ ἀριθμὸς πρὸς τὸν E ἀριθμὸν, οὕτως τὴν εὐθεΐαν πρὸς εὐθεΐαν. ἐὰν δὲ καὶ τῶν A , Z μέση ἀνάλογον ληφθῇ, ὡς ἡ B , ἔσται ὡς ἡ A πρὸς τὴν Z , οὕτως τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς B , τουτέστιν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον. ἀλλ' ὡς ἡ A πρὸς τὴν Z , οὕτως ἐστὶν ὁ Δ ἀριθμὸς πρὸς τὸν E ἀριθμὸν· γέγονεν ἄρα καὶ ὡς ὁ Δ ἀριθμὸς πρὸς τὸν E ἀριθμὸν, οὕτως τὸ ἀπὸ τῆς A εὐθείας πρὸς τὸ ἀπὸ τῆς B εὐθείας· ὅπερ ἔδει δεῖξαι.

ζ'.

Τὰ ἀσύμμετρα μεγέθη πρὸς ἄλληλα λόγον οὐκ ἔχει, ὅν ἀριθμὸς πρὸς ἀριθμὸν.

Ἐστω ἀσύμμετρα μεγέθη τὰ A , B · λέγω, ὅτι τὸ A πρὸς τὸ B λόγον οὐκ ἔχει, ὅν ἀριθμὸς πρὸς ἀριθμὸν.

For, as many units as there are in D , let A have been divided into so many equal (divisions). And let C be equal to one of them. And as many units as there are in E , let F be the sum of so many magnitudes equal to C .

Therefore, since as many units as there are in D , so many magnitudes equal to C are also in A , therefore whichever part a unit is of D , C is also the same part of A . Thus, as C is to A , so a unit (is) to D [Def. 7.20]. And a unit measures the number D . Thus, C also measures A . And since as C is to A , so a unit (is) to the [number] D , thus, inversely, as A (is) to C , so the number D (is) to a unit [Prop. 5.7 corr.]. Again, since as many units as there are in E , so many (magnitudes) equal to C are also in F , thus as C is to F , so a unit (is) to the [number] E [Def. 7.20]. And it was also shown that as A (is) to C , so D (is) to a unit. Thus, via equality, as A is to F , so D (is) to E [Prop. 5.22]. But, as D (is) to E , so A is to B . And thus as A (is) to B , so (it) also is to F [Prop. 5.11]. Thus, A has the same ratio to each of B and F . Thus, B is equal to F [Prop. 5.9]. And C measures F . Thus, it also measures B . But, in fact, (it) also (measures) A . Thus, C measures (both) A and B . Thus, A is commensurable with B [Def. 10.1].

Thus, if two magnitudes . . . to one another, and so on

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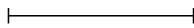
Corollary

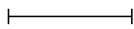
So it is clear, from this, that if there are two numbers, like D and E , and a straight-line, like A , then it is possible to contrive that as the number D (is) to the number E , so the straight-line (is) to (another) straight-line (i.e., F). And if the mean proportion, (say) B , is taken of A and F , then as A is to F , so the (square) on A (will be) to the (square) on B . That is to say, as the first (is) to the third, so the (figure) on the first (is) to the similar, and similarly described, (figure) on the second [Prop. 6.19 corr.]. But, as A (is) to F , so the number D is to the number E . Thus, it has also been contrived that as the number D (is) to the number E , so the (figure) on the straight-line A (is) to the (similar figure) on the straight-line B . (Which is) the very thing it was required to show.

Proposition 7

Incommensurable magnitudes do not have to one another the ratio which (some) number (has) to (some) number.

Let A and B be incommensurable magnitudes. I say that A does not have to B the ratio which (some) number (has) to (some) number.

A 

B 

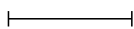
Εἰ γὰρ ἔχει τὸ A πρὸς τὸ B λόγον, ὃν ἀριθμὸς πρὸς ἀριθμὸν, σύμμετρον ἔσται τὸ A τῷ B . οὐκ ἔστι δέ· οὐκ ἄρα τὸ A πρὸς τὸ B λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμὸν.

Τὰ ἄρα ἀσύμμετρα μεγέθη πρὸς ἀλλήλα λόγον οὐκ ἔχει, καὶ τὰ ἐξῆς.

η'.

Ἐὰν δύο μεγέθη πρὸς ἀλλήλα λόγον μὴ ἔχῃ, ὃν ἀριθμὸς πρὸς ἀριθμὸν, ἀσύμμετρα ἔσται τὰ μεγέθη.

A 

B 

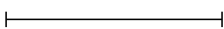
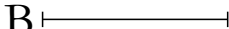
Δύο γὰρ μεγέθη τὰ A , B πρὸς ἀλλήλα λόγον μὴ ἔχέτω, ὃν ἀριθμὸς πρὸς ἀριθμὸν· λέγω, ὅτι ἀσύμμετρά ἐστι τὰ A , B μεγέθη.

Εἰ γὰρ ἔσται σύμμετρα, τὸ A πρὸς τὸ B λόγον ἔξει, ὃν ἀριθμὸς πρὸς ἀριθμὸν. οὐκ ἔχει δέ. ἀσύμμετρα ἄρα ἐστὶ τὰ A , B μεγέθη.

Ἐὰν ἄρα δύο μεγέθη πρὸς ἀλλήλα, καὶ τὰ ἐξῆς.

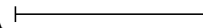
θ'.

Τὰ ἀπὸ τῶν μήκει συμμέτρων εὐθειῶν τετράγωνα πρὸς ἀλλήλα λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· καὶ τὰ τετράγωνα τὰ πρὸς ἀλλήλα λόγον ἔχοντα, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, καὶ τὰς πλευρὰς ἔξει μήκει συμμέτρους. τὰ δὲ ἀπὸ τῶν μήκει ἀσυμμέτρων εὐθειῶν τετράγωνα πρὸς ἀλλήλα λόγον οὐκ ἔχει, ὅνπερ τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· καὶ τὰ τετράγωνα τὰ πρὸς ἀλλήλα λόγον μὴ ἔχοντα, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, οὐδὲ τὰς πλευρὰς ἔξει μήκει συμμέτρους.

A  B 

Γ  Δ 

Ἐστωσαν γὰρ αἱ A , B μήκει σύμμετροι· λέγω, ὅτι τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς B τετράγωνον λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν.

A 

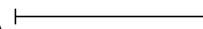
B 

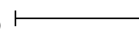
For if A has to B the ratio which (some) number (has) to (some) number then A will be commensurable with B [Prop. 10.6]. But it is not. Thus, A does not have to B the ratio which (some) number (has) to (some) number.

Thus, incommensurable numbers do not have to one another, and so on

Proposition 8

If two magnitudes do not have to one another the ratio which (some) number (has) to (some) number then the magnitudes will be incommensurable.

A 

B 

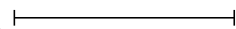
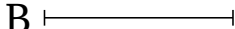
For let the two magnitudes A and B not have to one another the ratio which (some) number (has) to (some) number. I say that the magnitudes A and B are incommensurable.

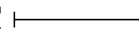
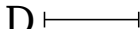
For if they are commensurable, A will have to B the ratio which (some) number (has) to (some) number [Prop. 10.5]. But it does not have (such a ratio). Thus, the magnitudes A and B are incommensurable.

Thus, if two magnitudes ... to one another, and so on

Proposition 9

Squares on straight-lines (which are) commensurable in length have to one another the ratio which (some) square number (has) to (some) square number. And squares having to one another the ratio which (some) square number (has) to (some) square number will also have sides (which are) commensurable in length. But squares on straight-lines (which are) incommensurable in length do not have to one another the ratio which (some) square number (has) to (some) square number. And squares not having to one another the ratio which (some) square number (has) to (some) square number will not have sides (which are) commensurable in length either.

A  B 

C  D 

For let A and B be (straight-lines which are) commensurable in length. I say that the square on A has to the square on B the ratio which (some) square number (has) to (some) square number.

Ἐπεὶ γὰρ σύμμετρος ἐστὶν ἡ A τῇ B μήκει, ἡ A ἄρα πρὸς τὴν B λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμὸν. ἐχέτω, ὃν ὁ Γ πρὸς τὸν Δ . ἐπεὶ οὖν ἐστὶν ὡς ἡ A πρὸς τὴν B , οὕτως ὁ Γ πρὸς τὸν Δ , ἀλλὰ τοῦ μὲν τῆς A πρὸς τὴν B λόγου διπλασίων ἐστὶν ὁ τοῦ ἀπὸ τῆς A τετραγώνου πρὸς τὸ ἀπὸ τῆς B τετράγωνον· τὰ γὰρ ὅμοια σχήματα ἐν διπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν· τοῦ δὲ τοῦ Γ [ἀριθμοῦ] πρὸς τὸν Δ [ἀριθμὸν] λόγου διπλασίων ἐστὶν ὁ τοῦ ἀπὸ τοῦ Γ τετραγώνου πρὸς τὸν ἀπὸ τοῦ Δ τετράγωνον· δύο γὰρ τετραγώνων ἀριθμῶν εἰς μέσος ἀνάλογόν ἐστιν ἀριθμὸς, καὶ ὁ τετράγωνος πρὸς τὸν τετράγωνον [ἀριθμὸν] διπλασίονα λόγον ἔχει, ἥπερ ἡ πλευρὰ πρὸς τὴν πλευράν· ἐστὶν ἄρα καὶ ὡς τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς B τετράγωνον, οὕτως ὁ ἀπὸ τοῦ Γ τετράγωνος [ἀριθμὸς] πρὸς τὸν ἀπὸ τοῦ Δ [ἀριθμοῦ] τετράγωνον [ἀριθμὸν].

Ἀλλὰ δὴ ἔστω ὡς τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς B , οὕτως ὁ ἀπὸ τοῦ Γ τετράγωνος πρὸς τὸν ἀπὸ τοῦ Δ [τετράγωνον]· λέγω, ὅτι σύμμετρος ἐστὶν ἡ A τῇ B μήκει.

Ἐπεὶ γὰρ ἐστὶν ὡς τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς B [τετράγωνον], οὕτως ὁ ἀπὸ τοῦ Γ τετράγωνος πρὸς τὸν ἀπὸ τοῦ Δ [τετράγωνον], ἀλλ' ὁ μὲν τοῦ ἀπὸ τῆς A τετραγώνου πρὸς τὸ ἀπὸ τῆς B [τετράγωνον] λόγος διπλασίων ἐστὶ τοῦ τῆς A πρὸς τὴν B λόγου, ὁ δὲ τοῦ ἀπὸ τοῦ Γ [ἀριθμοῦ] τετραγώνου [ἀριθμοῦ] πρὸς τὸν ἀπὸ τοῦ Δ [ἀριθμοῦ] τετράγωνον [ἀριθμὸν] λόγος διπλασίων ἐστὶ τοῦ τοῦ Γ [ἀριθμοῦ] πρὸς τὸν Δ [ἀριθμὸν] λόγου, ἐστὶν ἄρα καὶ ὡς ἡ A πρὸς τὴν B , οὕτως ὁ Γ [ἀριθμὸς] πρὸς τὸν Δ [ἀριθμὸν]. ἡ A ἄρα πρὸς τὴν B λόγον ἔχει, ὃν ἀριθμὸς ὁ Γ πρὸς ἀριθμὸν τὸν Δ · σύμμετρος ἄρα ἐστὶν ἡ A τῇ B μήκει.

Ἀλλὰ δὴ ἀσύμμετρος ἔστω ἡ A τῇ B μήκει· λέγω, ὅτι τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς B [τετράγωνον] λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν.

Εἰ γὰρ ἔχει τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς B [τετράγωνον] λόγον, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, σύμμετρος ἔσται ἡ A τῇ B . οὐκ ἔστι δέ· οὐκ ἄρα τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς B [τετράγωνον] λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν.

Πάλιν δὴ τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς B [τετράγωνον] λόγον μὴ ἐχέτω, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· λέγω, ὅτι ἀσύμμετρος ἐστὶν ἡ A τῇ B μήκει.

Εἰ γὰρ ἐστὶ σύμμετρος ἡ A τῇ B , ἔξει τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς B λόγον, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. οὐκ ἔχει δέ· οὐκ ἄρα σύμμετρος ἐστὶν ἡ A τῇ B μήκει.

Τὰ ἄρα ἀπὸ τῶν μήκει συμμέτρων, καὶ τὰ ἐξῆς.

For since A is commensurable in length with B , A thus has to B the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (that) which C (has) to D . Therefore, since as A is to B , so C (is) to D . But the (ratio) of the square on A to the square on B is the square of the ratio of A to B . For similar figures are in the squared ratio of (their) corresponding sides [Prop. 6.20 corr.]. And the (ratio) of the square on C to the square on D is the square of the ratio of the [number] C to the [number] D . For there exists one number in mean proportion to two square numbers, and (one) square (number) has to the (other) square [number] a squared ratio with respect to (that) the side (of the former has) to the side (of the latter) [Prop. 8.11]. And, thus, as the square on A is to the square on B , so the square [number] on the (number) C (is) to the square [number] on the [number] D .[†]

And so let the square on A be to the (square) on B as the square (number) on C (is) to the [square] (number) on D . I say that A is commensurable in length with B .

For since as the square on A is to the [square] on B , so the square (number) on C (is) to the [square] (number) on D . But, the ratio of the square on A to the (square) on B is the square of the (ratio) of A to B [Prop. 6.20 corr.]. And the (ratio) of the square [number] on the [number] C to the square [number] on the [number] D is the square of the ratio of the [number] C to the [number] D [Prop. 8.11]. Thus, as A is to B , so the [number] C also (is) to the [number] D . A , thus, has to B the ratio which the number C has to the number D . Thus, A is commensurable in length with B [Prop. 10.6].[‡]

And so let A be incommensurable in length with B . I say that the square on A does not have to the [square] on B the ratio which (some) square number (has) to (some) square number.

For if the square on A has to the [square] on B the ratio which (some) square number (has) to (some) square number then A will be commensurable (in length) with B . But it is not. Thus, the square on A does not have to the [square] on the B the ratio which (some) square number (has) to (some) square number.

So, again, let the square on A not have to the [square] on B the ratio which (some) square number (has) to (some) square number. I say that A is incommensurable in length with B .

For if A is commensurable (in length) with B then the (square) on A will have to the (square) on B the ratio which (some) square number (has) to (some) square number. But it does not have (such a ratio). Thus, A is not commensurable in length with B .

Thus, (squares) on (straight-lines which are) com-

measurable in length, and so on . . .

Πόρισμα.

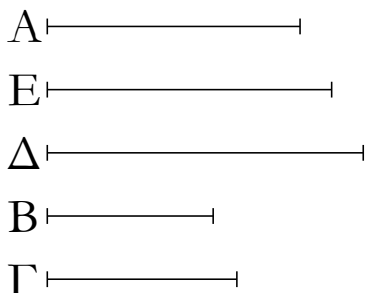
Καὶ φανερόν ἐκ τῶν δεδειγμένων ἔσται, ὅτι αἱ μήκει σύμμετροι πάντως καὶ δυνάμει, αἱ δὲ δυνάμει οὐ πάντως καὶ μήκει.

[†] There is an unstated assumption here that if $\alpha : \beta :: \gamma : \delta$ then $\alpha^2 : \beta^2 :: \gamma^2 : \delta^2$.

[‡] There is an unstated assumption here that if $\alpha^2 : \beta^2 :: \gamma^2 : \delta^2$ then $\alpha : \beta :: \gamma : \delta$.

ι'.

Τῇ προτεθείσῃ εὐθείᾳ προσευρεῖν δύο εὐθείας ἀσύμμετρος, τὴν μὲν μήκει μόνον, τὴν δὲ καὶ δυνάμει.



Ἐστω ἡ προτεθείσα εὐθεῖα ἡ A . δεῖ δὴ τῇ A προσευρεῖν δύο εὐθείας ἀσύμμετρος, τὴν μὲν μήκει μόνον, τὴν δὲ καὶ δυνάμει.

Ἐκκείσθωσαν γὰρ δύο ἀριθμοὶ οἱ B, Γ πρὸς ἀλλήλους λόγον μὴ ἔχοντες, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, τουτέστι μὴ ὅμοιοι ἐπίπεδοι, καὶ γεγονέντω ὡς ὁ B πρὸς τὸν Γ , οὕτως τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς Δ τετράγωνον· ἐμάθομεν γάρ· σύμμετρον ἄρα τὸ ἀπὸ τῆς A τῷ ἀπὸ τῆς Δ . καὶ ἐπεὶ ὁ B πρὸς τὸν Γ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ' ἄρα τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς Δ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ A τῇ Δ μήκει. εἰλήφθω τῶν A, Δ μέση ἀνάλογον ἡ E · ἔστιν ἄρα ὡς ἡ A πρὸς τὴν Δ , οὕτως τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς E . ἀσύμμετρος δὲ ἐστὶν ἡ A τῇ Δ μήκει· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς A τετράγωνον τῷ ἀπὸ τῆς E τετραγώνῳ· ἀσύμμετρος ἄρα ἐστὶν ἡ A τῇ E δυνάμει.

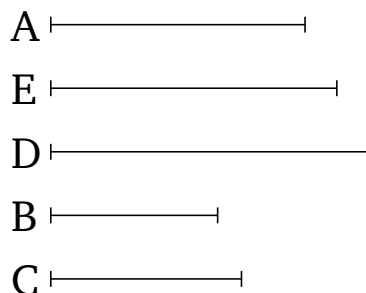
Τῇ ἄρα προτεθείσῃ εὐθείᾳ τῇ A προσεύρηται δύο εὐθεῖαι ἀσύμμετροι αἱ Δ, E , μήκει μὲν μόνον ἡ Δ , δυνάμει δὲ καὶ μήκει δηλαδὴ ἡ E [ὅπερ εἶδει δεῖξαι].

Corollary

And it will be clear, from (what) has been demonstrated, that (straight-lines) commensurable in length (are) always also (commensurable) in square, but (straight-lines commensurable) in square (are) not always also (commensurable) in length.

Proposition 10[†]

To find two straight-lines incommensurable with a given straight-line, the one (incommensurable) in length only, the other also (incommensurable) in square.



Let A be the given straight-line. So it is required to find two straight-lines incommensurable with A , the one (incommensurable) in length only, the other also (incommensurable) in square.

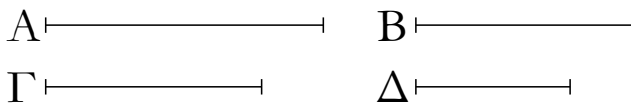
For let two numbers, B and C , not having to one another the ratio which (some) square number (has) to (some) square number—that is to say, not (being) similar plane (numbers)—have been taken. And let it be contrived that as B (is) to C , so the square on A (is) to the square on D . For we learned (how to do this) [Prop. 10.6 corr.]. Thus, the (square) on A (is) commensurable with the (square) on D [Prop. 10.6]. And since B does not have to C the ratio which (some) square number (has) to (some) square number, the (square) on A thus does not have to the (square) on D the ratio which (some) square number (has) to (some) square number either. Thus, A is incommensurable in length with D [Prop. 10.9]. Let the (straight-line) E (which is) in mean proportion to A and D have been taken [Prop. 6.13]. Thus, as A is to D , so the square on A (is) to the (square) on E [Def. 5.9]. And A is incommensurable in length with D . Thus, the square on A is also incommensurable with the square on E [Prop. 10.11]. Thus, A is incommensurable in square with E .

Thus, two straight-lines, D and E , (which are) incommensurable with the given straight-line A , have been found, the one, D , (incommensurable) in length only, the other, E , (incommensurable) in square, and, clearly, also in length. [(Which is) the very thing it was required to show.]

† This whole proposition is regarded by Heiberg as an interpolation into the original text.

ια'.

Ἐάν τέσσαρα μεγέθη ἀνάλογον ᾿ῃ, τὸ δὲ πρῶτον τῷ δευτέρῳ σύμμετρον ᾿ῃ, καὶ τὸ τρίτον τῷ τετάρτῳ σύμμετρον ᾿ῃ, καὶ τὸ πρῶτον τῷ δευτέρῳ ἀσύμμετρον ᾿ῃ, καὶ τὸ τρίτον τῷ τετάρτῳ ἀσύμμετρον ᾿ῃ.



Ἐστωσαν τέσσαρα μεγέθη ἀνάλογον τὰ A, B, Γ, Δ , ὡς τὸ A πρὸς τὸ B , οὕτως τὸ Γ πρὸς τὸ Δ , τὸ A δὲ τῷ B σύμμετρον ᾿ῃ, λέγω, ὅτι καὶ τὸ Γ τῷ Δ σύμμετρον ᾿ῃ.

Ἐπεὶ γὰρ σύμμετρον ᾿ῃ τὸ A τῷ B , τὸ A ἄρα πρὸς τὸ B λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. καὶ ἔστιν ὡς τὸ A πρὸς τὸ B , οὕτως τὸ Γ πρὸς τὸ Δ . καὶ τὸ Γ ἄρα πρὸς τὸ Δ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. σύμμετρον ἄρα ᾿ῃ τὸ Γ τῷ Δ .

Ἀλλὰ δὴ τὸ A τῷ B ἀσύμμετρον ᾿ῃ, λέγω, ὅτι καὶ τὸ Γ τῷ Δ ἀσύμμετρον ᾿ῃ. ἐπεὶ γὰρ ἀσύμμετρον ᾿ῃ τὸ A τῷ B , τὸ A ἄρα πρὸς τὸ B λόγον οὐκ ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. καὶ ἔστιν ὡς τὸ A πρὸς τὸ B , οὕτως τὸ Γ πρὸς τὸ Δ . οὐδὲ τὸ Γ ἄρα πρὸς τὸ Δ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. ἀσύμμετρον ἄρα ᾿ῃ τὸ Γ τῷ Δ .

Ἐάν ἄρα τέσσαρα μεγέθη, καὶ τὰ ἐξῆς.

ιβ'.

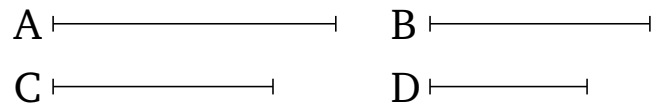
Τὰ τῷ αὐτῷ μεγέθει σύμμετρα καὶ ἀλλήλοις ᾿ῃ σύμμετρα.

Ἐκάτερον γὰρ τῶν A, B τῷ Γ ᾿ῃ, λέγω, ὅτι καὶ τὸ A τῷ B ᾿ῃ.

Ἐπεὶ γὰρ σύμμετρον ᾿ῃ τὸ A τῷ Γ , τὸ A ἄρα πρὸς τὸ Γ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. ἐχέτω, ὃν ὁ Δ πρὸς τὸν E . πάλιν, ἐπεὶ σύμμετρον ᾿ῃ τὸ Γ τῷ B , τὸ Γ ἄρα πρὸς τὸ B λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν. ἐχέτω, ὃν ὁ Z πρὸς τὸν H . καὶ λόγων δοθέντων ὁποσωνοῦν τοῦ τε, ὃν ἔχει ὁ Δ πρὸς τὸν E , καὶ ὁ Z πρὸς τὸν H εἰλήφθωσαν ἀριθμοὶ ἐξῆς ἐν τοῖς δοθεῖσι λόγοις οἱ Θ, K, Λ . ὥστε εἶναι

Proposition 11

If four magnitudes are proportional, and the first is commensurable with the second, then the third will also be commensurable with the fourth. And if the first is incommensurable with the second, then the third will also be incommensurable with the fourth.



Let A, B, C, D be four proportional magnitudes, (such that) as A (is) to B , so C (is) to D . And let A be commensurable with B . I say that C will also be commensurable with D .

For since A is commensurable with B , A thus has to B the ratio which (some) number (has) to (some) number [Prop. 10.5]. And as A is to B , so C (is) to D . Thus, C also has to D the ratio which (some) number (has) to (some) number. Thus, C is commensurable with D [Prop. 10.6].

And so let A be incommensurable with B . I say that C will also be incommensurable with D . For since A is incommensurable with B , A thus does not have to B the ratio which (some) number (has) to (some) number [Prop. 10.7]. And as A is to B , so C (is) to D . Thus, C does not have to D the ratio which (some) number (has) to (some) number either. Thus, C is incommensurable with D [Prop. 10.8].

Thus, if four magnitudes, and so on . . .

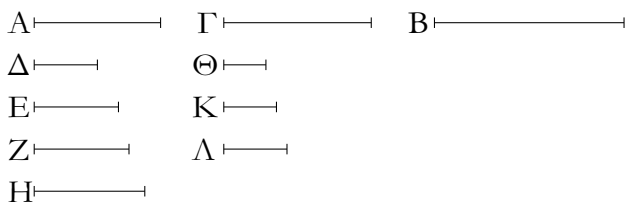
Proposition 12

(Magnitudes) commensurable with the same magnitude are also commensurable with one another.

For let A and B each be commensurable with C . I say that A is also commensurable with B .

For since A is commensurable with C , A thus has to C the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (the ratio) which D (has) to E . Again, since C is commensurable with B , C thus has to B the ratio which (some) number (has) to (some) number [Prop. 10.5]. Let it have (the ratio) which F (has) to G . And for any multitude whatsoever

ὥς μὲν τὸν Δ πρὸς τὸν Ε, οὕτως τὸν Θ πρὸς τὸν Κ, ὥς δὲ τὸν Ζ πρὸς τὸν Η, οὕτως τὸν Κ πρὸς τὸν Λ.

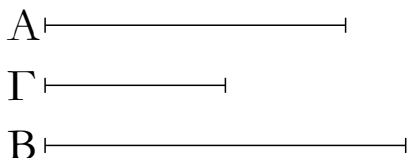


Ἐπεὶ οὖν ἐστὶν ὥς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Δ πρὸς τὸν Ε, ἀλλ' ὥς ὁ Δ πρὸς τὸν Ε, οὕτως ὁ Θ πρὸς τὸν Κ, ἔστιν ἄρα καὶ ὥς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Θ πρὸς τὸν Κ. πάλιν, ἐπεὶ ἐστὶν ὥς τὸ Γ πρὸς τὸ Β, οὕτως ὁ Ζ πρὸς τὸν Η, ἀλλ' ὥς ὁ Ζ πρὸς τὸν Η, [οὕτως] ὁ Κ πρὸς τὸν Λ, καὶ ὥς ἄρα τὸ Γ πρὸς τὸ Β, οὕτως ὁ Κ πρὸς τὸν Λ. ἔστι δὲ καὶ ὥς τὸ Α πρὸς τὸ Γ, οὕτως ὁ Θ πρὸς τὸν Κ· δι' ἴσου ἄρα ἐστὶν ὥς τὸ Α πρὸς τὸ Β, οὕτως ὁ Θ πρὸς τὸν Λ. τὸ Α ἄρα πρὸς τὸ Β λόγον ἔχει, ὃν ἀριθμὸς ὁ Θ πρὸς ἀριθμὸν τὸν Λ· σύμμετρον ἄρα ἐστὶ τὸ Α τῷ Β.

Τὰ ἄρα τῷ αὐτῷ μεγέθει σύμμετρα καὶ ἀλλήλοις ἐστὶ σύμμετρα· ὅπερ ἔδει δεῖξαι.

ιγ'.

Ἐὰν ἡ δύο μεγέθη σύμμετρα, τὸ δὲ ἕτερον αὐτῶν μεγέθει τινὶ ἀσύμμετρον ᾗ, καὶ τὸ λοιπὸν τῷ αὐτῷ ἀσύμμετρον ἔσται.



Ἐστω δύο μεγέθη σύμμετρα τὰ Α, Β, τὸ δὲ ἕτερον αὐτῶν τὸ Α ἄλλῳ τινὶ τῷ Γ ἀσύμμετρον ἔστω· λέγω, ὅτι καὶ τὸ λοιπὸν τὸ Β τῷ Γ ἀσύμμετρον ἐστίν.

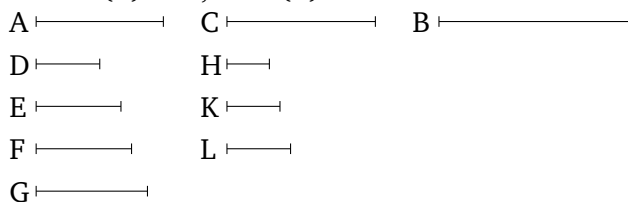
Εἰ γάρ ἐστι σύμμετρον τὸ Β τῷ Γ, ἀλλὰ καὶ τὸ Α τῷ Β σύμμετρον ἐστίν, καὶ τὸ Α ἄρα τῷ Γ σύμμετρον ἐστίν. ἀλλὰ καὶ ἀσύμμετρον· ὅπερ ἀδύνατον. οὐκ ἄρα σύμμετρον ἐστὶ τὸ Β τῷ Γ· ἀσύμμετρον ἄρα.

Ἐὰν ἄρα ἡ δύο μεγέθη σύμμετρα, καὶ τὰ ἐξῆς.

Λήμμα.

Δύο δοθεισῶν εὐθειῶν ἀνίσων εὑρεῖν, τίνι μείζον δύναται ἡ μείζων τῆς ἐλάσσονος.

of given ratios—(namely,) those which D has to E , and F to G —let the numbers H , K , L (which are) continuously (proportional) in the(se) given ratios have been taken [Prop. 8.4]. Hence, as D is to E , so H (is) to K , and as F (is) to G , so K (is) to L .

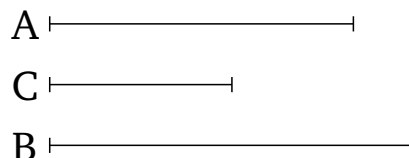


Therefore, since as A is to C , so D (is) to E , but as D (is) to E , so H (is) to K , thus also as A is to C , so H (is) to K [Prop. 5.11]. Again, since as C is to B , so F (is) to G , but as F (is) to G , [so] K (is) to L , thus also as C (is) to B , so K (is) to L [Prop. 5.11]. And also as A is to C , so H (is) to K . Thus, via equality, as A is to B , so H (is) to L [Prop. 5.22]. Thus, A has to B the ratio which the number H (has) to the number L . Thus, A is commensurable with B [Prop. 10.6].

Thus, (magnitudes) commensurable with the same magnitude are also commensurable with one another. (Which is) the very thing it was required to show.

Proposition 13

If two magnitudes are commensurable, and one of them is incommensurable with some magnitude, then the remaining (magnitude) will also be incommensurable with it.



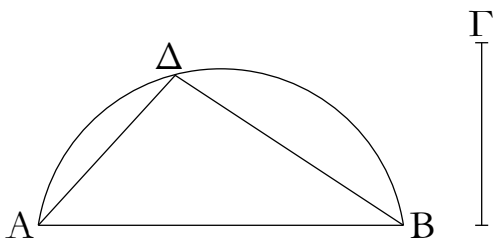
Let A and B be two commensurable magnitudes, and let one of them, A , be incommensurable with some other (magnitude), C . I say that the remaining (magnitude), B , is also incommensurable with C .

For if B is commensurable with C , but A is also commensurable with B , A is thus also commensurable with C [Prop. 10.12]. But, (it is) also incommensurable (with C). The very thing (is) impossible. Thus, B is not commensurable with C . Thus, (it is) incommensurable.

Thus, if two magnitudes are commensurable, and so on . . .

Lemma

For two given unequal straight-lines, to find by (the square on) which (straight-line) the square on the greater



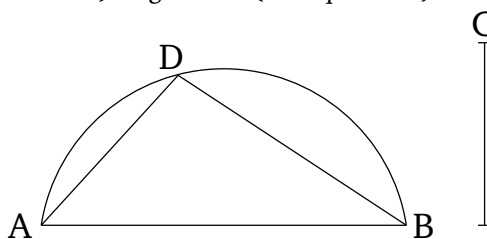
Ἐστωσαν αἱ δοθεῖσαι δύο ἄνιστοι εὐθεῖαι αἱ AB , Γ , ὧν μείζων ἔστω ἡ AB · δεῖ δὴ εὑρεῖν, τίνι μείζον δύναται ἡ AB τῆς Γ .

Γεγράφθω ἐπὶ τῆς AB ἡμικύκλιον τὸ ADB , καὶ εἰς αὐτὸ ἐνηρμόσθω τῇ Γ ἴση ἡ $A\Delta$, καὶ ἐπεζεύχθω ἡ ΔB . φανερόν δὴ, ὅτι ὀρθὴ ἐστὶν ἡ ὑπὸ $A\Delta B$ γωνία, καὶ ὅτι ἡ AB τῆς $A\Delta$, τουτέστι τῆς Γ , μείζον δύναται τῇ ΔB .

Ὅμοίως δὲ καὶ δύο δοθεισῶν εὐθειῶν ἡ δυναμένη αὐτὰς εὐρίσκεται οὕτως.

Ἐστωσαν αἱ δοθεῖσαι δύο εὐθεῖαι αἱ $A\Delta$, ΔB , καὶ δέον ἔστω εὑρεῖν τὴν δυναμένην αὐτάς. κείσθωσαν γάρ, ὥστε ὀρθὴν γωνίαν περιέχειν τὴν ὑπὸ $A\Delta$, ΔB , καὶ ἐπεζεύχθω ἡ AB · φανερόν πάλιν, ὅτι ἡ τὰς $A\Delta$, ΔB δυναμένη ἐστὶν ἡ AB · ὅπερ ἔδει δεῖξαι.

(straight-line is) larger than (the square on) the lesser.[†]



Let AB and C be the two given unequal straight-lines, and let AB be the greater of them. So it is required to find by (the square on) which (straight-line) the square on AB (is) greater than (the square on) C .

Let the semi-circle ADB have been described on AB . And let AD , equal to C , have been inserted into it [Prop. 4.1]. And let DB have been joined. So (it is) clear that the angle ADB is a right-angle [Prop. 3.31], and that the square on AB (is) greater than (the square on) AD —that is to say, (the square on) C —by (the square on) DB [Prop. 1.47].

And, similarly, the square-root of (the sum of the squares on) two given straight-lines is also found likewise.

Let AD and DB be the two given straight-lines. And let it be necessary to find the square-root of (the sum of the squares on) them. For let them have been laid down such as to encompass a right-angle—(namely), that (angle encompassed) by AD and DB . And let AB have been joined. (It is) again clear that AB is the square-root of (the sum of the squares on) AD and DB [Prop. 1.47]. (Which is) the very thing it was required to show.

[†] That is, if α and β are the lengths of two given straight-lines, with α being greater than β , to find a straight-line of length γ such that $\alpha^2 = \beta^2 + \gamma^2$. Similarly, we can also find γ such that $\gamma^2 = \alpha^2 + \beta^2$.

ιδ'.

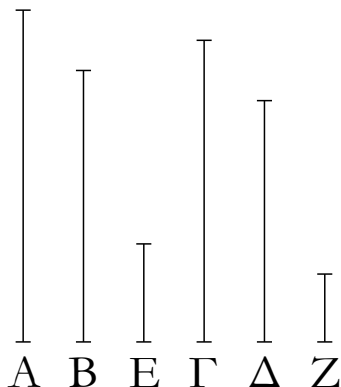
Proposition 14

Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ᾧσιν, δύνηται δὲ ἡ πρώτη τῆς δευτέρας μείζον τῷ ἀπὸ συμμετρου ἑαυτῇ [μήκει], καὶ ἡ τρίτη τῆς τετάρτης μείζον δυνήσεται τῷ ἀπὸ συμμετρου ἑαυτῇ [μήκει]. καὶ ἐὰν ἡ πρώτη τῆς δευτέρας μείζον δύνηται τῷ ἀπὸ ἀσυμμετρου ἑαυτῇ [μήκει], καὶ ἡ τρίτη τῆς τετάρτης μείζον δυνήσεται τῷ ἀπὸ ἀσυμμετρου ἑαυτῇ [μήκει].

Ἐστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ A , B , Γ , Δ , ὥς ἡ A πρὸς τὴν B , οὕτως ἡ Γ πρὸς τὴν Δ , καὶ ἡ A μὲν τῆς B μείζον δυνάσθω τῷ ἀπὸ τῆς E , ἡ δὲ Γ τῆς Δ μείζον δυνάσθω τῷ ἀπὸ τῆς Z · λέγω, ὅτι, εἴτε σύμμετρός ἐστὶν ἡ A τῇ E , σύμμετρός ἐστι καὶ ἡ Γ τῇ Z , εἴτε ἀσύμμετρός ἐστὶν ἡ A τῇ E , ἀσύμμετρός ἐστι καὶ ὁ Γ τῇ Z .

If four straight-lines are proportional, and the square on the first is greater than (the square on) the second by the (square) on (some straight-line) commensurable [in length] with the first, then the square on the third will also be greater than (the square on) the fourth by the (square) on (some straight-line) commensurable [in length] with the third. And if the square on the first is greater than (the square on) the second by the (square) on (some straight-line) incommensurable [in length] with the first, then the square on the third will also be greater than (the square on) the fourth by the (square) on (some straight-line) incommensurable [in length] with the third.

Let A , B , C , D be four proportional straight-lines, (such that) as A (is) to B , so C (is) to D . And let the square on A be greater than (the square on) B by the



Ἐπεὶ γὰρ ἐστὶν ὡς ἡ A πρὸς τὴν B , οὕτως ἡ Γ πρὸς τὴν Δ , ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς B , οὕτως τὸ ἀπὸ τῆς Γ πρὸς τὸ ἀπὸ τῆς Δ . ἀλλὰ τῷ μὲν ἀπὸ τῆς A ἴσα ἐστὶ τὰ ἀπὸ τῶν E, B , τῷ δὲ ἀπὸ τῆς Γ ἴσα ἐστὶ τὰ ἀπὸ τῶν Δ, Z . ἔστιν ἄρα ὡς τὰ ἀπὸ τῶν E, B πρὸς τὸ ἀπὸ τῆς B , οὕτως τὰ ἀπὸ τῶν Δ, Z πρὸς τὸ ἀπὸ τῆς Δ . διελόντι ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς E πρὸς τὸ ἀπὸ τῆς B , οὕτως τὸ ἀπὸ τῆς Z πρὸς τὸ ἀπὸ τῆς Δ . ἔστιν ἄρα καὶ ὡς ἡ E πρὸς τὴν B , οὕτως ἡ Z πρὸς τὴν Δ . ἀνάπαλιν ἄρα ἐστὶν ὡς ἡ B πρὸς τὴν E , οὕτως ἡ Δ πρὸς τὴν Z . ἐστὶ δὲ καὶ ὡς ἡ A πρὸς τὴν B , οὕτως ἡ Γ πρὸς τὴν Δ . δι' ἴσου ἄρα ἐστὶν ὡς ἡ A πρὸς τὴν E , οὕτως ἡ Γ πρὸς τὴν Z . εἴτε οὖν σύμμετρός ἐστιν ἡ A τῇ E , σύμμετρός ἐστι καὶ ἡ Γ τῇ Z , εἴτε ἀσύμμετρός ἐστιν ἡ A τῇ E , ἀσύμμετρός ἐστι καὶ ἡ Γ τῇ Z .

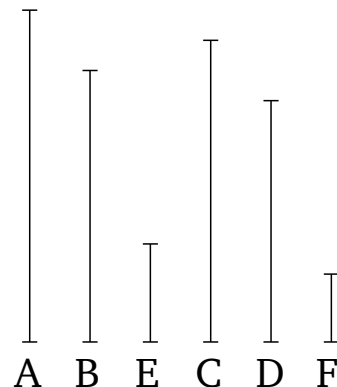
Ἐὰν ἄρα, καὶ τὰ ἐξῆς.

ιε'.

Ἐὰν δύο μεγέθη σύμμετρα συντεθῇ, καὶ τὸ ὅλον ἐκατέρῳ αὐτῶν σύμμετρον ἔσται· καὶ τὸ ὅλον ἐνὶ αὐτῶν σύμμετρον ἦ, καὶ τὰ ἐξ ἀρχῆς μεγέθη σύμμετρα ἔσται.

Συγκρίσθω γὰρ δύο μεγέθη σύμμετρα τὰ AB, BG · λέγω, ὅτι καὶ ὅλον τὸ AG ἐκατέρῳ τῶν AB, BG ἐστὶ σύμμετρον.

(square) on E , and let the square on C be greater than (the square on) D by the (square) on F . I say that A is either commensurable (in length) with E , and C is also commensurable with F , or A is incommensurable (in length) with E , and C is also incommensurable with F .



For since as A is to B , so C (is) to D , thus as the (square) on A is to the (square) on B , so the (square) on C (is) to the (square) on D [Prop. 6.22]. But the (sum of the squares) on E and B is equal to the (square) on A , and the (sum of the squares) on D and F is equal to the (square) on C . Thus, as the (sum of the squares) on E and B is to the (square) on B , so the (sum of the squares) on D and F (is) to the (square) on D . Thus, via separation, as the (square) on E is to the (square) on B , so the (square) on F (is) to the (square) on D [Prop. 5.17]. Thus, also, as E is to B , so F (is) to D [Prop. 6.22]. Thus, inversely, as B is to E , so D (is) to F [Prop. 5.7 corr.]. But, as A is to B , so C also (is) to D . Thus, via equality, as A is to E , so C (is) to F [Prop. 5.22]. Therefore, A is either commensurable (in length) with E , and C is also commensurable with F , or A is incommensurable (in length) with E , and C is also incommensurable with F [Prop. 10.11].

Thus, if, and so on . . .

Proposition 15

If two commensurable magnitudes are added together then the whole will also be commensurable with each of them. And if the whole is commensurable with one of them then the original magnitudes will also be commensurable (with one another).

For let the two commensurable magnitudes AB and BC be laid down together. I say that the whole AC is also commensurable with each of AB and BC .



Ἐπεὶ γὰρ σύμμετρά ἐστι τὰ AB, BG, μετρήσει τι αὐτὰ μέγεθος. μετρεῖτω, καὶ ἔστω τὸ Δ. ἐπεὶ οὖν τὸ Δ τὰ AB, BG μετρεῖ, καὶ ὅλον τὸ AG μετρήσει. μετρεῖ δὲ καὶ τὰ AB, BG. τὸ Δ ἄρα τὰ AB, BG, AG μετρεῖ· σύμμετρον ἄρα ἐστὶ τὸ AG ἑκατέρω τῶν AB, BG.

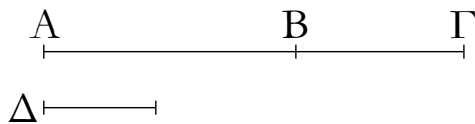
Ἀλλὰ δὴ τὸ AG ἔστω σύμμετρον τῷ AB· λέγω δὴ, ὅτι καὶ τὰ AB, BG σύμμετρά ἐστιν.

Ἐπεὶ γὰρ σύμμετρά ἐστι τὰ AG, AB, μετρήσει τι αὐτὰ μέγεθος. μετρεῖτω, καὶ ἔστω τὸ Δ. ἐπεὶ οὖν τὸ Δ τὰ GA, AB μετρεῖ, καὶ λοιπὸν ἄρα τὸ BG μετρήσει. μετρεῖ δὲ καὶ τὸ AB· τὸ Δ ἄρα τὰ AB, BG μετρήσει· σύμμετρα ἄρα ἐστὶ τὰ AB, BG.

Ἐὰν ἄρα δύο μεγέθη, καὶ τὰ ἐξῆς.

ιϛ'.

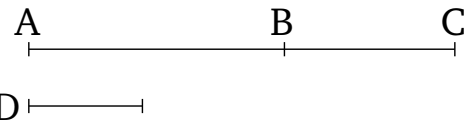
Ἐὰν δύο μεγέθη ἀσύμμετρα συντεθῇ, καὶ τὸ ὅλον ἑκατέρω αὐτῶν ἀσύμμετρον ἔσται· καὶ τὸ ὅλον ἐνὶ αὐτῶν ἀσύμμετρον ἦ, καὶ τὰ ἐξ ἀρχῆς μεγέθη ἀσύμμετρα ἔσται.



Συγκείσθω γὰρ δύο μεγέθη ἀσύμμετρα τὰ AB, BG· λέγω, ὅτι καὶ ὅλον τὸ AG ἑκατέρω τῶν AB, BG ἀσύμμετρόν ἐστιν.

Εἰ γὰρ μὴ ἐστὶν ἀσύμμετρα τὰ GA, AB, μετρήσει τι [αὐτὰ] μέγεθος. μετρεῖτω, εἰ δυνατόν, καὶ ἔστω τὸ Δ. ἐπεὶ οὖν τὸ Δ τὰ GA, AB μετρεῖ, καὶ λοιπὸν ἄρα τὸ BG μετρήσει. μετρεῖ δὲ καὶ τὸ AB· τὸ Δ ἄρα τὰ AB, BG μετρεῖ. σύμμετρα ἄρα ἐστὶ τὰ AB, BG· ὑπέκλειντο δὲ καὶ ἀσύμμετρα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ GA, AB μετρήσει τι μέγεθος· ἀσύμμετρα ἄρα ἐστὶ τὰ GA, AB. ὁμοίως δὴ δεῖξομεν, ὅτι καὶ τὰ AG, GB ἀσύμμετρά ἐστιν. τὸ AG ἄρα ἑκατέρω τῶν AB, BG ἀσύμμετρόν ἐστιν.

Ἀλλὰ δὴ τὸ AG ἐνὶ τῶν AB, BG ἀσύμμετρον ἔστω. ἔστω δὴ πρότερον τῷ AB· λέγω, ὅτι καὶ τὰ AB, BG ἀσύμμετρά ἐστιν. εἰ γὰρ ἔσται σύμμετρα, μετρήσει τι αὐτὰ μέγεθος. μετρεῖτω, καὶ ἔστω τὸ Δ. ἐπεὶ οὖν τὸ Δ τὰ AB, BG μετρεῖ, καὶ ὅλον ἄρα τὸ AG μετρήσει. μετρεῖ δὲ καὶ τὸ AB· τὸ Δ ἄρα τὰ GA, AB μετρεῖ. σύμμετρα ἄρα ἐστὶ τὰ



For since AB and BC are commensurable, some magnitude will measure them. Let it (so) measure (them), and let it be D . Therefore, since D measures (both) AB and BC , it will also measure the whole AC . And it also measures AB and BC . Thus, D measures AB , BC , and AC . Thus, AC is commensurable with each of AB and BC [Def. 10.1].

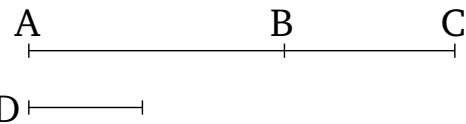
And so let AC be commensurable with AB . I say that AB and BC are also commensurable.

For since AC and AB are commensurable, some magnitude will measure them. Let it (so) measure (them), and let it be D . Therefore, since D measures (both) CA and AB , it will thus also measure the remainder BC . And it also measures AB . Thus, D will measure (both) AB and BC . Thus, AB and BC are commensurable [Def. 10.1].

Thus, if two magnitudes, and so on . . .

Proposition 16

If two incommensurable magnitudes are added together then the whole will also be incommensurable with each of them. And if the whole is incommensurable with one of them then the original magnitudes will also be incommensurable (with one another).



For let the two incommensurable magnitudes AB and BC be laid down together. I say that that the whole AC is also incommensurable with each of AB and BC .

For if CA and AB are not incommensurable then some magnitude will measure [them]. If possible, let it (so) measure (them), and let it be D . Therefore, since D measures (both) CA and AB , it will thus also measure the remainder BC . And it also measures AB . Thus, D measures (both) AB and BC . Thus, AB and BC are commensurable [Def. 10.1]. But they were also assumed (to be) incommensurable. The very thing is impossible. Thus, some magnitude cannot measure (both) CA and AB . Thus, CA and AB are incommensurable [Def. 10.1]. So, similarly, we can show that AC and CB are also incommensurable. Thus, AC is incommensurable with each of AB and BC .

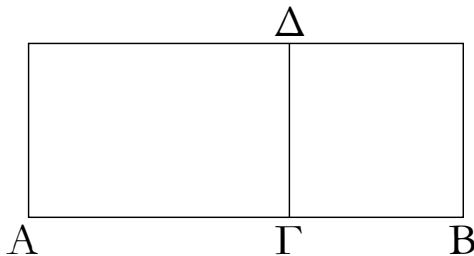
And so let AC be incommensurable with one of AB and BC . So let it, first of all, be incommensurable with

ΓΑ, ΑΒ· ὑπέκειτο δὲ καὶ ἀσύμμετρα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ ΑΒ, ΒΓ μετρήσει τι μέγεθος· ἀσύμμετρα ἄρα ἐστὶ τὰ ΑΒ, ΒΓ.

Ἐάν ἄρα δύο μεγέθη, καὶ τὰ ἐξῆς.

Λήμμα.

Ἐάν παρὰ τινὰ εὐθεΐαν παραβληθῇ παραλληλόγραμμον ἐλλείπον εἶδει τετραγώνω, τὸ παραβληθὲν ἴσον ἐστὶ τῷ ὑπὸ τῶν ἐκ τῆς παραβολῆς γενομένων τμημάτων τῆς εὐθείας.



Παρά γάρ εὐθεΐαν τὴν ΑΒ παραβεβλήσθω παραλληλόγραμμον τὸ ΑΔ ἐλλείπον εἶδει τετραγώνω τῷ ΔΒ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΔ τῷ ὑπὸ τῶν ΑΓ, ΓΒ.

Καὶ ἐστὶν αὐτόθεν φανερόν· ἐπεὶ γάρ τετραγώνον ἐστὶ τὸ ΔΒ, ἴση ἐστὶν ἡ ΔΓ τῇ ΓΒ, καὶ ἐστὶ τὸ ΑΔ τὸ ὑπὸ τῶν ΑΓ, ΓΔ, τοῦτέστι τὸ ὑπὸ τῶν ΑΓ, ΓΒ.

Ἐάν ἄρα παρὰ τινὰ εὐθεΐαν, καὶ τὰ ἐξῆς.

† Note that this lemma only applies to rectangular parallelograms.

ιζ'.

Ἐάν ᾧσι δύο εὐθεΐαι ἄνισοι, τῷ δὲ τετράτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἐλλείπον εἶδει τετραγώνω καὶ εἰς σύμμετρα αὐτὴν διαιρῇ μήκει, ἡ μείζων τῆς ἐλάσσονος μείζον δυνήσεται τῷ ἀπὸ συμμετρου ἑαυτῇ [μήκει]. καὶ ἐὰν ἡ μείζων τῆς ἐλάσσονος μείζον δύνηται τῷ ἀπὸ συμμετρου ἑαυτῇ [μήκει], τῷ δὲ τετράτῳ τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἐλλείπον εἶδει τετραγώνω, εἰς σύμμετρα αὐτὴν διαιρεῖ μήκει.

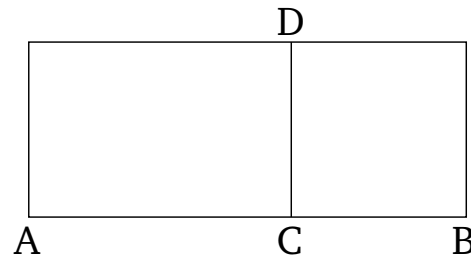
Ἐστωσαν δύο εὐθεΐαι ἄνισοι αἱ Α, ΒΓ, ὧν μείζων ἡ

ΑΒ. I say that ΑΒ and ΒΓ are also incommensurable. For if they are commensurable then some magnitude will measure them. Let it (so) measure (them), and let it be D. Therefore, since D measures (both) ΑΒ and ΒΓ, it will thus also measure the whole ΑΓ. And it also measures ΑΒ. Thus, D measures (both) ΓΑ and ΑΒ. Thus, ΓΑ and ΑΒ are commensurable [Def. 10.1]. But they were also assumed (to be) incommensurable. The very thing is impossible. Thus, some magnitude cannot measure (both) ΑΒ and ΒΓ. Thus, ΑΒ and ΒΓ are incommensurable [Def. 10.1].

Thus, if two... magnitudes, and so on . . .

Lemma

If a parallelogram,[†] falling short by a square figure, is applied to some straight-line then the applied (parallelogram) is equal (in area) to the (rectangle contained) by the pieces of the straight-line created via the application (of the parallelogram).



For let the parallelogram ΑΔ, falling short by the square figure ΔΒ, have been applied to the straight-line ΑΒ. I say that ΑΔ is equal to the (rectangle contained) by ΑΓ and ΓΒ.

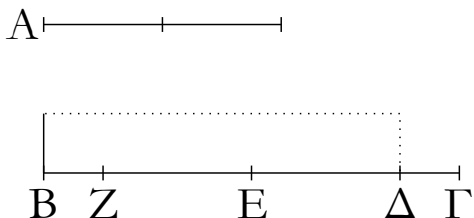
And it is immediately obvious. For since ΔΒ is a square, ΔC is equal to ΓΒ. And ΑΔ is the (rectangle contained) by ΑΓ and CD—that is to say, by ΑΓ and ΓΒ.

Thus, if . . . to some straight-line, and so on . . .

Proposition 17[†]

If there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) commensurable in length, then the square on the greater will be larger than (the square on) the lesser by the (square) on (some straight-line) commensurable [in length] with the greater. And if the square on the greater is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable [in length] with the

ΒΓ, τῷ δὲ τετράρτῳ μέρει τοῦ ἀπὸ ἐλάσσονος τῆς Α, τουτέστι τῷ ἀπὸ τῆς ἡμισείας τῆς Α, ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἐλλείπον εἶδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔ, ΔΓ, σύμμετρος δὲ ἔστω ἡ ΒΔ τῇ ΔΓ μήκει· λέγω, ὅτι ἡ ΒΓ τῆς Α μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ.



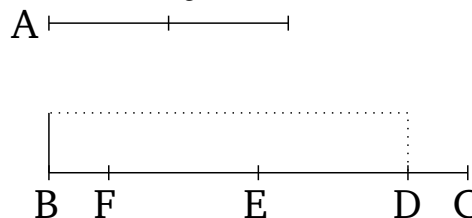
Τετμήσθω γάρ ἡ ΒΓ δίχα κατὰ τὸ Ε σημεῖον, καὶ κείσθω τῇ ΔΕ ἴση ἡ ΕΖ. λοιπὴ ἄρα ἡ ΔΓ ἴση ἐστὶ τῇ ΒΖ. καὶ ἐπεὶ εὐθεῖα ἡ ΒΓ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Ε, εἰς δὲ ἄνισα κατὰ τὸ Δ, τὸ ἄρα ὑπὸ ΒΔ, ΔΓ περιχόμενον ὀρθογώνιον μετὰ τοῦ ἀπὸ τῆς ΕΔ τετραγώνου ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΓ τετραγώνῳ· καὶ τὰ τετραπλάσια· τὸ ἄρα τετράκις ὑπὸ τῶν ΒΔ, ΔΓ μετὰ τοῦ τετραπλασίου τοῦ ἀπὸ τῆς ΔΕ ἴσον ἐστὶ τῷ τετράκις ἀπὸ τῆς ΕΓ τετραγώνῳ. ἀλλὰ τῷ μὲν τετραπλασίῳ τοῦ ὑπὸ τῶν ΒΔ, ΔΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς Α τετράγωνον, τῷ δὲ τετραπλασίῳ τοῦ ἀπὸ τῆς ΔΕ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΔΖ τετράγωνον· διπλασίων γάρ ἐστιν ἡ ΔΖ τῆς ΔΕ. τῷ δὲ τετραπλασίῳ τοῦ ἀπὸ τῆς ΕΓ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΒΓ τετράγωνον· διπλασίων γάρ ἐστι πάλιν ἡ ΒΓ τῆς ΓΕ. τὰ ἄρα ἀπὸ τῶν Α, ΔΖ τετράγωνα ἴσα ἐστὶ τῷ ἀπὸ τῆς ΒΓ τετράγωνῳ· ὥστε τὸ ἀπὸ τῆς ΒΓ τοῦ ἀπὸ τῆς Α μείζον ἐστὶ τῷ ἀπὸ τῆς ΔΖ· ἡ ΒΓ ἄρα τῆς Α μείζον δύναται τῇ ΔΖ. δεικτέον, ὅτι καὶ σύμμετρός ἐστιν ἡ ΒΓ τῇ ΔΖ. ἐπεὶ γὰρ σύμμετρός ἐστιν ἡ ΒΔ τῇ ΔΓ μήκει, σύμμετρος ἄρα ἐστὶ καὶ ἡ ΒΓ τῇ ΓΔ μήκει. ἀλλὰ ἡ ΓΔ ταῖς ΓΔ, ΒΖ ἐστὶ σύμμετρος μήκει· ἴση γάρ ἐστιν ἡ ΓΔ τῇ ΒΖ. καὶ ἡ ΒΓ ἄρα σύμμετρός ἐστι ταῖς ΒΖ, ΓΔ μήκει· ὥστε καὶ λοιπῇ τῇ ΖΔ σύμμετρός ἐστιν ἡ ΒΓ μήκει· ἡ ΒΓ ἄρα τῆς Α μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ.

Ἀλλὰ δὴ ἡ ΒΓ τῆς Α μείζον δυνάσθω τῷ ἀπὸ συμμέτρου ἑαυτῇ, τῷ δὲ τετράρτῳ τοῦ ἀπὸ τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἐλλείπον εἶδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔ, ΔΓ. δεικτέον, ὅτι σύμμετρός ἐστιν ἡ ΒΔ τῇ ΔΓ μήκει.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δεῖξομεν, ὅτι ἡ ΒΓ τῆς Α μείζον δύναται τῷ ἀπὸ τῆς ΖΔ. δύναται δὲ ἡ

greater, and a (rectangle) equal to the fourth (part) of the (square) on the lesser, falling short by a square figure, is applied to the greater, then it divides it into (parts which are) commensurable in length.

Let A and BC be two unequal straight-lines, of which (let) BC (be) the greater. And let a (rectangle) equal to the fourth part of the (square) on the lesser, A —that is, (equal) to the (square) on half of A —falling short by a square figure, have been applied to BC . And let it be the (rectangle contained) by BD and DC [see previous lemma]. And let BD be commensurable in length with DC . I say that that the square on BC is greater than the (square on) A by (the square on some straight-line) commensurable (in length) with (BC) .



For let BC have been cut in half at the point E [Prop. 1.10]. And let EF be made equal to DE [Prop. 1.3]. Thus, the remainder DC is equal to BF . And since the straight-line BC has been cut into equal (pieces) at E , and into unequal (pieces) at D , the rectangle contained by BD and DC , plus the square on ED , is thus equal to the square on EC [Prop. 2.5]. (The same) also (for) the quadruples. Thus, four times the (rectangle contained) by BD and DC , plus the quadruple of the (square) on DE , is equal to four times the square on EC . But, the square on A is equal to the quadruple of the (rectangle contained) by BD and DC , and the square on DF is equal to the quadruple of the (square) on DE . For DF is double DE . And the square on BC is equal to the quadruple of the (square) on EC . For, again, BC is double CE . Thus, the (sum of the) squares on A and DF is equal to the square on BC . Hence, the (square) on BC is greater than the (square) on A by the (square) on DF . Thus, BC is greater in square than A by DF . It must also be shown that BC is commensurable (in length) with DF . For since BD is commensurable in length with DC , BC is thus also commensurable in length with CD [Prop. 10.15]. But, CD is commensurable in length with CD plus BF . For CD is equal to BF [Prop. 10.6]. Thus, BC is also commensurable in length with BF plus CD [Prop. 10.12]. Hence, BC is also commensurable in length with the remainder FD [Prop. 10.15]. Thus, the square on BC is greater than (the square on) A by the (square) on (some straight-line) commensurable (in length) with (BC) .

ΒΓ τῆς Α μείζον τῷ ἀπὸ συμμετρου ἑαυτῇ. σύμμετρος ἄρα ἐστὶν ἡ ΒΓ τῇ ΖΔ μήκει· ὥστε καὶ λοιπῇ συναμφοτέρῳ τῇ ΒΖ, ΔΓ σύμμετρός ἐστὶν ἡ ΒΓ μήκει. ἀλλὰ συναμφοτέρος ἡ ΒΖ, ΔΓ σύμμετρός ἐστι τῇ ΔΓ [μήκει]. ὥστε καὶ ἡ ΒΓ τῇ ΓΔ σύμμετρός ἐστι μήκει· καὶ διελόντι ἄρα ἡ ΒΔ τῇ ΔΓ ἐστι σύμμετρος μήκει.

Ἐάν ἄρα ὥσι δύο εὐθεῖαι ἄνισοι, καὶ τὰ ἐξῆς.

And so let the square on BC be greater than the (square on) A by the (square) on (some straight-line) commensurable (in length) with (BC) . And let a (rectangle) equal to the fourth (part) of the (square) on A , falling short by a square figure, have been applied to BC . And let it be the (rectangle contained) by BD and DC . It must be shown that BD is commensurable in length with DC .

For, similarly, by the same construction, we can show that the square on BC is greater than the (square on) A by the (square) on FD . And the square on BC is greater than the (square on) A by the (square) on (some straight-line) commensurable (in length) with (BC) . Thus, BC is commensurable in length with FD . Hence, BC is also commensurable in length with the remaining sum of BF and DC [Prop. 10.15]. But, the sum of BF and DC is commensurable [in length] with DC [Prop. 10.6]. Hence, BC is also commensurable in length with CD [Prop. 10.12]. Thus, via separation, BD is also commensurable in length with DC [Prop. 10.15].

Thus, if there are two unequal straight-lines, and so on . . .

† This proposition states that if $\alpha x - x^2 = \beta^2/4$ (where $\alpha = BC$, $x = DC$, and $\beta = A$) then α and $\sqrt{\alpha^2 - \beta^2}$ are commensurable when $\alpha - x$ are x are commensurable, and vice versa.

ιη'.

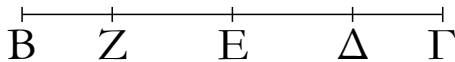
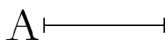
Proposition 18†

Ἐάν ὥσι δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἑλλεῖπον εἶδει τετραγώνῳ, καὶ εἰς ἀσύμμετρα αὐτὴν διαιρῇ [μήκει], ἡ μείζων τῆς ἐλάσσονος μείζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ ἐάν ἡ μείζων τῆς ἐλάσσονος μείζον δύνῃται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, τῷ δὲ τετάρτῳ τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἑλλεῖπον εἶδει τετραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διαιρεῖ [μήκει].

Ἐστῶσαν δύο εὐθεῖαι ἄνισοι αἱ Α, ΒΓ, ὧν μείζων ἡ ΒΓ, τῷ δὲ τετάρτῳ [μέρει] τοῦ ἀπὸ τῆς ἐλάσσονος τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἑλλεῖπον εἶδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔΓ, ἀσύμμετρος δὲ ἔστω ἡ ΒΔ τῇ ΔΓ μήκει· λέγω, ὅτι ἡ ΒΓ τῆς Α μείζον δύνανται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ.

If there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) incommensurable [in length], then the square on the greater will be larger than the (square on the) lesser by the (square) on (some straight-line) incommensurable (in length) with the greater. And if the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) incommensurable (in length) with the greater, and a (rectangle) equal to the fourth (part) of the (square) on the lesser, falling short by a square figure, is applied to the greater, then it divides it into (parts which are) incommensurable [in length].

Let A and BC be two unequal straight-lines, of which (let) BC (be) the greater. And let a (rectangle) equal to the fourth [part] of the (square) on the lesser, A , falling short by a square figure, have been applied to BC . And let it be the (rectangle contained) by BDC . And let BD be incommensurable in length with DC . I say that that the square on BC is greater than the (square on) A by the (square) on (some straight-line) incommensurable (in length) with (BC) .

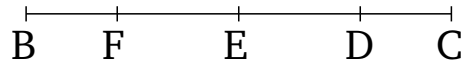
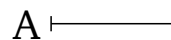


Τῶν γὰρ αὐτῶν κατασκευασθέντων τῷ πρότερον ὁμοίως δείξομεν, ὅτι ἡ ΒΓ τῆς Α μείζον δύναται τῷ ἀπὸ τῆς ΖΔ. δεικτέον [οὖν], ὅτι ἀσύμμετρός ἐστιν ἡ ΒΓ τῇ ΔΖ μήκει. ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ ΒΔ τῇ ΔΓ μήκει, ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΒΓ τῇ ΓΔ μήκει. ἀλλὰ ἡ ΔΓ σύμμετρός ἐστι συναμφοτέραις ταῖς ΒΖ, ΔΓ· καὶ ἡ ΒΓ ἄρα ἀσύμμετρός ἐστι συναμφοτέραις ταῖς ΒΖ, ΔΓ. ὥστε καὶ λοιπῇ τῇ ΖΔ ἀσύμμετρός ἐστιν ἡ ΒΓ μήκει. καὶ ἡ ΒΓ τῆς Α μείζον δύναται τῷ ἀπὸ τῆς ΖΔ· ἡ ΒΓ ἄρα τῆς Α μείζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ.

Δυνάσθω δὲ πάλιν ἡ ΒΓ τῆς Α μείζον τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, τῷ δὲ τετάρτῳ τοῦ ἀπὸ τῆς Α ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω ἐλλείπον εἶδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΒΔ, ΔΓ. δεικτέον, ὅτι ἀσύμμετρός ἐστιν ἡ ΒΔ τῇ ΔΓ μήκει.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ὁμοίως δείξομεν, ὅτι ἡ ΒΓ τῆς Α μείζον δύναται τῷ ἀπὸ τῆς ΖΔ. ἀλλὰ ἡ ΒΓ τῆς Α μείζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. ἀσύμμετρος ἄρα ἐστὶν ἡ ΒΓ τῇ ΖΔ μήκει· ὥστε καὶ λοιπῇ συναμφοτέρῳ τῇ ΒΖ, ΔΓ ἀσύμμετρός ἐστιν ἡ ΒΓ. ἀλλὰ συναμφοτέρος ἡ ΒΖ, ΔΓ τῇ ΔΓ σύμμετρός ἐστι μήκει· καὶ ἡ ΒΓ ἄρα τῇ ΔΓ ἀσύμμετρός ἐστι μήκει· ὥστε καὶ διελόντι ἡ ΒΔ τῇ ΔΓ ἀσύμμετρός ἐστι μήκει.

Ἐὰν ἄρα ὥσι δύο εὐθεῖαι, καὶ τὰ ἐξῆς.



For, similarly, by the same construction as before, we can show that the square on BC is greater than the (square on) A by the (square) on FD . [Therefore] it must be shown that BC is incommensurable in length with DF . For since BD is incommensurable in length with DC , BC is thus also incommensurable in length with CD [Prop. 10.16]. But, DC is commensurable (in length) with the sum of BF and DC [Prop. 10.6]. And, thus, BC is incommensurable (in length) with the sum of BF and DC [Prop. 10.13]. Hence, BC is also incommensurable in length with the remainder FD [Prop. 10.16]. And the square on BC is greater than the (square on) A by the (square) on FD . Thus, the square on BC is greater than the (square on) A by the (square) on (some straight-line) incommensurable (in length) with (BC) .

So, again, let the square on BC be greater than the (square on) A by the (square) on (some straight-line) incommensurable (in length) with (BC) . And let a (rectangle) equal to the fourth [part] of the (square) on A , falling short by a square figure, have been applied to BC . And let it be the (rectangle contained) by BD and DC . It must be shown that BD is incommensurable in length with DC .

For, similarly, by the same construction, we can show that the square on BC is greater than the (square) on A by the (square) on FD . But, the square on BC is greater than the (square) on A by the (square) on (some straight-line) incommensurable (in length) with (BC) . Thus, BC is incommensurable in length with FD . Hence, BC is also incommensurable (in length) with the remaining sum of BF and DC [Prop. 10.16]. But, the sum of BF and DC is commensurable in length with DC [Prop. 10.6]. Thus, BC is also incommensurable in length with DC [Prop. 10.13]. Hence, via separation, BD is also incommensurable in length with DC [Prop. 10.16].

Thus, if there are two . . . straight-lines, and so on . . .

† This proposition states that if $\alpha x - x^2 = \beta^2/4$ (where $\alpha = BC$, $x = DC$, and $\beta = A$) then α and $\sqrt{\alpha^2 - \beta^2}$ are incommensurable when $\alpha - x$ are incommensurable, and vice versa.

ιθ'.

Proposition 19

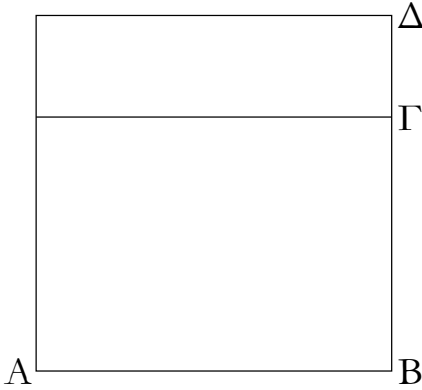
Τὸ ὑπὸ ῥητῶν μήκει συμμετρῶν εὐθειῶν περιεχόμενον ὀρθογώνιον ῥητόν ἐστιν.

Ἐὰν γὰρ ῥητῶν μήκει συμμετρῶν εὐθειῶν τῶν ΑΒ, ΒΓ

The rectangle contained by rational straight-lines (which are) commensurable in length is rational.

For let the rectangle AC have been enclosed by the

ὀρθογώνιον περιεχέσθω τὸ ΑΓ· λέγω, ὅτι ῥητόν ἐστι τὸ ΑΓ.

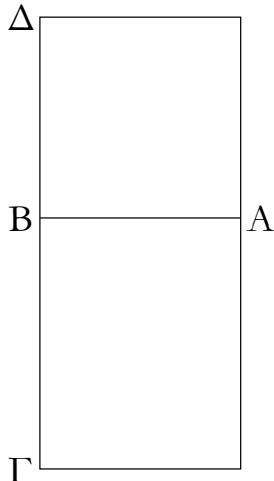


Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· ῥητόν ἄρα ἐστὶ τὸ ΑΔ. καὶ ἐπεὶ σύμμετρος ἐστὶν ἡ ΑΒ τῇ ΒΓ μήκει, ἴση δὲ ἐστὶν ἡ ΑΒ τῇ ΒΔ, σύμμετρος ἄρα ἐστὶν ἡ ΒΔ τῇ ΒΓ μήκει. καὶ ἐστὶν ὡς ἡ ΒΔ πρὸς τὴν ΒΓ, οὕτως τὸ ΔΑ πρὸς τὸ ΑΓ. σύμμετρον ἄρα ἐστὶ τὸ ΔΑ τῷ ΑΓ. ῥητόν δὲ τὸ ΔΑ· ῥητόν ἄρα ἐστὶ καὶ τὸ ΑΓ.

Τὸ ἄρα ὑπὸ ῥητῶν μήκει συμμέτρων, καὶ τὰ ἐξῆς.

κ'.

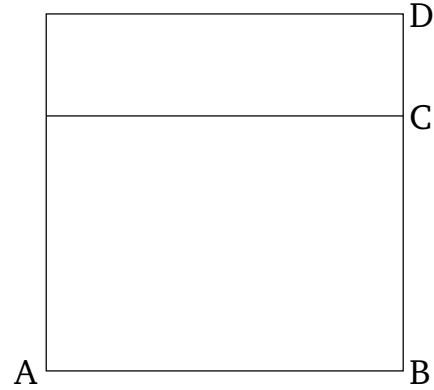
Ἐὰν ῥητόν παρὰ ῥητὴν παραβληθῇ, πλάτος ποιῇ ῥητὴν καὶ σύμμετρον τῇ, παρ' ἣν παράκειται, μήκει.



Ῥητόν γὰρ τὸ ΑΓ παρὰ ῥητὴν τὴν ΑΒ παραβλήσθω πλάτος ποιῶν τὴν ΒΓ· λέγω, ὅτι ῥητὴ ἐστὶν ἡ ΒΓ καὶ σύμμετρος τῇ ΒΑ μήκει.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· ῥητόν ἄρα ἐστὶ τὸ ΑΔ. ῥητόν δὲ καὶ τὸ ΑΓ· σύμμετρον ἄρα

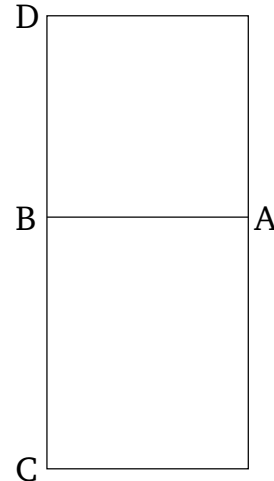
rational straight-lines AB and BC (which are) commensurable in length. I say that AC is rational.



For let the square AD have been described on AB . AD is thus rational [Def. 10.4]. And since AB is commensurable in length with BC , and AB is equal to BD , BD is thus commensurable in length with BC . And as BD is to BC , so DA (is) to AC [Prop. 6.1]. Thus, DA is commensurable with AC [Prop. 10.11]. And DA (is) rational. Thus, AC is also rational [Def. 10.4]. Thus, the ... by rational straight-lines ... commensurable, and so on

Proposition 20

If a rational (area) is applied to a rational (straight-line) then it produces as breadth a (straight-line which is) rational, and commensurable in length with the (straight-line) to which it is applied.



For let the rational (area) AC have been applied to the rational (straight-line) AB , producing the (straight-line) BC as breadth. I say that BC is rational, and commensurable in length with BA .

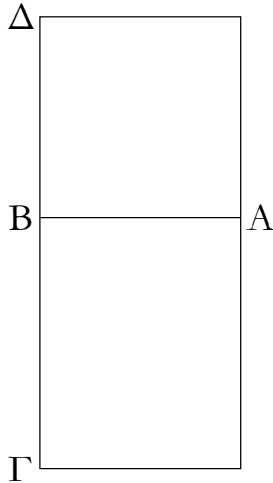
For let the square AD have been described on AB .

ἐστὶ τὸ ΔΑ τῷ ΑΓ. καὶ ἐστὶν ὡς τὸ ΔΑ πρὸς τὸ ΑΓ, οὕτως ἡ ΔΒ πρὸς τὴν ΒΓ. σύμμετρος ἄρα ἐστὶ καὶ ἡ ΔΒ τῇ ΒΓ· ἴση δὲ ἡ ΔΒ τῇ ΒΑ· σύμμετρος ἄρα καὶ ἡ ΑΒ τῇ ΒΓ. ῥητὴ δὲ ἐστὶν ἡ ΑΒ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΒΓ καὶ σύμμετρος τῇ ΑΒ μήκει.

Ἐάν ἄρα ῥητὸν παρὰ ῥητὴν παραβληθῇ, καὶ τὰ ἐξῆς.

κα'.

Τὸ ὑπὸ ῥητῶν δυνάμει μόνον συμμέτρων εὐθειῶν περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλεῖσθω δὲ μέση.



ὑπὸ γὰρ ῥητῶν δυνάμει μόνον συμμέτρων εὐθειῶν τῶν ΑΒ, ΒΓ ὀρθογώνιον περιεχέσθω τὸ ΑΓ· λέγω, ὅτι ἄλογόν ἐστι τὸ ΑΓ, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλεῖσθω δὲ μέση.

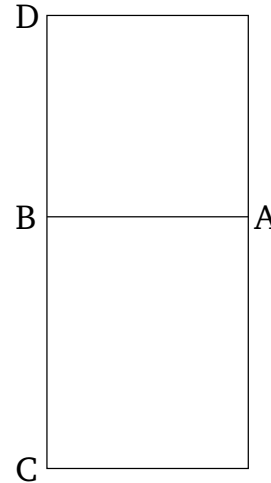
Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔ· ῥητὸν ἄρα ἐστὶ τὸ ΑΔ. καὶ ἐπεὶ ἀσύμμετρος ἐστὶν ἡ ΑΒ τῇ ΒΓ μήκει· δυνάμει γὰρ μόνον ὑπόκεινται σύμμετροι· ἴση δὲ ἡ ΑΒ τῇ ΒΔ, ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΔΒ τῇ ΒΓ μήκει. καὶ ἐστὶν ὡς ἡ ΔΒ πρὸς τὴν ΒΓ, οὕτως τὸ ΑΔ πρὸς τὸ ΑΓ· ἀσύμμετρον ἄρα [ἐστὶ] τὸ ΔΑ τῷ ΑΓ. ῥητὸν δὲ τὸ ΔΑ· ἄλογον ἄρα ἐστὶ τὸ ΑΓ· ὥστε καὶ ἡ δυναμένη τὸ ΑΓ [τουτέστιν ἡ ἴσον αὐτῷ τετράγωνον δυναμένη] ἄλογός ἐστιν, καλεῖσθω δὲ μέση· ὅπερ ἔδει δεῖξαι.

AD is thus rational [Def. 10.4]. And AC (is) also rational. DA is thus commensurable with AC . And as DA is to AC , so DB (is) to BC [Prop. 6.1]. Thus, DB is also commensurable (in length) with BC [Prop. 10.11]. And DB (is) equal to BA . Thus, AB (is) also commensurable (in length) with BC . And AB is rational. Thus, BC is also rational, and commensurable in length with AB [Def. 10.3].

Thus, if a rational (area) is applied to a rational (straight-line), and so on . . .

Proposition 21

The rectangle contained by rational straight-lines (which are) commensurable in square only is irrational, and its square-root is irrational—let it be called medial.[†]



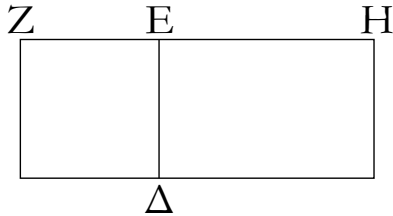
For let the rectangle AC be contained by the rational straight-lines AB and BC (which are) commensurable in square only. I say that AC is irrational, and its square-root is irrational—let it be called medial.

For let the square AD have been described on AB . AD is thus rational [Def. 10.4]. And since AB is incommensurable in length with BC . For they were assumed to be commensurable in square only. And AB (is) equal to BD . DB is thus also incommensurable in length with BC . And as DB is to BC , so AD (is) to AC [Prop. 6.1]. Thus, DA [is] incommensurable with AC [Prop. 10.11]. And DA (is) rational. Thus, AC is irrational [Def. 10.4]. Hence, its square-root [that is to say, the square-root of the square equal to it] is also irrational [Def. 10.4]—let it be called medial. (Which is) the very thing it was required to show.

[†] Thus, a medial straight-line has a length expressible as $k^{1/4}$.

Λήμμα.

Ἐάν ὦσι δύο εὐθεῖαι, ἔστιν ὡς ἡ πρώτη πρὸς τὴν δευτέραν, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ὑπὸ τῶν δύο εὐθειῶν.

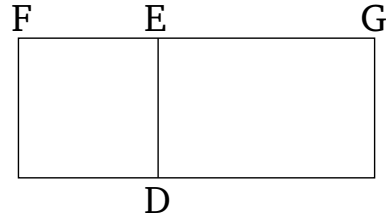


Ἐστωσαν δύο εὐθεῖαι αἱ ZE, EH. λέγω, ὅτι ἔστιν ὡς ἡ ZE πρὸς τὴν EH, οὕτως τὸ ἀπὸ τῆς ZE πρὸς τὸ ὑπὸ τῶν ZE, EH.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ZE τετράγωνον τὸ ΔΖ, καὶ συμπληρώσθω τὸ ΗΔ. ἐπεὶ οὖν ἔστιν ὡς ἡ ZE πρὸς τὴν EH, οὕτως τὸ ΖΔ πρὸς τὸ ΔΗ, καὶ ἔστι τὸ μὲν ΖΔ τὸ ἀπὸ τῆς ZE, τὸ δὲ ΔΗ τὸ ὑπὸ τῶν ΔΕ, ΕΗ, τουτέστι τὸ ὑπὸ τῶν ZE, ΕΗ, ἔστιν ἄρα ὡς ἡ ZE πρὸς τὴν EH, οὕτως τὸ ἀπὸ τῆς ZE πρὸς τὸ ὑπὸ τῶν ZE, ΕΗ. ὁμοίως δὲ καὶ ὡς τὸ ὑπὸ τῶν HE, ΕΖ πρὸς τὸ ἀπὸ τῆς ΕΖ, τουτέστιν ὡς τὸ ΗΔ πρὸς τὸ ΖΔ, οὕτως ἡ HE πρὸς τὴν ΕΖ· ὅπερ ἔδει δεῖξαι.

Lemma

If there are two straight-lines then as the first is to the second, so the (square) on the first (is) to the (rectangle contained) by the two straight-lines.

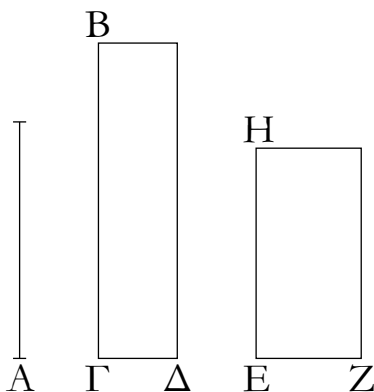


Let FE and EG be two straight-lines. I say that as FE is to EG , so the (square) on FE (is) to the (rectangle contained) by FE and EG .

For let the square DF have been described on FE . And let GD have been completed. Therefore, since as FE is to EG , so FD (is) to DG [Prop. 6.1], and FD is the (square) on FE , and DG the (rectangle contained) by DE and EG —that is to say, the (rectangle contained) by FE and EG —thus as FE is to EG , so the (square) on FE (is) to the (rectangle contained) by FE and EG . And also, similarly, as the (rectangle contained) by GE and EF is to the (square on) EF —that is to say, as GD (is) to FD —so GE (is) to EF . (Which is) the very thing it was required to show.

κβ'.

Τὸ ἀπὸ μέσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ῥητὴν καὶ ἀσύμμετρον τῇ, παρ' ἣν παράκειται, μήκει.

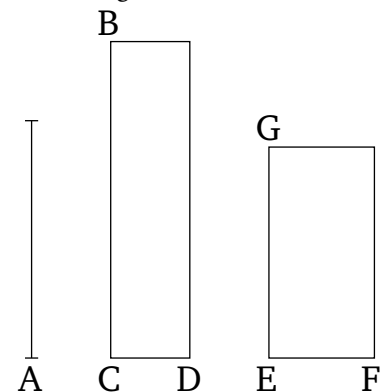


Ἐστω μέση μὲν ἡ A, ῥητὴ δὲ ἡ ΓΒ, καὶ τῷ ἀπὸ τῆς A ἴσον παρὰ τὴν ΒΓ παραβεβλήσθω χωρίον ὀρθογώνιον τὸ ΒΔ πλάτος ποιοῦν τὴν ΓΔ· λέγω, ὅτι ῥητὴ ἔστιν ἡ ΓΔ καὶ ἀσύμμετρος τῇ ΓΒ μήκει.

Ἐπεὶ γὰρ μέση ἔστιν ἡ A, δύναται χωρίον περιεχόμενον ὑπὸ ῥητῶν δυνάμει μόνον συμμετρῶν. δυνάσθω τὸ ΗΖ.

Proposition 22

The square on a medial (straight-line), being applied to a rational (straight-line), produces as breadth a (straight-line which is) rational, and incommensurable in length with the (straight-line) to which it is applied.



Let A be a medial (straight-line), and CB a rational (straight-line), and let the rectangular area BD , equal to the (square) on A , have been applied to BC , producing CD as breadth. I say that CD is rational, and incommensurable in length with CB .

For since A is medial, the square on it is equal to a

δύναται δὲ καὶ τὸ ΒΔ· ἴσον ἄρα ἐστὶ τὸ ΒΔ τῷ ΗΖ. ἔστι δὲ αὐτῷ καὶ ἰσογώνιον· τῶν δὲ ἴσων τε καὶ ἰσογώνιων παραλληλογράμμων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας· ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΓ πρὸς τὴν ΕΗ, οὕτως ἡ ΕΖ πρὸς τὴν ΓΔ. ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς ΒΓ πρὸς τὸ ἀπὸ τῆς ΕΗ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΓΔ. σύμμετρον δὲ ἐστὶ τὸ ἀπὸ τῆς ΓΒ τῷ ἀπὸ τῆς ΕΗ· ῥητὴ γάρ ἐστιν ἑκατέρω αὐτῶν· σύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΓΔ. ῥητὸν δὲ ἐστὶ τὸ ἀπὸ τῆς ΕΖ· ῥητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΓΔ· ῥητὴ ἄρα ἐστὶν ἡ ΓΔ. καὶ ἐπεὶ ἀσύμμετρος ἐστὶν ἡ ΕΖ τῇ ΕΗ μήκει· δυνάμει γὰρ μόνον εἰσὶ σύμμετροι· ὡς δὲ ἡ ΕΖ πρὸς τὴν ΕΗ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ὑπὸ τῶν ΖΕ, ΕΗ, ἀσύμμετρον ἄρα [ἐστὶ] τὸ ἀπὸ τῆς ΕΖ τῷ ὑπὸ τῶν ΖΕ, ΕΗ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΕΖ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΓΔ· ῥηταὶ γὰρ εἰσι δυνάμει· τῷ δὲ ὑπὸ τῶν ΖΕ, ΕΗ σύμμετρόν ἐστι τὸ ὑπὸ τῶν ΔΓ, ΓΒ· ἴσα γὰρ ἐστὶ τῷ ἀπὸ τῆς Α· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΓΔ τῷ ὑπὸ τῶν ΔΓ, ΓΒ. ὡς δὲ τὸ ἀπὸ τῆς ΓΔ πρὸς τὸ ὑπὸ τῶν ΔΓ, ΓΒ, οὕτως ἐστὶν ἡ ΔΓ πρὸς τὴν ΓΒ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΔΓ τῇ ΓΒ μήκει. ῥητὴ ἄρα ἐστὶν ἡ ΓΔ καὶ ἀσύμμετρος τῇ ΓΒ μήκει· ὅπερ ἔδει δεῖξαι.

(rectangular) area contained by rational (straight-lines which are) commensurable in square only [Prop. 10.21]. Let the square on (A) be equal to GF . And the square on (A) is also equal to BD . Thus, BD is equal to GF . And (BD) is also equiangular with (GF) . And for equal and equiangular parallelograms, the sides about the equal angles are reciprocally proportional [Prop. 6.14]. Thus, proportionally, as BC is to EG , so EF (is) to CD . And, also, as the (square) on BC is to the (square) on EG , so the (square) on EF (is) to the (square) on CD [Prop. 6.22]. And the (square) on CB is commensurable with the (square) on EG . For they are each rational. Thus, the (square) on EF is also commensurable with the (square) on CD [Prop. 10.11]. And the (square) on EF is rational. Thus, the (square) on CD is also rational [Def. 10.4]. Thus, CD is rational. And since EF is incommensurable in length with EG . For they are commensurable in square only. And as EF (is) to EG , so the (square) on EF (is) to the (rectangle contained) by FE and EG [see previous lemma]. The (square) on EF [is] thus incommensurable with the (rectangle contained) by FE and EG [Prop. 10.11]. But, the (square) on CD is commensurable with the (square) on EF . For they are rational in square. And the (rectangle contained) by DC and CB is commensurable with the (rectangle contained) by FE and EG . For they are (both) equal to the (square) on A . Thus, the (square) on CD is also incommensurable with the (rectangle contained) by DC and CB [Prop. 10.13]. And as the (square) on CD (is) to the (rectangle contained) by DC and CB , so DC is to CB [see previous lemma]. Thus, DC is incommensurable in length with CB [Prop. 10.11]. Thus, CD is rational, and incommensurable in length with CB . (Which is) the very thing it was required to show.

† Literally, “rational”.

κγ'.

Ἡ τῇ μέσῃ σύμμετρος μέση ἐστίν.

Ἐστω μέση ἡ Α, καὶ τῇ Α σύμμετρος ἔστω ἡ Β· λέγω, ὅτι καὶ ἡ Β μέση ἐστίν.

Ἐκκείσθω γὰρ ῥητὴ ἡ ΓΔ, καὶ τῷ μὲν ἀπὸ τῆς Α ἴσον παρὰ τὴν ΓΔ παραβελήσθω χωρίον ὀρθογώνιον τὸ ΓΕ πλάτος ποιοῦν τὴν ΕΔ· ῥητὴ ἄρα ἐστὶν ἡ ΕΔ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. τῷ δὲ ἀπὸ τῆς Β ἴσον παρὰ τὴν ΓΔ παραβελήσθω χωρίον ὀρθογώνιον τὸ ΓΖ πλάτος ποιοῦν τὴν ΔΖ. ἐπεὶ οὖν σύμμετρος ἐστὶν ἡ Α τῇ Β, σύμμετρόν ἐστι καὶ τὸ ἀπὸ τῆς Α τῷ ἀπὸ τῆς Β. ἀλλὰ τῷ μὲν ἀπὸ τῆς Α ἴσον ἐστὶ τὸ ΕΓ, τῷ δὲ ἀπὸ τῆς Β ἴσον ἐστὶ τὸ ΓΖ· σύμμετρον ἄρα ἐστὶ τὸ ΕΓ τῷ ΓΖ. καὶ ἐστὶν ὡς τὸ ΕΓ πρὸς τὸ ΓΖ, οὕτως ἡ ΕΔ πρὸς τὴν ΔΖ.

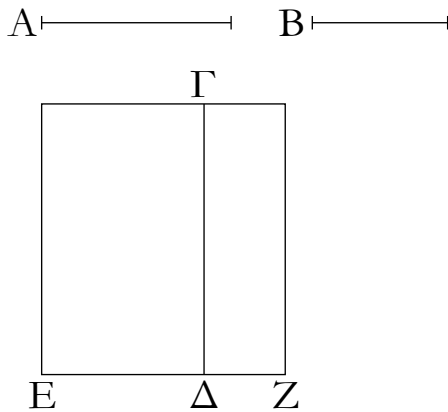
Proposition 23

A (straight-line) commensurable with a medial (straight-line) is medial.

Let A be a medial (straight-line), and let B be commensurable with A . I say that B is also a medial (straight-line).

Let the rational (straight-line) CD be set out, and let the rectangular area CE , equal to the (square) on A , have been applied to CD , producing ED as width. ED is thus rational, and incommensurable in length with CD [Prop. 10.22]. And let the rectangular area CF , equal to the (square) on B , have been applied to CD , producing DF as width. Therefore, since A is commensurable with B , the (square) on A is also commensurable with

σύμμετρος ἄρα ἐστὶν ἡ $ΕΔ$ τῇ $ΔΖ$ μήκει. ῥητὴ δὲ ἐστὶν ἡ $ΕΔ$ καὶ ἀσύμμετρος τῇ $ΔΓ$ μήκει· ῥητὴ ἄρα ἐστὶ καὶ ἡ $ΔΖ$ καὶ ἀσύμμετρος τῇ $ΔΓ$ μήκει· αἱ $ΓΔ$, $ΔΖ$ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι. ἡ δὲ τὸ ὑπὸ ῥητῶν δυνάμει μόνον συμμέτρων δυναμένη μέση ἐστίν. ἡ ἄρα τὸ ὑπὸ τῶν $ΓΔ$, $ΔΖ$ δυναμένη μέση ἐστίν· καὶ δύνανται τὸ ὑπὸ τῶν $ΓΔ$, $ΔΖ$ ἢ $Β$ · μέση ἄρα ἐστὶν ἡ $Β$.



Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι τὸ τῷ μέσῳ χωρίῳ σύμμετρον μέσον ἐστίν.

† A medial area is equal to the square on some medial straight-line. Hence, a medial area is expressible as $k^{1/2}$.

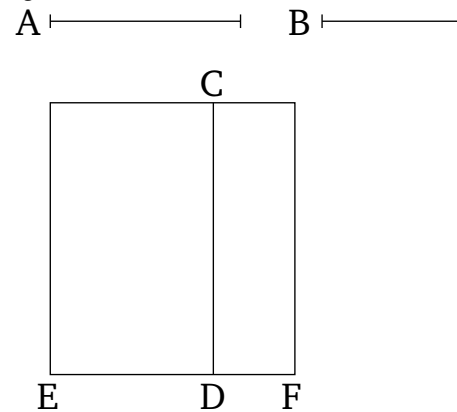
καὶ δ'.

Τὸ ὑπὸ μέσων μήκει συμμέτρων εὐθειῶν περιεχόμενον ὀρθογώνιον μέσον ἐστίν.

Ὑπὸ γὰρ μέσων μήκει συμμέτρων εὐθειῶν τῶν $ΑΒ$, $ΒΓ$ περιεχέσθω ὀρθογώνιον τὸ $ΑΓ$ · λέγω, ὅτι τὸ $ΑΓ$ μέσον ἐστίν.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς $ΑΒ$ τετράγωνον τὸ $ΑΔ$ · μέσον ἄρα ἐστὶ τὸ $ΑΔ$. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ $ΑΒ$ τῇ $ΒΓ$ μήκει, ἴση δὲ ἡ $ΑΒ$ τῇ $ΒΔ$, σύμμετρος ἄρα ἐστὶ καὶ ἡ $ΔΒ$ τῇ $ΒΓ$ μήκει· ὥστε καὶ τὸ $ΔΑ$ τῷ $ΑΓ$ σύμμετρόν ἐστιν. μέσον δὲ τὸ $ΔΑ$ · μέσον ἄρα καὶ τὸ $ΑΓ$ · ὅπερ εἶδει δεῖξαι.

the (square) on B . But, EC is equal to the (square) on A , and CF is equal to the (square) on B . Thus, EC is commensurable with CF . And as EC is to CF , so ED (is) to DF [Prop. 6.1]. Thus, ED is commensurable in length with DF [Prop. 10.11]. And ED is rational, and incommensurable in length with CD . DF is thus also rational [Def. 10.3], and incommensurable in length with DC [Prop. 10.13]. Thus, CD and DF are rational, and commensurable in square only. And the square-root of a (rectangle contained) by rational (straight-lines which are) commensurable in square only is medial [Prop. 10.21]. Thus, the square-root of the (rectangle contained) by CD and DF is medial. And the square on B is equal to the (rectangle contained) by CD and DF . Thus, B is a medial (straight-line).



Corollary

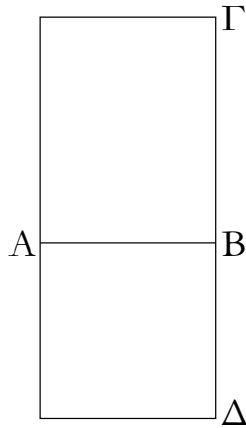
And (it is) clear, from this, that an (area) commensurable with a medial area[†] is medial.

Proposition 24

A rectangle contained by medial straight-lines (which are) commensurable in length is medial.

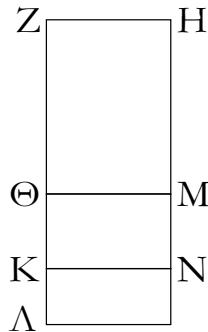
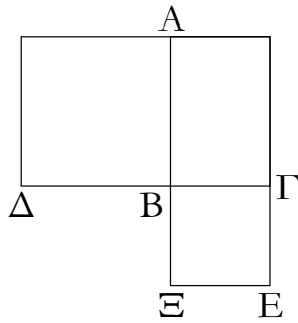
For let the rectangle AC be contained by the medial straight-lines AB and BC (which are) commensurable in length. I say that AC is medial.

For let the square AD have been described on AB . AD is thus medial [see previous footnote]. And since AB is commensurable in length with BC , and AB (is) equal to BD , DB is thus also commensurable in length with BC . Hence, DA is also commensurable with AC [Props. 6.1, 10.11]. And DA (is) medial. Thus, AC (is) also medial [Prop. 10.23 corr.]. (Which is) the very thing it was required to show.



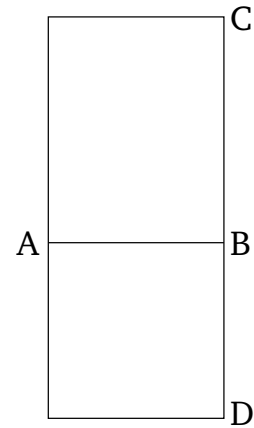
κε'.

Τὸ ὑπὸ μέσων δυνάμει μόνον συμμετρων εὐθειῶν περιεχόμενον ὀρθογώνιον ἤτοι ῥητὸν ἢ μέσον ἐστίν.



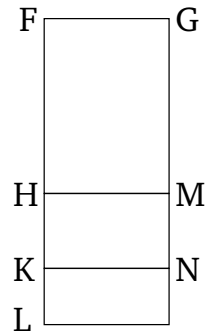
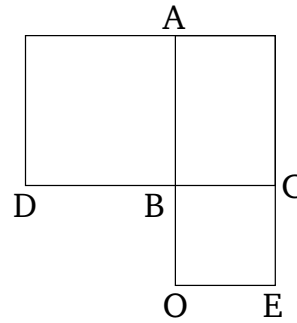
Ὑπὸ γὰρ μέσων δυνάμει μόνον συμμετρων εὐθειῶν τῶν AB, BΓ ὀρθογώνιον περιεχέσθω τὸ AΓ· λέγω, ὅτι τὸ AΓ ἤτοι ῥητὸν ἢ μέσον ἐστίν.

Ἀναγεγράφθω γὰρ ἀπὸ τῶν AB, BΓ τετραγώνια τὰ AΔ, BE· μέσον ἄρα ἐστὶν ἑκάτερον τῶν AΔ, BE. καὶ ἐκκείσθω ῥητὴ ἡ ZH, καὶ τῷ μὲν AΔ ἴσον παρὰ τὴν ZH παραβελήσθω ὀρθογώνιον παραλληλόγραμμον τὸ HΘ πλάτος ποιοῦν τὴν ZΘ, τῷ δὲ AΓ ἴσον παρὰ τὴν ΘM παραβελήσθω ὀρθογώνιον παραλληλόγραμμον τὸ MK πλάτος ποιοῦν τὴν ΘK, καὶ ἔτι τῷ BE ἴσον ὁμοίως παρὰ τὴν KN παραβελήσθω τὸ NL πλάτος ποιοῦν τὴν KΛ· ἐπ' εὐθείας ἄρα εἰσὶν αἱ ZΘ, ΘK, KΛ. ἐπεὶ οὖν μέσον ἐστὶν ἑκάτερον τῶν AΔ, BE, καὶ ἐστὶν ἴσον τὸ μὲν AΔ τῷ HΘ, τὸ δὲ BE τῷ NL, μέσον ἄρα καὶ ἑκάτερον τῶν HΘ, NL. καὶ παρὰ ῥητὴν τὴν ZH παράκειται· ῥητὴ ἄρα ἐστὶν ἑκατέρω τῶν ZΘ, KΛ καὶ ἀσύμμετρος τῇ ZH μήκει. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ AΔ τῷ BE, σύμμετρον ἄρα ἐστὶ καὶ τὸ HΘ τῷ NL. καὶ ἐστὶν ὡς τὸ HΘ πρὸς τὸ NL, οὕτως ἡ ZΘ πρὸς τὴν KΛ· σύμμετρος ἄρα ἐστὶν ἡ ZΘ τῇ KΛ μήκει. αἱ ZΘ, KΛ ἄρα ῥηταὶ εἰσι μήκει σύμμετροι· ῥητὸν ἄρα ἐστὶ τὸ ὑπὸ τῶν ZΘ, KΛ. καὶ



Proposition 25

The rectangle contained by medial straight-lines (which are) commensurable in square only is either rational or medial.



For let the rectangle AC be contained by the medial straight-lines AB and BC (which are) commensurable in square only. I say that AC is either rational or medial.

For let the squares AD and BE have been described on (the straight-lines) AB and BC (respectively). AD and BE are thus each medial. And let the rational (straight-line) FG be laid out. And let the rectangular parallelogram GH , equal to AD , have been applied to FG , producing FH as breadth. And let the rectangular parallelogram MK , equal to AC , have been applied to HM , producing HK as breadth. And, finally, let NL , equal to BE , have similarly been applied to KN , producing KL as breadth. Thus, FH , HK , and KL are in a straight-line. Therefore, since AD and BE are each medial, and AD is equal to GH , and BE to NL , GH and NL (are) thus each also medial. And they are applied to the rational (straight-line) FG . FH and KL are thus each rational, and incommensurable in length with FG [Prop. 10.22]. And since AD is commensurable with BE , GH is thus also commensurable with NL . And as

ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔB τῇ BA , ἡ δὲ ΞB τῇ $B\Gamma$, ἔστιν ἄρα ὡς ἡ ΔB πρὸς τὴν $B\Gamma$, οὕτως ἡ AB πρὸς τὴν $B\Xi$. ἀλλ' ὡς μὲν ἡ ΔB πρὸς τὴν $B\Gamma$, οὕτως τὸ ΔA πρὸς τὸ AG . ὡς δὲ ἡ AB πρὸς τὴν $B\Xi$, οὕτως τὸ AG πρὸς τὸ $\Gamma\Xi$. ἔστιν ἄρα ὡς τὸ ΔA πρὸς τὸ AG , οὕτως τὸ AG πρὸς τὸ $\Gamma\Xi$. ἴσον δὲ ἐστὶ τὸ μὲν AD τῷ $H\Theta$, τὸ δὲ AG τῷ MK , τὸ δὲ $\Gamma\Xi$ τῷ $N\Lambda$. ἔστιν ἄρα ὡς τὸ $H\Theta$ πρὸς τὸ MK , οὕτως τὸ MK πρὸς τὸ $N\Lambda$. ἔστιν ἄρα καὶ ὡς ἡ $Z\Theta$ πρὸς τὴν ΘK , οὕτως ἡ ΘK πρὸς τὴν $K\Lambda$. τὸ ἄρα ὑπὸ τῶν $Z\Theta$, $K\Lambda$ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΘK . ῥητὸν δὲ τὸ ὑπὸ τῶν $Z\Theta$, $K\Lambda$. ῥητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΘK . ῥητὴ ἄρα ἐστὶν ἡ ΘK . καὶ εἰ μὲν σύμμετρος ἐστὶ τῇ ZH μήκει, ῥητὸν ἐστὶ τὸ ΘN . εἰ δὲ ἀσύμμετρος ἐστὶ τῇ ZH μήκει, αἱ $K\Theta$, ΘM ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα τὸ ΘN . τὸ ΘN ἄρα ἦτοι ῥητὸν ἢ μέσον ἐστίν. ἴσον δὲ τὸ ΘN τῷ AG . τὸ AG ἄρα ἦτοι ῥητὸν ἢ μέσον ἐστίν.

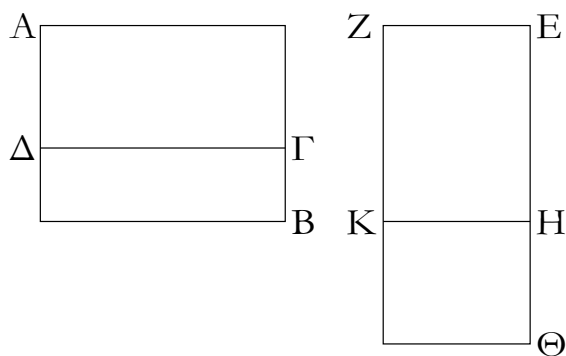
Τὸ ἄρα ὑπὸ μέσων δυνάμει μόνον συμμέτρων, καὶ τὰ ἐξῆς.

GH is to NL , so FH (is) to KL [Prop. 6.1]. Thus, FH is commensurable in length with KL [Prop. 10.11]. Thus, FH and KL are rational (straight-lines which are) commensurable in length. Thus, the (rectangle contained) by FH and KL is rational [Prop. 10.19]. And since DB is equal to BA , and OB to BC , thus as DB is to BC , so AB (is) to BO . But, as DB (is) to BC , so DA (is) to AC [Props. 6.1]. And as AB (is) to BO , so AC (is) to CO [Prop. 6.1]. Thus, as DA is to AC , so AC (is) to CO . And AD is equal to GH , and AC to MK , and CO to NL . Thus, as GH is to MK , so MK (is) to NL . Thus, also, as FH is to HK , so HK (is) to KL [Props. 6.1, 5.11]. Thus, the (rectangle contained) by FH and KL is equal to the (square) on HK [Prop. 6.17]. And the (rectangle contained) by FH and KL (is) rational. Thus, the (square) on HK is also rational. Thus, HK is rational. And if it is commensurable in length with FG then HN is rational [Prop. 10.19]. And if it is incommensurable in length with FG then KH and HM are rational (straight-lines which are) commensurable in square only: thus, HN is medial [Prop. 10.21]. Thus, HN is either rational or medial. And HN (is) equal to AC . Thus, AC is either rational or medial.

Thus, the . . . by medial straight-lines (which are) commensurable in square only, and so on . . .

κτ'.

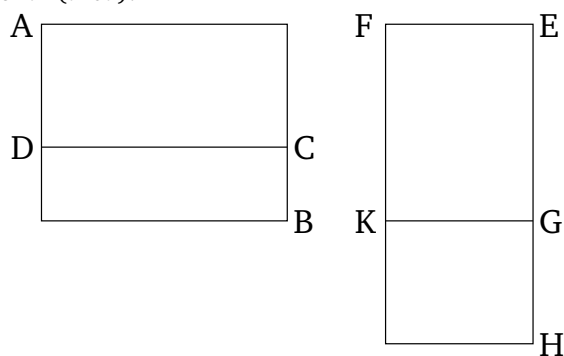
Μέσον μέσου οὐχ ὑπερέχει ῥητῷ.



Εἰ γὰρ δυνατόν, μέσον τὸ AB μέσου τοῦ AG ὑπερεχέτω ῥητῷ τῷ ΔB , καὶ ἐκλείσθω ῥητὴ ἡ EZ , καὶ τῷ AB ἴσον παρὰ τὴν EZ παραβεβλήσθω παραλληλόγραμμον ὀρθογώνιον τὸ $Z\Theta$ πλάτος ποιῶν τὴν $E\Theta$, τῷ δὲ AG ἴσον ἀφηρήσθω τὸ ZH . λοιπὸν ἄρα τὸ $B\Delta$ λοιπῷ τῷ $K\Theta$ ἐστὶν ἴσον. ῥητὸν δὲ ἐστὶ τὸ ΔB . ῥητὸν ἄρα ἐστὶ καὶ τὸ $K\Theta$. ἐπεὶ οὖν μέσον ἐστὶν ἐκάτερον τῶν AB , AG , καὶ ἐστὶ τὸ μὲν AB τῷ $Z\Theta$ ἴσον, τὸ δὲ AG τῷ ZH , μέσον ἄρα καὶ ἐκάτερον τῶν $Z\Theta$, ZH . καὶ παρὰ ῥητὴν τὴν EZ παράκειται· ῥητὴ ἄρα ἐστὶν ἐκάτερα τῶν ΘE , EH καὶ ἀσύμμετρος τῇ EZ μήκει. καὶ ἐπεὶ ῥητὸν ἐστὶ

Proposition 26

A medial (area) does not exceed a medial (area) by a rational (area).[†]



For, if possible, let the medial (area) AB exceed the medial (area) AC by the rational (area) DB . And let the rational (straight-line) EF be laid down. And let the rectangular parallelogram FH , equal to AB , have been applied to to EF , producing EH as breadth. And let FG , equal to AC , have been cut off (from FH). Thus, the remainder BD is equal to the remainder KH . And DB is rational. Thus, KH is also rational. Therefore, since AB and AC are each medial, and AB is equal to FH , and AC to FG , FH and FG are thus each also medial.

τὸ ΔΒ καὶ ἐστὶν ἴσον τῷ ΚΘ, ῥητὸν ἄρα ἐστὶ καὶ τὸ ΚΘ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ ΗΘ καὶ σύμμετρος τῇ ΕΖ μήκει. ἀλλὰ καὶ ἡ ΕΗ ῥητὴ ἐστὶ καὶ ἀσύμμετρος τῇ ΕΖ μήκει· ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΗ τῇ ΗΘ μήκει. καὶ ἐστὶν ὡς ἡ ΕΗ πρὸς τὴν ΗΘ, οὕτως τὸ ἀπὸ τῆς ΕΗ πρὸς τὸ ὑπὸ τῶν ΕΗ, ΗΘ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΕΗ τῷ ὑπὸ τῶν ΕΗ, ΗΘ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΕΗ σύμμετρά ἐστὶ τὰ ἀπὸ τῶν ΕΗ, ΗΘ τετράγωνα· ῥητὰ γὰρ ἀμφοτέρω· τῷ δὲ ὑπὸ τῶν ΕΗ, ΗΘ σύμμετρόν ἐστὶ τὸ δις ὑπὸ τῶν ΕΗ, ΗΘ· διπλάσιον γάρ ἐστιν αὐτοῦ· ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΕΗ, ΗΘ τῷ δις ὑπὸ τῶν ΕΗ, ΗΘ· καὶ συναμφοτέρα ἄρα τὰ τε ἀπὸ τῶν ΕΗ, ΗΘ καὶ τὸ δις ὑπὸ τῶν ΕΗ, ΗΘ, ὅπερ ἐστὶ τὸ ἀπὸ τῆς ΕΘ, ἀσύμμετρόν ἐστι τοῖς ἀπὸ τῶν ΕΗ, ΗΘ. ῥητὰ δὲ τὰ ἀπὸ τῶν ΕΗ, ΗΘ· ἄλογον ἄρα τὸ ἀπὸ τῆς ΕΘ. ἄλογος ἄρα ἐστὶν ἡ ΕΘ. ἀλλὰ καὶ ῥηρή· ὅπερ ἐστὶν ἀδύνατον.

Μέσον ἄρα μέσου οὐχ ὑπερέχει ῥητῷ· ὅπερ ἔδει δεῖξαι.

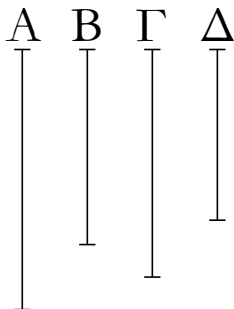
And they are applied to the rational (straight-line) EF . Thus, HE and EG are each rational, and incommensurable in length with EF [Prop. 10.22]. And since DB is rational, and is equal to KH , KH is thus also rational. And (KH) is applied to the rational (straight-line) EF . GH is thus rational, and commensurable in length with EF [Prop. 10.20]. But, EG is also rational, and incommensurable in length with EF . Thus, EG is incommensurable in length with GH [Prop. 10.13]. And as EG is to GH , so the (square) on EG (is) to the (rectangle contained) by EG and GH [Prop. 10.13 lem.]. Thus, the (square) on EG is incommensurable with the (rectangle contained) by EG and GH [Prop. 10.11]. But, the (sum of the) squares on EG and GH is commensurable with the (square) on EG . For $(EG$ and GH are) both rational. And twice the (rectangle contained) by EG and GH is commensurable with the (rectangle contained) by EG and GH [Prop. 10.6]. For (the former) is double the latter. Thus, the (sum of the squares) on EG and GH is incommensurable with twice the (rectangle contained) by EG and GH [Prop. 10.13]. And thus the sum of the (squares) on EG and GH plus twice the (rectangle contained) by EG and GH , that is the (square) on EH [Prop. 2.4], is incommensurable with the (sum of the squares) on EG and GH [Prop. 10.16]. And the (sum of the squares) on EG and GH (is) rational. Thus, the (square) on EH is irrational [Def. 10.4]. Thus, EH is irrational [Def. 10.4]. But, (it is) also rational. The very thing is impossible.

Thus, a medial (area) does not exceed a medial (area) by a rational (area). (Which is) the very thing it was required to show.

† In other words, $\sqrt{k} - \sqrt{k'} \neq k''$.

κζ'.

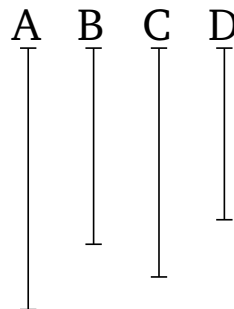
Μέσας εὐρεῖν δυνάμει μόνον συμμετρους ῥητὸν περιεχούσας.



Ἐκκείσθωσαν δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ Α, Β, καὶ εἰληφθῶ τῶν Α, Β μέση ἀνάλογον ἡ Γ, καὶ γεγόνετω ὡς ἡ Α πρὸς τὴν Β, οὕτως ἡ Γ πρὸς τὴν Δ.

Proposition 27

To find (two) medial (straight-lines), containing a rational (area), (which are) commensurable in square only.



Let the two rational (straight-lines) A and B , (which are) commensurable in square only, be laid down. And let C —the mean proportional (straight-line) to A and B —

Καὶ ἐπεὶ αἱ A, B ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, τὸ ἄρα ὑπὸ τῶν A, B , τουτέστι τὸ ἀπὸ τῆς Γ , μέσον ἐστίν. μέση ἄρα ἡ Γ . καὶ ἐπεὶ ἐστὶν ὡς ἡ A πρὸς τὴν B , [οὕτως] ἡ Γ πρὸς τὴν Δ , αἱ δὲ A, B δυνάμει μόνον [εἰσὶ] σύμμετροι, καὶ αἱ Γ, Δ ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. καὶ ἐστὶ μέση ἡ Γ · μέση ἄρα καὶ ἡ Δ . αἱ Γ, Δ ἄρα μέσαι εἰσι δυνάμει μόνον σύμμετροι. λέγω, ὅτι καὶ ῥητὸν περιέχουσιν. ἐπεὶ γάρ ἐστὶν ὡς ἡ A πρὸς τὴν B , οὕτως ἡ Γ πρὸς τὴν Δ , ἐναλλάξ ἄρα ἐστὶν ὡς ἡ A πρὸς τὴν Γ , ἡ B πρὸς τὴν Δ . ἀλλ' ὡς ἡ A πρὸς τὴν Γ , ἡ Γ πρὸς τὴν B · καὶ ὡς ἄρα ἡ Γ πρὸς τὴν B , οὕτως ἡ B πρὸς τὴν Δ · τὸ ἄρα ὑπὸ τῶν Γ, Δ ἴσον ἐστὶ τῷ ἀπὸ τῆς B . ῥητὸν δὲ τὸ ἀπὸ τῆς B · ῥητὸν ἄρα [ἐστὶ] καὶ τὸ ὑπὸ τῶν Γ, Δ .

Εὐρηγνται ἄρα μέσαι δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι· ὅπερ ἔδει δείξαι.

have been taken [Prop. 6.13]. And let it be contrived that as A (is) to B , so C (is) to D [Prop. 6.12].

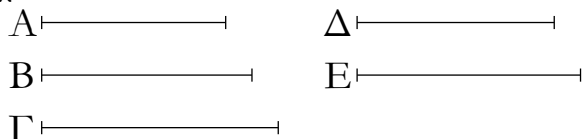
And since the rational (straight-lines) A and B are commensurable in square only, the (rectangle contained) by A and B —that is to say, the (square) on C [Prop. 6.17]—is thus medial [Prop 10.21]. Thus, C is medial [Prop. 10.21]. And since as A is to B , [so] C (is) to D , and A and B [are] commensurable in square only, C and D are thus also commensurable in square only [Prop. 10.11]. And C is medial. Thus, D is also medial [Prop. 10.23]. Thus, C and D are medial (straight-lines which are) commensurable in square only. I say that they also contain a rational (area). For since as A is to B , so C (is) to D , thus, alternately, as A is to C , so B (is) to D [Prop. 5.16]. But, as A (is) to C , (so) C (is) to B . And thus as C (is) to B , so B (is) to D [Prop. 5.11]. Thus, the (rectangle contained) by C and D is equal to the (square) on B [Prop. 6.17]. And the (square) on B (is) rational. Thus, the (rectangle contained) by C and D [is] also rational.

Thus, (two) medial (straight-lines, C and D), containing a rational (area), (which are) commensurable in square only, have been found.[†] (Which is) the very thing it was required to show.

[†] C and D have lengths $k^{1/4}$ and $k^{3/4}$ times that of A , respectively, where the length of B is $k^{1/2}$ times that of A .

κη'.

Μέσας εὐρεῖν δυνάμει μόνον συμμέτρους μέσον περιεχούσας.

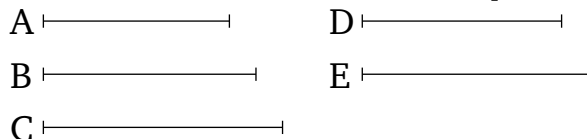


Ἐκκείσθωσαν [τρεῖς] ῥηταὶ δυνάμει μόνον σύμμετροι αἱ A, B, Γ , καὶ εἰλήφθω τῶν A, B μέση ἀνάλογον ἡ Δ , καὶ γεγονέτω ὡς ἡ B πρὸς τὴν Γ , ἡ Δ πρὸς τὴν E .

Ἐπεὶ αἱ A, B ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, τὸ ἄρα ὑπὸ τῶν A, B , τουτέστι τὸ ἀπὸ τῆς Δ , μέσον ἐστίν. μέση ἄρα ἡ Δ . καὶ ἐπεὶ αἱ B, Γ δυνάμει μόνον εἰσὶ σύμμετροι, καὶ ἐστὶν ὡς ἡ B πρὸς τὴν Γ , ἡ Δ πρὸς τὴν E , καὶ αἱ Δ, E ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. μέση δὲ ἡ Δ · μέση ἄρα καὶ ἡ E . αἱ Δ, E ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω δὴ, ὅτι καὶ μέσον περιέχουσιν. ἐπεὶ γάρ ἐστὶν ὡς ἡ B πρὸς τὴν Γ , ἡ Δ πρὸς τὴν E , ἐναλλάξ ἄρα ὡς ἡ B πρὸς τὴν Δ , ἡ Γ πρὸς τὴν E . ὡς δὲ ἡ B πρὸς τὴν Δ , ἡ Δ πρὸς τὴν A · καὶ ὡς ἄρα ἡ Δ πρὸς τὴν A , ἡ Γ πρὸς τὴν E · τὸ ἄρα ὑπὸ τῶν A, Γ ἴσον ἐστὶ τῷ ὑπὸ τῶν Δ, E . μέσον δὲ τὸ ὑπὸ τῶν A, E . Εὐρηγνται ἄρα μέσαι δυνάμει μόνον σύμμετροι μέσον

Proposition 28

To find (two) medial (straight-lines), containing a medial (area), (which are) commensurable in square only.



Let the [three] rational (straight-lines) A, B , and C , (which are) commensurable in square only, be laid down. And let, D , the mean proportional (straight-line) to A and B , have been taken [Prop. 6.13]. And let it be contrived that as B (is) to C , (so) D (is) to E [Prop. 6.12].

Since the rational (straight-lines) A and B are commensurable in square only, the (rectangle contained) by A and B —that is to say, the (square) on D [Prop. 6.17]—is medial [Prop. 10.21]. Thus, D (is) medial [Prop. 10.21]. And since B and C are commensurable in square only, and as B is to C , (so) D (is) to E , D and E are thus commensurable in square only [Prop. 10.11]. And D (is) medial. E (is) thus also medial [Prop. 10.23]. Thus, D and E are medial (straight-lines which are) commensurable in square only. So, I say that they also enclose a medial (area). For since as B is to C , (so) D (is) to E , thus,

περιέχουσαι· ὅπερ ἔδει δεῖξαι.

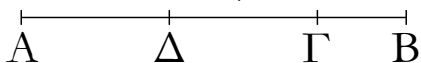
alternately, as B (is) to D , (so) C (is) to E [Prop. 5.16]. And as B (is) to D , (so) D (is) to A . And thus as D (is) to A , (so) C (is) to E . Thus, the (rectangle contained) by A and C is equal to the (rectangle contained) by D and E [Prop. 6.16]. And the (rectangle contained) by A and C is medial [Prop. 10.21]. Thus, the (rectangle contained) by D and E (is) also medial.

Thus, (two) medial (straight-lines, D and E), containing a medial (area), (which are) commensurable in square only, have been found. (Which is) the very thing it was required to show.

† D and E have lengths $k^{1/4}$ and $k^{1/2}/k^{1/4}$ times that of A , respectively, where the lengths of B and C are $k^{1/2}$ and $k^{1/2}$ times that of A , respectively.

Λήμμα α'.

Εὕρειν δύο τετραγώνους ἀριθμούς, ὥστε καὶ τὸν συγχείμενον ἐξ αὐτῶν εἶναι τετράγωνον.

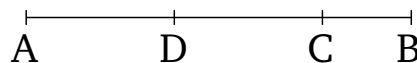


Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ AB , $BΓ$, ἔστωσαν δὲ ἦτοι ἄρτιοι ἢ περιττοί. καὶ ἐπεὶ, ἐάν τε ἀπὸ ἀρτίου ἄρτιος ἀφαιρεθῇ, ἐάν τε ἀπὸ περισσοῦ περισσός, ὁ λοιπὸς ἄρτιός ἐστιν, ὁ λοιπὸς ἄρα ὁ $ΑΓ$ ἄρτιός ἐστιν. τετμήσθω ὁ $ΑΓ$ δίχα κατὰ τὸ $Δ$. ἔστωσαν δὲ καὶ οἱ AB , $BΓ$ ἦτοι ὅμοιοι ἐπίπεδοι ἢ τετράγωνοι, οἱ καὶ αὐτοὶ ὅμοιοι εἰσιν ἐπίπεδοι· ὁ ἄρα ἐκ τῶν AB , $BΓ$ μετὰ τοῦ ἀπὸ [τοῦ] $ΓΔ$ τετραγώνου ἴσος ἐστὶ τῷ ἀπὸ τοῦ $BΔ$ τετραγώνῳ. καὶ ἐστὶ τετράγωνος ὁ ἐκ τῶν AB , $BΓ$, ἐπειδὴ περ ἐδείχθη, ὅτι, ἐὰν δύο ὅμοιοι ἐπίπεδοι πολλαπλασιάσαντες ἀλλήλους ποιῶσι τινα, ὁ γενόμενος τετράγωνός ἐστιν. εὕρηται ἄρα δύο τετράγωνοι ἀριθμοὶ ὃ τε ἐκ τῶν AB , $BΓ$ καὶ ὁ ἀπὸ τοῦ $ΓΔ$, οἱ συντεθέντες ποιῶσι τὸν ἀπὸ τοῦ $BΔ$ τετράγωνον.

Καὶ φανερόν, ὅτι εὕρηται πάλιν δύο τετράγωνοι ὃ τε ἀπὸ τοῦ $BΔ$ καὶ ὁ ἀπὸ τοῦ $ΓΔ$, ὥστε τὴν ὑπεροχὴν αὐτῶν τὸν ὑπὸ AB , $BΓ$ εἶναι τετράγωνον, ὅταν οἱ AB , $BΓ$ ὅμοιοι ὦσιν ἐπίπεδοι. ὅταν δὲ μὴ ὦσιν ὅμοιοι ἐπίπεδοι, εὕρηται δύο τετράγωνοι ὃ τε ἀπὸ τοῦ $BΔ$ καὶ ὁ ἀπὸ τοῦ $ΔΓ$, ὧν ἡ ὑπεροχὴ ὁ ὑπὸ τῶν AB , $BΓ$ οὐκ ἐστὶ τετράγωνος· ὅπερ ἔδει δεῖξαι.

Lemma I

To find two square numbers such that the sum of them is also square.

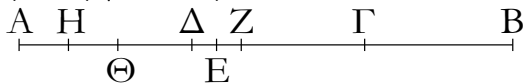


Let the two numbers AB and BC be laid down. And let them be either (both) even or (both) odd. And since, if an even (number) is subtracted from an even (number), or if an odd (number) is subtracted from an odd (number), then the remainder is even [Props. 9.24, 9.26], the remainder AC is thus even. Let AC have been cut in half at D . And let AB and BC also be either similar plane (numbers), or square (numbers)—which are themselves also similar plane (numbers). Thus, the (number created) from (multiplying) AB and BC , plus the square on CD , is equal to the square on BD [Prop. 2.6]. And the (number created) from (multiplying) AB and BC is square—inasmuch as it was shown that if two similar plane (numbers) make some (number) by multiplying one another then the (number so) created is square [Prop. 9.1]. Thus, two square numbers have been found—(namely,) the (number created) from (multiplying) AB and BC , and the (square) on CD —which, (when) added (together), make the square on BD .

And (it is) clear that two square (numbers) have again been found—(namely,) the (square) on BD , and the (square) on CD —such that their difference—(namely,) the (rectangle) contained by AB and BC —is square whenever AB and BC are similar plane (numbers). But, when they are not similar plane numbers, two square (numbers) have been found—(namely,) the (square) on BD , and the (square) on DC —between which the difference—(namely,) the (rectangle) contained by AB and BC —is not square. (Which is) the very thing it was required to show.

Λήμμα β'.

Εὑρεῖν δύο τετραγώνους ἀριθμούς, ὥστε τὸν ἐξ αὐτῶν συγκεῖμενον μὴ εἶναι τετράγωνον.

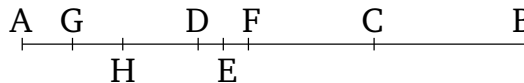


Ἐστω γὰρ ὁ ἐκ τῶν AB, BΓ, ὡς ἔφαμεν, τετράγωνος, καὶ ἄρτιος ὁ ΓΑ, καὶ τετμήσθω ὁ ΓΑ δίχα τῷ Δ. φανερόν δὴ, ὅτι ὁ ἐκ τῶν AB, BΓ τετράγωνος μετὰ τοῦ ἀπὸ [τοῦ] ΓΔ τετραγώνου ἴσος ἐστὶ τῷ ἀπὸ [τοῦ] ΒΔ τετραγώνῳ. ἀφηρήσθω μονὰς ἡ ΔΕ· ὁ ἄρα ἐκ τῶν AB, BΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ ἐλάσσων ἐστὶ τοῦ ἀπὸ [τοῦ] ΒΔ τετραγώνου. λέγω οὖν, ὅτι ὁ ἐκ τῶν AB, BΓ τετράγωνος μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ οὐκ ἔσται τετράγωνος.

Εἰ γὰρ ἔσται τετράγωνος, ἦτοι ἴσος ἐστὶ τῷ ἀπὸ [τοῦ] ΒΕ ἢ ἐλάσσων τοῦ ἀπὸ [τοῦ] ΒΕ, οὐκέτι δὲ καὶ μείζων, ἵνα μὴ τμηθῇ ἡ μονὰς. ἔστω, εἰ δυνατόν, πρότερον ὁ ἐκ τῶν AB, BΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος τῷ ἀπὸ ΒΕ, καὶ ἔστω τῆς ΔΕ μονάδος διπλασίων ὁ ΗΑ. ἐπεὶ οὖν ὅλος ὁ ΑΓ ὅλου τοῦ ΓΔ ἐστὶ διπλασίων, ὧν ὁ ΑΗ τοῦ ΔΕ ἐστὶ διπλασίων, καὶ λοιπὸς ἄρα ὁ ΗΓ λοιποῦ τοῦ ΕΓ ἐστὶ διπλασίων· δίχα ἄρα τέτμηται ὁ ΗΓ τῷ Ε. ὁ ἄρα ἐκ τῶν ΗΒ, BΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ τῷ ἀπὸ ΒΕ τετραγώνῳ. ἀλλὰ καὶ ὁ ἐκ τῶν AB, BΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ὑπόκειται τῷ ἀπὸ [τοῦ] ΒΕ τετραγώνῳ· ὁ ἄρα ἐκ τῶν ΗΒ, BΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ τῷ ἐκ τῶν AB, BΓ μετὰ τοῦ ἀπὸ ΓΕ. καὶ κοινοῦ ἀφαιρεθέντος τοῦ ἀπὸ ΓΕ συνάγεται ὁ ΑΒ ἴσος τῷ ΗΒ· ὅπερ ἄτοπον. οὐκ ἄρα ὁ ἐκ τῶν AB, BΓ μετὰ τοῦ ἀπὸ [τοῦ] ΓΕ ἴσος ἐστὶ τῷ ἀπὸ ΒΕ. λέγω δὴ, ὅτι οὐδὲ ἐλάσσων τοῦ ἀπὸ ΒΕ. εἰ γὰρ δυνατόν, ἔστω τῷ ἀπὸ ΒΖ ἴσος, καὶ τοῦ ΔΖ διπλασίων ὁ ΘΑ. καὶ συναχθήσεται πάλιν διπλασίων ὁ ΘΓ τοῦ ΓΖ· ὥστε καὶ τὸν ΓΘ δίχα τετμήσθαι κατὰ τὸ Ζ, καὶ διὰ τοῦτο τὸν ἐκ τῶν ΘΒ, BΓ μετὰ τοῦ ἀπὸ ΖΓ ἴσον γίνεσθαι τῷ ἀπὸ ΒΖ. ὑπόκειται δὲ καὶ ὁ ἐκ τῶν AB, BΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος τῷ ἀπὸ ΒΖ. ὥστε καὶ ὁ ἐκ τῶν ΘΒ, BΓ μετὰ τοῦ ἀπὸ ΓΖ ἴσος ἔσται τῷ ἐκ τῶν AB, BΓ μετὰ τοῦ ἀπὸ ΓΕ· ὅπερ ἄτοπον. οὐκ ἄρα ὁ ἐκ τῶν AB, BΓ μετὰ τοῦ ἀπὸ ΓΕ ἴσος ἐστὶ [τῷ] ἐλάσσωνι τοῦ ἀπὸ ΒΕ. ἐδείχθη δέ, ὅτι οὐδὲ [αὐτῷ] τῷ ἀπὸ ΒΕ. οὐκ ἄρα ὁ ἐκ τῶν AB, BΓ μετὰ τοῦ ἀπὸ ΓΕ τετράγωνός ἐστιν. ὅπερ ἔδει δεῖξαι.

Lemma II

To find two square numbers such that the sum of them is not square.



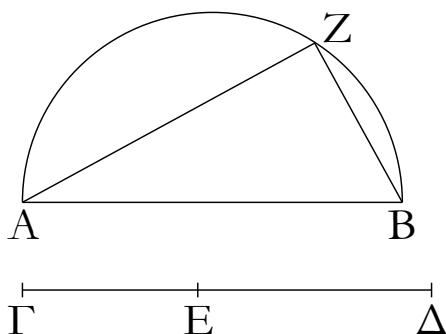
For let the (number created) from (multiplying) AB and BC , as we said, be square. And (let) CA (be) even. And let CA have been cut in half at D . So it is clear that the square (number created) from (multiplying) AB and BC , plus the square on CD , is equal to the square on BD [see previous lemma]. Let the unit DE have been subtracted (from BD). Thus, the (number created) from (multiplying) AB and BC , plus the (square) on CE , is less than the square on BD . I say, therefore, that the square (number created) from (multiplying) AB and BC , plus the (square) on CE , is not square.

For if it is square, it is either equal to the (square) on BE , or less than the (square) on BE , but cannot any more be greater (than the square on BE), lest the unit be divided. First of all, if possible, let the (number created) from (multiplying) AB and BC , plus the (square) on CE , be equal to the (square) on BE . And let GA be double the unit DE . Therefore, since the whole of AC is double the whole of CD , of which AG is double DE , the remainder GC is thus double the remainder EC . Thus, GC has been cut in half at E . Thus, the (number created) from (multiplying) GB and BC , plus the (square) on CE , is equal to the square on BE [Prop. 2.6]. But, the (number created) from (multiplying) AB and BC , plus the (square) on CE , was also assumed (to be) equal to the square on BE . Thus, the (number created) from (multiplying) GB and BC , plus the (square) on CE , is equal to the (number created) from (multiplying) AB and BC , plus the (square) on CE . And subtracting the (square) on CE from both, AB is inferred (to be) equal to GB . The very thing is absurd. Thus, the (number created) from (multiplying) AB and BC , plus the (square) on CE , is not equal to the (square) on BE . So I say that (it is) not less than the (square) on BE either. For, if possible, let it be equal to the (square) on BF . And (let) HA (be) double DF . And it can again be inferred that HC (is) double CF . Hence, CH has also been cut in half at F . And, on account of this, the (number created) from (multiplying) HB and BC , plus the (square) on FC , becomes equal to the (square) on BF [Prop. 2.6]. And the (number created) from (multiplying) AB and BC , plus the (square) on CE , was also assumed (to be) equal to the (square) on BF . Hence, the (number created) from (multiplying) HB and BC , plus the (square) on CF , will also be equal to the (number created) from (multiplying) AB and BC ,

plus the (square) on CE . The very thing is absurd. Thus, the (number created) from (multiplying) AB and BC , plus the (square) on CE , is not equal to less than the (square) on BE . And it was shown that (is it) not equal to the (square) on BE either. Thus, the (number created) from (multiplying) AB and BC , plus the square on CE , is not square. (Which is) the very thing it was required to show.

κθ'.

Εὑρεῖν δύο ῥητὰς δυνάμει μόνον συμμετρους, ὥστε τὴν μείζονα τῆς ἐλάσσονος μεῖζον δύνασθαι τῷ ἀπὸ συμμετρου ἑαυτῇ μήκει.

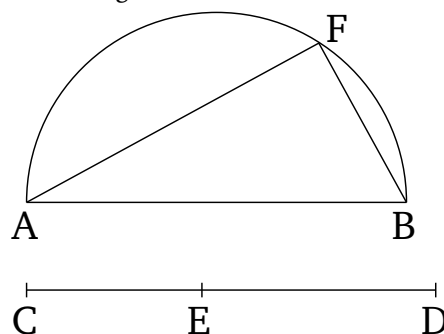


Ἐκκείσθω γάρ τις ῥητὴ ἡ AB καὶ δύο τετράγωνοι ἀριθμοὶ οἱ $\Gamma\Delta$, ΔE , ὥστε τὴν ὑπεροχὴν αὐτῶν τὸν GE μὴ εἶναι τετράγωνον, καὶ γεγράφθω ἐπὶ τῆς AB ἡμικύκλιον τὸ AZB , καὶ πεποιήσθω ὡς ὁ $\Delta\Gamma$ πρὸς τὸν GE , οὕτως τὸ ἀπὸ τῆς BA τετράγωνον πρὸς τὸ ἀπὸ τῆς AZ τετράγωνον, καὶ ἐπεζεύχθω ἡ ZB .

Ἐπεὶ [οὖν] ἐστὶν ὡς τὸ ἀπὸ τῆς BA πρὸς τὸ ἀπὸ τῆς AZ , οὕτως ὁ $\Delta\Gamma$ πρὸς τὸν GE , τὸ ἀπὸ τῆς BA ἄρα πρὸς τὸ ἀπὸ τῆς AZ λόγον ἔχει, ὃν ἀριθμὸς ὁ $\Delta\Gamma$ πρὸς ἀριθμὸν τὸν GE . σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς BA τῷ ἀπὸ τῆς AZ . ῥητὸν δὲ τὸ ἀπὸ τῆς AB . ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς AZ . ῥητὴ ἄρα καὶ ἡ AZ . καὶ ἐπεὶ ὁ $\Delta\Gamma$ πρὸς τὸν GE λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, οὐδὲ τὸ ἀπὸ τῆς BA ἄρα πρὸς τὸ ἀπὸ τῆς AZ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. ἀσύμμετρος ἄρα ἐστὶν ἡ AB τῇ AZ μήκει· αἱ BA , AZ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι. καὶ ἐπεὶ [ἐστὶν] ὡς ὁ $\Delta\Gamma$ πρὸς τὸν GE , οὕτως τὸ ἀπὸ τῆς BA πρὸς τὸ ἀπὸ τῆς AZ , ἀναστρέψαντι ἄρα ὡς ὁ $\Gamma\Delta$ πρὸς τὸν ΔE , οὕτως τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς BZ . ὁ δὲ $\Gamma\Delta$ πρὸς τὸν ΔE λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· καὶ τὸ ἀπὸ τῆς AB ἄρα πρὸς τὸ ἀπὸ τῆς BZ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. σύμμετρος ἄρα ἐστὶν ἡ AB τῇ BZ μήκει. καὶ ἐστὶ τὸ ἀπὸ τῆς AB ἴσον τοῖς ἀπὸ τῶν AZ , ZB . ἡ AB ἄρα τῆς AZ μεῖζον δύναται τῇ BZ συμμέτρῳ

Proposition 29

To find two rational (straight-lines which are) commensurable in square only, such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line which is) commensurable in length with the greater.



For let some rational (straight-line) AB be laid down, and two square numbers, CD and DE , such that the difference between them, CE , is not square [Prop. 10.28 lem. I]. And let the semi-circle AFB have been drawn on AB . And let it be contrived that as DC (is) to CE , so the square on BA (is) to the square on AF [Prop. 10.6 corr.]. And let FB have been joined.

[Therefore,] since as the (square) on BA is to the (square) on AF , so DC (is) to CE , the (square) on BA thus has to the (square) on AF the ratio which the number DC (has) to the number CE . Thus, the (square) on BA is commensurable with the (square) on AF [Prop. 10.6]. And the (square) on AB (is) rational [Def. 10.4]. Thus, the (square) on AF (is) also rational. Thus, AF (is) also rational. And since DC does not have to CE the ratio which (some) square number (has) to (some) square number, the (square) on BA thus does not have to the (square) on AF the ratio which (some) square number has to (some) square number either. Thus, AB is incommensurable in length with AF [Prop. 10.9]. Thus, the rational (straight-lines) BA and AF are commensurable in square only. And since as DC [is] to CE , so the (square) on BA (is) to the (square) on AF , thus, via conversion, as CD (is) to DE , so the (square) on AB (is) to the (square) on

ἑαυτῇ.

Εὕρηται ἄρα δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ BA , AZ , ὥστε τὴν μείζονα τὴν AB τῆς ἐλάσσονος τῆς AZ μείζον δύνασθαι τῷ ἀπὸ τῆς BZ συμμέτρου ἑαυτῇ μήκει· ὅπερ ἔδει δεῖξαι.

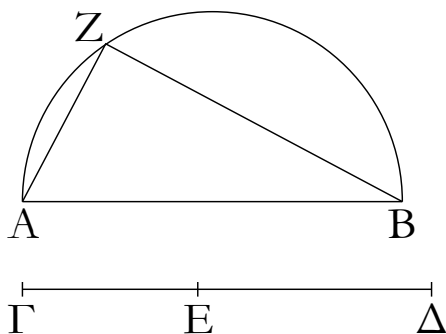
BF [Props. 5.19 corr., 3.31, 1.47]. And CD has to DE the ratio which (some) square number (has) to (some) square number. Thus, the (square) on AB also has to the (square) on BF the ratio which (some) square number has to (some) square number. AB is thus commensurable in length with BF [Prop. 10.9]. And the (square) on AB is equal to the (sum of the squares) on AF and FB [Prop. 1.47]. Thus, the square on AB is greater than (the square on) AF by (the square on) BF , (which is) commensurable (in length) with (AB) .

Thus, two rational (straight-lines), BA and AF , commensurable in square only, have been found such that the square on the greater, AB , is larger than (the square on) the lesser, AF , by the (square) on BF , (which is) commensurable in length with (AB) .[†] (Which is) the very thing it was required to show.

[†] BA and AF have lengths 1 and $\sqrt{1-k^2}$ times that of AB , respectively, where $k = \sqrt{DE/CD}$.

λ'.

Εὕρεῖν δύο ῥητὰς δυνάμει μόνον συμμέτρους, ὥστε τὴν μείζονα τῆς ἐλάσσονος μείζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει.

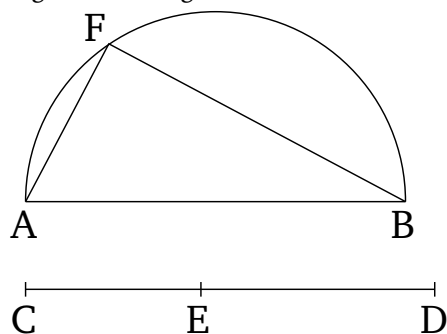


Ἐκκείσθω ῥητὴ ἡ AB καὶ δύο τετράγωνοι ἀριθμοὶ οἱ ΓE , $E\Delta$, ὥστε τὸν συγκείμενον ἐξ αὐτῶν τὸν $\Gamma\Delta$ μὴ εἶναι τετράγωνον, καὶ γεγράφθω ἐπὶ τῆς AB ἡμικύκλιον τὸ AZB , καὶ πεποιήσθω ὡς ὁ $\Delta\Gamma$ πρὸς τὸν ΓE , οὕτως τὸ ἀπὸ τῆς BA πρὸς τὸ ἀπὸ τῆς AZ , καὶ ἐπεζεύχθω ἡ ZB .

Ὅμοιως δὴ δείξομεν τῷ πρὸ τούτου, ὅτι αἱ BA , AZ ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι. καὶ ἐπεὶ ἐστὶν ὡς ὁ $\Delta\Gamma$ πρὸς τὸν ΓE , οὕτως τὸ ἀπὸ τῆς BA πρὸς τὸ ἀπὸ τῆς AZ , ἀναστρέψαντι ἄρα ὡς ὁ $\Gamma\Delta$ πρὸς τὸν ΔE , οὕτως τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς BZ . ὁ δὲ $\Gamma\Delta$ πρὸς τὸν ΔE λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ' ἄρα τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς BZ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ AB τῇ BZ μήκει. καὶ δύναται ἡ AB τῆς AZ μείζον τῷ ἀπὸ τῆς ZB ἀσυμμέτρου ἑαυτῇ.

Proposition 30

To find two rational (straight-lines which are) commensurable in square only, such that the square on the greater is larger than the (the square on) lesser by the (square) on (some straight-line which is) incommensurable in length with the greater.



Let the rational (straight-line) AB be laid out, and the two square numbers, CE and ED , such that the sum of them, CD , is not square [Prop. 10.28 lem. II]. And let the semi-circle AFB have been drawn on AB . And let it be contrived that as DC (is) to CE , so the (square) on BA (is) to the (square) on AF [Prop. 10.6 corr]. And let FB have been joined.

So, similarly to the (proposition) before this, we can show that BA and AF are rational (straight-lines which are) commensurable in square only. And since as DC is to CE , so the (square) on BA (is) to the (square) on AF , thus, via conversion, as CD (is) to DE , so the (square) on AB (is) to the (square) on BF [Props. 5.19 corr., 3.31, 1.47]. And CD does not have to DE the ratio which (some) square number (has) to (some) square number.

Αἱ AB , AZ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ AB τῆς AZ μείζον δύναται τῷ ἀπὸ τῆς ZB ἀσυμμέτρου ἑαυτῇ μήκει· ὅπερ εἶδει δεῖξαι.

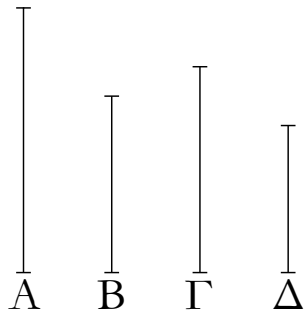
Thus, the (square) on AB does not have to the (square) on BZ the ratio which (some) square number has to (some) square number either. Thus, AB is incommensurable in length with BZ [Prop. 10.9]. And the square on AB is greater than the (square on) AZ by the (square) on ZB [Prop. 1.47], (which is) incommensurable (in length) with (AB) .

Thus, AB and AZ are rational (straight-lines which are) commensurable in square only, and the square on AB is greater than the (square on) AZ by the (square) on ZB , (which is) incommensurable (in length) with (AB) .[†] (Which is) the very thing it was required to show.

[†] AB and AZ have lengths 1 and $1/\sqrt{1+k^2}$ times that of AB , respectively, where $k = \sqrt{DE/CE}$.

λα'.

Εὐρεῖν δύο μέσας δυνάμει μόνον συμμέτρους ῥητὸν περιεχούσας, ὥστε τὴν μείζονα τῆς ἐλάσσονος μείζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει.

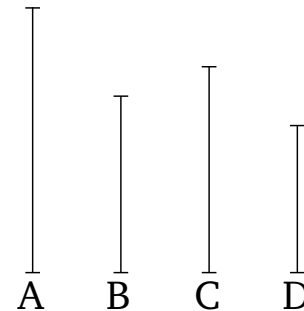


Ἐκκεῖσθωσαν δύο ῥηταί δυνάμει μόνον σύμμετροι αἱ A , B , ὥστε τὴν A μείζονα οὖσαν τῆς ἐλάσσονος τῆς B μείζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καὶ τῷ ὑπὸ τῶν A , B ἴσον ἔστω τὸ ἀπὸ τῆς Γ . μέσον δὲ τὸ ὑπὸ τῶν A , B · μέσον ἄρα καὶ τὸ ἀπὸ τῆς Γ · μέση ἄρα καὶ ἡ Γ . τῷ δὲ ἀπὸ τῆς B ἴσον ἔστω τὸ ὑπὸ τῶν Γ , Δ · ῥητὸν δὲ τὸ ἀπὸ τῆς B · ῥητὸν ἄρα καὶ τὸ ὑπὸ τῶν Γ , Δ . καὶ ἐπεὶ ἔστιν ὡς ἡ A πρὸς τὴν B , οὕτως τὸ ὑπὸ τῶν A , B πρὸς τὸ ἀπὸ τῆς B , ἀλλὰ τῷ μὲν ὑπὸ τῶν A , B ἴσον ἔστι τὸ ἀπὸ τῆς Γ , τῷ δὲ ἀπὸ τῆς B ἴσον τὸ ὑπὸ τῶν Γ , Δ , ὡς ἄρα ἡ A πρὸς τὴν B , οὕτως τὸ ἀπὸ τῆς Γ πρὸς τὸ ὑπὸ τῶν Γ , Δ . ὡς δὲ τὸ ἀπὸ τῆς Γ πρὸς τὸ ὑπὸ τῶν Γ , Δ , οὕτως ἡ Γ πρὸς τὴν Δ · καὶ ὡς ἄρα ἡ A πρὸς τὴν B , οὕτως ἡ Γ πρὸς τὴν Δ . σύμμετρος δὲ ἡ A τῇ B δυνάμει μόνον· σύμμετρος ἄρα καὶ ἡ Γ τῇ Δ δυνάμει μόνον. καὶ ἔστι μέση ἡ Γ · μέση ἄρα καὶ ἡ Δ . καὶ ἐπεὶ ἔστιν ὡς ἡ A πρὸς τὴν B , ἡ Γ πρὸς τὴν Δ , ἡ δὲ A τῆς B μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ Γ ἄρα τῆς Δ μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ.

Εὕρηται ἄρα δύο μέσας δυνάμει μόνον σύμμετροι αἱ Γ ,

Proposition 31

To find two medial (straight-lines), commensurable in square only, (and) containing a rational (area), such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable in length with the greater.



Let two rational (straight-lines), A and B , commensurable in square only, be laid out, such that the square on the greater A is larger than the (square on the) lesser B by the (square) on (some straight-line) commensurable in length with (A) [Prop. 10.29]. And let the (square) on C be equal to the (rectangle contained) by A and B . And the (rectangle contained) by A and B (is) medial [Prop. 10.21]. Thus, the (square) on C (is) also medial. Thus, C (is) also medial [Prop. 10.21]. And let the (rectangle contained) by C and D be equal to the (square) on B . And the (square) on B (is) rational. Thus, the (rectangle contained) by C and D (is) also rational. And since as A is to B , so the (rectangle contained) by A and B (is) to the (square) on B [Prop. 10.21 lem.], but the (square) on C is equal to the (rectangle contained) by A and B , and the (rectangle contained) by C and D to the (square) on B , thus as A (is) to B , so the (square) on C (is) to the (rectangle contained) by C and D . And as the (square) on C (is) to the (rectangle contained) by

Δ ῥητὸν περιέχουσαι, καὶ ἡ Γ τῆς Δ μείζον δυνάται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει.

Ὅμοίως δὴ δειχθήσεται καὶ τῷ ἀπὸ ἀσυμμέτρου, ὅταν ἡ A τῆς B μείζον δύνῃται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ.

C and D , so C (is) to D [Prop. 10.21 lem.]. And thus as A (is) to B , so C (is) to D . And A is commensurable in square only with B . Thus, C (is) also commensurable in square only with D [Prop. 10.11]. And C is medial. Thus, D (is) also medial [Prop. 10.23]. And since as A is to B , (so) C (is) to D , and the square on A is greater than (the square on) B by the (square) on (some straight-line) commensurable (in length) with (A) , the square on C is thus also greater than (the square on) D by the (square) on (some straight-line) commensurable (in length) with (C) [Prop. 10.14].

Thus, two medial (straight-lines), C and D , commensurable in square only, (and) containing a rational (area), have been found. And the square on C is greater than (the square on) D by the (square) on (some straight-line) commensurable in length with (C) .[†]

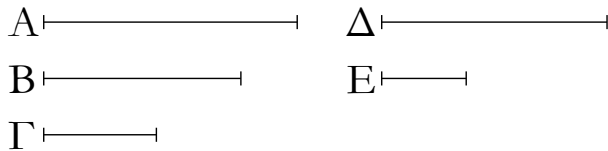
So, similarly, (the proposition) can also be demonstrated for (some straight-line) incommensurable (in length with C), provided that the square on A is greater than (the square on B) by the (square) on (some straight-line) incommensurable (in length) with (A) [Prop. 10.30].[‡]

[†] C and D have lengths $(1 - k^2)^{1/4}$ and $(1 - k^2)^{3/4}$ times that of A , respectively, where k is defined in the footnote to Prop. 10.29.

[‡] C and D would have lengths $1/(1 + k^2)^{1/4}$ and $1/(1 + k^2)^{3/4}$ times that of A , respectively, where k is defined in the footnote to Prop. 10.30.

λβ'.

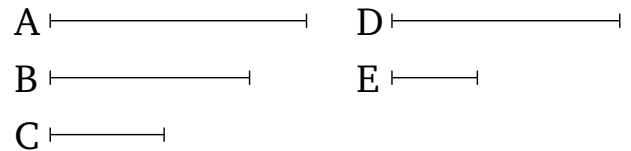
Εὕρεῖν δύο μέσας δυνάμει μόνον συμέτρους μέσον περιεχούσας, ὥστε τὴν μείζονα τῆς ἐλάσσονος μείζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ.



Ἐκκείσθωσαν τρεῖς ῥηταὶ δυνάμει μόνον σύμμετροι αἱ A , B , Γ , ὥστε τὴν A τῆς Γ μείζον δύνασθαι τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ τῷ μὲν ὑπὸ τῶν A , B ἴσον ἔστω τὸ ἀπὸ τῆς Δ . μέσον ἄρα τὸ ἀπὸ τῆς Δ · καὶ ἡ Δ ἄρα μέση ἐστίν. τῷ δὲ ὑπὸ τῶν B , Γ ἴσον ἔστω τὸ ὑπὸ τῶν Δ , E . καὶ ἐπεὶ ἐστὶν ὡς τὸ ὑπὸ τῶν A , B πρὸς τὸ ὑπὸ τῶν B , Γ , οὕτως ἡ A πρὸς τὴν Γ , ἀλλὰ τῷ μὲν ὑπὸ τῶν A , B ἴσον ἐστὶ τὸ ἀπὸ τῆς Δ , τῷ δὲ ὑπὸ τῶν B , Γ ἴσον τὸ ὑπὸ τῶν Δ , E , ἔστιν ἄρα ὡς ἡ A πρὸς τὴν Γ , οὕτως τὸ ἀπὸ τῆς Δ πρὸς τὸ ὑπὸ τῶν Δ , E . ὡς δὲ τὸ ἀπὸ τῆς Δ πρὸς τὸ ὑπὸ τῶν Δ , E , οὕτως ἡ Δ πρὸς τὴν E · καὶ ὡς ἄρα ἡ A πρὸς τὴν Γ , οὕτως ἡ Δ πρὸς τὴν E . σύμμετρος δὲ ἡ A τῇ Γ δυνάμει [μόνον]. σύμμετρος ἄρα καὶ ἡ Δ τῇ E δυνάμει μόνον. μέση

Proposition 32

To find two medial (straight-lines), commensurable in square only, (and) containing a medial (area), such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater.



Let three rational (straight-lines), A , B and C , commensurable in square only, be laid out such that the square on A is greater than (the square on C) by the (square) on (some straight-line) commensurable (in length) with (A) [Prop. 10.29]. And let the (square) on D be equal to the (rectangle contained) by A and B . Thus, the (square) on D (is) medial. Thus, D is also medial [Prop. 10.21]. And let the (rectangle contained) by D and E be equal to the (rectangle contained) by B and C . And since as the (rectangle contained) by A and B is to the (rectangle contained) by B and C , so A (is) to C [Prop. 10.21 lem.], but the (square) on D is equal to the (rectangle contained) by A and B , and the (rectangle

δὲ ἡ Δ · μέση ἄρα καὶ ἡ E . καὶ ἐπεὶ ἐστὶν ὡς ἡ A πρὸς τὴν Γ , ἡ Δ πρὸς τὴν E , ἡ δὲ A τῆς Γ μείζον δύνανται τῷ ἀπὸ συμμετρου ἑαυτῇ, καὶ ἡ Δ ἄρα τῆς E μείζον δυνήσεται τῷ ἀπὸ συμμετρου ἑαυτῇ. λέγω δὴ, ὅτι καὶ μέσον ἐστὶ τὸ ὑπὸ τῶν Δ , E . ἐπεὶ γὰρ ἴσον ἐστὶ τὸ ὑπὸ τῶν B , Γ τῷ ὑπὸ τῶν Δ , E , μέσον δὲ τὸ ὑπὸ τῶν B , Γ [αἱ γὰρ B , Γ ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι], μέσον ἄρα καὶ τὸ ὑπὸ τῶν Δ , E .

Εὐρηγται ἄρα δύο μέσαι δυνάμει μόνον σύμμετροι αἱ Δ , E μέσον περιέχουσιν, ὥστε τὴν μείζονα τῆς ἐλάσσονος μείζον δύνασθαι τῷ ἀπὸ συμμετρου ἑαυτῇ.

Ὅμοίως δὲ πάλιν διεχθήσεται καὶ τῷ ἀπὸ ἀσυμμετρου, ὅταν ἡ A τῆς Γ μείζον δύνηται τῷ ἀπὸ ἀσυμμετρου ἑαυτῇ.

contained) by D and E to the (rectangle contained) by B and C , thus as A is to C , so the (square) on D (is) to the (rectangle contained) by D and E . And as the (square) on D (is) to the (rectangle contained) by D and E , so D (is) to E [Prop. 10.21 lem.]. And thus as A (is) to C , so D (is) to E . And A (is) commensurable in square [only] with C . Thus, D (is) also commensurable in square only with E [Prop. 10.11]. And D (is) medial. Thus, E (is) also medial [Prop. 10.23]. And since as A is to C , (so) D (is) to E , and the square on A is greater than (the square on) C by the (square) on (some straight-line) commensurable (in length) with (A), the square on D will thus also be greater than (the square on) E by the (square) on (some straight-line) commensurable (in length) with (D) [Prop. 10.14]. So, I also say that the (rectangle contained) by D and E is medial. For since the (rectangle contained) by B and C is equal to the (rectangle contained) by D and E , and the (rectangle contained) by B and C (is) medial [for B and C are rational (straight-lines which are) commensurable in square only] [Prop. 10.21], the (rectangle contained) by D and E (is) thus also medial.

Thus, two medial (straight-lines), D and E , commensurable in square only, (and) containing a medial (area), have been found such that the square on the greater is larger than the (square on the) lesser by the (square) on (some straight-line) commensurable (in length) with the greater.[†]

So, similarly, (the proposition) can again also be demonstrated for (some straight-line) incommensurable (in length with the greater), provided that the square on A is greater than (the square on) C by the (square) on (some straight-line) incommensurable (in length) with (A) [Prop. 10.30].[‡]

[†] D and E have lengths $k^{1/4}$ and $k^{1/4}\sqrt{1-k^2}$ times that of A , respectively, where the length of B is $k^{1/2}$ times that of A , and k is defined in the footnote to Prop. 10.29.

[‡] D and E would have lengths $k^{1/4}$ and $k^{1/4}/\sqrt{1+k^2}$ times that of A , respectively, where the length of B is $k^{1/2}$ times that of A , and k is defined in the footnote to Prop. 10.30.

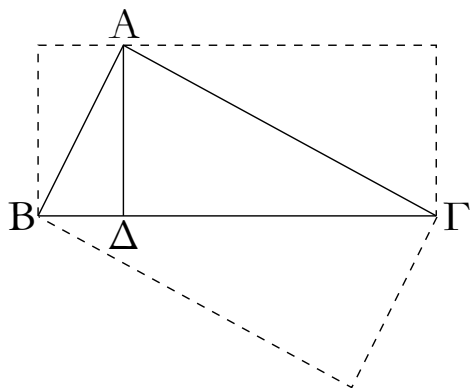
Λήμμα.

Ἐστω τρίγωνον ὀρθογώνιον τὸ $AB\Gamma$ ὀρθὴν ἔχον τὴν A , καὶ ῥητὴν κάθετος ἡ AD . λέγω, ὅτι τὸ μὲν ὑπὸ τῶν $\Gamma B\Delta$ ἴσον ἐστὶ τῷ ἀπὸ τῆς BA , τὸ δὲ ὑπὸ τῶν $B\Gamma A$ ἴσον τῷ ἀπὸ τῆς ΓA , καὶ τὸ ὑπὸ τῶν $B\Delta$, $\Delta\Gamma$ ἴσον τῷ ἀπὸ τῆς $A\Delta$, καὶ ἔτι τὸ ὑπὸ τῶν $B\Gamma$, $A\Delta$ ἴσον [ἐστὶ] τῷ ὑπὸ τῶν BA , $A\Gamma$.

Καὶ πρῶτον, ὅτι τὸ ὑπὸ τῶν $\Gamma B\Delta$ ἴσον [ἐστὶ] τῷ ἀπὸ τῆς BA .

Lemma

Let ABC be a right-angled triangle having the (angle) A a right-angle. And let the perpendicular AD have been drawn. I say that the (rectangle contained) by CBD is equal to the (square) on BA , and the (rectangle contained) by BCD (is) equal to the (square) on CA , and the (rectangle contained) by BD and DC (is) equal to the (square) on AD , and, further, the (rectangle contained) by BC and AD [is] equal to the (rectangle contained) by BA and AC .



Ἐπεὶ γὰρ ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἦται ἡ AD , τὰ ABD , ADG ἄρα τρίγωνα ὁμοιά ἐστι τῷ τε ὅλῳ τῷ ABG καὶ ἀλλήλοις. καὶ ἐπεὶ ὁμοιόν ἐστι τὸ ABG τριγώνον τῷ ABD τριγώνῳ, ἔστιν ἄρα ὡς ἡ GB πρὸς τὴν BA , οὕτως ἡ BA πρὸς τὴν BD . τὸ ἄρα ὑπὸ τῶν GBD ἴσον ἐστὶ τῷ ἀπὸ τῆς AB .

Δ ιὰ τὰ αὐτὰ δὴ καὶ τὸ ὑπὸ τῶν BGD ἴσον ἐστὶ τῷ ἀπὸ τῆς AG .

Καὶ ἐπεὶ, ἐὰν ἐν ὀρθογωνίῳ τριγώνῳ ἀπὸ τῆς ὀρθῆς γωνίας ἐπὶ τὴν βάσιν κάθετος ἀχθῇ, ἡ ἀχθεῖσα τῶν τῆς βάσεως τμημάτων μέση ἀνάλογόν ἐστιν, ἔστιν ἄρα ὡς ἡ BA πρὸς τὴν DA , οὕτως ἡ AD πρὸς τὴν DG . τὸ ἄρα ὑπὸ τῶν BAD , DAG ἴσον ἐστὶ τῷ ἀπὸ τῆς DA .

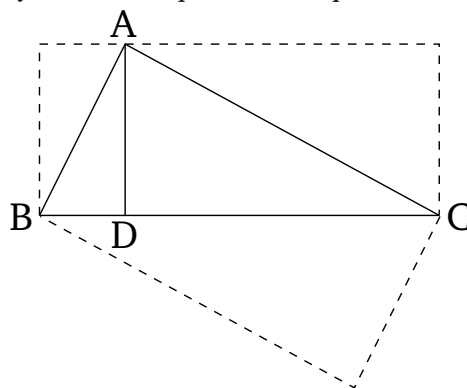
Λέγω, ὅτι καὶ τὸ ὑπὸ τῶν BG , AD ἴσον ἐστὶ τῷ ὑπὸ τῶν BA , AG . ἐπεὶ γὰρ, ὡς ἔφαμεν, ὁμοιόν ἐστι τὸ ABG τῷ ABD , ἔστιν ἄρα ὡς ἡ BG πρὸς τὴν GA , οὕτως ἡ BA πρὸς τὴν AD . τὸ ἄρα ὑπὸ τῶν BG , AD ἴσον ἐστὶ τῷ ὑπὸ τῶν BA , AG . ὅπερ ἔδει δεῖξαι.

λγ'.

Εὐρεῖν δύο εὐθείας δυνάμει ἀσυμμέτρους ποιούσας τὸ μὲν συγκεῖμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δ' ὑπ' αὐτῶν μέσον.

Ἐκκεῖσθωσαν δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ AB , BC , ὥστε τὴν μείζονα τὴν AB τῆς ἐλάσσονος τῆς BC μείζον δύνασθαι τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ τετμήσθω ἡ BC δίχα κατὰ τὸ D , καὶ τῷ ἀφ' ὁποτέρως τῶν BD , DC ἴσον παρὰ τὴν AB παραβεβλήσθω παραλληλόγραμμον ἐλλείπον εἶδος τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν AEB , καὶ γεγράφθω ἐπὶ τῆς AB ἡμικύκλιον τὸ AZB , καὶ ἦχθω τῇ AB πρὸς

And, first of all, (let us prove) that the (rectangle contained) by CBD [is] equal to the (square) on BA .



For since AD has been drawn from the right-angle in a right-angled triangle, perpendicular to the base, ABD and ADC are thus triangles (which are) similar to the whole, ABC , and to one another [Prop. 6.8]. And since triangle ABC is similar to triangle ABD , thus as CB is to BA , so BA (is) to BD [Prop. 6.4]. Thus, the (rectangle contained) by CBD is equal to the (square) on AB [Prop. 6.17].

So, for the same (reasons), the (rectangle contained) by BCD is also equal to the (square) on AC .

And since if a (straight-line) is drawn from the right-angle in a right-angled triangle, perpendicular to the base, the (straight-line so) drawn is the mean proportional to the pieces of the base [Prop. 6.8 corr.], thus as BD is to DA , so AD (is) to DC . Thus, the (rectangle contained) by BD and DC is equal to the (square) on DA [Prop. 6.17].

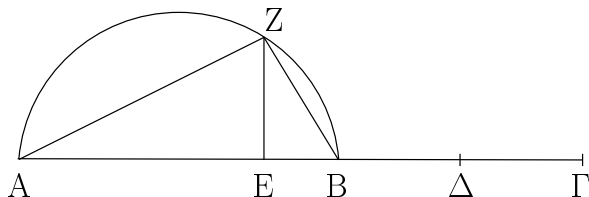
I also say that the (rectangle contained) by BC and AD is equal to the (rectangle contained) by BA and AC . For since, as we said, ABC is similar to ABD , thus as BC is to CA , so BA (is) to AD [Prop. 6.4]. Thus, the (rectangle contained) by BC and AD is equal to the (rectangle contained) by BA and AC [Prop. 6.16]. (Which is) the very thing it was required to show.

Proposition 33

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial.

Let the two rational (straight-lines) AB and BC , (which are) commensurable in square only, be laid out such that the square on the greater, AB , is larger than (the square on) the lesser, BC , by the (square) on (some straight-line which is) incommensurable (in length) with (AB) [Prop. 10.30]. And let BC have been cut in half at D . And let a parallelogram equal to the (square) on ei-

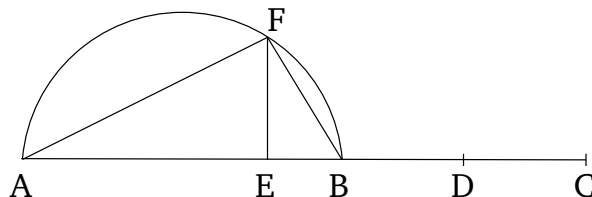
ὀρθὰς ἡ EZ, καὶ ἐπεζεύχθωσαν αἱ AZ, ZB.



Καὶ ἐπεὶ [δύο] εὐθεῖαι ἄνισοί εἰσιν αἱ AB, BΓ, καὶ ἡ AB τῆς BΓ μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ, τῷ δὲ τετάρτῳ τοῦ ἀπὸ τῆς BΓ, τουτέστι τῷ ἀπὸ τῆς ἡμισείας αὐτῆς, ἴσον παρὰ τὴν AB παραβέβληται παραλληλόγραμμον ἐλλείπον εἶδει τετραγώνῳ καὶ ποιεῖ τὸ ὑπὸ τῶν AEB, ἀσύμμετρος ἄρα ἐστὶν ἡ AE τῇ EB. καὶ ἐστὶν ὡς ἡ AE πρὸς EB, οὕτως τὸ ὑπὸ τῶν BA, AE πρὸς τὸ ὑπὸ τῶν AB, BE, ἴσον δὲ τὸ μὲν ὑπὸ τῶν BA, AE τῷ ἀπὸ τῆς AZ, τὸ δὲ ὑπὸ τῶν AB, BE τῷ ἀπὸ τῆς BZ· ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς AZ τῷ ἀπὸ τῆς BZ· αἱ AZ, ZB ἄρα δυνάμει εἰσὶν ἀσύμμετροι. καὶ ἐπεὶ ἡ AB ῥητὴ ἐστὶν, ῥητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς AB· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AZ, ZB ῥητὸν ἐστὶν. καὶ ἐπεὶ πάλιν τὸ ὑπὸ τῶν AE, EB ἴσον ἐστὶ τῷ ἀπὸ τῆς EZ, ὑπόκειται δὲ τὸ ὑπὸ τῶν AE, EB καὶ τῷ ἀπὸ τῆς BΔ ἴσον, ἴση ἄρα ἐστὶν ἡ ZE τῇ BΔ· διπλῇ ἄρα ἡ BΓ τῆς ZE· ὥστε καὶ τὸ ὑπὸ τῶν AB, BΓ σύμμετρόν ἐστι τῷ ὑπὸ τῶν AB, EZ. μέσον δὲ τὸ ὑπὸ τῶν AB, BΓ· μέσον ἄρα καὶ τὸ ὑπὸ τῶν AB, EZ. ἴσον δὲ τὸ ὑπὸ τῶν AB, EZ τῷ ὑπὸ τῶν AZ, ZB· μέσον ἄρα καὶ τὸ ὑπὸ τῶν AZ, ZB. ἐδείχθη δὲ καὶ ῥητὸν τὸ συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων.

Εὕρηται ἄρα δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ AZ, ZB ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητὸν, τὸ δὲ ὑπ' αὐτῶν μέσον· ὅπερ ἔδει δεῖξαι.

ther of BD or DC , (and) falling short by a square figure, have been applied to AB [Prop. 6.28], and let it be the (rectangle contained) by AEB . And let the semi-circle AFB have been drawn on AB . And let EF have been drawn at right-angles to AB . And let AF and FB have been joined.



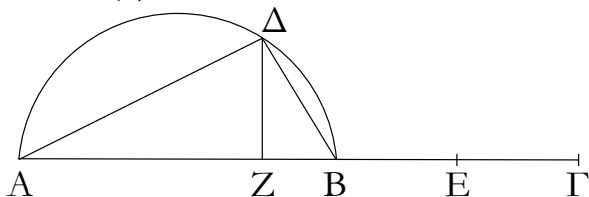
And since AB and BC are [two] unequal straight-lines, and the square on AB is greater than (the square on) BC by the (square) on (some straight-line which is) incommensurable (in length) with (AB). And a parallelogram, equal to one quarter of the (square) on BC —that is to say, (equal) to the (square) on half of it—and falling short by a square figure, has been applied to AB , and makes the (rectangle contained) by AEB . AE is thus incommensurable (in length) with EB [Prop. 10.18]. And as AE is to EB , so the (rectangle contained) by BA and AE (is) to the (rectangle contained) by AB and BE . And the (rectangle contained) by BA and AE (is) equal to the (square) on AF , and the (rectangle contained) by AB and BE to the (square) on BF [Prop. 10.32 lem.]. The (square) on AF is thus incommensurable with the (square) on FB [Prop. 10.11]. Thus, AF and FB are incommensurable in square. And since AB is rational, the (square) on AB is also rational. Hence, the sum of the (squares) on AF and FB is also rational [Prop. 1.47]. And, again, since the (rectangle contained) by AE and EB is equal to the (square) on EF , and the (rectangle contained) by AE and EB was assumed (to be) equal to the (square) on BD , FE is thus equal to BD . Thus, BC is double FE . And hence the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and EF [Prop. 10.6]. And the (rectangle contained) by AB and BC (is) medial [Prop. 10.21]. Thus, the (rectangle contained) by AB and EF (is) also medial [Prop. 10.23 corr.]. And the (rectangle contained) by AB and EF (is) equal to the (rectangle contained) by AF and FB [Prop. 10.32 lem.]. Thus, the (rectangle contained) by AF and FB (is) also medial. And the sum of the squares on them was also shown (to be) rational.

Thus, the two straight-lines, AF and FB , (which are) incommensurable in square, have been found, making the sum of the squares on them rational, and the (rectangle contained) by them medial. (Which is) the very thing it was required to show.

[†] AF and FB have lengths $\sqrt{[1 + k/(1 + k^2)^{1/2}]/2}$ and $\sqrt{[1 - k/(1 + k^2)^{1/2}]/2}$ times that of AB , respectively, where k is defined in the footnote to Prop. 10.30.

λδ'.

Εὑρεῖν δύο εὐθείας δυνάμει ἀσύμμετρος ποιούσας τὸ μὲν συγχείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δ' ὑπ' αὐτῶν ῥητόν.



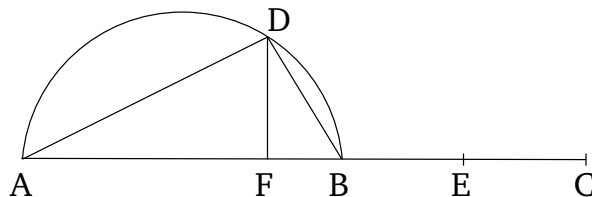
Ἐκκείσθωσαν δύο μέσαι δυνάμει μόνον σύμμετροι αἱ AB , $BΓ$ ῥητόν περιέχουσαι τὸ ὑπ' αὐτῶν, ὥστε τὴν AB τῆς $BΓ$ μείζον δύνασθαι τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ, καὶ γεγράφθω ἐπὶ τῆς AB τὸ $AΔB$ ἡμικύκλιον, καὶ τετμήσθω ἡ $BΓ$ δίχα κατὰ τὸ E , καὶ παραβεβλήσθω παρὰ τὴν AB τῷ ἀπὸ τῆς BE ἴσον παραλληλόγραμμον ἐλλείπον εἶδει τετραγώνῳ τὸ ὑπὸ τῶν AZB · ἀσύμμετρος ἄρα [ἐστίν] ἡ AZ τῇ ZB μήκει. καὶ ἤχθω ἀπὸ τοῦ Z τῇ AB πρὸς ὀρθὰς ἡ $ZΔ$, καὶ ἐπεζεύχθωσαν αἱ $AΔ$, $ΔB$.

Ἐπεὶ ἀσύμμετρός ἐστιν ἡ AZ τῇ ZB , ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν BA , AZ τῷ ὑπὸ τῶν AB , BZ . ἴσον δὲ τὸ μὲν ὑπὸ τῶν BA , AZ τῷ ἀπὸ τῆς $AΔ$, τὸ δὲ ὑπὸ τῶν AB , BZ τῷ ἀπὸ τῆς $ΔB$ · ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς $AΔ$ τῷ ἀπὸ τῆς $ΔB$. καὶ ἐπεὶ μέσον ἐστὶ τὸ ἀπὸ τῆς AB , μέσον ἄρα καὶ τὸ συγχείμενον ἐκ τῶν ἀπὸ τῶν $AΔ$, $ΔB$. καὶ ἐπεὶ διπλὴ ἐστὶν ἡ $BΓ$ τῆς $ΔZ$, διπλάσιον ἄρα καὶ τὸ ὑπὸ τῶν AB , $BΓ$ τοῦ ὑπὸ τῶν AB , $ZΔ$. ῥητόν δὲ τὸ ὑπὸ τῶν AB , $BΓ$ · ῥητόν ἄρα καὶ τὸ ὑπὸ τῶν AB , $ZΔ$. τὸ δὲ ὑπὸ τῶν AB , $ZΔ$ ἴσον τῷ ὑπὸ τῶν $AΔ$, $ΔB$ · ὥστε καὶ τὸ ὑπὸ τῶν $AΔ$, $ΔB$ ῥητόν ἐστίν.

Εὕρηται ἄρα δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ $AΔ$, $ΔB$ ποιούσαι τὸ [μὲν] συγχείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δ' ὑπ' αὐτῶν ῥητόν· ὅπερ ἔδει δεῖξαι.

Proposition 34

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational.



Let the two medial (straight-lines) AB and BC , (which are) commensurable in square only, be laid out having the (rectangle contained) by them rational, (and) such that the square on AB is greater than (the square on) BC by the (square) on (some straight-line) incommensurable (in length) with (AB) [Prop. 10.31]. And let the semi-circle ADB have been drawn on AB . And let BC have been cut in half at E . And let a (rectangular) parallelogram equal to the (square) on BE , (and) falling short by a square figure, have been applied to AB , (and let it be) the (rectangle contained by) AFB [Prop. 6.28]. Thus, AF [is] incommensurable in length with FB [Prop. 10.18]. And let FD have been drawn from F at right-angles to AB . And let AD and DB have been joined.

Since AF is incommensurable (in length) with FB , the (rectangle contained) by BA and AF is thus also incommensurable with the (rectangle contained) by AB and BF [Prop. 10.11]. And the (rectangle contained) by BA and AF (is) equal to the (square) on AD , and the (rectangle contained) by AB and BF to the (square) on DB [Prop. 10.32 lem.]. Thus, the (square) on AD is also incommensurable with the (square) on DB . And since the (square) on AB is medial, the sum of the (squares) on AD and DB (is) thus also medial [Props. 3.31, 1.47]. And since BC is double DF [see previous proposition], the (rectangle contained) by AB and BC (is) thus also double the (rectangle contained) by AB and FD . And the (rectangle contained) by AB and BC (is) rational. Thus, the (rectangle contained) by AB and FD (is) also rational [Prop. 10.6, Def. 10.4]. And the (rectangle contained) by AB and FD (is) equal to the (rectangle contained) by AD and DB [Prop. 10.32 lem.]. And hence the (rectangle contained) by AD and DB is rational.

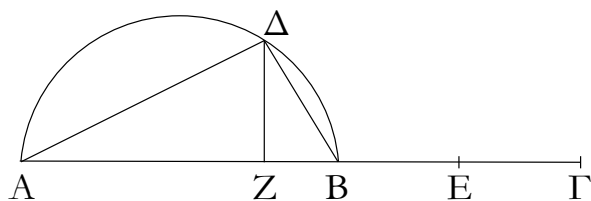
Thus, two straight-lines, AD and DB , (which are) incommensurable in square, have been found, making the sum of the squares on them medial, and the (rectangle

contained) by them rational.[†] (Which is) the very thing it was required to show.

[†] AD and DB have lengths $\sqrt{[(1+k^2)^{1/2} + k]/[2(1+k^2)]}$ and $\sqrt{[(1+k^2)^{1/2} - k]/[2(1+k^2)]}$ times that of AB , respectively, where k is defined in the footnote to Prop. 10.29.

λε'.

Εὑρεῖν δύο εὐθείας δυνάμει ἀσύμμετρος ποιούσας τό τε συγχείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ' αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγχειμένῳ ἐκ τῶν ἀπ' αὐτῶν τετραγώνων.



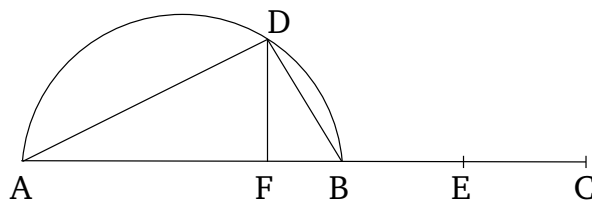
Ἐκκείσθωσαν δύο μέσαι δυνάμει μόνον σύμμετροι αἱ AB , $BΓ$ μέσον περιέχουσαι, ὥστε τὴν AB τῆς $BΓ$ μείζον δύνασθαι τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ, καὶ γεγράφθω ἐπὶ τῆς AB ἡμικύκλιον τὸ $AΔB$, καὶ τὰ λοιπὰ γεγονέτω τοῖς ἐπάνω ὁμοίως.

Καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ AZ τῇ ZB μήκει, ἀσύμμετρός ἐστι καὶ ἡ $AΔ$ τῇ $ΔB$ δυνάμει. καὶ ἐπεὶ μέσον ἐστὶ τὸ ἀπὸ τῆς AB , μέσον ἄρα καὶ τὸ συγχείμενον ἐκ τῶν ἀπὸ τῶν $AΔ$, $ΔB$. καὶ ἐπεὶ τὸ ὑπὸ τῶν AZ , ZB ἴσον ἐστὶ τῷ ἀφ' ἑκατέρως τῶν BE , $ΔZ$, ἴση ἄρα ἐστὶν ἡ BE τῇ $ΔZ$ · διπλῇ ἄρα ἡ $BΓ$ τῆς $ZΔ$ · ὥστε καὶ τὸ ὑπὸ τῶν AB , $BΓ$ διπλάσιόν ἐστι τοῦ ὑπὸ τῶν AB , $ZΔ$. μέσον δὲ τὸ ὑπὸ τῶν AB , $BΓ$ · μέσον ἄρα καὶ τὸ ὑπὸ τῶν AB , $ZΔ$. καὶ ἐστὶν ἴσον τῷ ὑπὸ τῶν $AΔ$, $ΔB$ · μέσον ἄρα καὶ τὸ ὑπὸ τῶν $AΔ$, $ΔB$. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ AB τῇ $BΓ$ μήκει, σύμμετρος δὲ ἡ $ΓB$ τῇ BE , ἀσύμμετρος ἄρα καὶ ἡ AB τῇ BE μήκει· ὥστε καὶ τὸ ἀπὸ τῆς AB τῷ ὑπὸ τῶν AB , BE ἀσύμμετρόν ἐστιν. ἀλλὰ τῷ μὲν ἀπὸ τῆς AB ἴσα ἐστὶ τὰ ἀπὸ τῶν $AΔ$, $ΔB$, τῷ δὲ ὑπὸ τῶν AB , BE ἴσον ἐστὶ τὸ ὑπὸ τῶν AB , $ZΔ$, τουτέστι τὸ ὑπὸ τῶν $AΔ$, $ΔB$ · ἀσύμμετρον ἄρα ἐστὶ τὸ συγχείμενον ἐκ τῶν ἀπὸ τῶν $AΔ$, $ΔB$ τῷ ὑπὸ τῶν $AΔ$, $ΔB$.

Εὐρηγνται ἄρα δύο εὐθεῖαι αἱ $AΔ$, $ΔB$ δυνάμει ἀσύμμετροι ποιοῦσαι τό τε συγχείμενον ἐκ τῶν ἀπ' αὐτῶν μέσον καὶ τὸ ὑπ' αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγχειμένῳ ἐκ τῶν ἀπ' αὐτῶν τετραγώνων· ὅπερ ἔδει δεῖξαι.

Proposition 35

To find two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them.



Let the two medial (straight-lines) AB and BC , (which are) commensurable in square only, be laid out containing a medial (area), such that the square on AB is greater than (the square on) BC by the (square) on (some straight-line) incommensurable (in length) with (AB) [Prop. 10.32]. And let the semi-circle ADB have been drawn on AB . And let the remainder (of the figure) be generated similarly to the above (proposition).

And since AF is incommensurable in length with FB [Prop. 10.18], AD is also incommensurable in square with DB [Prop. 10.11]. And since the (square) on AB is medial, the sum of the (squares) on AD and DB (is) thus also medial [Props. 3.31, 1.47]. And since the (rectangle contained) by AF and FB is equal to the (square) on each of BE and DF , BE is thus equal to DF . Thus, BC (is) double FD . And hence the (rectangle contained) by AB and BC is double the (rectangle) contained by AB and FD . And the (rectangle contained) by AB and BC (is) medial. Thus, the (rectangle contained) by AB and FD (is) also medial. And it is equal to the (rectangle contained) by AD and DB [Prop. 10.32 lem.]. Thus, the (rectangle contained) by AD and DB (is) also medial. And since AB is incommensurable in length with BC , and CB (is) commensurable (in length) with BE , AB (is) thus also incommensurable in length with BE [Prop. 10.13]. And hence the (square) on AB is also incommensurable with the (rectangle contained) by AB and BE [Prop. 10.11]. But the (sum of the squares) on AD and DB is equal to the (square) on AB [Prop. 1.47]. And the (rectangle contained) by AB and FD —that is to say, the (rectangle contained) by AD and DB —is equal to the (rectangle contained) by AB and BE . Thus, the

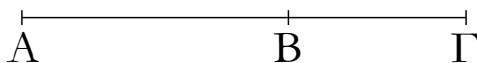
sum of the (squares) on AD and DB is incommensurable with the (rectangle contained) by AD and DB .

Thus, two straight-lines, AD and DB , (which are) incommensurable in square, have been found, making the sum of the (squares) on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them.[†] (Which is) the very thing it was required to show.

[†] AD and DB have lengths $k^{1/4}\sqrt{[1 + k/(1 + k^2)^{1/2}]/2}$ and $k'^{1/4}\sqrt{[1 - k/(1 + k^2)^{1/2}]/2}$ times that of AB , respectively, where k and k' are defined in the footnote to Prop. 10.32.

λαζ'.

Ἐὰν δύο ῥηταὶ δυνάμει μόνον σύμμετροι συντεθῶσιν, ἡ ὅλη ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο ὀνομάτων.

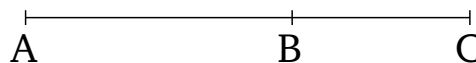


Συγκείσθωσαν γὰρ δύο ῥηταὶ δυνάμει μόνον σύμμετροι αἱ AB , BG . λέγω, ὅτι ὅλη ἡ AG ἄλογός ἐστιν.

Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ AB τῇ BG μήκει· δυνάμει γὰρ μόνον εἰσὶ σύμμετροι· ὥς δὲ ἡ AB πρὸς τὴν BG , οὕτως τὸ ὑπὸ τῶν ABG πρὸς τὸ ἀπὸ τῆς BG , ἀσύμμετρον ἄρα ἐστὶ τὸ ὑπὸ τῶν AB , BG τῷ ἀπὸ τῆς BG . ἀλλὰ τῷ μὲν ὑπὸ τῶν AB , BG σύμμετρόν ἐστι τὸ δις ὑπὸ τῶν AB , BG , τῷ δὲ ἀπὸ τῆς BG σύμμετρά ἐστι τὰ ἀπὸ τῶν AB , BG . αἱ γὰρ AB , BG ῥηταὶ εἰσὶ δυνάμει μόνον σύμμετροι· ἀσύμμετρον ἄρα ἐστὶ τὸ δις ὑπὸ τῶν AB , BG τοῖς ἀπὸ τῶν AB , BG . καὶ συνθέντι τὸ δις ὑπὸ τῶν AB , BG μετὰ τῶν ἀπὸ τῶν AB , BG , τουτέστι τὸ ἀπὸ τῆς AG , ἀσύμμετρόν ἐστι τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν AB , BG . ῥητὸν δὲ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AB , BG . ἄλογον ἄρα [ἐστὶ] τὸ ἀπὸ τῆς AG . ὥστε καὶ ἡ AG ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο ὀνομάτων· ὅπερ ἔδει δείξαι.

Proposition 36

If two rational (straight-lines which are) commensurable in square only are added together then the whole (straight-line) is irrational—let it be called a binomial (straight-line).[†]



For let the two rational (straight-lines), AB and BC , (which are) commensurable in square only, be laid down together. I say that the whole (straight-line), AC , is irrational. For since AB is incommensurable in length with BC —for they are commensurable in square only—and as AB (is) to BC , so the (rectangle contained) by ABC (is) to the (square) on BC , the (rectangle contained) by AB and BC is thus incommensurable with the (square) on BC [Prop. 10.11]. But, twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. And (the sum of) the (squares) on AB and BC is commensurable with the (square) on BC —for the rational (straight-lines) AB and BC are commensurable in square only [Prop. 10.15]. Thus, twice the (rectangle contained) by AB and BC is incommensurable with (the sum of) the (squares) on AB and BC [Prop. 10.13]. And, via composition, twice the (rectangle contained) by AB and BC , plus (the sum of) the (squares) on AB and BC —that is to say, the (square) on AC [Prop. 2.4]—is incommensurable with the sum of the (squares) on AB and BC [Prop. 10.16]. And the sum of the (squares) on AB and BC (is) rational. Thus, the (square) on AC [is] irrational [Def. 10.4]. Hence, AC is also irrational [Def. 10.4]—let it be called a binomial (straight-line).[‡] (Which is) the very thing it was required to show.

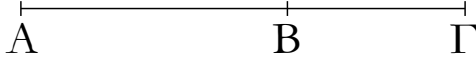
[†] Literally, “from two names”.

[‡] Thus, a binomial straight-line has a length expressible as $1 + k^{1/2}$ [or, more generally, $\rho(1 + k^{1/2})$, where ρ is rational—the same proviso applies to the definitions in the following propositions]. The binomial and the corresponding apotome, whose length is expressible as $1 - k^{1/2}$

(see Prop. 10.73), are the positive roots of the quartic $x^4 - 2(1+k)x^2 + (1-k)^2 = 0$.

λζ'.

Ἐὰν δύο μέσαι δυνάμει μόνον σύμμετροι συντεθῶσι ῥητὸν περιέχουσai, ἡ ὅλη ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο μέσων πρώτη.

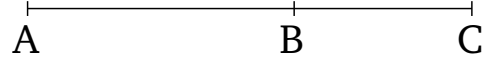


Συγκείσθωσαν γὰρ δύο μέσαι δυνάμει μόνον σύμμετροι αἱ AB, BG ῥητὸν περιέχουσai· λέγω, ὅτι ὅλη ἡ AG ἄλογός ἐστιν.

Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ AB τῇ BG μήκει, καὶ τὰ ἀπὸ τῶν AB, BG ἄρα ἀσύμμετρά ἐστι τῷ δις ὑπὸ τῶν AB, BG· καὶ συνθέντι τὰ ἀπὸ τῶν AB, BG μετὰ τοῦ δις ὑπὸ τῶν AB, BG, ὅπερ ἐστὶ τὸ ἀπὸ τῆς AG, ἀσύμμετρόν ἐστι τῷ ὑπὸ τῶν AB, BG. ῥητὸν δὲ τὸ ὑπὸ τῶν AB, BG· ὑπόκεινται γὰρ αἱ AB, BG ῥητὸν περιέχουσai· ἄλογον ἄρα τὸ ἀπὸ τῆς AG· ἄλογος ἄρα ἡ AG, καλείσθω δὲ ἐκ δύο μέσων πρώτη· ὅπερ εἶδει δεῖξαι.

Proposition 37

If two medial (straight-lines), commensurable in square only, which contain a rational (area), are added together then the whole (straight-line) is irrational—let it be called a first bimedial (straight-line).[†]



For let the two medial (straight-lines), AB and BC , commensurable in square only, (and) containing a rational (area), be laid down together. I say that the whole (straight-line), AC , is irrational.

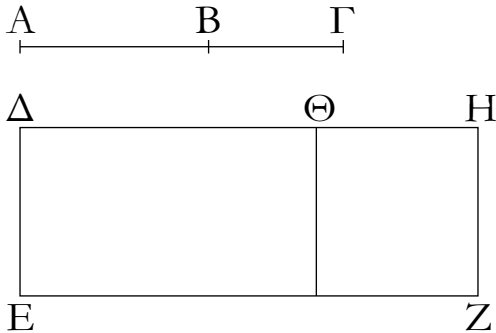
For since AB is incommensurable in length with BC , (the sum of) the (squares) on AB and BC is thus also incommensurable with twice the (rectangle contained) by AB and BC [see previous proposition]. And, via composition, (the sum of) the (squares) on AB and BC , plus twice the (rectangle contained) by AB and BC —that is, the (square) on AC [Prop. 2.4]—is incommensurable with the (rectangle contained) by AB and BC [Prop. 10.16]. And the (rectangle contained) by AB and BC (is) rational—for AB and BC were assumed to enclose a rational (area). Thus, the (square) on AC (is) irrational. Thus, AC (is) irrational [Def. 10.4]—let it be called a first bimedial (straight-line).[‡] (Which is) the very thing it was required to show.

[†] Literally, “first from two medials”.

[‡] Thus, a first bimedial straight-line has a length expressible as $k^{1/4} + k^{3/4}$. The first bimedial and the corresponding first apotome of a medial, whose length is expressible as $k^{1/4} - k^{3/4}$ (see Prop. 10.74), are the positive roots of the quartic $x^4 - 2\sqrt{k}(1+k)x^2 + k(1-k)^2 = 0$.

λη'.

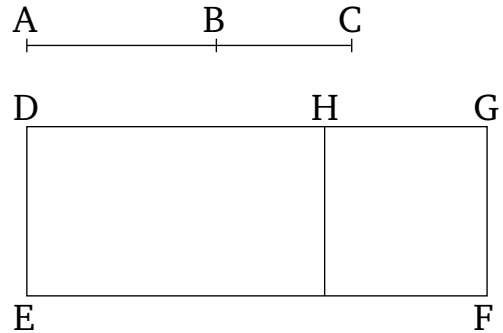
Ἐὰν δύο μέσαι δυνάμει μόνον σύμμετροι συντεθῶσι μέσον περιέχουσai, ἡ ὅλη ἄλογός ἐστιν, καλείσθω δὲ ἐκ δύο μέσων δευτέρα.



Συγκείσθωσαν γὰρ δύο μέσαι δυνάμει μόνον σύμμετροι αἱ AB, BG μέσον περιέχουσai· λέγω, ὅτι ἄλογός ἐστιν ἡ

Proposition 38

If two medial (straight-lines), commensurable in square only, which contain a medial (area), are added together then the whole (straight-line) is irrational—let it be called a second bimedial (straight-line).



For let the two medial (straight-lines), AB and BC , commensurable in square only, (and) containing a medial

ΑΓ.

Ἐκκεῖσθω γὰρ ῥητὴ ἡ ΔΕ, καὶ τῷ ἀπὸ τῆς ΑΓ ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΖ πλάτος ποιοῦν τὴν ΔΗ. καὶ ἐπεὶ τὸ ἀπὸ τῆς ΑΓ ἴσον ἐστὶ τοῖς τε ἀπὸ τῶν ΑΒ, ΒΓ καὶ τῷ δις ὑπὸ τῶν ΑΒ, ΒΓ, παραβεβλήσθω δὴ τοῖς ἀπὸ τῶν ΑΒ, ΒΓ παρὰ τὴν ΔΕ ἴσον τὸ ΕΘ· λοιπὸν ἄρα τὸ ΘΖ ἴσον ἐστὶ τῷ δις ὑπὸ τῶν ΑΒ, ΒΓ. καὶ ἐπεὶ μέση ἐστὶν ἑκατέρω ΑΒ, ΒΓ, μέσα ἄρα ἐστὶ καὶ τὰ ἀπὸ τῶν ΑΒ, ΒΓ. μέσον δὲ ὑπόκειται καὶ τὸ δις ὑπὸ τῶν ΑΒ, ΒΓ. καὶ ἐστὶ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΕΘ, τῷ δὲ δις ὑπὸ τῶν ΑΒ, ΒΓ ἴσον τὸ ΖΘ· μέσον ἄρα ἑκάτερον τῶν ΕΘ, ΘΖ. καὶ παρὰ ῥητὴν τὴν ΔΕ παράκειται ῥητὴ ἄρα ἐστὶν ἑκατέρω τῶν ΔΘ, ΘΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. ἐπεὶ οὖν ἀσύμμετρός ἐστιν ἡ ΑΒ τῇ ΒΓ μήκει, καὶ ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ὑπὸ τῶν ΑΒ, ΒΓ, ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΒ τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ σύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τετραγώνων, τῷ δὲ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δις ὑπὸ τῶν ΑΒ, ΒΓ. ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΒ, ΒΓ τῷ δις ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τὸ ΕΘ, τῷ δὲ δις ὑπὸ τῶν ΑΒ, ΒΓ ἴσον ἐστὶ τὸ ΘΖ. ἀσύμμετρον ἄρα ἐστὶ τὸ ΕΘ τῷ ΘΖ· ὥστε καὶ ἡ ΔΘ τῇ ΘΗ ἐστὶν ἀσύμμετρος μήκει. αἱ ΔΘ, ΘΗ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι. ὥστε ἡ ΔΗ ἄλογός ἐστιν. ῥητὴ δὲ ἡ ΔΕ· τὸ δὲ ὑπὸ ἀλόγου καὶ ῥητῆς περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν· ἄλογον ἄρα ἐστὶ τὸ ΔΖ χωρίον, καὶ ἡ δυναμένη [αὐτὸ] ἄλογός ἐστιν. δύναται δὲ τὸ ΔΖ ἡ ΑΓ· ἄλογος ἄρα ἐστὶν ἡ ΑΓ, καλεῖσθω δὲ ἐκ δύο μέσων δευτέρα. ὅπερ ἔδει δεῖξαι.

(area), be laid down together [Prop. 10.28]. I say that AC is irrational.

For let the rational (straight-line) DE be laid down, and let (the rectangle) DF , equal to the (square) on AC , have been applied to DE , making DG as breadth [Prop. 1.44]. And since the (square) on AC is equal to (the sum of) the (squares) on AB and BC , plus twice the (rectangle contained) by AB and BC [Prop. 2.4], so let (the rectangle) EH , equal to (the sum of) the squares on AB and BC , have been applied to DE . The remainder HF is thus equal to twice the (rectangle contained) by AB and BC . And since AB and BC are each medial, (the sum of) the squares on AB and BC is thus also medial.[†] And twice the (rectangle contained) by AB and BC was also assumed (to be) medial. And EH is equal to (the sum of) the squares on AB and BC , and FH (is) equal to twice the (rectangle contained) by AB and BC . Thus, EH and HF (are) each medial. And they were applied to the rational (straight-line) DE . Thus, DH and HG are each rational, and incommensurable in length with DE [Prop. 10.22]. Therefore, since AB is incommensurable in length with BC , and as AB is to BC , so the (square) on AB (is) to the (rectangle contained) by AB and BC [Prop. 10.21 lem.], the (square) on AB is thus incommensurable with the (rectangle contained) by AB and BC [Prop. 10.11]. But, the sum of the squares on AB and BC is commensurable with the (square) on AB [Prop. 10.15], and twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. Thus, the sum of the (squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC [Prop. 10.13]. But, EH is equal to (the sum of) the squares on AB and BC , and HF is equal to twice the (rectangle) contained by AB and BC . Thus, EH is incommensurable with HF . Hence, DH is also incommensurable in length with HG [Props. 6.1, 10.11]. Thus, DH and HG are rational (straight-lines which are) commensurable in square only. Hence, DG is irrational [Prop. 10.36]. And DE (is) rational. And the rectangle contained by irrational and rational (straight-lines) is irrational [Prop. 10.20]. The area DF is thus irrational, and (so) the square-root [of it] is irrational [Def. 10.4]. And AC is the square-root of DF . AC is thus irrational—let it be called a second bimedral (straight-line).[§] (Which is) the very thing it was required to show.

[†] Literally, “second from two medials”.

[‡] Since, by hypothesis, the squares on AB and BC are commensurable—see Props. 10.15, 10.23.

[§] Thus, a second bimedral straight-line has a length expressible as $k^{1/4} + k'^{1/2}/k^{1/4}$. The second bimedral and the corresponding second apotome of a medial, whose length is expressible as $k^{1/4} - k'^{1/2}/k^{1/4}$ (see Prop. 10.75), are the positive roots of the quartic $x^4 - 2[(k + k')/\sqrt{k}]x^2 +$

$$[(k - k')^2/k] = 0.$$

λθ'.

Ἐάν δύο εὐθεΐαι δυνάμει ἀσύμμετροι συντεθῶσι ποιούσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δ' ὑπ' αὐτῶν μέσον, ἢ ὅλη εὐθεΐα ἄλογός ἐστιν, καλείσθω δὲ μείζων.



Συγκείσθωσαν γὰρ δύο εὐθεΐαι δυνάμει ἀσύμμετροι αἱ AB, BG ποιούσαι τὰ προκείμενα· λέγω, ὅτι ἄλογός ἐστιν ἡ AG.

Ἐπεὶ γὰρ τὸ ὑπὸ τῶν AB, BG μέσον ἐστίν, καὶ τὸ δις [ἄρα] ὑπὸ τῶν AB, BG μέσον ἐστίν. τὸ δὲ συγκείμενον ἐκ τῶν ἀπὸ τῶν AB, BG ῥητόν· ἀσύμμετρον ἄρα ἐστὶ τὸ δις ὑπὸ τῶν AB, BG τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν AB, BG· ὥστε καὶ τὰ ἀπὸ τῶν AB, BG μετὰ τοῦ δις ὑπὸ τῶν AB, BG, ὅπερ ἐστὶ τὸ ἀπὸ τῆς AG, ἀσύμμετρόν ἐστι τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν AB, BG [ῥητόν δὲ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AB, BG]· ἄλογον ἄρα ἐστὶ τὸ ἀπὸ τῆς AG. ὥστε καὶ ἡ AG ἄλογός ἐστιν, καλείσθω δὲ μείζων. ὅπερ εἶδει δεῖξαι.

Proposition 39

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial, are added together then the whole straight-line is irrational—let it be called a major (straight-line).



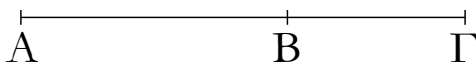
For let the two straight-lines, AB and BC , incommensurable in square, and fulfilling the prescribed (conditions), be laid down together [Prop. 10.33]. I say that AC is irrational.

For since the (rectangle contained) by AB and BC is medial, twice the (rectangle contained) by AB and BC is [thus] also medial [Props. 10.6, 10.23 corr.]. And the sum of the (squares) on AB and BC (is) rational. Thus, twice the (rectangle contained) by AB and BC is incommensurable with the sum of the (squares) on AB and BC [Def. 10.4]. Hence, (the sum of) the squares on AB and BC , plus twice the (rectangle contained) by AB and BC —that is, the (square) on AC [Prop. 2.4]—is also incommensurable with the sum of the (squares) on AB and BC [Prop. 10.16] [and the sum of the (squares) on AB and BC (is) rational]. Thus, the (square) on AC is irrational. Hence, AC is also irrational [Def. 10.4]—let it be called a major (straight-line).[†] (Which is) the very thing it was required to show.

[†] Thus, a major straight-line has a length expressible as $\sqrt{[1 + k/(1 + k^2)^{1/2}]/2} + \sqrt{[1 - k/(1 + k^2)^{1/2}]/2}$. The major and the corresponding minor, whose length is expressible as $\sqrt{[1 + k/(1 + k^2)^{1/2}]/2} - \sqrt{[1 - k/(1 + k^2)^{1/2}]/2}$ (see Prop. 10.76), are the positive roots of the quartic $x^4 - 2x^2 + k^2/(1 + k^2) = 0$.

μ'.

Ἐάν δύο εὐθεΐαι δυνάμει ἀσύμμετροι συντεθῶσι ποιούσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δ' ὑπ' αὐτῶν ῥητόν, ἢ ὅλη εὐθεΐα ἄλογός ἐστιν, καλείσθω δὲ ῥητόν καὶ μέσον δυναμένη.

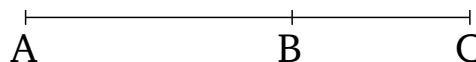


Συγκείσθωσαν γὰρ δύο εὐθεΐαι δυνάμει ἀσύμμετροι αἱ AB, BG ποιούσαι τὰ προκείμενα· λέγω, ὅτι ἄλογός ἐστιν ἡ AG.

Ἐπεὶ γὰρ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AB, BG μέσον ἐστίν, τὸ δὲ δις ὑπὸ τῶν AB, BG ῥητόν, ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AB, BG τῷ δις

Proposition 40

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational, are added together then the whole straight-line is irrational—let it be called the square-root of a rational plus a medial (area).



For let the two straight-lines, AB and BC , incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.34]. I say that AC is irrational.

For since the sum of the (squares) on AB and BC is medial, and twice the (rectangle contained) by AB and

ὑπὸ τῶν AB , $BΓ$ ὥστε καὶ τὸ ἀπὸ τῆς $ΑΓ$ ἀσύμμετρόν ἐστι τῷ δις ὑπὸ τῶν AB , $BΓ$. ῥητὸν δὲ τὸ δις ὑπὸ τῶν AB , $BΓ$ ἄλογον ἄρα τὸ ἀπὸ τῆς $ΑΓ$. ἄλογος ἄρα ἡ $ΑΓ$, καλείσθω δὲ ῥητὸν καὶ μέσον δυναμένη. ὅπερ ἔδει δεῖξαι.

BC (is) rational, the sum of the (squares) on AB and BC is thus incommensurable with twice the (rectangle contained) by AB and BC . Hence, the (square) on AC is also incommensurable with twice the (rectangle contained) by AB and BC [Prop. 10.16]. And twice the (rectangle contained) by AB and BC (is) rational. The (square) on AC (is) thus irrational. Thus, AC (is) irrational [Def. 10.4]—let it be called the square-root of a rational plus a medial (area).[†] (Which is) the very thing it was required to show.

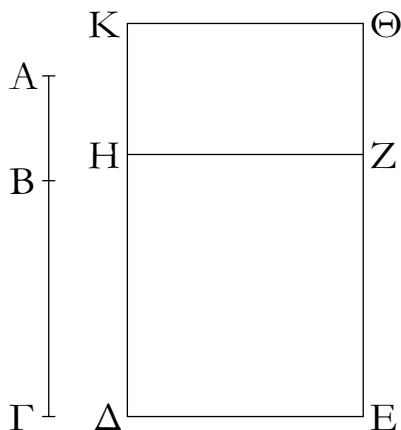
[†] Thus, the square-root of a rational plus a medial (area) has a length expressible as $\sqrt{[(1+k^2)^{1/2} + k]/[2(1+k^2)]} + \sqrt{[(1+k^2)^{1/2} - k]/[2(1+k^2)]}$. This and the corresponding irrational with a minus sign, whose length is expressible as $\sqrt{[(1+k^2)^{1/2} + k]/[2(1+k^2)]} - \sqrt{[(1+k^2)^{1/2} - k]/[2(1+k^2)]}$ (see Prop. 10.77), are the positive roots of the quartic $x^4 - (2/\sqrt{1+k^2})x^2 + k^2/(1+k^2)^2 = 0$.

μα'.

Ἐὰν δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιούσαι τὸ τε συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ' αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκείμενῳ ἐκ τῶν ἀπ' αὐτῶν τετραγώνων, ἡ ὅλη εὐθεῖα ἄλογός ἐστιν, καλείσθω δὲ δύο μέσα δυναμένη.

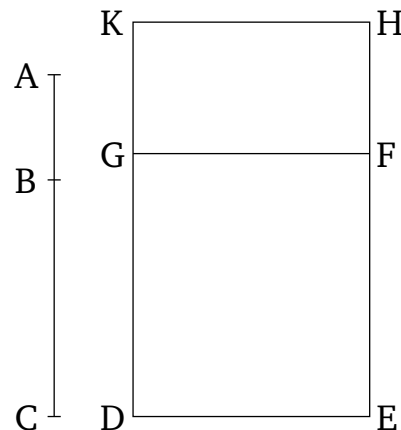
Proposition 41

If two straight-lines (which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the squares on them, are added together then the whole straight-line is irrational—let it be called the square-root of (the sum of) two medial (areas).



Συγκείμεθωσαν γὰρ δύο εὐθεῖαι δυνάμει ἀσύμμετροι αἱ AB , $BΓ$ ποιούσαι τὰ προκείμενα· λέγω, ὅτι ἡ $ΑΓ$ ἄλογός ἐστιν.

Ἐκκείσθω ῥητὴ ἡ $ΔΕ$, καὶ παραβεβλήσθω παρὰ τὴν $ΔΕ$ τοῖς μὲν ἀπὸ τῶν AB , $BΓ$ ἴσον τὸ $ΔΖ$, τῷ δὲ δις ὑπὸ τῶν AB , $BΓ$ ἴσον τὸ $ΗΘ$. ὅλον ἄρα τὸ $ΔΘ$ ἴσον ἐστὶ τῷ ἀπὸ τῆς $ΑΓ$ τετραγώνῳ. καὶ ἐπεὶ μέσον ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AB , $BΓ$, καὶ ἐστὶν ἴσον τῷ $ΔΖ$, μέσον ἄρα ἐστὶ καὶ τὸ $ΔΖ$. καὶ παρὰ ῥητὴν τὴν $ΔΕ$ παράκειται ῥητὴ ἄρα ἐστὶν ἡ $ΔΗ$ καὶ ἀσύμμετρος τῇ $ΔΕ$ μήκει. διὰ τὰ αὐτὰ δὲ καὶ ἡ $ΗΚ$ ῥητὴ ἐστὶ καὶ ἀσύμμετρος τῇ $ΗΖ$, τουτέστι τῇ $ΔΕ$, μήκει. καὶ ἐπεὶ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν AB , $BΓ$ τῷ δις ὑπὸ τῶν AB , $BΓ$, ἀσύμμετρόν ἐστι τὸ $ΔΖ$ τῷ $ΗΘ$.



For let the two straight-lines, AB and BC , incommensurable in square, (and) fulfilling the prescribed (conditions), be laid down together [Prop. 10.35]. I say that AC is irrational.

Let the rational (straight-line) DE be laid out, and let (the rectangle) DF , equal to (the sum of) the (squares) on AB and BC , and (the rectangle) GH , equal to twice the (rectangle contained) by AB and BC , have been applied to DE . Thus, the whole of DH is equal to the square on AC [Prop. 2.4]. And since the sum of the (squares) on AB and BC is medial, and is equal to DF , DF is thus also medial. And it is applied to the rational (straight-line) DE . Thus, DG is rational, and incommen-

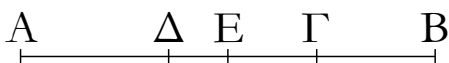
ὥστε καὶ ἡ ΔΗ τῇ ΗΚ ἀσύμμετρος ἐστίν. καὶ εἰσι ῥηταί· αἱ ΔΗ, ΗΚ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἄλλογος ἄρα ἐστὶν ἡ ΔΚ ἡ καλουμένη ἐκ δύο ὀνομάτων. ῥητὴ δὲ ἡ ΔΕ· ἄλλογον ἄρα ἐστὶ τὸ ΔΘ καὶ ἡ δυναμένη αὐτὸ ἄλλογός ἐστιν. δύνανται δὲ τὸ ΘΔ ἡ ΑΓ· ἄλλογος ἄρα ἐστὶν ἡ ΑΓ, καλείσθω δὲ δύο μέσα δυναμένη. ὅπερ ἔδει δεῖξαι.

surable in length with DE [Prop. 10.22]. So, for the same (reasons), GK is also rational, and incommensurable in length with GF —that is to say, DE . And since (the sum of) the (squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC , DF is incommensurable with GH . Hence, DG is also incommensurable (in length) with GK [Props. 6.1, 10.11]. And they are rational. Thus, DG and GK are rational (straight-lines which are) commensurable in square only. Thus, DK is irrational, and that (straight-line which is) called binomial [Prop. 10.36]. And DE (is) rational. Thus, DH is irrational, and its square-root is irrational [Def. 10.4]. And AC (is) the square-root of HD . Thus, AC is irrational—let it be called the square-root of (the sum of) two medial (areas).[†] (Which is) the very thing it was required to show.

[†] Thus, the square-root of (the sum of) two medial (areas) has a length expressible as $k^{1/4} \left(\sqrt{[1 + k/(1 + k^2)^{1/2}]/2} + \sqrt{[1 - k/(1 + k^2)^{1/2}]/2} \right)$. This and the corresponding irrational with a minus sign, whose length is expressible as $k^{1/4} \left(\sqrt{[1 + k/(1 + k^2)^{1/2}]/2} - \sqrt{[1 - k/(1 + k^2)^{1/2}]/2} \right)$ (see Prop. 10.78), are the positive roots of the quartic $x^4 - 2k^{1/2}x^2 + k'k^2/(1 + k^2) = 0$.

Λήμμα.

Ὅτι δὲ αἱ εἰρημέναι ἄλλογοι μοναχῶς διαιροῦνται εἰς τὰς εὐθείας, ἐξ ὧν σύγκεινται ποιουσῶν τὰ προκείμενα εἶδη, δείξομεν ἥδη προεκθέμενοι λημμάτιον τοιοῦτον·

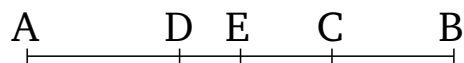


Ἐκκεῖσθω εὐθεῖα ἡ AB καὶ τετμήσθω ἡ ὅλη εἰς ἄνισα καθ' ἑκάτερον τῶν Γ , Δ , ὑποκείσθω δὲ μείζων ἡ $ΑΓ$ τῆς ΔB · λέγω, ὅτι τὰ ἀπὸ τῶν $ΑΓ$, ΓB μείζονα ἐστὶ τῶν ἀπὸ τῶν $\Delta\Delta$, ΔB .

Τετμήσθω γὰρ ἡ AB δίχα κατὰ τὸ E . καὶ ἐπεὶ μείζων ἐστὶν ἡ $ΑΓ$ τῆς ΔB , κοινῇ ἀφηρήσθω ἡ $\Delta\Gamma$ · λοιπὴ ἄρα ἡ $\Delta\Delta$ λοιπῆς τῆς ΓB μείζων ἐστίν. ἴση δὲ ἡ AE τῇ EB · ἐλάττωσιν ἄρα ἡ ΔE τῆς $E\Gamma$ · τὰ Γ , Δ ἄρα σημεία οὐκ ἴσον ἀπέχουσι τῆς διχοτομίας. καὶ ἐπεὶ τὸ ὑπὸ τῶν $ΑΓ$, ΓB μετὰ τοῦ ἀπὸ τῆς $E\Gamma$ ἴσον ἐστὶ τῷ ἀπὸ τῆς EB , ἀλλὰ μὴν καὶ τὸ ὑπὸ τῶν $\Delta\Delta$, ΔB μετὰ τοῦ ἀπὸ ΔE ἴσον ἐστὶ τῷ ἀπὸ τῆς EB , τὸ ἄρα ὑπὸ τῶν $ΑΓ$, ΓB μετὰ τοῦ ἀπὸ τῆς $E\Gamma$ ἴσον ἐστὶ τῷ ὑπὸ τῶν $\Delta\Delta$, ΔB μετὰ τοῦ ἀπὸ τῆς ΔE · ὧν τὸ ἀπὸ τῆς ΔE ἑλασσόν ἐστὶ τοῦ ἀπὸ τῆς $E\Gamma$ · καὶ λοιπὸν ἄρα τὸ ὑπὸ τῶν $ΑΓ$, ΓB ἑλασσόν ἐστὶ τοῦ ὑπὸ τῶν $\Delta\Delta$, ΔB . ὥστε καὶ τὸ δις ὑπὸ τῶν $ΑΓ$, ΓB ἑλασσόν ἐστὶ τοῦ δις ὑπὸ τῶν $\Delta\Delta$, ΔB . καὶ λοιπὸν ἄρα τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν $ΑΓ$, ΓB μείζον ἐστὶ τοῦ συγκειμένου ἐκ τῶν ἀπὸ τῶν $\Delta\Delta$, ΔB . ὅπερ ἔδει δεῖξαι.

Lemma

We will now demonstrate that the aforementioned irrational (straight-lines) are uniquely divided into the straight-lines of which they are the sum, and which produce the prescribed types, (after) setting forth the following lemma.



Let the straight-line AB be laid out, and let the whole (straight-line) have been cut into unequal parts at each of the (points) C and D . And let AC be assumed (to be) greater than DB . I say that (the sum of) the (squares) on AC and CB is greater than (the sum of) the (squares) on AD and DB .

For let AB have been cut in half at E . And since AC is greater than DB , let DC have been subtracted from both. Thus, the remainder AD is greater than the remainder CB . And AE (is) equal to EB . Thus, DE (is) less than EC . Thus, points C and D are not equally far from the point of bisection. And since the (rectangle contained) by AC and CB , plus the (square) on EC , is equal to the (square) on EB [Prop. 2.5], but, moreover, the (rectangle contained) by AD and DB , plus the (square) on DE , is also equal to the (square) on EB [Prop. 2.5], the (rectangle contained) by AC and CB , plus the (square) on EC , is thus equal to the (rectangle contained) by AD and DB , plus the (square) on DE . And, of these, the (square) on DE is less than the (square) on EC . And, thus, the

remaining (rectangle contained) by AC and CB is less than the (rectangle contained) by AD and DB . And, hence, twice the (rectangle contained) by AC and CB is less than twice the (rectangle contained) by AD and DB . And thus the remaining sum of the (squares) on AC and CB is greater than the sum of the (squares) on AD and DB .[†] (Which is) the very thing it was required to show.

[†] Since, $AC^2 + CB^2 + 2ACCB = AD^2 + DB^2 + 2ADDB = AB^2$.

μβ'.

Ἡ ἐκ δύο ὀνομάτων κατὰ ἓν μόνον σημεῖον διαιρεῖται εἰς τὰ ὀνόματα.



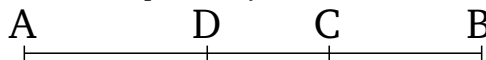
Ἐστω ἐκ δύο ὀνομάτων ἡ AB διηρημένη εἰς τὰ ὀνόματα κατὰ τὸ Γ . αἱ AG , GB ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι. λέγω, ὅτι ἡ AB κατ' ἄλλο σημεῖον οὐ διαιρεῖται εἰς δύο ῥητάς δυνάμει μόνον συμμέτρους.

Εἰ γὰρ δυνατόν, διηρήσθω καὶ κατὰ τὸ Δ , ὥστε καὶ τὰς $A\Delta$, ΔB ῥητάς εἶναι δυνάμει μόνον συμμέτρους. φανερόν δὴ, ὅτι ἡ AG τῇ ΔB οὐκ ἔστιν ἡ αὐτή. εἰ γὰρ δυνατόν, ἔστω. ἔσται δὴ καὶ ἡ $A\Delta$ τῇ GB ἡ αὐτή· καὶ ἔσται ὡς ἡ AG πρὸς τὴν GB , οὕτως ἡ $B\Delta$ πρὸς τὴν ΔA , καὶ ἔσται ἡ AB κατὰ τὸ αὐτὸ τῇ κατὰ τὸ Γ διαιρέσει διαιρεθεῖσα καὶ κατὰ τὸ Δ . ὅπερ οὐχ ὑπόκειται. οὐκ ἄρα ἡ AG τῇ ΔB ἔστιν ἡ αὐτή. διὰ δὴ τοῦτο καὶ τὰ Γ , Δ σημεῖα οὐκ ἴσον ἀπέχουσι τῆς διχοτομίας. ὅ ἄρα διαφέρει τὰ ἀπὸ τῶν AG , GB τῶν ἀπὸ τῶν $A\Delta$, ΔB , τούτῳ διαφέρει καὶ τὸ δις ὑπὸ τῶν $A\Delta$, ΔB τοῦ δις ὑπὸ τῶν AG , GB διὰ τὸ καὶ τὰ ἀπὸ τῶν AG , GB μετὰ τοῦ δις ὑπὸ τῶν AG , GB καὶ τὰ ἀπὸ τῶν $A\Delta$, ΔB μετὰ τοῦ δις ὑπὸ τῶν $A\Delta$, ΔB ἴσα εἶναι τῷ ἀπὸ τῆς AB . ἀλλὰ τὰ ἀπὸ τῶν AG , GB τῶν ἀπὸ τῶν $A\Delta$, ΔB διαφέρει ῥητῷ· ῥητὰ γὰρ ἀμφοτέρω· καὶ τὸ δις ἄρα ὑπὸ τῶν $A\Delta$, ΔB τοῦ δις ὑπὸ τῶν AG , GB διαφέρει ῥητῷ μέσῳ ὄντι· ὅπερ ἄτοπον· μέσον γὰρ μέσου οὐχ ὑπερέχει ῥητῷ.

Οὐχ ἄρα ἡ ἐκ δύο ὀνομάτων κατ' ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται· καθ' ἓν ἄρα μόνον· ὅπερ ἔδει δεῖξαι.

Proposition 42

A binomial (straight-line) can be divided into its (component) terms at one point only.[†]



Let AB be a binomial (straight-line) which has been divided into its (component) terms at C . AC and CB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. I say that AB cannot be divided at another point into two rational (straight-lines which are) commensurable in square only.

For, if possible, let it also have been divided at D , such that AD and DB are also rational (straight-lines which are) commensurable in square only. So, (it is) clear that AC is not the same as DB . For, if possible, let it be (the same). So, AD will also be the same as CB . And as AC will be to CB , so BD (will be) to DA . And AB will (thus) also be divided at D in the same (manner) as the division at C . The very opposite was assumed. Thus, AC is not the same as DB . So, on account of this, points C and D are not equally far from the point of bisection. Thus, by whatever (amount the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB , twice the (rectangle contained) by AD and DB also differs from twice the (rectangle contained) by AC and CB by this (same amount)—on account of both (the sum of) the (squares) on AC and CB , plus twice the (rectangle contained) by AC and CB , and (the sum of) the (squares) on AD and DB , plus twice the (rectangle contained) by AD and DB , being equal to the (square) on AB [Prop. 2.4]. But, (the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB by a rational (area). For (they are) both rational (areas). Thus, twice the (rectangle contained) by AD and DB also differs from twice the (rectangle contained) by AC and CB by a rational (area, despite both) being medial (areas) [Prop. 10.21]. The very thing is absurd. For a medial (area) cannot exceed a medial (area) by a rational (area) [Prop. 10.26].

Thus, a binomial (straight-line) cannot be divided (into its component terms) at different points. Thus, (it can be so divided) at one point only. (Which is) the very thing it was required to show.

† In other words, $k + k'^{1/2} = k'' + k'''^{1/2}$ has only one solution: i.e., $k'' = k$ and $k''' = k'$. Likewise, $k^{1/2} + k'^{1/2} = k''^{1/2} + k'''^{1/2}$ has only one solution: i.e., $k'' = k$ and $k''' = k'$ (or, equivalently, $k'' = k'$ and $k''' = k$).

μγ'.

Ἡ ἐκ δύο μέσων πρώτη καθ' ἐν μόνον σημείον διαιρεῖται.



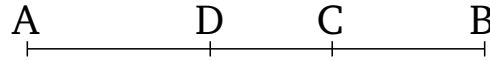
Ἐστω ἐκ δύο μέσων πρώτη ἡ AB διηρημένη κατὰ τὸ Γ , ὥστε τὰς AG , GB μέσας εἶναι δυνάμει μόνον συμμετρους ῥητὸν περιεχούσας· λέγω, ὅτι ἡ AB κατ' ἄλλο σημείον οὐ διαιρεῖται.

Εἰ γὰρ δυνατόν διηρήσθω καὶ κατὰ τὸ Δ , ὥστε καὶ τὰς $A\Delta$, ΔB μέσας εἶναι δυνάμει μόνον συμμετρους ῥητὸν περιεχούσας. ἐπεὶ οὖν, ὅς διαφέρει τὸ δις ὑπὸ τῶν $A\Delta$, ΔB τοῦ δις ὑπὸ τῶν AG , GB , τοῦτω διαφέρει τὰ ἀπὸ τῶν AG , GB τῶν ἀπὸ τῶν $A\Delta$, ΔB , ῥητῶ δὲ διαφέρει τὸ δις ὑπὸ τῶν $A\Delta$, ΔB τοῦ δις ὑπὸ τῶν AG , GB · ῥητὰ γὰρ ἀμφοτέρω· ῥητῶ ἄρα διαφέρει καὶ τὰ ἀπὸ τῶν AG , GB τῶν ἀπὸ τῶν $A\Delta$, ΔB μέσα ὄντα· ὅπερ ἄτοπον.

Οὐκ ἄρα ἡ ἐκ δύο μέσων πρώτη κατ' ἄλλο καὶ ἄλλο σημείον διαιρεῖται εἰς τὰ ὀνόματα· καθ' ἐν ἄρα μόνον· ὅπερ ἔδει δεῖξαι.

Proposition 43

A first binomial (straight-line) can be divided (into its component terms) at one point only.†



Let AB be a first binomial (straight-line) which has been divided at C , such that AC and CB are medial (straight-lines), commensurable in square only, (and) containing a rational (area) [Prop. 10.37]. I say that AB cannot be (so) divided at another point.

For, if possible, let it also have been divided at D , such that AD and DB are also medial (straight-lines), commensurable in square only, (and) containing a rational (area). Since, therefore, by whatever (amount) twice the (rectangle contained) by AD and DB differs from twice the (rectangle contained) by AC and CB , (the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB by this (same amount) [Prop. 10.41 lem.]. And twice the (rectangle contained) by AD and DB differs from twice the (rectangle contained) by AC and CB by a rational (area). For (they are) both rational (areas). (The sum of) the (squares) on AC and CB thus differs from (the sum of) the (squares) on AD and DB by a rational (area, despite both) being medial (areas). The very thing is absurd [Prop. 10.26].

Thus, a first binomial (straight-line) cannot be divided into its (component) terms at different points. Thus, (it can be so divided) at one point only. (Which is) the very thing it was required to show.

† In other words, $k^{1/4} + k^{3/4} = k'^{1/4} + k'^{3/4}$ has only one solution: i.e., $k' = k$.

μδ'.

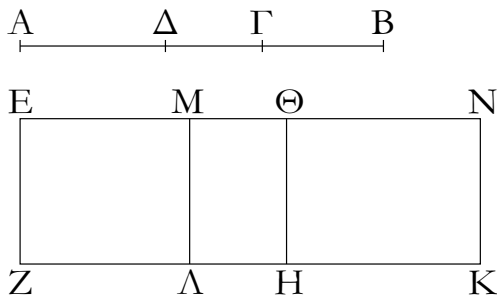
Ἡ ἐκ δύο μέσων δευτέρα καθ' ἐν μόνον σημείον διαιρεῖται.

Ἐστω ἐκ δύο μέσων δευτέρα ἡ AB διηρημένη κατὰ τὸ Γ , ὥστε τὰς AG , GB μέσας εἶναι δυνάμει μόνον συμμετρους μέσον περιεχούσας· φανερόν δὲ, ὅτι τὸ Γ οὐκ ἔστι κατὰ τῆς διχοτομίας, ὅτι οὐκ εἰσὶ μήκει σύμμετροι. λέγω, ὅτι ἡ AB κατ' ἄλλο σημείον οὐ διαιρεῖται.

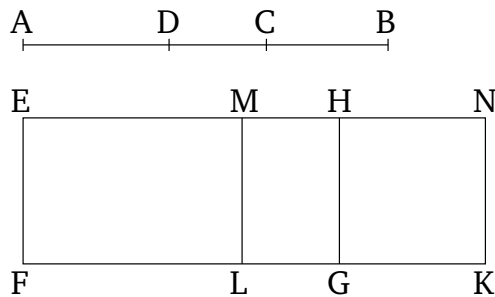
Proposition 44

A second binomial (straight-line) can be divided (into its component terms) at one point only.†

Let AB be a second binomial (straight-line) which has been divided at C , so that AC and BC are medial (straight-lines), commensurable in square only, (and) containing a medial (area) [Prop. 10.38]. So, (it is) clear that C is not (located) at the point of bisection, since (AC and BC) are not commensurable in length. I say that AB cannot be (so) divided at another point.



Εἰ γὰρ δυνατόν, διηρήσθω καὶ κατὰ τὸ Δ , ὥστε τὴν $ΑΓ$ τῇ ΔB μὴ εἶναι τὴν αὐτὴν, ἀλλὰ μείζονα καθ' ὑπόθεσιν τὴν $ΑΓ$ · δῆλον δὴ, ὅτι καὶ τὰ ἀπὸ τῶν $ΑΔ$, ΔB , ὡς ἐπάνω ἐδείξαμεν, ἐλάσσονα τῶν ἀπὸ τῶν $ΑΓ$, $ΓB$ · καὶ τὰς $ΑΔ$, ΔB μέσας εἶναι δυνάμει μόνον συμμετρους μέσον περιεχούσας. καὶ ἐκχείσθω ῥητὴ ἡ EZ , καὶ τῷ μὲν ἀπὸ τῆς AB ἴσον παρὰ τὴν EZ παραλληλόγραμμον ὀρθογώνιον παραβεβλήσθω τὸ EK , τοῖς δὲ ἀπὸ τῶν $ΑΓ$, $ΓB$ ἴσον ἀφηρήσθω τὸ EH · λοιπὸν ἄρα τὸ $ΘK$ ἴσον ἐστὶ τῷ δις ὑπὸ τῶν $ΑΓ$, $ΓB$. πάλιν δὴ τοῖς ἀπὸ τῶν $ΑΔ$, ΔB , ἄπερ ἐλάσσονα ἐδείχθη τῶν ἀπὸ τῶν $ΑΓ$, $ΓB$, ἴσον ἀφηρήσθω τὸ $ΕΛ$ · καὶ λοιπὸν ἄρα τὸ MK ἴσον τῷ δις ὑπὸ τῶν $ΑΔ$, ΔB . καὶ ἐπεὶ μέσα ἐστὶ τὰ ἀπὸ τῶν $ΑΓ$, $ΓB$, μέσον ἄρα [καὶ] τὸ EH . καὶ παρὰ ῥητὴν τὴν EZ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ $EΘ$ καὶ ἀσύμμετρος τῇ EZ μήκει. διὰ τὰ αὐτὰ δὴ καὶ ἡ $ΘN$ ῥητὴ ἐστὶ καὶ ἀσύμμετρος τῇ EZ μήκει. καὶ ἐπεὶ αἱ $ΑΓ$, $ΓB$ μέσα εἰσι δυνάμει μόνον σύμμετροι, ἀσύμμετρος ἄρα ἐστὶν ἡ $ΑΓ$ τῇ $ΓB$ μήκει. ὡς δὲ ἡ $ΑΓ$ πρὸς τὴν $ΓB$, οὕτως τὸ ἀπὸ τῆς $ΑΓ$ πρὸς τὸ ὑπὸ τῶν $ΑΓ$, $ΓB$ · ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς $ΑΓ$ τῷ ὑπὸ τῶν $ΑΓ$, $ΓB$. ἀλλὰ τῷ μὲν ἀπὸ τῆς $ΑΓ$ σύμμετρά ἐστι τὰ ἀπὸ τῶν $ΑΓ$, $ΓB$ · δυνάμει γάρ εἰσι σύμμετροι αἱ $ΑΓ$, $ΓB$. τῷ δὲ ὑπὸ τῶν $ΑΓ$, $ΓB$ σύμμετρόν ἐστι τὸ δις ὑπὸ τῶν $ΑΓ$, $ΓB$. καὶ τὰ ἀπὸ τῶν $ΑΓ$, $ΓB$ ἄρα ἀσύμμετρά ἐστι τῷ δις ὑπὸ τῶν $ΑΓ$, $ΓB$. ἀλλὰ τοῖς μὲν ἀπὸ τῶν $ΑΓ$, $ΓB$ ἴσον ἐστὶ τὸ EH , τῷ δὲ δις ὑπὸ τῶν $ΑΓ$, $ΓB$ ἴσον τὸ $ΘK$ · ἀσύμμετρον ἄρα ἐστὶ τὸ EH τῷ $ΘK$ · ὥστε καὶ ἡ $EΘ$ τῇ $ΘN$ ἀσύμμετρός ἐστι μήκει. καὶ εἰσι ῥηταί· αἱ $EΘ$, $ΘN$ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἐὰν δὲ δύο ῥηταὶ δυνάμει μόνον σύμμετροι συντεθῶσιν, ἡ ὅλη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο ὀνομάτων· ἡ EN ἄρα ἐκ δύο ὀνομάτων ἐστὶ διηρημένη κατὰ τὸ $Θ$. κατὰ τὰ αὐτὰ δὴ δειχθήσονται καὶ αἱ EM , MN ῥηταὶ δυνάμει μόνον σύμμετροι· καὶ ἔσται ἡ EN ἐκ δύο ὀνομάτων κατ' ἄλλο καὶ ἄλλο διηρημένη τό τε $Θ$ καὶ τὸ M , καὶ οὐκ ἔστιν ἡ $EΘ$ τῇ MN ἡ αὐτὴ, ὅτι τὰ ἀπὸ τῶν $ΑΓ$, $ΓB$ μείζονά ἐστι τῶν ἀπὸ τῶν $ΑΔ$, ΔB . ἀλλὰ τὰ ἀπὸ τῶν $ΑΔ$, ΔB μείζονά ἐστι τοῦ δις ὑπὸ $ΑΔ$, ΔB · πολλῷ ἄρα καὶ τὰ ἀπὸ τῶν $ΑΓ$, $ΓB$, τουτέστι τὸ EH , μείζον ἐστὶ τοῦ δις ὑπὸ τῶν $ΑΔ$, ΔB , τουτέστι τοῦ MK · ὥστε καὶ ἡ $EΘ$ τῆς MN μείζων ἐστίν. ἡ ἄρα $EΘ$ τῇ MN οὐκ ἔστιν ἡ αὐτὴ· ὅπερ ἔδει δεῖξαι.



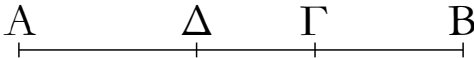
For, if possible, let it also have been (so) divided at D , so that AC is not the same as DB , but AC (is), by hypothesis, greater. So, (it is) clear that (the sum of) the (squares) on AD and DB is also less than (the sum of) the (squares) on AC and CB , as we showed above [Prop. 10.41 lem.]. And AD and DB are medial (straight-lines), commensurable in square only, (and) containing a medial (area). And let the rational (straight-line) EF be laid down. And let the rectangular parallelogram EK , equal to the (square) on AB , have been applied to EF . And let EG , equal to (the sum of) the (squares) on AC and CB , have been cut off (from EK). Thus, the remainder, HK , is equal to twice the (rectangle contained) by AC and CB [Prop. 2.4]. So, again, let EL , equal to (the sum of) the (squares) on AD and DB —which was shown (to be) less than (the sum of) the (squares) on AC and CB —have been cut off (from EK). And, thus, the remainder, MK , (is) equal to twice the (rectangle contained) by AD and DB . And since (the sum of) the (squares) on AC and CB is medial, EG (is) thus [also] medial. And it is applied to the rational (straight-line) EF . Thus, EH is rational, and incommensurable in length with EF [Prop. 10.22]. So, for the same (reasons), HN is also rational, and incommensurable in length with EF . And since AC and CB are medial (straight-lines which are) commensurable in square only, AC is thus incommensurable in length with CB . And as AC (is) to CB , so the (square) on AC (is) to the (rectangle contained) by AC and CB [Prop. 10.21 lem.]. Thus, the (square) on AC is incommensurable with the (rectangle contained) by AC and CB [Prop. 10.11]. But, (the sum of) the (squares) on AC and CB is commensurable with the (square) on AC . For, AC and CB are commensurable in square [Prop. 10.15]. And twice the (rectangle contained) by AC and CB is commensurable with the (rectangle contained) by AC and CB [Prop. 10.6]. And thus (the sum of) the squares on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB [Prop. 10.13]. But, EG is equal to (the sum of) the (squares) on AC and CB , and HK equal to twice the (rectangle contained) by AC and CB . Thus, EG is incommensurable with HK . Hence, EH is also incom-

measurable in length with HN [Props. 6.1, 10.11]. And (they are) rational (straight-lines). Thus, EH and HN are rational (straight-lines which are) commensurable in square only. And if two rational (straight-lines which are) commensurable in square only are added together then the whole (straight-line) is that irrational called binomial [Prop. 10.36]. Thus, EN is a binomial (straight-line) which has been divided (into its component terms) at H . So, according to the same (reasoning), EM and MN can be shown (to be) rational (straight-lines which are) commensurable in square only. And EN will (thus) be a binomial (straight-line) which has been divided (into its component terms) at the different (points) H and M (which is absurd [Prop. 10.42]). And EH is not the same as MN , since (the sum of) the (squares) on AC and CB is greater than (the sum of) the (squares) on AD and DB . But, (the sum of) the (squares) on AD and DB is greater than twice the (rectangle contained) by AD and DB [Prop. 10.59 lem.]. Thus, (the sum of) the (squares) on AC and CB —that is to say, EG —is also much greater than twice the (rectangle contained) by AD and DB —that is to say, MK . Hence, EH is also greater than MN [Prop. 6.1]. Thus, EH is not the same as MN . (Which is) the very thing it was required to show.

† In other words, $k^{1/4} + k^{1/2}/k^{1/4} = k'^{1/4} + k''^{1/2}/k'^{1/4}$ has only one solution: i.e., $k'' = k$ and $k''' = k'$.

με'.

Ἡ μείζων κατὰ τὸ αὐτὸ μόνον σημεῖον διαιρεῖται.

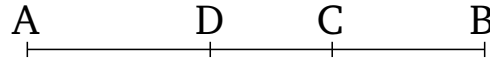


Ἐστω μείζων ἡ AB διηρημένη κατὰ τὸ Γ , ὥστε τὰς AG , GB δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν AG , GB τετραγώνων ῥητόν, τὸ δ' ὑπὸ τῶν AG , GB μέσον· λέγω, ὅτι ἡ AB κατ' ἄλλο σημεῖον οὐ διαιρεῖται.

Εἰ γὰρ δυνατόν, διηρήσθω καὶ κατὰ τὸ Δ , ὥστε καὶ τὰς $A\Delta$, ΔB δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν $A\Delta$, ΔB ῥητόν, τὸ δ' ὑπ' αὐτῶν μέσον. καὶ ἐπεὶ, ὥ διαφέρει τὰ ἀπὸ τῶν AG , GB τῶν ἀπὸ τῶν $A\Delta$, ΔB , τούτῳ διαφέρει καὶ τὸ δις ὑπὸ τῶν $A\Delta$, ΔB τοῦ δις ὑπὸ τῶν AG , GB , ἀλλὰ τὰ ἀπὸ τῶν AG , GB τῶν ἀπὸ τῶν $A\Delta$, ΔB ὑπερέχει ῥητῶ· ῥητὰ γὰρ ἀμφοτέρω· καὶ τὸ δις ὑπὸ τῶν $A\Delta$, ΔB ἄρα τοῦ δις ὑπὸ τῶν AG , GB ὑπερέχει ῥητῶ μέσα ὄντα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ μείζων κατ' ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται· κατὰ τὸ αὐτὸ ἄρα μόνον διαιρεῖται· ὅπερ ἔδει δεῖξαι.

Proposition 45

A major (straight-line) can only be divided (into its component terms) at the same point.†



Let AB be a major (straight-line) which has been divided at C , so that AC and CB are incommensurable in square, making the sum of the squares on AC and CB rational, and the (rectangle contained) by AC and CB medial [Prop. 10.39]. I say that AB cannot be (so) divided at another point.

For, if possible, let it also have been divided at D , such that AD and DB are also incommensurable in square, making the sum of the (squares) on AD and DB rational, and the (rectangle contained) by them medial. And since, by whatever (amount the sum of) the (squares) on AC and CB differs from (the sum of) the (squares) on AD and DB , twice the (rectangle contained) by AD and DB also differs from twice the (rectangle contained) by AC and CB by this (same amount). But, (the sum of) the (squares) on AC and CB exceeds (the sum of) the (squares) on AD and DB by a rational (area). For (they are) both rational (areas). Thus, twice the (rectangle

contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area), (despite both) being medial (areas). The very thing is impossible [Prop. 10.26]. Thus, a major (straight-line) cannot be divided (into its component terms) at different points. Thus, it can only be (so) divided at the same (point). (Which is) the very thing it was required to show.

[†] In other words, $\sqrt{[1 + k/(1 + k^2)^{1/2}]/2} + \sqrt{[1 - k/(1 + k^2)^{1/2}]/2} = \sqrt{[1 + k'/(1 + k'^2)^{1/2}]/2} + \sqrt{[1 - k'/(1 + k'^2)^{1/2}]/2}$ has only one solution: i.e., $k' = k$.

μζ'.

Ἡ ῥητὸν καὶ μέσον δυναμένη καθ' ἓν μόνον σημεῖον διαιρεῖται.

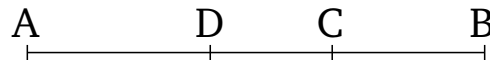


Ἐστω ῥητὸν καὶ μέσον δυναμένη ἡ AB διηρημένη κατὰ τὸ Γ , ὥστε τὰς $A\Gamma$, ΓB δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγχείμενον ἐκ τῶν ἀπὸ τῶν $A\Gamma$, ΓB μέσον, τὸ δὲ δις ὑπὸ τῶν $A\Gamma$, ΓB ῥητόν· λέγω, ὅτι ἡ AB κατ' ἄλλο σημεῖον οὐ διαιρεῖται.

Εἰ γὰρ δυνατόν, διηρήσθω καὶ κατὰ τὸ Δ , ὥστε καὶ τὰς $A\Delta$, ΔB δυνάμει ἀσυμμέτρους εἶναι ποιούσας τὸ μὲν συγχείμενον ἐκ τῶν ἀπὸ τῶν $A\Delta$, ΔB μέσον, τὸ δὲ δις ὑπὸ τῶν $A\Delta$, ΔB ῥητόν. ἐπεὶ οὖν, ὅς διαφέρει τὸ δις ὑπὸ τῶν $A\Gamma$, ΓB τοῦ δις ὑπὸ τῶν $A\Delta$, ΔB , τούτῳ διαφέρει καὶ τὰ ἀπὸ τῶν $A\Delta$, ΔB τῶν ἀπὸ τῶν $A\Gamma$, ΓB , τὸ δὲ δις ὑπὸ τῶν $A\Gamma$, ΓB τοῦ δις ὑπὸ τῶν $A\Delta$, ΔB ὑπερέχει ῥητῶ, καὶ τὰ ἀπὸ τῶν $A\Delta$, ΔB ἄρα τῶν ἀπὸ τῶν $A\Gamma$, ΓB ὑπερέχει ῥητῶ μέσα ὄντα· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ῥητὸν καὶ μέσον δυναμένη κατ' ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται. κατὰ ἓν ἄρα σημεῖον διαιρεῖται· ὅπερ εἶδει δεῖξαι.

Proposition 46

The square-root of a rational plus a medial (area) can be divided (into its component terms) at one point only.[†]



Let AB be the square-root of a rational plus a medial (area) which has been divided at C , so that AC and CB are incommensurable in square, making the sum of the (squares) on AC and CB medial, and twice the (rectangle contained) by AC and CB rational [Prop. 10.40]. I say that AB cannot be (so) divided at another point.

For, if possible, let it also have been divided at D , so that AD and DB are also incommensurable in square, making the sum of the (squares) on AD and DB medial, and twice the (rectangle contained) by AD and DB rational. Therefore, since by whatever (amount) twice the (rectangle contained) by AC and CB differs from twice the (rectangle contained) by AD and DB , (the sum of) the (squares) on AD and DB also differs from (the sum of) the (squares) on AC and CB by this (same amount). And twice the (rectangle contained) by AC and CB exceeds twice the (rectangle contained) by AD and DB by a rational (area). (The sum of) the (squares) on AD and DB thus also exceeds (the sum of) the (squares) on AC and CB by a rational (area), (despite both) being medial (areas). The very thing is impossible [Prop. 10.26]. Thus, the square-root of a rational plus a medial (area) cannot be divided (into its component terms) at different points. Thus, it can be (so) divided at one point (only). (Which is) the very thing it was required to show.

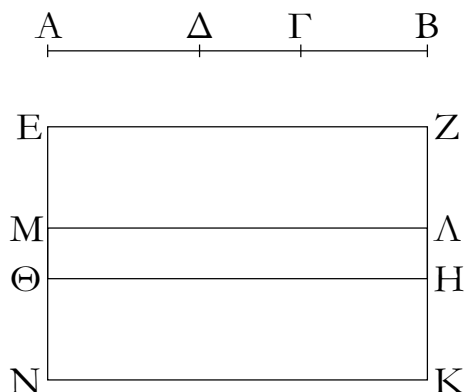
[†] In other words, $\sqrt{[(1 + k^2)^{1/2} + k]/[2(1 + k^2)]} + \sqrt{[(1 + k^2)^{1/2} - k]/[2(1 + k^2)]} = \sqrt{[(1 + k'^2)^{1/2} + k']/[2(1 + k'^2)]} + \sqrt{[(1 + k'^2)^{1/2} - k']/[2(1 + k'^2)]}$ has only one solution: i.e., $k' = k$.

μζ'.

Ἡ δύο μέσα δυναμένη καθ' ἓν μόνον σημεῖον διαιρεῖται.

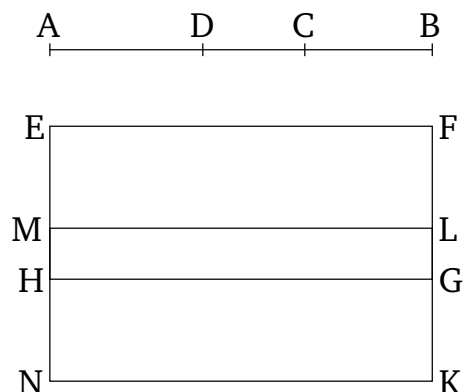
Proposition 47

The square-root of (the sum of) two medial (areas) can be divided (into its component terms) at one point only.[†]



Ἐστω [δύο μέσα δυναμένη] ἡ AB διηρημένη κατὰ τὸ Γ , ὥστε τὰς AG , GB δυνάμει ἀσύμμετρος εἶναι ποιούσας τό τε συγχείμενον ἐκ τῶν ἀπὸ τῶν AG , GB μέσον καὶ τὸ ὑπὸ τῶν AG , GB μέσον καὶ ἔτι ἀσύμμετρον τῷ συγχείμενῳ ἐκ τῶν ἀπ' αὐτῶν. λέγω, ὅτι ἡ AB κατ' ἄλλο σημεῖον οὐ διαιρεῖται ποιούσα τὰ προκείμενα.

Εἰ γὰρ δυνατόν, διηρήσθω κατὰ τὸ Δ , ὥστε πάλιν δηλονότι τὴν AG τῇ ΔB μὴ εἶναι τὴν αὐτήν, ἀλλὰ μείζονα καθ' ὑπόθεσιν τὴν AG , καὶ ἐκκείσθω ῥητὴ ἡ EZ , καὶ παραβελήσθω παρὰ τὴν EZ τοῖς μὲν ἀπὸ τῶν AG , GB ἴσον τὸ EH , τῷ δὲ δις ὑπὸ τῶν AG , GB ἴσον τὸ ΘK . ὅλον ἄρα τὸ EK ἴσον ἐστὶ τῷ ἀπὸ τῆς AB τετραγώνῳ. πάλιν δὲ παραβελήσθω παρὰ τὴν EZ τοῖς ἀπὸ τῶν $A\Delta$, ΔB ἴσον τὸ EL . λοιπὸν ἄρα τὸ δις ὑπὸ τῶν $A\Delta$, ΔB λοιπῷ τῷ MK ἴσον ἐστίν. καὶ ἐπεὶ μέσον ὑπόκειται τὸ συγχείμενον ἐκ τῶν ἀπὸ τῶν AG , GB , μέσον ἄρα ἐστὶ καὶ τὸ EH . καὶ παρὰ ῥητὴν τὴν EZ παράκειται ῥητὴ ἄρα ἐστὶν ἡ ΘE καὶ ἀσύμμετρος τῇ EZ μήκει. διὰ τὰ αὐτὰ δὲ καὶ ἡ ΘN ῥητὴ ἐστὶ καὶ ἀσύμμετρος τῇ EZ μήκει. καὶ ἐπεὶ ἀσύμμετρον ἐστὶ τὸ συγχείμενον ἐκ τῶν ἀπὸ τῶν AG , GB τῷ δις ὑπὸ τῶν AG , GB , καὶ τὸ EH ἄρα τῷ HN ἀσύμμετρον ἐστίν· ὥστε καὶ ἡ $E\Theta$ τῇ ΘN ἀσύμμετρος ἐστίν. καὶ εἰσι ῥηταί· αἱ $E\Theta$, ΘN ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ EN ἄρα ἐκ δύο ὀνομάτων ἐστὶ διηρημένη κατὰ τὸ Θ . ὁμοίως δὲ δεῖξομεν, ὅτι καὶ κατὰ τὸ M διήρηται. καὶ οὐκ ἔστιν ἡ $E\Theta$ τῇ MN ἡ αὐτὴ· ἡ ἄρα ἐκ δύο ὀνομάτων κατ' ἄλλο καὶ ἄλλο σημεῖον διήρηται· ὅπερ ἐστὶν ἄτοπον. οὐκ ἄρα ἡ δύο μέσα δυναμένη κατ' ἄλλο καὶ ἄλλο σημεῖον διαιρεῖται· καθ' ἓν ἄρα μόνον [σημεῖον] διαιρεῖται.



Let AB be [the square-root of (the sum of) two medial (areas)] which has been divided at C , such that AC and CB are incommensurable in square, making the sum of the (squares) on AC and CB medial, and the (rectangle contained) by AC and CB medial, and, moreover, incommensurable with the sum of the (squares) on $(AC$ and $CB)$ [Prop. 10.41]. I say that AB cannot be divided at another point fulfilling the prescribed (conditions).

For, if possible, let it have been divided at D , such that AC is again manifestly not the same as DB , but AC (is), by hypothesis, greater. And let the rational (straight-line) EF be laid down. And let EG , equal to (the sum of) the (squares) on AC and CB , and HK , equal to twice the (rectangle contained) by AC and CB , have been applied to EF . Thus, the whole of EK is equal to the square on AB [Prop. 2.4]. So, again, let EL , equal to (the sum of) the (squares) on AD and DB , have been applied to EF . Thus, the remainder—twice the (rectangle contained) by AD and DB —is equal to the remainder, MK . And since the sum of the (squares) on AC and CB was assumed (to be) medial, EG is also medial. And it is applied to the rational (straight-line) EF . HE is thus rational, and incommensurable in length with EF [Prop. 10.22]. So, for the same (reasons), HN is also rational, and incommensurable in length with EF . And since the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB , EG is thus also incommensurable with GN . Hence, EH is also incommensurable with HN [Props. 6.1, 10.11]. And they are (both) rational (straight-lines). Thus, EH and HN are rational (straight-lines which are) commensurable in square only. Thus, EN is a binomial (straight-line) which has been divided (into its component terms) at H [Prop. 10.36]. So, similarly, we can show that it has also been (so) divided at M . And EH is not the same as MN . Thus, a binomial (straight-line) has been divided (into its component terms) at different points. The very thing is absurd [Prop. 10.42]. Thus, the square-root of (the sum of) two medial (areas) cannot be divided (into

its component terms) at different points. Thus, it can be (so) divided at one [point] only.

† In other words, $k'^{1/4}\sqrt{[1+k/(1+k^2)^{1/2}]/2} + k'^{1/4}\sqrt{[1-k/(1+k^2)^{1/2}]/2} = k''^{1/4}\sqrt{[1+k''/(1+k''^2)^{1/2}]/2} + k''^{1/4}\sqrt{[1-k''/(1+k''^2)^{1/2}]/2}$ has only one solution: i.e., $k'' = k$ and $k''' = k'$.

Ὅροι δεύτεροι.

ε'. Ὑποκειμένης ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων διηρημένης εἰς τὰ ὀνόματα, ἥς τὸ μείζον ὄνομα τοῦ ἐλάσσονος μείζον δύναται τῷ ἀπὸ συμμετρου ἐαυτῇ μήκει, ἐὰν μὲν τὸ μείζον ὄνομα σύμμετρον ᾖ τῇ μήκει τῆς ἐκκειμένης ῥητῆς, καλεῖσθω [ἡ ὅλη] ἐκ δύο ὀνομάτων πρώτη.

ς'. Ἐὰν δὲ τὸ ἐλάσσον ὄνομα σύμμετρον ᾖ τῇ μήκει τῆς ἐκκειμένης ῥητῆς, καλεῖσθω ἐκ δύο ὀνομάτων δευτέρα.

ζ'. Ἐὰν δὲ μηδέτερον τῶν ὀνομάτων σύμμετρον ᾖ τῇ μήκει τῆς ἐκκειμένης ῥητῆς, καλεῖσθω ἐκ δύο ὀνομάτων τρίτη.

η'. Πάλιν δὲ ἐὰν τὸ μείζον ὄνομα [τοῦ ἐλάσσονος] μείζον δύνηται τῷ ἀπὸ ἀσυμμετρου ἐαυτῇ μήκει, ἐὰν μὲν τὸ μείζον ὄνομα σύμμετρον ᾖ τῇ μήκει τῆς ἐκκειμένης ῥητῆς, καλεῖσθω ἐκ δύο ὀνομάτων τετάρτη.

θ'. Ἐὰν δὲ τὸ ἐλάσσον, πέμπτη.

ι'. Ἐὰν δὲ μηδέτερον, ἕκτη.

Definitions II

5. Given a rational (straight-line), and a binomial (straight-line) which has been divided into its (component) terms, of which the square on the greater term is larger than (the square on) the lesser by the (square) on (some straight-line) commensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let [the whole] (straight-line) be called a first binomial (straight-line).

6. And if the lesser term is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a second binomial (straight-line).

7. And if neither of the terms is commensurable in length with the rational (straight-line previously) laid out then let (the whole straight-line) be called a third binomial (straight-line).

8. So, again, if the square on the greater term is larger than (the square on) [the lesser] by the (square) on (some straight-line) incommensurable in length with (the greater) then, if the greater term is commensurable in length with the rational (straight-line previously) laid out, let (the whole straight-line) be called a fourth binomial (straight-line).

9. And if the lesser (term is commensurable), a fifth (binomial straight-line).

10. And if neither (term is commensurable), a sixth (binomial straight-line).

μη'.

Εὐρεῖν τὴν ἐκ δύο ὀνομάτων πρώτην.

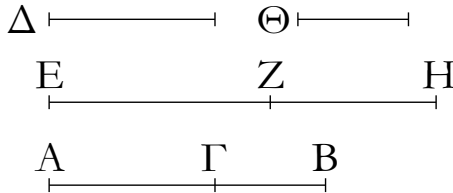
Ἐκκεῖσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν συγχεόμενον ἐξ αὐτῶν τὸν ΑΒ πρὸς μὲν τὸν ΒΓ λόγον ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, πρὸς δὲ τὸν ΓΑ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, καὶ ἐκκεῖσθω τις ῥητὴ ἡ Δ, καὶ τῇ Δ σύμμετρος ἔστω μήκει ἡ ΕΖ. ῥητὴ ἄρα ἐστὶ καὶ ἡ ΕΖ. καὶ γεγρονέτω ὡς ὁ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ. ὁ δὲ ΑΒ πρὸς τὸν ΑΓ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· καὶ τὸ ἀπὸ τῆς ΕΖ ἄρα πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμόν· ὥστε σύμμετρον ἐστὶ τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς

Proposition 48

To find a first binomial (straight-line).

Let two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to CA the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let some rational (straight-line) D be laid down. And let EF be commensurable in length with D . EF is thus also rational [Def. 10.3]. And let it have been contrived that as the number BA (is) to AC , so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.]. And AB has to AC the ratio which (some) number (has) to (some) num-

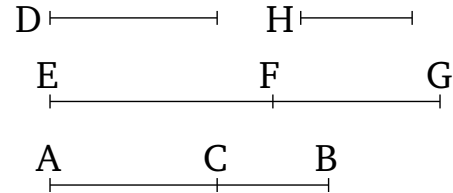
ZH. καὶ ἐστὶ ῥητὴ ἡ EZ· ῥητὴ ἄρα καὶ ἡ ZH. καὶ ἐπεὶ ὁ BA πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, οὐδὲ τὸ ἀπὸ τῆς EZ ἄρα πρὸς τὸ ἀπὸ τῆς ZH λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἀσύμμετρος ἄρα ἐστὶν ἡ EZ τῇ ZH μήκει. αἱ EZ, ZH ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ EH. λέγω, ὅτι καὶ πρώτη.



Ἐπεὶ γάρ ἐστιν ὡς ὁ BA ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς EZ πρὸς τὸ ἀπὸ τῆς ZH, μείζων δὲ ὁ BA τοῦ ΑΓ, μείζον ἄρα καὶ τὸ ἀπὸ τῆς EZ τοῦ ἀπὸ τῆς ZH. ἔστω οὖν τῷ ἀπὸ τῆς EZ ἴσα τὰ ἀπὸ τῶν ZH, Θ. καὶ ἐπεὶ ἐστὶν ὡς ὁ BA πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς EZ πρὸς τὸ ἀπὸ τῆς ZH, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ AB πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς EZ πρὸς τὸ ἀπὸ τῆς Θ. ὁ δὲ AB πρὸς τὸν ΒΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. καὶ τὸ ἀπὸ τῆς EZ ἄρα πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. σύμμετρος ἄρα ἐστὶν ἡ EZ τῇ Θ μήκει· ἡ EZ ἄρα τῆς ZH μείζων δύνανται τῷ ἀπὸ συμέτρου ἑαυτῇ. καὶ εἰσι ῥηταὶ αἱ EZ, ZH, καὶ σύμμετρος ἡ EZ τῇ Δ μήκει.

Ἡ EH ἄρα ἐκ δύο ὀνομάτων ἐστὶ πρώτη· ὅπερ ἔδει δεῖξαι.

ber. Thus, the (square) on EF also has to the (square) on FG the ratio which (some) number (has) to (some) number. Hence, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6]. And EF is rational. Thus, FG (is) also rational. And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, thus the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with FG [Prop. 10.9]. EF and FG are thus rational (straight-lines which are) commensurable in square only. Thus, EG is a binomial (straight-line) [Prop. 10.36]. I say that (it is) also a first (binomial straight-line).



For since as the number BA is to AC , so the (square) on EF (is) to the (square) on FG , and BA (is) greater than AC , the (square) on EF (is) thus also greater than the (square) on FG [Prop. 5.14]. Therefore, let (the sum of) the (squares) on FG and H be equal to the (square) on EF . And since as BA is to AC , so the (square) on EF (is) to the (square) on FG , thus, via conversion, as AB is to BC , so the (square) on EF (is) to the (square) on H [Prop. 5.19 corr.]. And AB has to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on EF also has to the (square) on H the ratio which (some) square number (has) to (some) square number. Thus, EF is commensurable in length with H [Prop. 10.9]. Thus, the square on EF is greater than (the square on) FG by the (square) on (some straight-line) commensurable (in length) with (EF) . And EF and FG are rational (straight-lines). And EF (is) commensurable in length with D .

Thus, EG is a first binomial (straight-line) [Def. 10.5].[†] (Which is) the very thing it was required to show.

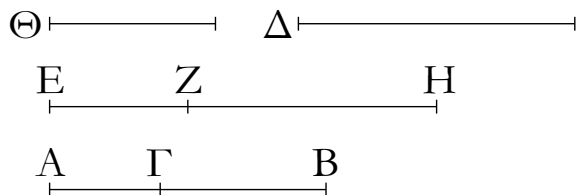
[†]If the rational straight-line has unit length then the length of a first binomial straight-line is $k + k\sqrt{1 - k'^2}$. This, and the first apotome, whose length is $k - k\sqrt{1 - k'^2}$ [Prop. 10.85], are the roots of $x^2 - 2kx + k^2 k'^2 = 0$.

μθ'.

Εὐρεῖν τὴν ἐκ δύο ὀνομάτων δευτέραν.

Proposition 49

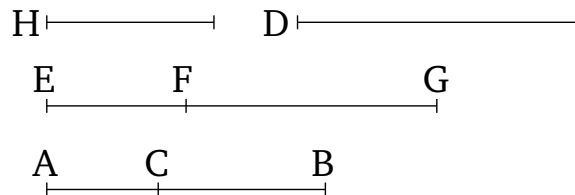
To find a second binomial (straight-line).



Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν συγκείμενον ἐξ αὐτῶν τὸν ΑΒ πρὸς μὲν τὸν ΒΓ λόγον ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, πρὸς δὲ τὸν ΑΓ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, καὶ ἐκκείσθω ῥητὴ ἡ Δ, καὶ τῇ Δ σύμμετρος ἔστω ἡ ΕΖ μήκει· ῥητὴ ἄρα ἐστὶν ἡ ΕΖ. γεγονέτω δὴ καὶ ὡς ὁ ΓΑ ἀριθμὸς πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΕΖ τῷ ἀπὸ τῆς ΖΗ. ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ ΓΑ ἀριθμὸς πρὸς τὸν ΑΒ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, οὐδὲ τὸ ἀπὸ τῆς ΕΖ πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. ἀσύμμετρος ἄρα ἐστὶν ἡ ΕΖ τῇ ΖΗ μήκει· αἱ ΕΖ, ΖΗ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΕΗ. δεῖκτέον δὴ, ὅτι καὶ δευτέρα.

Ἐπεὶ γὰρ ἀνάπαλιν ἐστὶν ὡς ὁ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΗΖ πρὸς τὸ ἀπὸ τῆς ΖΕ, μεῖζων δὲ ὁ ΒΑ τοῦ ΑΓ, μεῖζον ἄρα [καὶ] τὸ ἀπὸ τῆς ΗΖ τοῦ ἀπὸ τῆς ΖΕ. ἔστω τῷ ἀπὸ τῆς ΗΖ ἴσα τὰ ἀπὸ τῶν ΕΖ, Θ· ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΑΒ πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Θ. ἀλλ' ὁ ΑΒ πρὸς τὸν ΒΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· καὶ τὸ ἀπὸ τῆς ΖΗ ἄρα πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. σύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ Θ μήκει· ὥστε ἡ ΖΗ τῆς ΖΕ μεῖζον δύναται τῷ ἀπὸ συμέτρου ἑαυτῇ. καὶ εἰσι ῥηταὶ αἱ ΖΗ, ΖΕ δυνάμει μόνον σύμμετροι, καὶ τὸ ΕΖ ἔλασσον ὄνομα τῇ ἐκκειμένῃ ῥητῇ σύμμετρόν ἐστι τῇ Δ μήκει.

Ἡ ΕΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ δευτέρα· ὅπερ ἔδει δεῖξαι.



Let the two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to AC the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let the rational (straight-line) D be laid down. And let EF be commensurable in length with D . EF is thus a rational (straight-line). So, let it also have been contrived that as the number CA (is) to AB , so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6]. Thus, FG is also a rational (straight-line). And since the number CA does not have to AB the ratio which (some) square number (has) to (some) square number, the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with FG [Prop. 10.9]. EF and FG are thus rational (straight-lines which are) commensurable in square only. Thus, EG is a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a second (binomial straight-line).

For since, inversely, as the number BA is to AC , so the (square) on GF (is) to the (square) on FE [Prop. 5.7 corr.], and BA (is) greater than AC , the (square) on GF (is) thus [also] greater than the (square) on FE [Prop. 5.14]. Let (the sum of) the (squares) on EF and H be equal to the (square) on GF . Thus, via conversion, as AB is to BC , so the (square) on FG (is) to the (square) on H [Prop. 5.19 corr.]. But, AB has to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG also has to the (square) on H the ratio which (some) square number (has) to (some) square number. Thus, FG is commensurable in length with H [Prop. 10.9]. Hence, the square on FG is greater than (the square on) FE by the (square) on (some straight-line) commensurable in length with (FG) . And FG and FE are rational (straight-lines which are) commensurable in square only. And the lesser term EF is commensurable in length with the rational (straight-line) D (previously) laid down.

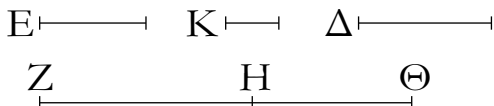
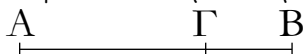
Thus, EG is a second binomial (straight-line) [Def. 10.6].[†] (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then the length of a second binomial straight-line is $k/\sqrt{1-k'^2} + k$. This, and the second apotome,

whose length is $k/\sqrt{1-k'^2} - k$ [Prop. 10.86], are the roots of $x^2 - (2k/\sqrt{1-k'^2})x + k^2 [k'^2/(1-k'^2)] = 0$.

ν'.

Εὕρεῖν τὴν ἐκ δύο ὀνομάτων τρίτην.

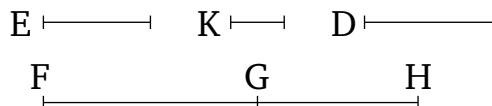


Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΑΓ, ΓΒ, ὥστε τὸν συγχείμενον ἐξ αὐτῶν τὸν ΑΒ πρὸς μὲν τὸν ΒΓ λόγον ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, πρὸς δὲ τὸν ΑΓ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. ἐκκείσθω δὲ τις καὶ ἄλλος μὴ τετράγωνος ἀριθμὸς ὁ Δ, καὶ πρὸς ἑκάτερον τῶν ΒΑ, ΑΓ λόγον μὴ ἔχέτω, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· καὶ ἐκκείσθω τις ῥητὴ εὐθεῖα ἡ Ε, καὶ γεγονέτω ὡς ὁ Δ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς Ε τῷ ἀπὸ τῆς ΖΗ. καὶ ἐστὶ ῥητὴ ἡ Ε· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΗ. καὶ ἐπεὶ ὁ Δ πρὸς τὸν ΑΒ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, οὐδὲ τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἀσύμμετρος ἄρα ἐστὶν ἡ Ε τῇ ΖΗ μήκει. γεγονέτω δὴ πάλιν ὡς ἡ ΒΑ ἀριθμὸς πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΗΘ. ῥητὴ δὲ ἡ ΖΗ· ῥητὴ ἄρα καὶ ἡ ΗΘ. καὶ ἐπεὶ ὁ ΒΑ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, οὐδὲ τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἀσύμμετρος ἄρα ἐστὶν ἡ ΖΗ τῇ ΗΘ μήκει. αἱ ΖΗ, ΗΘ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἡ ΖΘ ἄρα ἐκ δύο ὀνομάτων ἐστίν. λέγω δὴ, ὅτι καὶ τρίτη.

Ἐπεὶ γάρ ἐστιν ὡς ὁ Δ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ, ὡς δὲ ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, δι' ἴσου ἄρα ἐστὶν ὡς ὁ Δ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΗΘ. ὁ δὲ Δ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· οὐδὲ τὸ ἀπὸ τῆς Ε ἄρα πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἀσύμμετρος ἄρα ἐστὶν ἡ Ε τῇ ΗΘ μήκει. καὶ ἐπεὶ ἐστὶν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, μείζον ἄρα τὸ ἀπὸ τῆς ΖΗ τοῦ ἀπὸ τῆς ΗΘ. ἔστω οὖν τῷ ἀπὸ τῆς ΖΗ ἴσα τὰ ἀπὸ τῶν ΗΘ, Κ· ἀναστρέψαντι ἄρα [ἐστὶν] ὡς ὁ ΑΒ πρὸς τὸν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς Κ. ὁ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς

Proposition 50

To find a third binomial (straight-line).



Let the two numbers AC and CB be laid down such that their sum AB has to BC the ratio which (some) square number (has) to (some) square number, and does not have to AC the ratio which (some) square number (has) to (some) square number. And let some other non-square number D also be laid down, and let it not have to each of BA and AC the ratio which (some) square number (has) to (some) square number. And let some rational straight-line E be laid down, and let it have been contrived that as D (is) to AB , so the (square) on E (is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on E is commensurable with the (square) on FG [Prop. 10.6]. And E is a rational (straight-line). Thus, FG is also a rational (straight-line). And since D does not have to AB the ratio which (some) square number has to (some) square number, the (square) on E does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. E is thus incommensurable in length with FG [Prop. 10.9]. So, again, let it have been contrived that as the number BA (is) to AC , so the (square) on FG (is) to the (square) on GH [Prop. 10.6 corr.]. Thus, the (square) on FG is commensurable with the (square) on GH [Prop. 10.6]. And FG (is) a rational (straight-line). Thus, GH (is) also a rational (straight-line). And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, the (square) on FG does not have to the (square) on HG the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. FG and GH are thus rational (straight-lines which are) commensurable in square only. Thus, FH is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a third (binomial straight-line).

For since as D is to AB , so the (square) on E (is) to the (square) on FG , and as BA (is) to AC , so the (square) on FG (is) to the (square) on GH , thus, via equality, as D (is) to AC , so the (square) on E (is) to the (square) on GH [Prop. 5.22]. And D does not

τετράγωνον ἀριθμόν· καὶ τὸ ἀπὸ τῆς ZH ἄρα πρὸς τὸ ἀπὸ τῆς K λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα [ἐστὶν] ἡ ZH τῇ K μήκει. ἡ ZH ἄρα τῆς $H\Theta$ μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ· καὶ εἰσιν αἱ ZH , $H\Theta$ ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα αὐτῶν σύμμετρός ἐστι τῇ E μήκει.

Ἡ $Z\Theta$ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τρίτη· ὅπερ ἔδει δεῖξαι.

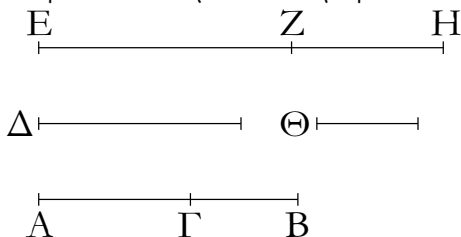
have to AC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on E does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. Thus, E is incommensurable in length with GH [Prop. 10.9]. And since as BA is to AC , so the (square) on FG (is) to the (square) on GH , the (square) on FG (is) thus greater than the (square) on GH [Prop. 5.14]. Therefore, let (the sum of) the (squares) on GH and K be equal to the (square) on FG . Thus, via conversion, as AB [is] to BC , so the (square) on FG (is) to the (square) on K [Prop. 5.19 corr.]. And AB has to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG also has to the (square) on K the ratio which (some) square number (has) to (some) square number. Thus, FG [is] commensurable in length with K [Prop. 10.9]. Thus, the square on FG is greater than (the square on) GH by the (square) on (some straight-line) commensurable (in length) with (FG). And FG and GH are rational (straight-lines which are) commensurable in square only, and neither of them is commensurable in length with E .

Thus, FH is a third binomial (straight-line) [Def. 10.7].[†] (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then the length of a third binomial straight-line is $k^{1/2}(1 + \sqrt{1 - k'^2})$. This, and the third apotome, whose length is $k^{1/2}(1 - \sqrt{1 - k'^2})$ [Prop. 10.87], are the roots of $x^2 - 2k^{1/2}x + kk'^2 = 0$.

να'.

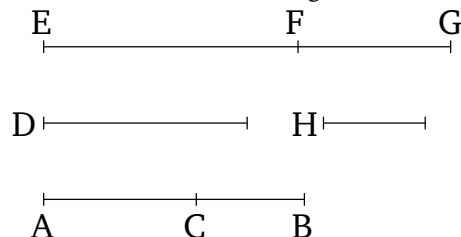
Εὐρεῖν τὴν ἐκ δύο ὀνομάτων τετάρτην.



Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ $ΑΓ$, $ΓΒ$, ὥστε τὸν $ΑΒ$ πρὸς τὸν $ΒΓ$ λόγον μὴ ἔχειν μήτε μὴν πρὸς τὸν $ΑΓ$, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ ἐκκείσθω ῥητὴ ἡ $Δ$, καὶ τῇ $Δ$ σύμμετρος ἔστω μήκει ἡ $ΕΖ$ · ῥητὴ ἄρα ἐστὶ καὶ ἡ $ΕΖ$. καὶ γεγονέτω ὡς ὁ $ΒΑ$ ἀριθμὸς πρὸς τὸν $ΑΓ$, οὕτως τὸ ἀπὸ τῆς $ΕΖ$ πρὸς τὸ ἀπὸ τῆς $ΖΗ$ · σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς $ΕΖ$ τῷ ἀπὸ τῆς $ΖΗ$ · ῥητὴ ἄρα ἐστὶ καὶ ἡ $ΖΗ$. καὶ ἐπεὶ ὁ $ΒΑ$ πρὸς τὸν $ΑΓ$ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδὲ τὸ ἀπὸ τῆς $ΕΖ$ πρὸς τὸ ἀπὸ τῆς $ΖΗ$ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ $ΕΖ$ τῇ $ΖΗ$ μήκει. αἱ $ΕΖ$, $ΖΗ$ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ὥστε ἡ $ΕΗ$ ἐκ δύο ὀνομάτων ἐστίν. λέγω δὴ,

Proposition 51

To find a fourth binomial (straight-line).



Let the two numbers AC and CB be laid down such that AB does not have to BC , or to AC either, the ratio which (some) square number (has) to (some) square number [Prop. 10.28 lem. I]. And let the rational (straight-line) D be laid down. And let EF be commensurable in length with D . Thus, EF is also a rational (straight-line). And let it have been contrived that as the number BA (is) to AC , so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on EF is commensurable with the (square) on FG [Prop. 10.6]. Thus, FG is also a rational (straight-line). And since BA does not have to AC the ratio which (some) square number (has) to (some) square number,

ὅτι καὶ τετάρτη.

Ἐπεὶ γὰρ ἐστὶν ὡς ὁ BA πρὸς τὸν AG , οὕτως τὸ ἀπὸ τῆς EZ πρὸς τὸ ἀπὸ τῆς ZH [μείζων δὲ ὁ BA τοῦ AG], μείζον ἄρα τὸ ἀπὸ τῆς EZ τοῦ ἀπὸ τῆς ZH . ἔστω οὖν τῷ ἀπὸ τῆς EZ ἴσα τὰ ἀπὸ τῶν ZH , Θ · ἀναστρέψαντι ἄρα ὡς ὁ AB ἀριθμὸς πρὸς τὸν $B\Gamma$, οὕτως τὸ ἀπὸ τῆς EZ πρὸς τὸ ἀπὸ τῆς Θ . ὁ δὲ AB πρὸς τὸν $B\Gamma$ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· οὐδ' ἄρα τὸ ἀπὸ τῆς EZ πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. ἀσύμμετρος ἄρα ἐστὶν ἡ EZ τῇ Θ μήκει· ἡ EZ ἄρα τῆς HZ μείζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ εἰσιν αἱ EZ , ZH ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ ἡ EZ τῇ Δ σύμμετρός ἐστι μήκει.

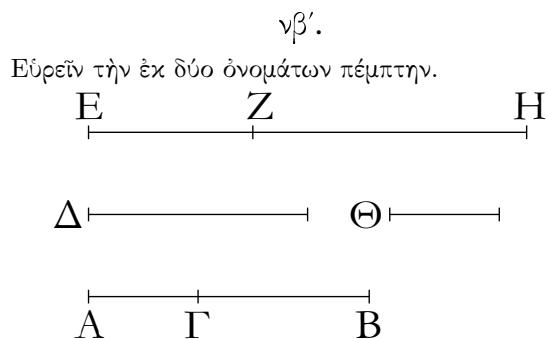
Ἡ EH ἄρα ἐκ δύο ὀνομάτων ἐστὶ τετάρτη· ὅπερ ἔδει δεῖξαι.

the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with FG [Prop. 10.9]. Thus, EF and FG are rational (straight-lines which are) commensurable in square only. Hence, EG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fourth (binomial straight-line).

For since as BA is to AC , so the (square) on EF (is) to the (square) on FG [and BA (is) greater than AC], the (square) on EF (is) thus greater than the (square) on FG [Prop. 5.14]. Therefore, let (the sum of) the squares on FG and H be equal to the (square) on EF . Thus, via conversion, as the number AB (is) to BC , so the (square) on EF (is) to the (square) on H [Prop. 5.19 corr.]. And AB does not have to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on EF does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, EF is incommensurable in length with H [Prop. 10.9]. Thus, the square on EF is greater than (the square on) GF by the (square) on (some straight-line) incommensurable (in length) with (EF). And EF and FG are rational (straight-lines which are) commensurable in square only. And EF is commensurable in length with D .

Thus, EG is a fourth binomial (straight-line) [Def. 10.8].[†] (Which is) the very thing it was required to show.

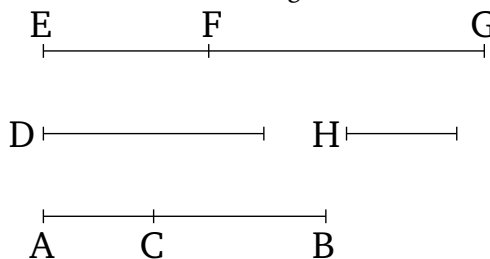
[†] If the rational straight-line has unit length then the length of a fourth binomial straight-line is $k(1 + 1/\sqrt{1+k'})$. This, and the fourth apotome, whose length is $k(1 - 1/\sqrt{1+k'})$ [Prop. 10.88], are the roots of $x^2 - 2kx + k^2k'/(1+k') = 0$.



Ἐκκεῖσθωσαν δύο ἀριθμοὶ οἱ AG , GB , ὥστε τὸν AB πρὸς ἑκάτερον αὐτῶν λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, καὶ ἐκκεῖσθω ῥητὴ τις εὐθεῖα ἡ Δ , καὶ τῇ Δ σύμμετρος ἔστω [μήκει] ἡ EZ · ῥητὴ ἄρα ἡ EZ . καὶ γεγονένω ὡς ὁ GA πρὸς τὸν AB , οὕτως τὸ ἀπὸ τῆς EZ πρὸς τὸ ἀπὸ τῆς ZH . ὁ δὲ GA πρὸς τὸν AB λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· οὐδὲ τὸ ἀπὸ τῆς EZ ἄρα πρὸς τὸ ἀπὸ τῆς ZH λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. αἱ

Proposition 52

To find a fifth binomial straight-line.



Let the two numbers AC and CB be laid down such that AB does not have to either of them the ratio which (some) square number (has) to (some) square number [Prop. 10.38 lem.]. And let some rational straight-line D be laid down. And let EF be commensurable [in length] with D . Thus, EF (is) a rational (straight-line). And let it have been contrived that as CA (is) to AB , so the (square) on EF (is) to the (square) on FG [Prop. 10.6 corr.]. And CA does not have to AB the ra-

EZ , ZH ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ EH . λέγω δὴ, ὅτι καὶ πέμπτη.

Ἐπεὶ γάρ ἐστιν ὡς ὁ $ΓΑ$ πρὸς τὸν $ΑΒ$, οὕτως τὸ ἀπὸ τῆς EZ πρὸς τὸ ἀπὸ τῆς ZH , ἀνάπαλιν ὡς ὁ $ΒΑ$ πρὸς τὸν $ΑΓ$, οὕτως τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς ZE . μείζον ἄρα τὸ ἀπὸ τῆς HZ τοῦ ἀπὸ τῆς ZE . ἔστω οὖν τῷ ἀπὸ τῆς HZ ἴσα τὰ ἀπὸ τῶν EZ , $Θ$ · ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ $ΑΒ$ ἀριθμὸς πρὸς τὸν $ΒΓ$, οὕτως τὸ ἀπὸ τῆς HZ πρὸς τὸ ἀπὸ τῆς $Θ$. ὁ δὲ $ΑΒ$ πρὸς τὸν $ΒΓ$ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· οὐδ' ἄρα τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς $Θ$ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ ZH τῇ $Θ$ μήκει· ὥστε ἡ ZH τῆς ZE μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ. καὶ εἰσιν αἱ HZ , ZE ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ τὸ EZ ἑλαττον ὄνομα σύμμετρόν ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ $Δ$ μήκει.

Ἡ EH ἄρα ἐκ δύο ὀνομάτων ἐστὶ πέμπτη· ὅπερ ἔδει δεῖξαι.

tio which (some) square number (has) to (some) square number. Thus, the (square) on EF does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, EF and FG are rational (straight-lines which are) commensurable in square only [Prop. 10.9]. Thus, EG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fifth (binomial straight-line).

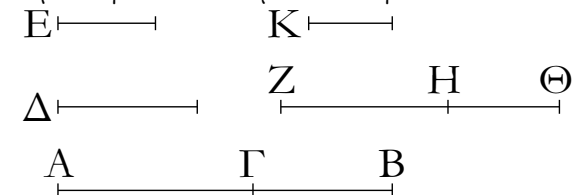
For since as CA is to AB , so the (square) on EF (is) to the (square) on FG , inversely, as BA (is) to AC , so the (square) on FG (is) to the (square) on FE [Prop. 5.7 corr.]. Thus, the (square) on GF (is) greater than the (square) on FE [Prop. 5.14]. Therefore, let (the sum of) the (squares) on EF and H be equal to the (square) on GF . Thus, via conversion, as the number AB is to BC , so the (square) on GF (is) to the (square) on H [Prop. 5.19 corr.]. And AB does not have to BC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with H [Prop. 10.9]. Hence, the square on FG is greater than (the square on) FE by the (square) on (some straight-line) incommensurable (in length) with (FG). And GF and FE are rational (straight-lines which are) commensurable in square only. And the lesser term EF is commensurable in length with the rational (straight-line previously) laid down, D .

Thus, EG is a fifth binomial (straight-line).[†] (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then the length of a fifth binomial straight-line is $k(\sqrt{1+k'}+1)$. This, and the fifth apotome, whose length is $k(\sqrt{1+k'}-1)$ [Prop. 10.89], are the roots of $x^2 - 2k\sqrt{1+k'}x + k^2k' = 0$.

νγ'.

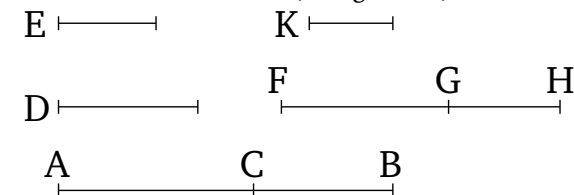
Εὐρεῖν τὴν ἐκ δύο ὀνομάτων ἑκτὴν.



Ἐκκείσθωσαν δύο ἀριθμοὶ οἱ $ΑΓ$, $ΓΒ$, ὥστε τὸν $ΑΒ$ πρὸς ἑκάτερον αὐτῶν λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἔστω δὲ καὶ ἕτερος ἀριθμὸς ὁ $Δ$ μὴ τετράγωνος ὢν μηδὲ πρὸς ἑκάτερον τῶν $ΒΑ$, $ΑΓ$ λόγον ἔχων, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ ἐκκείσθω τις ῥητὴ εὐθεῖα ἡ E , καὶ γεγονέτω ὡς ὁ $Δ$ πρὸς τὸν $ΑΒ$, οὕτως τὸ ἀπὸ τῆς E πρὸς τὸ ἀπὸ τῆς ZH · σύμμετρον ἄρα τὸ ἀπὸ τῆς E τῷ ἀπὸ

Proposition 53

To find a sixth binomial (straight-line).



Let the two numbers AC and CB be laid down such that AB does not have to each of them the ratio which (some) square number (has) to (some) square number. And let D also be another number, which is not square, and does not have to each of BA and AC the ratio which (some) square number (has) to (some) square number either [Prop. 10.28 lem. I]. And let some rational straight-line E be laid down. And let it have been contrived that

τῆς ΖΗ. καὶ ἐστὶ ῥητὴ ἢ Ε· ῥητὴ ἄρα καὶ ἡ ΖΗ. καὶ ἐπεὶ οὐκ ἔχει ὁ Δ πρὸς τὸν ΑΒ λόγον, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, οὐδὲ τὸ ἀπὸ τῆς Ε ἄρα πρὸς τὸ ἀπὸ τῆς ΖΗ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἀσύμμετρος ἄρα ἢ Ε τῇ ΖΗ μήκει. γεγονέντω δὴ πάλιν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ. σύμμετρον ἄρα τὸ ἀπὸ τῆς ΖΗ τῷ ἀπὸ τῆς ΗΘ. ῥητὸν ἄρα τὸ ἀπὸ τῆς ΗΘ· ῥητὴ ἄρα ἢ ΗΘ. καὶ ἐπεὶ ὁ ΒΑ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, οὐδὲ τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἀσύμμετρος ἄρα ἐστὶν ἢ ΖΗ τῇ ΗΘ μήκει. αἱ ΖΗ, ΗΘ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΖΘ. δεικτέον δὴ, ὅτι καὶ ἔκτε.

Ἐπεὶ γάρ ἐστιν ὡς ὁ Δ πρὸς τὸν ΑΒ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΖΗ, ἔστι δὲ καὶ ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, δι' ἴσου ἄρα ἐστὶν ὡς ὁ Δ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς Ε πρὸς τὸ ἀπὸ τῆς ΗΘ. ὁ δὲ Δ πρὸς τὸν ΑΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· οὐδὲ τὸ ἀπὸ τῆς Ε ἄρα πρὸς τὸ ἀπὸ τῆς ΗΘ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἀσύμμετρος ἄρα ἐστὶν ἢ Ε τῇ ΗΘ μήκει. ἐδείχθη δὲ καὶ τῇ ΖΗ ἀσύμμετρος· ἑκατέρα ἄρα τῶν ΖΗ, ΗΘ ἀσύμμετρος ἐστὶ τῇ Ε μήκει. καὶ ἐπεὶ ἐστὶν ὡς ὁ ΒΑ πρὸς τὸν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΖΗ πρὸς τὸ ἀπὸ τῆς ΗΘ, μείζον ἄρα τὸ ἀπὸ τῆς ΖΗ τοῦ ἀπὸ τῆς ΗΘ. ἔστω οὖν τῷ ἀπὸ [τῆς] ΖΗ ἴσα τὰ ἀπὸ τῶν ΗΘ, Κ· ἀναστρέψαντι ἄρα ὡς ὁ ΑΒ πρὸς ΒΓ, οὕτως τὸ ἀπὸ ΖΗ πρὸς τὸ ἀπὸ τῆς Κ. ὁ δὲ ΑΒ πρὸς τὸν ΒΓ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ὥστε οὐδὲ τὸ ἀπὸ ΖΗ πρὸς τὸ ἀπὸ τῆς Κ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. ἀσύμμετρος ἄρα ἐστὶν ἢ ΖΗ τῇ Κ μήκει· ἢ ΖΗ ἄρα τῆς ΗΘ μείζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ εἰσιν αἱ ΖΗ, ΗΘ ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα αὐτῶν σύμμετρος ἐστὶ μήκει τῇ ἐκκειμένῃ ῥητῇ τῇ Ε.

Ἡ ΖΘ ἄρα ἐκ δύο ὀνομάτων ἐστὶν ἔκτε· ὅπερ ἔδει δεῖξαι.

as D (is) to AB , so the (square) on E (is) to the (square) on FG [Prop. 10.6 corr.]. Thus, the (square) on E (is) commensurable with the (square) on FG [Prop. 10.6]. And E is rational. Thus, FG (is) also rational. And since D does not have to AB the ratio which (some) square number (has) to (some) square number, the (square) on E thus does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, E (is) incommensurable in length with FG [Prop. 10.9]. So, again, let it have be contrived that as BA (is) to AC , so the (square) on FG (is) to the (square) on GH [Prop. 10.6 corr.]. The (square) on FG (is) thus commensurable with the (square) on HG [Prop. 10.6]. The (square) on HG (is) thus rational. Thus, HG (is) rational. And since BA does not have to AC the ratio which (some) square number (has) to (some) square number, the (square) on FG does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. Thus, FG and GH are rational (straight-lines which are) commensurable in square only. Thus, FH is a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a sixth (binomial straight-line).

For since as D is to AB , so the (square) on E (is) to the (square) on FG , and also as BA is to AC , so the (square) on FG (is) to the (square) on GH , thus, via equality, as D is to AC , so the (square) on E (is) to the (square) on GH [Prop. 5.22]. And D does not have to AC the ratio which (some) square number (has) to (some) square number. Thus, the (square) on E does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. E is thus incommensurable in length with GH [Prop. 10.9]. And (E) was also shown (to be) incommensurable (in length) with FG . Thus, FG and GH are each incommensurable in length with E . And since as BA is to AC , so the (square) on FG (is) to the (square) on GH , the (square) on FG (is) thus greater than the (square) on GH [Prop. 5.14]. Therefore, let (the sum of) the (squares) on GH and K be equal to the (square) on FG . Thus, via conversion, as AB (is) to BC , so the (square) on FG (is) to the (square) on K [Prop. 5.19 corr.]. And AB does not have to BC the ratio which (some) square number (has) to (some) square number. Hence, the (square) on FG does not have to the (square) on K the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with K [Prop. 10.9]. The square on FG is thus greater than (the square on) GH by the (square) on (some straight-line which is) incom-

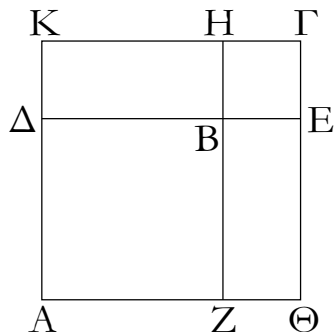
measurable (in length) with (FG) . And FG and GH are rational (straight-lines which are) commensurable in square only, and neither of them is commensurable in length with the rational (straight-line) E (previously) laid down.

Thus, FH is a sixth binomial (straight-line) [Def. 10.10].[†] (Which is) the very thing it was required to show.

[†] If the rational straight-line has unit length then the length of a sixth binomial straight-line is $\sqrt{k} + \sqrt{k'}$. This, and the sixth apotome, whose length is $\sqrt{k} - \sqrt{k'}$ [Prop. 10.90], are the roots of $x^2 - 2\sqrt{k}x + (k - k') = 0$.

Λήμμα.

Ἐστω δύο τετράγωνα τὰ AB , $BΓ$ καὶ κείσθωσαν ὥστε ἐπ' εὐθείας εἶναι τὴν $ΔB$ τῇ BE · ἐπ' εὐθείας ἄρα ἐστὶ καὶ ἡ ZB τῇ BH . καὶ συμπληρώσω τὸ $ΑΓ$ παραλληλόγραμμον· λέγω, ὅτι τετράγωνόν ἐστι τὸ $ΑΓ$, καὶ ὅτι τῶν AB , $BΓ$ μέσον ἀνάλογόν ἐστι τὸ $ΔH$, καὶ ἔτι τῶν $ΑΓ$, $ΓB$ μέσον ἀνάλογόν ἐστι τὸ $ΔΓ$.



Ἐπεὶ γὰρ ἴση ἐστὶν ἡ μὲν $ΔB$ τῇ BZ , ἡ δὲ BE τῇ BH , ὅλη ἄρα ἡ $ΔE$ ὅλη τῇ ZH ἐστὶν ἴση. ἀλλ' ἡ μὲν $ΔE$ ἑκατέρω τῶν $ΑΘ$, $KΓ$ ἐστὶν ἴση, ἡ δὲ ZH ἑκατέρω τῶν AK , $ΘΓ$ ἐστὶν ἴση· καὶ ἑκατέρω ἄρα τῶν $ΑΘ$, $KΓ$ ἑκατέρω τῶν AK , $ΘΓ$ ἐστὶν ἴση. ἰσόπλευρον ἄρα ἐστὶ τὸ $ΑΓ$ παραλληλόγραμμον· ἔστι δὲ καὶ ὀρθογώνιον· τετράγωνον ἄρα ἐστὶ τὸ $ΑΓ$.

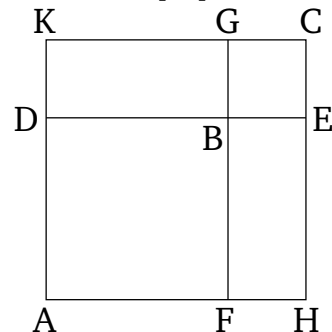
Καὶ ἐπεὶ ἐστὶν ὡς ἡ ZB πρὸς τὴν BH , οὕτως ἡ $ΔB$ πρὸς τὴν BE , ἀλλ' ὡς μὲν ἡ ZB πρὸς τὴν BH , οὕτως τὸ AB πρὸς τὸ $ΔH$, ὡς δὲ ἡ $ΔB$ πρὸς τὴν BE , οὕτως τὸ $ΔH$ πρὸς τὸ $BΓ$, καὶ ὡς ἄρα τὸ AB πρὸς τὸ $ΔH$, οὕτως τὸ $ΔH$ πρὸς τὸ $BΓ$. τῶν AB , $BΓ$ ἄρα μέσον ἀνάλογόν ἐστι τὸ $ΔH$.

Λέγω δὴ, ὅτι καὶ τῶν $ΑΓ$, $ΓB$ μέσον ἀνάλογόν [ἐστὶ] τὸ $ΔΓ$.

Ἐπεὶ γὰρ ἐστὶν ὡς ἡ $ΑΔ$ πρὸς τὴν $ΔK$, οὕτως ἡ KH πρὸς τὴν $HΓ$ · ἴση γὰρ [ἐστὶν] ἑκατέρω ἑκατέρω· καὶ συνθέντι ὡς ἡ AK πρὸς $KΔ$, οὕτως ἡ $KΓ$ πρὸς $ΓH$, ἀλλ' ὡς μὲν ἡ AK πρὸς $KΔ$, οὕτως τὸ $ΑΓ$ πρὸς τὸ $ΓΔ$, ὡς δὲ ἡ $KΓ$ πρὸς $ΓH$, οὕτως τὸ $ΔΓ$ πρὸς $ΓB$, καὶ ὡς ἄρα τὸ $ΑΓ$ πρὸς $ΔΓ$, οὕτως τὸ $ΔΓ$ πρὸς τὸ $BΓ$. τῶν $ΑΓ$, $ΓB$ ἄρα μέσον ἀνάλογόν ἐστι τὸ $ΔΓ$ · ὃ προέκειτο δεῖξαι.

Lemma

Let AB and BC be two squares, and let them be laid down such that DB is straight-on to BE . FB is, thus, also straight-on to BG . And let the parallelogram AC have been completed. I say that AC is a square, and that DG is the mean proportional to AB and BC , and, moreover, DC is the mean proportional to AC and CB .



For since DB is equal to BF , and BE to BG , the whole of DE is thus equal to the whole of FG . But DE is equal to each of AH and KC , and FG is equal to each of AK and HC [Prop. 1.34]. Thus, AH and KC are also equal to AK and HC , respectively. Thus, the parallelogram AC is equilateral. And (it is) also right-angled. Thus, AC is a square.

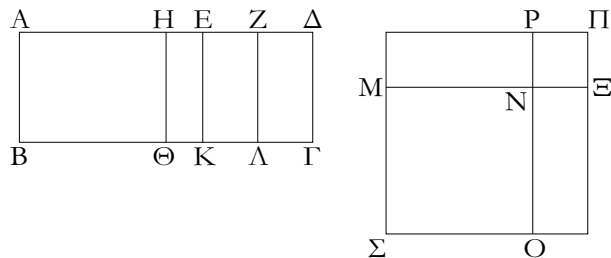
And since as FB is to BG , so DB (is) to BE , but as FB (is) to BG , so AB (is) to DG , and as DB (is) to BE , so DG (is) to BC [Prop. 6.1], thus also as AB (is) to DG , so DG (is) to BC [Prop. 5.11]. Thus, DG is the mean proportional to AB and BC .

So I also say that DC [is] the mean proportional to AC and CB .

For since as AD is to DK , so KG (is) to GC . For [they are] respectively equal. And, via composition, as AK (is) to KD , so KC (is) to CG [Prop. 5.18]. But as AK (is) to KD , so AC (is) to CD , and as KC (is) to CG , so DC (is) to CB [Prop. 6.1]. Thus, also, as AC (is) to DC , so DC (is) to CB [Prop. 5.11]. Thus, DC is the mean proportional to AC and CB . Which (is the very thing) it

νδ'.

Ἐάν χωρίον περιέχεται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων πρώτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο ὀνομάτων.



Χωρίον γὰρ τὸ ΑΓ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων πρώτης τῆς ΑΔ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο ὀνομάτων.

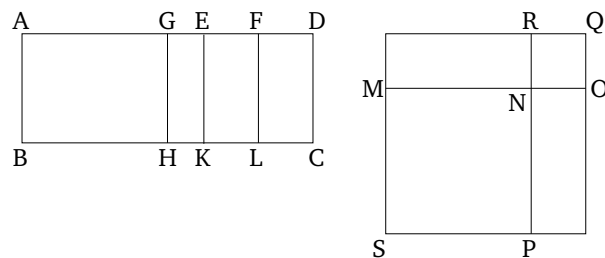
Ἐπεὶ γὰρ ἐκ δύο ὀνομάτων ἐστὶ πρώτη ἡ ΑΔ, διηρήσθω εἰς τὰ ὀνόματα κατὰ τὸ Ε, καὶ ἔστω τὸ μείζον ὄνομα τὸ ΑΕ. φανερόν δὴ, ὅτι αἱ ΑΕ, ΕΔ ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΑΕ τῆς ΕΔ μείζον δύναται τῷ ἀπὸ συμέτρου ἑαυτῆς, καὶ ἡ ΑΕ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΑΒ μήκει. τετμήσθω δὲ ἡ ΕΔ δίχα κατὰ τὸ Ζ σημεῖον. καὶ ἐπεὶ ἡ ΑΕ τῆς ΕΔ μείζον δύναται τῷ ἀπὸ συμέτρου ἑαυτῆς, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος, τουτέστι τῷ ἀπὸ τῆς ΕΖ, ἴσον παρὰ τὴν μείζονα τὴν ΑΕ παραβληθῇ ἐλλείπον εἶδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διαιρεῖ. παραβεβλήσθω οὖν παρὰ τὴν ΑΕ τῷ ἀπὸ τῆς ΕΖ ἴσον τὸ ὑπὸ ΑΗ, ΗΕ· σύμμετρος ἄρα ἐστὶν ἡ ΑΗ τῇ ΕΗ μήκει. καὶ ἤχθωσαν ἀπὸ τῶν Η, Ε, Ζ ὁποτέρᾳ τῶν ΑΒ, ΓΔ παράλληλοι αἱ ΗΘ, ΕΚ, ΖΛ· καὶ τῷ μὲν ΑΘ παραλληλογράμμῳ ἴσον τετράγωνον συνεστάτω τὸ ΣΝ, τῷ δὲ ΗΚ ἴσον τὸ ΝΠ, καὶ κείσθω ὥστε ἐπ' εὐθείας εἶναι τὴν ΜΝ τῇ ΝΞ· ἐπ' εὐθείας ἄρα ἐστὶ καὶ ἡ ΡΝ τῇ ΝΟ. καὶ συμπληρώσθω τὸ ΣΠ παραλληλόγραμμον· τετράγωνον ἄρα ἐστὶ τὸ ΣΠ. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΗ, ΗΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΖ, ἔστιν ἄρα ὡς ἡ ΑΗ πρὸς ΕΖ, οὕτως ἡ ΖΕ πρὸς ΕΗ· καὶ ὡς ἄρα τὸ ΑΘ πρὸς ΕΛ, τὸ ΕΛ πρὸς ΚΗ· τῶν ΑΘ, ΗΚ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΕΛ. ἀλλὰ τὸ μὲν ΑΘ ἴσον ἐστὶ τῷ ΣΝ, τὸ δὲ ΗΚ ἴσον τῷ ΝΠ· τῶν ΣΝ, ΝΠ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΕΛ. ἔστι δὲ τῶν αὐτῶν τῶν ΣΝ, ΝΠ μέσον ἀνάλογον καὶ τὸ ΜΡ· ἴσον ἄρα ἐστὶ τὸ ΕΛ τῷ ΜΡ· ὥστε καὶ τῷ ΟΞ ἴσον ἐστίν. ἔστι δὲ καὶ τὰ ΑΘ, ΗΚ τοῖς ΣΝ, ΝΠ ἴσα· ὅλον ἄρα τὸ ΑΓ ἴσον ἐστὶν ὅλῳ τῷ ΣΠ, τουτέστι τῷ ἀπὸ τῆς ΜΞ τετραγώνῳ· τὸ ΑΓ ἄρα δύναται ἡ ΜΞ. λέγω, ὅτι ἡ ΜΞ ἐκ δύο ὀνομάτων ἐστίν.

Ἐπεὶ γὰρ σύμμετρός ἐστιν ἡ ΑΗ τῇ ΗΕ, σύμμετρός ἐστι καὶ ἡ ΑΕ ἐκατέρᾳ τῶν ΑΗ, ΗΕ. ὑπόκειται δὲ καὶ ἡ ΑΕ τῇ

was prescribed to show.

Proposition 54

If an area is contained by a rational (straight-line) and a first binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called binomial.[†]



For let the area AC be contained by the rational (straight-line) AB and by the first binomial (straight-line) AD . I say that square-root of area AC is the irrational (straight-line which is) called binomial.

For since AD is a first binomial (straight-line), let it have been divided into its (component) terms at E , and let AE be the greater term. So, (it is) clear that AE and ED are rational (straight-lines which are) commensurable in square only, and that the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE) , and that AE is commensurable (in length) with the rational (straight-line) AB (first) laid out [Def. 10.5]. So, let ED have been cut in half at point F . And since the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE) , thus if a (rectangle) equal to the fourth part of the (square) on the lesser (term)—that is to say, the (square) on EF —falling short by a square figure, is applied to the greater (term) AE , then it divides it into (terms which are) commensurable (in length) [Prop 10.17]. Therefore, let the (rectangle contained) by AG and GE , equal to the (square) on EF , have been applied to AE . AG is thus commensurable in length with EG . And let GH , EK , and FL have been drawn from (points) G , E , and F (respectively), parallel to either of AB or CD . And let the square SN , equal to the parallelogram AH , have been constructed, and (the square) NQ , equal to (the parallelogram) GK [Prop. 2.14]. And let MN be laid down so as to be straight-on to NO . RN is thus also straight-on to NP . And let the parallelogram SQ have been completed. SQ is thus a square [Prop. 10.53 lem.]. And since the (rectangle contained) by AG and GE is equal to the (square) on EF , thus as AG is to EF , so FE (is) to EG [Prop. 6.17]. And thus as AH (is) to EL , (so) EL (is)

AB σύμμετρος· καὶ αἱ AH, HE ἄρα τῇ AB σύμμετροί εἰσιν. καὶ ἐστὶ ῥητὴ ἡ AB· ῥητὴ ἄρα ἐστὶ καὶ ἑκατέρω τῶν AH, HE· ῥητὸν ἄρα ἐστὶν ἑκάτερον τῶν ΑΘ, ΗΚ, καὶ ἐστὶ σύμμετρον τὸ ΑΘ τῷ ΗΚ. ἀλλὰ τὸ μὲν ΑΘ τῷ ΣΝ ἴσον ἐστίν, τὸ δὲ ΗΚ τῷ ΝΠ· καὶ τὰ ΣΝ, ΝΠ ἄρα, τουτέστι τὰ ἀπὸ τῶν ΜΝ, ΝΞ, ῥητά ἐστὶ καὶ σύμμετρα. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΕ τῇ ΕΔ μήκει, ἀλλ' ἡ μὲν ΑΕ τῇ ΑΗ ἐστὶ σύμμετρος, ἡ δὲ ΔΕ τῇ ΕΖ σύμμετρος, ἀσύμμετρος ἄρα καὶ ἡ ΑΗ τῇ ΕΖ· ὥστε καὶ τὸ ΑΘ τῷ ΕΛ ἀσύμμετρον ἐστίν. ἀλλὰ τὸ μὲν ΑΘ τῷ ΣΝ ἐστὶν ἴσον, τὸ δὲ ΕΛ τῷ ΜΡ· καὶ τὸ ΣΝ ἄρα τῷ ΜΡ ἀσύμμετρον ἐστίν. ἀλλ' ὥς τὸ ΣΝ πρὸς ΜΡ, ἡ ΟΝ πρὸς τὴν ΝΡ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΟΝ τῇ ΝΡ. ἴση δὲ ἡ μὲν ΟΝ τῇ ΜΝ, ἡ δὲ ΝΡ τῇ ΝΞ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΜΝ τῇ ΝΞ. καὶ ἐστὶ τὸ ἀπὸ τῆς ΜΝ σύμμετρον τῷ ἀπὸ τῆς ΝΞ, καὶ ῥητὸν ἑκάτερον· αἱ ΜΝ, ΝΞ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι.

Ἡ ΜΞ ἄρα ἐκ δύο ὀνομάτων ἐστὶ καὶ δύνανται τὸ ΑΓ· ὅπερ ἔδει δεῖξαι.

to KG [Prop. 6.1]. Thus, EL is the mean proportional to AH and GK . But, AH is equal to SN , and GK (is) equal to NQ . EL is thus the mean proportional to SN and NQ . And MR is also the mean proportional to the same—(namely), SN and NQ [Prop. 10.53 lem.]. EL is thus equal to MR . Hence, it is also equal to PO [Prop. 1.43]. And AH plus GK is equal to SN plus NQ . Thus, the whole of AC is equal to the whole of SQ —that is to say, to the square on MO . Thus, MO (is) the square-root of (area) AC . I say that MO is a binomial (straight-line).

For since AG is commensurable (in length) with GE , AE is also commensurable (in length) with each of AG and GE [Prop. 10.15]. And AE was also assumed (to be) commensurable (in length) with AB . Thus, AG and GE are also commensurable (in length) with AB [Prop. 10.12]. And AB is rational. AG and GE are thus each also rational. Thus, AH and GK are each rational (areas), and AH is commensurable with GK [Prop. 10.19]. But, AH is equal to SN , and GK to NQ . SN and NQ —that is to say, the (squares) on MN and NO (respectively)—are thus also rational and commensurable. And since AE is incommensurable in length with ED , but AE is commensurable (in length) with AG , and DE (is) commensurable (in length) with EF , AG (is) thus also incommensurable (in length) with EF [Prop. 10.13]. Hence, AH is also incommensurable with EL [Props. 6.1, 10.11]. But, AH is equal to SN , and EL to MR . Thus, SN is also incommensurable with MR . But, as SN (is) to MR , (so) PN (is) to NR [Prop. 6.1]. PN is thus incommensurable (in length) with NR [Prop. 10.11]. And PN (is) equal to MN , and NR to NO . Thus, MN is incommensurable (in length) with NO . And the (square) on MN is commensurable with the (square) on NO , and each (is) rational. MN and NO are thus rational (straight-lines which are) commensurable in square only.

Thus, MO is (both) a binomial (straight-line) [Prop. 10.36], and the square-root of AC . (Which is) the very thing it was required to show.

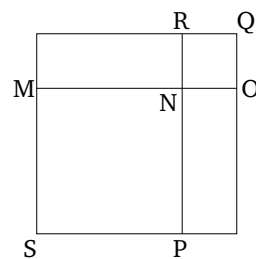
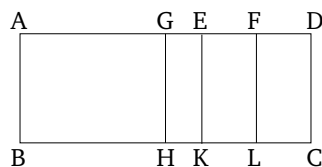
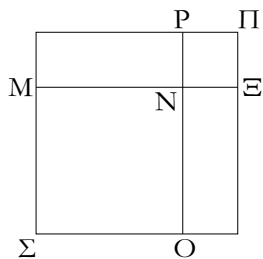
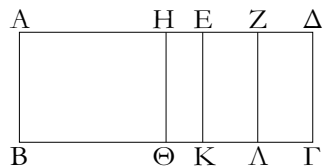
† If the rational straight-line has unit length then this proposition states that the square-root of a first binomial straight-line is a binomial straight-line: i.e., a first binomial straight-line has a length $k + k'\sqrt{1 - k'^2}$ whose square-root can be written $\rho(1 + \sqrt{k''})$, where $\rho = \sqrt{k(1 + k')}/2$ and $k'' = (1 - k')/(1 + k')$. This is the length of a binomial straight-line (see Prop. 10.36), since ρ is rational.

νε'.

Proposition 55

Ἐὰν χωρίον περιέχεται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων δευτέρας, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο μέσων πρώτη.

If an area is contained by a rational (straight-line) and a second binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called first bimedral.†



Περιεχέσθω γὰρ χωρίον τὸ $AB\Gamma\Delta$ ὑπὸ ῥητῆς τῆς AB καὶ τῆς ἐκ δύο ὀνομάτων δευτέρας τῆς AD . λέγω, ὅτι ἡ τὸ AG χωρίον δυναμένη ἐκ δύο μέσων πρώτη ἐστίν.

Ἐπεὶ γὰρ ἐκ δύο ὀνομάτων δευτέρα ἐστὶν ἡ AD , διηρήσθω εἰς τὰ ὀνόματα κατὰ τὸ E , ὥστε τὸ μείζον ὄνομα εἶναι τὸ AE . αἱ AE , ED ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ AE τῆς ED μείζον δύναται τῷ ἀπὸ συμμετρου ἑαυτῇ, καὶ τὸ ἐλάττω ὄνομα ἡ ED σύμμετρόν ἐστι τῇ AB μήκει. τεμήσθω ἡ ED δίχα κατὰ τὸ Z , καὶ τῷ ἀπὸ τῆς EZ ἴσον παρὰ τὴν AE παραβεβλήσθω ἐλλείπον εἶδει τετραγώνῳ τὸ ὑπὸ τῶν AHE . σύμμετρος ἄρα ἡ AH τῇ HE μήκει. καὶ διὰ τῶν H , E , Z παράλληλοι ἤχθωσαν ταῖς AB , $\Gamma\Delta$ αἱ $H\Theta$, EK , $Z\Lambda$, καὶ τῷ μὲν $A\Theta$ παραλληλογράμμῳ ἴσον τετράγωνον συνεστάτω τὸ ΣN , τῷ δὲ HK ἴσον τετράγωνον τὸ $N\Pi$, καὶ κείσθω ὥστε ἐπ' εὐθείας εἶναι τὴν MN τῇ $N\Xi$. ἐπ' εὐθείας ἄρα [ἐστὶ] καὶ ἡ PN τῇ NO . καὶ συμπεπληρώσθω τὸ $\Sigma\Pi$ τετράγωνον· φανερόν δὲ ἐκ τοῦ προδεδειγμένου, ὅτι τὸ MP μέσον ἀνάλογόν ἐστι τῶν ΣN , $N\Pi$, καὶ ἴσον τῷ EL , καὶ ὅτι τὸ AG χωρίον δύναται ἡ $M\Xi$. δεικτέον δὲ, ὅτι ἡ $M\Xi$ ἐκ δύο μέσων ἐστὶ πρώτη.

Ἐπεὶ ἀσύμμετρός ἐστιν ἡ AE τῇ ED μήκει, σύμμετρος δὲ ἡ ED τῇ AB , ἀσύμμετρος ἄρα ἡ AE τῇ AB . καὶ ἐπεὶ σύμμετρός ἐστιν ἡ AH τῇ EH , σύμμετρός ἐστι καὶ ἡ AE ἑκάτερά τῶν AH , HE . ἀλλὰ ἡ AE ἀσύμμετρος τῇ AB μήκει· καὶ αἱ AH , HE ἄρα ἀσύμμετροί εἰσι τῇ AB . αἱ BA , AH , HE ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ὥστε μέσον ἐστὶν ἑκάτερον τῶν $A\Theta$, HK . ὥστε καὶ ἑκάτερον τῶν ΣN , $N\Pi$ μέσον ἐστίν. καὶ αἱ MN , $N\Xi$ ἄρα μέσαι εἰσίν. καὶ ἐπεὶ σύμμετρος ἡ AH τῇ HE μήκει, σύμμετρόν ἐστι καὶ τὸ $A\Theta$ τῷ HK , τουτέστι τὸ ΣN τῷ $N\Pi$, τουτέστι τὸ ἀπὸ τῆς MN τῷ ἀπὸ τῆς $N\Xi$ [ὥστε δυνάμει εἰσι σύμμετροι αἱ MN , $N\Xi$]. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ AE τῇ ED μήκει, ἀλλ' ἡ μὲν AE σύμμετρός ἐστι τῇ AH , ἡ δὲ ED τῇ EZ σύμμετρος, ἀσύμμετρος ἄρα ἡ AH τῇ EZ . ὥστε καὶ τὸ $A\Theta$ τῷ EL ἀσύμμετρόν ἐστιν, τουτέστι τὸ ΣN τῷ MP , τουτέστιν ὁ ON τῇ NP , τουτέστιν ἡ MN τῇ $N\Xi$ ἀσύμμετρός ἐστι μήκει. ἐδείχθησαν δὲ αἱ MN , $N\Xi$ καὶ μέσαι οὕσαι καὶ δυνάμει σύμμετροι· αἱ MN , $N\Xi$ ἄρα μέσαι εἰσι δυνάμει μόνον σύμμετροι. λέγω δὲ, ὅτι καὶ ῥητὸν περιέχουσιν. ἐπεὶ γὰρ ἡ ΔE ὑπόκειται ἑκάτερά τῶν AB , EZ σύμμετρος, σύμμετρος ἄρα καὶ ἡ EZ τῇ EK . καὶ ῥητὴ ἑκάτερα αὐτῶν· ῥητὸν ἄρα τὸ EL , τουτέστι τὸ MP . τὸ δὲ MP ἐστὶ τὸ ὑπὸ τῶν $MN\Xi$. ἐὰν δὲ δύο μέσαι δυνάμει μόνον σύμμετροι συντεθῶσι ῥητὸν

For let the area $ABCD$ be contained by the rational (straight-line) AB and by the second binomial (straight-line) AD . I say that the square-root of area AC is a first binomial (straight-line).

For since AD is a second binomial (straight-line), let it have been divided into its (component) terms at E , such that AE is the greater term. Thus, AE and ED are rational (straight-lines which are) commensurable in square only, and the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE) , and the lesser term ED is commensurable in length with AB [Def. 10.6]. Let ED have been cut in half at F . And let the (rectangle contained) by AGE , equal to the (square) on EF , have been applied to AE , falling short by a square figure. AG (is) thus commensurable in length with GE [Prop. 10.17]. And let GH , EK , and FL have been drawn through (points) G , E , and F (respectively), parallel to AB and CD . And let the square SN , equal to the parallelogram AH , have been constructed, and the square NQ , equal to GK . And let MN be laid down so as to be straight-on to NO . Thus, RN [is] also straight-on to NP . And let the square SQ have been completed. So, (it is) clear from what has been previously demonstrated [Prop. 10.53 lem.] that MR is the mean proportional to SN and NQ , and (is) equal to EL , and that MO is the square-root of the area AC . So, we must show that MO is a first binomial (straight-line).

Since AE is incommensurable in length with ED , and ED (is) commensurable (in length) with AB , AE (is) thus incommensurable (in length) with AB [Prop. 10.13]. And since AG is commensurable (in length) with EG , AE is also commensurable (in length) with each of AG and GE [Prop. 10.15]. But, AE is incommensurable in length with AB . Thus, AG and GE are also (both) incommensurable (in length) with AB [Prop. 10.13]. Thus, BA , AG , and $(BA, \text{ and } GE)$ are (pairs of) rational (straight-lines which are) commensurable in square only. And, hence, each of AH and GK is a medial (area) [Prop. 10.21]. Hence, each of SN and NQ is also a medial (area). Thus, MN and NO are medial (straight-lines). And since AG (is) commensurable in length with GE , AH is also commensurable

περιέχουσαι, ἡ ὅλη ἄλογός ἐστιν, καλεῖται δὲ ἐκ δύο μέσων πρώτη.

Ἡ ἄρα ΜΞ ἐκ δύο μέσων ἐστὶ πρώτη· ὅπερ ἔδει δεῖξαι.

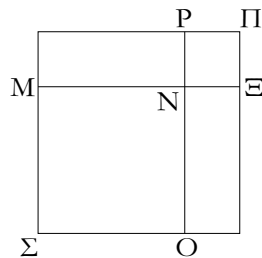
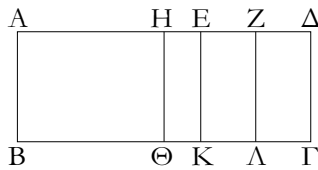
with GK —that is to say, SN with NQ —that is to say, the (square) on MN with the (square) on NO [hence, MN and NO are commensurable in square] [Props. 6.1, 10.11]. And since AE is incommensurable in length with ED , but AE is commensurable (in length) with AG , and ED commensurable (in length) with EF , AG (is) thus incommensurable (in length) with EF [Prop. 10.13]. Hence, AH is also incommensurable with EL —that is to say, SN with MR —that is to say, PN with NR —that is to say, MN is incommensurable in length with NO [Props. 6.1, 10.11]. But MN and NO have also been shown to be medial (straight-lines) which are commensurable in square. Thus, MN and NO are medial (straight-lines which are) commensurable in square only. So, I say that they also contain a rational (area). For since DE was assumed (to be) commensurable (in length) with each of AB and EF , EF (is) thus also commensurable with EK [Prop. 10.12]. And they (are) each rational. Thus, EL —that is to say, MR —(is) rational [Prop. 10.19]. And MR is the (rectangle contained) by MNO . And if two medial (straight-lines), commensurable in square only, which contain a rational (area), are added together, then the whole is (that) irrational (straight-line which is) called first bimedral [Prop. 10.37].

Thus, MO is a first bimedral (straight-line). (Which is) the very thing it was required to show.

† If the rational straight-line has unit length then this proposition states that the square-root of a second binomial straight-line is a first bimedral straight-line: i.e., a second binomial straight-line has a length $k/\sqrt{1-k'^2} + k$ whose square-root can be written $\rho(k''^{1/4} + k'^{3/4})$, where $\rho = \sqrt{(k/2)(1+k')/(1-k')}$ and $k'' = (1-k')/(1+k')$. This is the length of a first bimedral straight-line (see Prop. 10.37), since ρ is rational.

νζ'.

Ἐάν χωρίον περιέχεται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων τρίτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο μέσων δευτέρα.

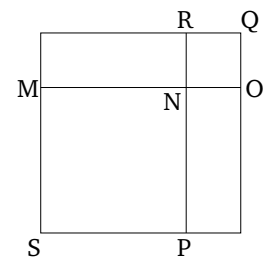
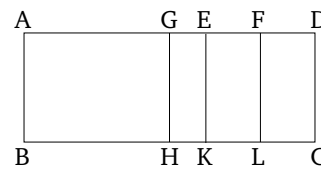


Χωρίον γὰρ τὸ ΑΒΓΔ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων τρίτης τῆς ΑΔ διηρημένης εἰς τὰ ὀνόματα κατὰ τὸ Ε, ὧν μείζον ἐστὶ τὸ ΑΕ· λέγω, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ἐκ δύο μέσων δευτέρα.

Κατεσκευάσθω γὰρ τὰ αὐτὰ τοῖς πρότερον. καὶ ἐπεὶ

Proposition 56

If an area is contained by a rational (straight-line) and a third binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called second bimedral.†



For let the area $ABCD$ be contained by the rational (straight-line) AB and by the third binomial (straight-line) AD , which has been divided into its (component) terms at E , of which AE is the greater. I say that the square-root of area AC is the irrational (straight-line which is) called second bimedral.

ἐκ δύο ὀνομάτων ἐστὶ τρίτη ἡ AD , αἱ AE , ED ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ AE τῆς ED μείζον δύναται τῷ ἀπὸ συμμετρου ἑαυτῇ, καὶ οὐδετέρα τῶν AE , ED σύμμετρός [ἐστὶ] τῇ AB μήκει. ὁμοίως δὲ τοῖς προδεδειγμένοις δείξομεν, ὅτι ἡ ME ἐστὶν ἡ τὸ AG χωρίον δυναμένη, καὶ αἱ MN , NE μέσαι εἰσι δυνάμει μόνον σύμμετροι· ὥστε ἡ ME ἐκ δύο μέσων ἐστίν. δεικτέον δὲ, ὅτι καὶ δευτέρα.

[Καὶ] ἐπεὶ ἀσύμμετρός ἐστιν ἡ DE τῇ AB μήκει, τούτεστι τῇ EK , σύμμετρος δὲ ἡ DE τῇ EZ , ἀσύμμετρος ἄρα ἐστὶν ἡ EZ τῇ EK μήκει. καὶ εἰσι ῥηταί· αἱ ZE , EK ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι. μέσον ἄρα [ἐστὶ] τὸ EL , τούτεστι τὸ MP · καὶ περιέχεται ὑπὸ τῶν MNE · μέσον ἄρα ἐστὶ τὸ ὑπὸ τῶν MNE .

Ἡ ME ἄρα ἐκ δύο μέσων ἐστὶ δευτέρα· ὅπερ ἔδει δείξαι.

For let the same construction be made as previously. And since AD is a third binomial (straight-line), AE and ED are thus rational (straight-lines which are) commensurable in square only, and the square on AE is greater than (the square on) ED by the (square) on (some straight-line) commensurable (in length) with (AE) , and neither of AE and ED [is] commensurable in length with AB [Def. 10.7]. So, similarly to that which has been previously demonstrated, we can show that MO is the square-root of area AC , and MN and NO are medial (straight-lines which are) commensurable in square only. Hence, MO is bimedral. So, we must show that (it is) also second (bimedral).

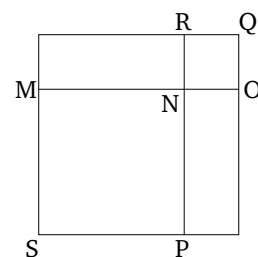
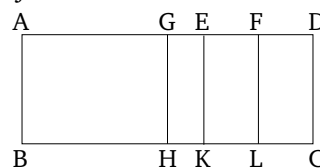
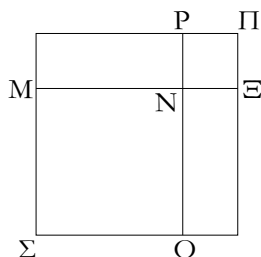
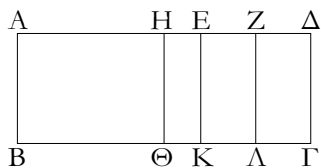
[And] since DE is incommensurable in length with AB —that is to say, with EK —and DE (is) commensurable (in length) with EF , EF is thus incommensurable in length with EK [Prop. 10.13]. And they are (both) rational (straight-lines). Thus, FE and EK are rational (straight-lines which are) commensurable in square only. EL —that is to say, MR —[is] thus medial [Prop. 10.21]. And it is contained by MNO . Thus, the (rectangle contained) by MNO is medial.

Thus, MO is a second bimedral (straight-line) [Prop. 10.38]. (Which is) the very thing it was required to show.

† If the rational straight-line has unit length then this proposition states that the square-root of a third binomial straight-line is a second bimedral straight-line: i.e., a third binomial straight-line has a length $k^{1/2}(1 + \sqrt{1 - k'^2})$ whose square-root can be written $\rho(k^{1/4} + k'^{1/2}/k^{1/4})$, where $\rho = \sqrt{(1 + k')/2}$ and $k'' = k(1 - k')/(1 + k')$. This is the length of a second bimedral straight-line (see Prop. 10.38), since ρ is rational.

νζ'.

Ἐὰν χωρίον περιέχεται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων τετάρτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη μείζων.



Χωρίον γὰρ τὸ AG περιεχέσθω ὑπὸ ῥητῆς τῆς AB καὶ τῆς ἐκ δύο ὀνομάτων τετάρτης τῆς AD διηρημένης εἰς τὰ ὀνόματα κατὰ τὸ E , ὧν μείζον ἔστω τὸ AE · λέγω, ὅτι ἡ τὸ AG χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη μείζων.

Ἐπεὶ γὰρ ἡ AD ἐκ δύο ὀνομάτων ἐστὶ τετάρτη, αἱ AE , ED ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ AE τῆς ED μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ, καὶ ἡ AE τῇ AB σύμμετρός [ἐστὶ] μήκει. τετμήσθω ἡ DE δίχα κατὰ

Proposition 57

If an area is contained by a rational (straight-line) and a fourth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called major.†

For let the area AC be contained by the rational (straight-line) AB and the fourth binomial (straight-line) AD , which has been divided into its (component) terms at E , of which let AE be the greater. I say that the square-root of AC is the irrational (straight-line which is) called major.

For since AD is a fourth binomial (straight-line), AE and ED are thus rational (straight-lines which are) com-

τὸ Ζ, καὶ τῷ ἀπὸ τῆς ΕΖ ἴσον παρὰ τὴν ΑΕ παραβεβλήσθω παραλληλόγραμμον τὸ ὑπὸ ΑΗ, ΗΕ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΗ τῇ ΗΕ μήκει. ἤχθωσαν παράλληλοι τῇ ΑΒ αἱ ΗΘ, ΕΚ, ΖΛ, καὶ τὰ λοιπὰ τὰ αὐτὰ τοῖς πρὸ τούτου γεγονέντω· φανερόν δὴ, ὅτι ἡ τὸ ΑΓ χωρίον δυναμένη ἐστὶν ἡ ΜΞ. δεικτέον δὴ, ὅτι ἡ ΜΞ ἄλογός ἐστιν ἡ καλουμένη μείζων.

Ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΑΗ τῇ ΕΗ μήκει, ἀσύμμετρόν ἐστι καὶ τὸ ΑΘ τῷ ΗΚ, τουτέστι τὸ ΣΝ τῷ ΝΗ· αἱ ΜΝ, ΝΞ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΑΕ τῇ ΑΒ μήκει, ῥητόν ἐστι τὸ ΑΚ· καὶ ἐστὶν ἴσον τοῖς ἀπὸ τῶν ΜΝ, ΝΞ· ῥητόν ἄρα [ἐστὶ] καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ. καὶ ἐπεὶ ἀσύμμετρός [ἐστὶν] ἡ ΔΕ τῇ ΑΒ μήκει, τουτέστι τῇ ΕΚ, ἀλλὰ ἡ ΔΕ σύμμετρός ἐστι τῇ ΕΖ, ἀσύμμετρος ἄρα ἡ ΕΖ τῇ ΕΚ μήκει. αἱ ΕΚ, ΕΖ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα τὸ ΑΕ, τουτέστι τὸ ΜΡ. καὶ περιέχεται ὑπὸ τῶν ΜΝ, ΝΞ· μέσον ἄρα ἐστὶ τὸ ὑπὸ τῶν ΜΝ, ΝΞ. καὶ ῥητόν τὸ [συγκείμενον] ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ, καὶ εἰσὶν ἀσύμμετροι αἱ ΜΝ, ΝΞ δυνάμει. ἐὰν δὲ δύο εὐθεῖαι δυνάμει ἀσύμμετροι συντεθῶσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δ' ὑπ' αὐτῶν μέσον, ἡ ὅλη ἄλογός ἐστιν, καλεῖται δὲ μείζων.

Ἡ ΜΞ ἄρα ἄλογός ἐστιν ἡ καλουμένη μείζων, καὶ δύναται τὸ ΑΓ χωρίον· ὅπερ εἶδει δεῖξαι.

measurable in square only, and the square on AE is greater than (the square on) ED by the (square) on (some straight-line) incommensurable (in length) with (AE), and AE [is] commensurable in length with AB [Def. 10.8]. Let DE have been cut in half at F , and let the parallelogram (contained by) AG and GE , equal to the (square) on EF , (and falling short by a square figure) have been applied to AE . AG is thus incommensurable in length with GE [Prop. 10.18]. Let GH , EK , and FL have been drawn parallel to AB , and let the rest (of the construction) have been made the same as the (proposition) before this. So, it is clear that MO is the square-root of area AC . So, we must show that MO is the irrational (straight-line which is) called major.

Since AG is incommensurable in length with EG , AH is also incommensurable with GK —that is to say, SN with NQ [Props. 6.1, 10.11]. Thus, MN and NO are incommensurable in square. And since AE is commensurable in length with AB , AK is rational [Prop. 10.19]. And it is equal to the (sum of the squares) on MN and NO . Thus, the sum of the (squares) on MN and NO [is] also rational. And since DE [is] incommensurable in length with AB [Prop. 10.13]—that is to say, with EK —but DE is commensurable (in length) with EF , EF (is) thus incommensurable in length with EK [Prop. 10.13]. Thus, EK and EF are rational (straight-lines which are) commensurable in square only. LE —that is to say, MR —(is) thus medial [Prop. 10.21]. And it is contained by MN and NO . The (rectangle contained) by MN and NO is thus medial. And the [sum] of the (squares) on MN and NO (is) rational, and MN and NO are incommensurable in square. And if two straight-lines (which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial, are added together, then the whole is the irrational (straight-line which is) called major [Prop. 10.39].

Thus, MO is the irrational (straight-line which is) called major. And (it is) the square-root of area AC . (Which is) the very thing it was required to show.

† If the rational straight-line has unit length then this proposition states that the square-root of a fourth binomial straight-line is a major straight-line: i.e., a fourth binomial straight-line has a length $k(1 + 1/\sqrt{1+k'})$ whose square-root can be written $\rho\sqrt{[1 + k''/(1 + k''^2)^{1/2}]/2} + \rho\sqrt{[1 - k''/(1 + k''^2)^{1/2}]/2}$, where $\rho = \sqrt{k}$ and $k''^2 = k'$. This is the length of a major straight-line (see Prop. 10.39), since ρ is rational.

νη'.

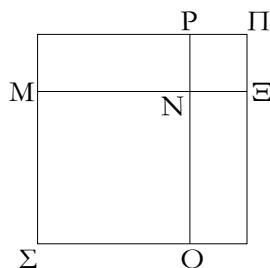
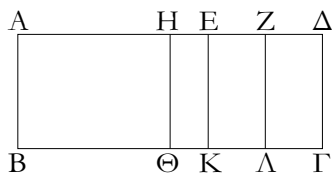
Proposition 58

Ἐὰν χωρίον περιέχεται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων πέμπτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ῥητόν καὶ μέσον δυναμένη.

Χωρίον γὰρ τὸ ΑΓ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ

If an area is contained by a rational (straight-line) and a fifth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called the square-root of a rational plus a medial (area).†

τῆς ἐκ δύο ὀνομάτων πέμπτης τῆς AD διηρημένης εἰς τὰ ὀνόματα κατὰ τὸ E , ὥστε τὸ μείζον ὄνομα εἶναι τὸ AE · λέγω $[δῆ]$, ὅτι ἡ τὸ AG χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη ῥητὸν καὶ μέσον δυναμένη.

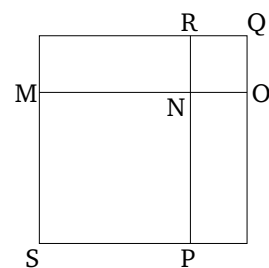
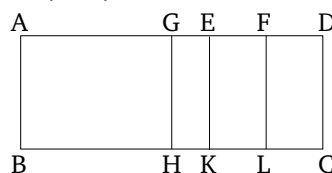


Κατεσκευάσθω γὰρ τὰ αὐτὰ τοῖς πρότερον δεδειγμένοις· φανερόν δῆ, ὅτι ἡ τὸ AG χωρίον δυναμένη ἐστὶν ἡ ME . δεικτέον δῆ, ὅτι ἡ ME ἐστὶν ἡ ῥητὸν καὶ μέσον δυναμένη.

Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ AH τῇ HE , ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ $AΘ$ τῷ $ΘΕ$, τουτέστι τὸ ἀπὸ τῆς MN τῷ ἀπὸ τῆς NE · αἱ MN , NE ἄρα δυνάμει εἰσὶν ἀσύμμετροι. καὶ ἐπεὶ ἡ AD ἐκ δύο ὀνομάτων ἐστὶ πέμπτη, καὶ $[ἐστὶν]$ ἔλασσον αὐτῆς τμήμα τὸ ED , σύμμετρος ἄρα ἡ ED τῇ AB μήκει. ἀλλὰ ἡ AE τῇ ED ἐστὶν ἀσύμμετρος· καὶ ἡ AB ἄρα τῇ AE ἐστὶν ἀσύμμετρος μήκει $[αἱ BA, AE$ ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι]· μέσον ἄρα ἐστὶ τὸ AK , τουτέστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν MN , NE . καὶ ἐπεὶ σύμμετρός ἐστιν ἡ DE τῇ AB μήκει, τουτέστι τῇ EK , ἀλλὰ ἡ DE τῇ EZ σύμμετρός ἐστιν, καὶ ἡ EZ ἄρα τῇ EK σύμμετρός ἐστιν. καὶ ῥητὴ ἡ EK · ῥητὸν ἄρα καὶ τὸ EL , τουτέστι τὸ MP , τουτέστι τὸ ὑπὸ MNE · αἱ MN , NE ἄρα δυνάμει ἀσύμμετροί εἰσι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δ' ὑπ' αὐτῶν ῥητόν.

Ἡ ME ἄρα ῥητὸν καὶ μέσον δυναμένη ἐστὶ καὶ δύναται τὸ AG χωρίον· ὅπερ ἔδει δεῖξαι.

For let the area AC be contained by the rational (straight-line) AB and the fifth binomial (straight-line) AD , which has been divided into its (component) terms at E , such that AE is the greater term. [So] I say that the square-root of area AC is the irrational (straight-line which is) called the square-root of a rational plus a medial (area).



For let the same construction be made as that shown previously. So, (it is) clear that MO is the square-root of area AC . So, we must show that MO is the square-root of a rational plus a medial (area).

For since AG is incommensurable (in length) with GE [Prop. 10.18], AH is thus also incommensurable with HE —that is to say, the (square) on MN with the (square) on NO [Props. 6.1, 10.11]. Thus, MN and NO are incommensurable in square. And since AD is a fifth binomial (straight-line), and ED [is] its lesser segment, ED (is) thus commensurable in length with AB [Def. 10.9]. But, AE is incommensurable (in length) with ED . Thus, AB is also incommensurable in length with AE [BA and AE are rational (straight-lines which are) commensurable in square only] [Prop. 10.13]. Thus, AK —that is to say, the sum of the (squares) on MN and NO —is medial [Prop. 10.21]. And since DE is commensurable in length with AB —that is to say, with EK —but, DE is commensurable (in length) with EF , EF is thus also commensurable (in length) with EK [Prop. 10.12]. And EK (is) rational. Thus, EL —that is to say, MR —that is to say, the (rectangle contained) by MNO —(is) also rational [Prop. 10.19]. MN and NO are thus (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational.

Thus, MO is the square-root of a rational plus a medial (area) [Prop. 10.40]. And (it is) the square-root of area AC . (Which is) the very thing it was required to show.

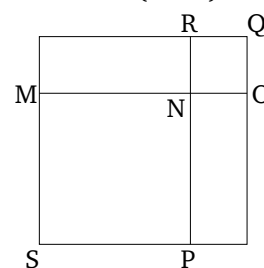
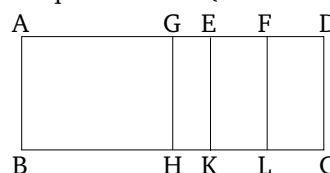
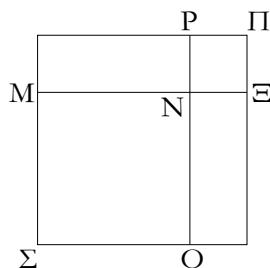
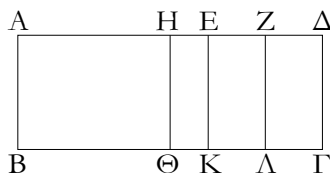
[†] If the rational straight-line has unit length then this proposition states that the square-root of a fifth binomial straight-line is the square root of a rational plus a medial area: i.e., a fifth binomial straight-line has a length $k(\sqrt{1+k'}+1)$ whose square-root can be written

$\rho\sqrt{[(1+k''^2)^{1/2}+k'']/[2(1+k''^2)]} + \rho\sqrt{[(1+k''^2)^{1/2}-k'']/[2(1+k''^2)]}$, where $\rho = \sqrt{k(1+k''^2)}$ and $k''^2 = k'$. This is the length of

the square root of a rational plus a medial area (see Prop. 10.40), since ρ is rational.

νθ'.

Ἐάν χωρίον περιέχεται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων ἔκτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἡ καλουμένη δύο μέσα δυναμένη.



Χωρίον γὰρ τὸ ΑΒΓΔ περιεχέσθω ὑπὸ ῥητῆς τῆς ΑΒ καὶ τῆς ἐκ δύο ὀνομάτων ἔκτης τῆς ΑΔ διηρημένης εἰς τὰ ὀνόματα κατὰ τὸ Ε, ὥστε τὸ μείζον ὄνομα εἶναι τὸ ΑΕ· λέγω, ὅτι ἡ τὸ ΑΓ δυναμένη ἡ δύο μέσα δυναμένη ἐστίν.

Κατεσκευάσθω [γὰρ] τὰ αὐτὰ τοῖς προοδεδειγμένοις. φανερόν δὴ, ὅτι [ἡ] τὸ ΑΓ δυναμένη ἐστίν ἡ ΜΞ, καὶ ὅτι ἀσύμμετρός ἐστιν ἡ ΜΝ τῇ ΝΞ δυνάμει. καὶ ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΕΑ τῇ ΑΒ μήκει, αἱ ΕΑ, ΑΒ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα ἐστὶ τὸ ΑΚ, τουτέστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ. πάλιν, ἐπεὶ ἀσύμμετρός ἐστιν ἡ ΕΔ τῇ ΑΒ μήκει, ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΖΕ τῇ ΕΚ· αἱ ΖΕ, ΕΚ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· μέσον ἄρα ἐστὶ τὸ ΕΛ, τουτέστι τὸ ΜΡ, τουτέστι τὸ ὑπὸ τῶν ΜΝΞ. καὶ ἐπεὶ ἀσύμμετρος ἡ ΑΕ τῇ ΕΖ, καὶ τὸ ΑΚ τῷ ΕΛ ἀσύμμετρόν ἐστιν. ἀλλὰ τὸ μὲν ΑΚ ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝ, ΝΞ, τὸ δὲ ΕΛ ἐστὶ τὸ ὑπὸ τῶν ΜΝΞ· ἀσύμμετρον ἄρα ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΜΝΞ τῷ ὑπὸ τῶν ΜΝΞ. καὶ ἐστὶ μέσον ἑκάτερον αὐτῶν, καὶ αἱ ΜΝ, ΝΞ δυνάμει εἰσὶν ἀσύμμετροι.

Ἡ ΜΞ ἄρα δύο μέσα δυναμένη ἐστὶ καὶ δύναται τὸ ΑΓ· ὅπερ ἔδει δεῖξαι.

Proposition 59

If an area is contained by a rational (straight-line) and a sixth binomial (straight-line) then the square-root of the area is the irrational (straight-line which is) called the square-root of (the sum of) two medial (areas).[†]

For let the area $ABCD$ be contained by the rational (straight-line) AB and the sixth binomial (straight-line) AD , which has been divided into its (component) terms at E , such that AE is the greater term. So, I say that the square-root of AC is the square-root of (the sum of) two medial (areas).

[For] let the same construction be made as that shown previously. So, (it is) clear that MO is the square-root of AC , and that MN is incommensurable in square with NO . And since EA is incommensurable in length with AB [Def. 10.10], EA and AB are thus rational (straight-lines which are) commensurable in square only. Thus, AK —that is to say, the sum of the (squares) on MN and NO —is medial [Prop. 10.21]. Again, since ED is incommensurable in length with AB [Def. 10.10], FE is thus also incommensurable (in length) with EK [Prop. 10.13]. Thus, FE and EK are rational (straight-lines which are) commensurable in square only. Thus, EL —that is to say, MR —that is to say, the (rectangle contained) by MNO —is medial [Prop. 10.21]. And since AE is incommensurable (in length) with EF , AK is also incommensurable with EL [Props. 6.1, 10.11]. But, AK is the sum of the (squares) on MN and NO , and EL is the (rectangle contained) by MNO . Thus, the sum of the (squares) on MNO is incommensurable with the (rectangle contained) by MNO . And each of them is medial. And MN and NO are incommensurable in square.

Thus, MO is the square-root of (the sum of) two medial (areas) [Prop. 10.41]. And (it is) the square-root of AC . (Which is) the very thing it was required to show.

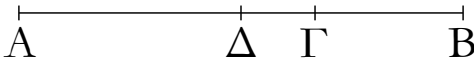
[†] If the rational straight-line has unit length then this proposition states that the square-root of a sixth binomial straight-line is the square root of the sum of two medial areas: i.e., a sixth binomial straight-line has a length $\sqrt{k} + \sqrt{k'}$ whose square-root can be written

$k^{1/4} \left(\sqrt{[1 + k''/(1 + k''^2)^{1/2}]/2} + \sqrt{[1 - k''/(1 + k''^2)^{1/2}]/2} \right)$, where $k''^2 = (k - k')/k'$. This is the length of the square-root of the sum of

two medial areas (see Prop. 10.41).

Λήμμα.

Ἐάν εὐθεῖα γραμμὴ τμηθῇ εἰς ἄνισα, τὰ ἀπὸ τῶν ἀνίσων τετράγωνα μείζονά ἐστι τοῦ δις ὑπὸ τῶν ἀνίσων περιεχομένου ὀρθογωνίου.

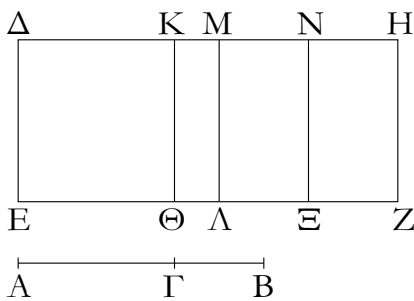


Ἐστω εὐθεῖα ἡ AB καὶ τετμήσθω εἰς ἄνισα κατὰ τὸ Γ , καὶ ἔστω μείζων ἡ AG . λέγω, ὅτι τὰ ἀπὸ τῶν AG , GB μείζονά ἐστι τοῦ δις ὑπὸ τῶν AG , GB .

Τετμήσθω γὰρ ἡ AB δίχα κατὰ τὸ Δ . ἐπεὶ οὖν εὐθεῖα γραμμὴ τέτμηται εἰς μὲν ἴσα κατὰ τὸ Δ , εἰς δὲ ἄνισα κατὰ τὸ Γ , τὸ ἄρα ὑπὸ τῶν AG , GB μετὰ τοῦ ἀπὸ $\Gamma\Delta$ ἴσον ἐστὶ τῷ ἀπὸ $A\Delta$. ὥστε τὸ ὑπὸ τῶν AG , GB ἑλαττόν ἐστι τοῦ ἀπὸ $A\Delta$. τὸ ἄρα δις ὑπὸ τῶν AG , GB ἑλαττόν ἢ διπλάσιόν ἐστι τοῦ ἀπὸ $A\Delta$. ἀλλὰ τὰ ἀπὸ τῶν AG , GB διπλάσιά [ἐστι] τῶν ἀπὸ τῶν $A\Delta$, $\Delta\Gamma$. τὰ ἄρα ἀπὸ τῶν AG , GB μείζονά ἐστι τοῦ δις ὑπὸ τῶν AG , GB . ὅπερ ἔδει δεῖξαι.

ξ'.

Τὸ ἀπὸ τῆς ἐκ δύο ὀνομάτων παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πρώτην.

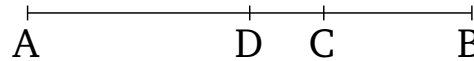


Ἐστω ἐκ δύο ὀνομάτων ἡ AB διηρημένη εἰς τὰ ὀνόματα κατὰ τὸ Γ , ὥστε τὸ μείζον ὄνομα εἶναι τὸ AG , καὶ ἐκκείσθω ῥητὴ ἡ DE , καὶ τῷ ἀπὸ τῆς AB ἴσον παρὰ τὴν DE παραβελήσθω τὸ $DEZH$ πλάτος ποιῶν τὴν ΔH . λέγω, ὅτι ἡ ΔH ἐκ δύο ὀνομάτων ἐστὶ πρώτη.

Παραβελήσθω γὰρ παρὰ τὴν DE τῷ μὲν ἀπὸ τῆς AG ἴσον τὸ $\Delta\Theta$, τῷ δὲ ἀπὸ τῆς GB ἴσον τὸ KL . λοιπὸν ἄρα τὸ δις ὑπὸ τῶν AG , GB ἴσον ἐστὶ τῷ MZ . τετμήσθω ἡ MH δίχα κατὰ τὸ N , καὶ παράλληλος ἦχθω ἡ $NΞ$ [ἐκατέρω

Lemma

If a straight-line is cut unequally then (the sum of) the squares on the unequal (parts) is greater than twice the rectangle contained by the unequal (parts).

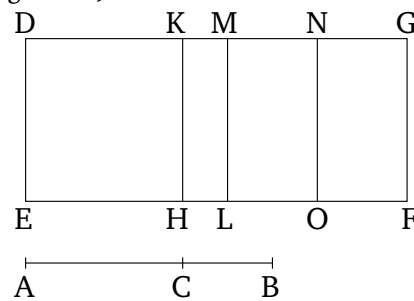


Let AB be a straight-line, and let it have been cut unequally at C , and let AC be greater (than CB). I say that (the sum of) the (squares) on AC and CB is greater than twice the (rectangle contained) by AC and CB .

For let AB have been cut in half at D . Therefore, since a straight-line has been cut into equal (parts) at D , and into unequal (parts) at C , the (rectangle contained) by AC and CB , plus the (square) on CD , is thus equal to the (square) on AD [Prop. 2.5]. Hence, the (rectangle contained) by AC and CB is less than the (square) on AD . Thus, twice the (rectangle contained) by AC and CB is less than double the (square) on AD . But, (the sum of) the (squares) on AC and CB [is] double (the sum of) the (squares) on AD and DC [Prop. 2.9]. Thus, (the sum of) the (squares) on AC and CB is greater than twice the (rectangle contained) by AC and CB . (Which is) the very thing it was required to show.

Proposition 60

The square on a binomial (straight-line) applied to a rational (straight-line) produces as breadth a first binomial (straight-line).[†]



Let AB be a binomial (straight-line), having been divided into its (component) terms at C , such that AC is the greater term. And let the rational (straight-line) DE be laid down. And let the (rectangle) $DEFG$, equal to the (square) on AB , have been applied to DE , producing DG as breadth. I say that DG is a first binomial (straight-line).

For let DH , equal to the (square) on AC , and KL , equal to the (square) on BC , have been applied to DE .

τῶν $ΜΑ$, $ΗΖ$]. ἑκάτερον ἄρα τῶν $ΜΞ$, $ΝΖ$ ἴσον ἐστὶ τῷ ἀπαξ ὑπὸ τῶν $ΑΓΒ$. καὶ ἐπεὶ ἐκ δύο ὀνομάτων ἐστὶν ἡ $ΑΒ$ διηρημένη εἰς τὰ ὀνόματα κατὰ τὸ $Γ$, αἱ $ΑΓ$, $ΓΒ$ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· τὰ ἄρα ἀπὸ τῶν $ΑΓ$, $ΓΒ$ ῥητὰ ἐστὶ καὶ σύμμετρα ἀλλήλοις· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν $ΑΓ$, $ΓΒ$. καὶ ἐστὶν ἴσον τῷ $ΔΛ$ · ῥητὸν ἄρα ἐστὶ τὸ $ΔΛ$. καὶ παρὰ ῥητὴν τὴν $ΔΕ$ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ $ΔΜ$ καὶ σύμμετρος τῇ $ΔΕ$ μήκει. πάλιν, ἐπεὶ αἱ $ΑΓ$, $ΓΒ$ ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, μέσον ἄρα ἐστὶ τὸ δις ὑπὸ τῶν $ΑΓ$, $ΓΒ$, τουτέστι τὸ $ΜΖ$. καὶ παρὰ ῥητὴν τὴν $ΜΑ$ παράκειται· ῥητὴ ἄρα καὶ ἡ $ΜΗ$ καὶ ἀσύμμετρος τῇ $ΜΑ$, τουτέστι τῇ $ΔΕ$, μήκει. ἐστὶ δὲ καὶ ἡ $ΜΔ$ ῥητὴ καὶ τῇ $ΔΕ$ μήκει σύμμετρος· ἀσύμμετρος ἄρα ἐστὶν ἡ $ΔΜ$ τῇ $ΜΗ$ μήκει. καὶ εἰσι ῥηταὶ· αἱ $ΔΜ$, $ΜΗ$ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ $ΔΗ$. δεϊκτέον δὴ, ὅτι καὶ πρώτη.

Ἐπεὶ τῶν ἀπὸ τῶν $ΑΓ$, $ΓΒ$ μέσον ἀνάλογόν ἐστὶ τὸ ὑπὸ τῶν $ΑΓΒ$, καὶ τῶν $ΔΘ$, $ΚΛ$ ἄρα μέσον ἀνάλογόν ἐστὶ τὸ $ΜΞ$. ἐστὶν ἄρα ὡς τὸ $ΔΘ$ πρὸς τὸ $ΜΞ$, οὕτως τὸ $ΜΞ$ πρὸς τὸ $ΚΛ$, τουτέστιν ὡς ἡ $ΔΚ$ πρὸς τὴν $ΜΝ$, ἡ $ΜΝ$ πρὸς τὴν $ΜΚ$ · τὸ ἄρα ὑπὸ τῶν $ΔΚ$, $ΚΜ$ ἴσον ἐστὶ τῷ ἀπὸ τῆς $ΜΝ$. καὶ ἐπεὶ σύμμετρόν ἐστὶ τὸ ἀπὸ τῆς $ΑΓ$ τῷ ἀπὸ τῆς $ΓΒ$, σύμμετρόν ἐστὶ καὶ τὸ $ΔΘ$ τῷ $ΚΛ$ · ὥστε καὶ ἡ $ΔΚ$ τῇ $ΚΜ$ σύμμετρος ἐστὶν. καὶ ἐπεὶ μείζονά ἐστὶ τὰ ἀπὸ τῶν $ΑΓ$, $ΓΒ$ τοῦ δις ὑπὸ τῶν $ΑΓ$, $ΓΒ$, μείζον ἄρα καὶ τὸ $ΔΛ$ τοῦ $ΜΖ$ · ὥστε καὶ ἡ $ΔΜ$ τῆς $ΜΗ$ μείζων ἐστίν. καὶ ἐστὶν ἴσον τὸ ὑπὸ τῶν $ΔΚ$, $ΚΜ$ τῷ ἀπὸ τῆς $ΜΝ$, τουτέστι τῷ τετάρτῳ τοῦ ἀπὸ τῆς $ΜΗ$, καὶ σύμμετρος ἡ $ΔΚ$ τῇ $ΚΜ$. ἐὰν δὲ ὡς δύο εὐθεῖαι ἄνισοι, τῷ δὲ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παρὰ τὴν μείζονα παραβληθῇ ἐλλείπον εἶδει τετραγώνῳ καὶ εἰς σύμμετρα αὐτὴν διαιρῇ, ἡ μείζων τῆς ἐλάσσονος μείζον δύναται τῷ ἀπὸ συμέτρου ἑαυτῇ· ἡ $ΔΜ$ ἄρα τῆς $ΜΗ$ μείζον δύναται τῷ ἀπὸ συμέτρου ἑαυτῇ. καὶ εἰσι ῥηταὶ αἱ $ΔΜ$, $ΜΗ$, καὶ ἡ $ΔΜ$ μείζον ὄνομα οὕσα σύμμετρος ἐστὶ τῇ ἐκκειμένη ῥητῇ τῇ $ΔΕ$ μήκει.

Ἡ $ΔΗ$ ἄρα ἐκ δύο ὀνομάτων ἐστὶ πρώτη· ὁπερ ἔδει δεῖξαι.

Thus, the remaining twice the (rectangle contained) by AC and CB is equal to MF [Prop. 2.4]. Let MG have been cut in half at N , and let NO have been drawn parallel [to each of ML and GF]. MO and NF are thus each equal to once the (rectangle contained) by ACB . And since AB is a binomial (straight-line), having been divided into its (component) terms at C , AC and CB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. Thus, the (squares) on AC and CB are rational, and commensurable with one another. And hence the sum of the (squares) on AC and CB (is rational) [Prop. 10.15], and is equal to DL . Thus, DL is rational. And it is applied to the rational (straight-line) DE . DM is thus rational, and commensurable in length with DE [Prop. 10.20]. Again, since AC and CB are rational (straight-lines which are) commensurable in square only, twice the (rectangle contained) by AC and CB —that is to say, MF —is thus medial [Prop. 10.21]. And it is applied to the rational (straight-line) ML . MG is thus also rational, and incommensurable in length with ML —that is to say, with DE [Prop. 10.22]. And MD is also rational, and commensurable in length with DE . Thus, DM is incommensurable in length with MG [Prop. 10.13]. And they are rational. DM and MG are thus rational (straight-lines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a first (binomial straight-line).

Since the (rectangle contained) by ACB is the mean proportional to the squares on AC and CB [Prop. 10.53 lem.], MO is thus also the mean proportional to DH and KL . Thus, as DH is to MO , so MO (is) to KL —that is to say, as DK (is) to MN , (so) MN (is) to MK [Prop. 6.1]. Thus, the (rectangle contained) by DK and KM is equal to the (square) on MN [Prop. 6.17]. And since the (square) on AC is commensurable with the (square) on CB , DH is also commensurable with KL . Hence, DK is also commensurable with KM [Props. 6.1, 10.11]. And since (the sum of) the squares on AC and CB is greater than twice the (rectangle contained) by AC and CB [Prop. 10.59 lem.], DL (is) thus also greater than MF . Hence, DM is also greater than MG [Props. 6.1, 5.14]. And the (rectangle contained) by DK and KM is equal to the (square) on MN —that is to say, to one quarter the (square) on MG . And DK (is) commensurable (in length) with KM . And if there are two unequal straight-lines, and a (rectangle) equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) commensurable (in length), then the square on the greater is larger

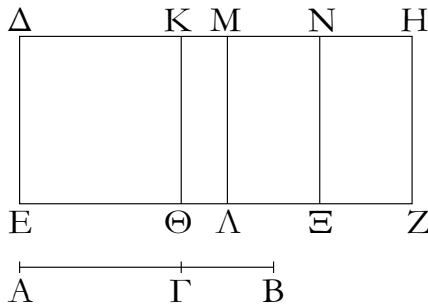
than (the square on) the lesser by the (square) on (some straight-line) commensurable (in length) with the greater [Prop. 10.17]. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) commensurable (in length) with (DM) . And DM and MG are rational. And DM , which is the greater term, is commensurable in length with the (previously) laid down rational (straight-line) DE .

Thus, DG is a first binomial (straight-line) [Def. 10.5]. (Which is) the very thing it was required to show.

† In other words, the square of a binomial is a first binomial. See Prop. 10.54.

ξά'.

Τὸ ἀπὸ τῆς ἐκ δύο μέσων πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων δευτέραν.



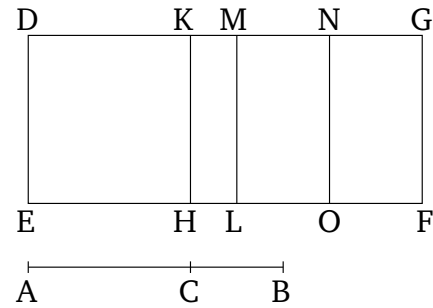
Ἐστω ἐκ δύο μέσων πρώτη ἡ AB διηρημένη εἰς τὰς μέσας κατὰ τὸ Γ , ὧν μείζων ἡ AG , καὶ ἐκκείσθω ῥητὴ ἡ DE , καὶ παραβεβλήσθω παρὰ τὴν DE τῷ ἀπὸ τῆς AB ἴσον παραλληλόγραμμον τὸ DZ πλάτος ποιοῦν τὴν ΔH . λέγω, ὅτι ἡ ΔH ἐκ δύο ὀνομάτων ἐστὶ δευτέρα.

Κατεσκευάσθω γὰρ τὰ αὐτὰ τοῖς πρὸ τούτου. καὶ ἐπεὶ ἡ AB ἐκ δύο μέσων ἐστὶ πρώτη διηρημένη κατὰ τὸ Γ , αἱ AG , GB ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι· ὥστε καὶ τὰ ἀπὸ τῶν AG , GB μέσα ἐστίν. μέσον ἄρα ἐστὶ τὸ ΔA . καὶ παρὰ ῥητὴν τὴν DE παραβεβλήται· ῥητὴ ἄρα ἐστὶν ἡ $M\Delta$ καὶ ἀσύμμετρος τῇ ΔE μήκει. πάλιν, ἐπεὶ ῥητὸν ἐστὶ τὸ δις ὑπὸ τῶν AG , GB , ῥητὸν ἐστὶ καὶ τὸ MZ . καὶ παρὰ ῥητὴν τὴν MA παράκειται· ῥητὴ ἄρα [ἐστὶ] καὶ ἡ MH καὶ μήκει σύμμετρος τῇ MA , τουτέστι τῇ ΔE · ἀσύμμετρος ἄρα ἐστὶν ἡ ΔM τῇ MH μήκει. καὶ εἰσι ῥηταί· αἱ ΔM , MH ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔH . δεικτέον δὴ, ὅτι καὶ δευτέρα.

Ἐπεὶ γὰρ τὰ ἀπὸ τῶν AG , GB μείζονά ἐστι τοῦ δις ὑπὸ τῶν AG , GB , μείζον ἄρα καὶ τὸ ΔA τοῦ MZ · ὥστε καὶ ἡ ΔM τῆς MH . καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς AG τῷ ἀπὸ τῆς GB , σύμμετρόν ἐστι καὶ τὸ $\Delta\Theta$ τῷ $ΚΛ$ · ὥστε καὶ ἡ ΔK τῇ KM σύμμετρός ἐστιν. καὶ ἐστὶ τὸ ὑπὸ τῶν ΔKM ἴσον τῷ ἀπὸ τῆς MN · ἡ ΔM ἄρα τῆς MH μείζον δύναται τῷ

Proposition 61

The square on a first binomial (straight-line) applied to a rational (straight-line) produces as breadth a second binomial (straight-line).†



Let AB be a first binomial (straight-line) having been divided into its (component) medial (straight-lines) at C , of which AC (is) the greater. And let the rational (straight-line) DE be laid down. And let the parallelogram DF , equal to the (square) on AB , have been applied to DE , producing DG as breadth. I say that DG is a second binomial (straight-line).

For let the same construction have been made as in the (proposition) before this. And since AB is a first binomial (straight-line), having been divided at C , AC and CB are thus medial (straight-lines) commensurable in square only, and containing a rational (area) [Prop. 10.37]. Hence, the (squares) on AC and CB are also medial [Prop. 10.21]. Thus, DL is medial [Props. 10.15, 10.23 corr.]. And it has been applied to the rational (straight-line) DE . MD is thus rational, and incommensurable in length with DE [Prop. 10.22]. Again, since twice the (rectangle contained) by AC and CB is rational, MF is also rational. And it is applied to the rational (straight-line) ML . Thus, MG [is] also rational, and commensurable in length with ML —that is to say, with DE [Prop. 10.20]. DM is thus incommensurable in length with MG [Prop. 10.13]. And they are rational. DM and MG are thus rational, and commensu-

ἀπὸ συμμετρου ἐαυτῇ. καὶ ἐστὶν ἡ MH σύμμετρος τῇ ΔE μήκει.

Ἡ ΔH ἄρα ἐκ δύο ὀνομάτων ἐστὶ δευτέρα.

able in square only. DG is thus a binomial (straight-line) [Prop. 10.36]. So, we must show that (it is) also a second (binomial straight-line).

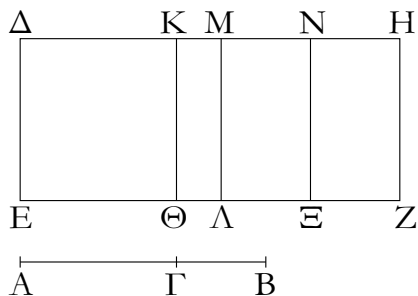
For since (the sum of) the squares on AC and CB is greater than twice the (rectangle contained) by AC and CB [Prop. 10.59], DL (is) thus also greater than MF . Hence, DM (is) also (greater) than MG [Prop. 6.1]. And since the (square) on AC is commensurable with the (square) on CB , DH is also commensurable with KL . Hence, DK is also commensurable (in length) with KM [Props. 6.1, 10.11]. And the (rectangle contained) by DKM is equal to the (square) on MN . Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) commensurable (in length) with (DM) [Prop. 10.17]. And MG is commensurable in length with DE .

Thus, DG is a second binomial (straight-line) [Def. 10.6].

† In other words, the square of a first binomial is a second binomial. See Prop. 10.55.

ξβ'.

Τὸ ἀπὸ τῆς ἐκ δύο μέσων δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τρίτην.

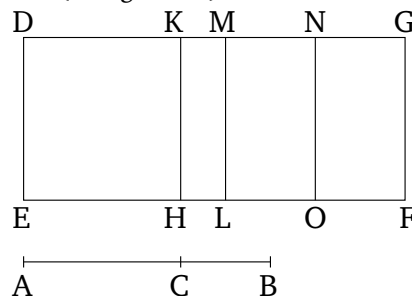


Ἐστω ἐκ δύο μέσων δευτέρα ἡ AB διηρημένη εἰς τὰς μέσας κατὰ τὸ Γ , ὥστε τὸ μείζον τμήμα εἶναι τὸ $ΑΓ$, ῥητὴ δέ τις ἔστω ἡ ΔE , καὶ παρὰ τὴν ΔE τῷ ἀπὸ τῆς AB ἴσον παραλληλόγραμμον παραβεβλήσθω τὸ ΔZ πλάτος ποιῶν τὴν ΔH . λέγω, ὅτι ἡ ΔH ἐκ δύο ὀνομάτων ἐστὶ τρίτη.

Κατεσκευάσθω τὰ αὐτὰ τοῖς προδεδειγμένοις. καὶ ἐπεὶ ἐκ δύο μέσων δευτέρα ἐστὶν ἡ AB διηρημένη κατὰ τὸ Γ , αἱ $ΑΓ$, $ΓΒ$ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι· ὥστε καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν $ΑΓ$, $ΓΒ$ μέσον ἐστίν. καὶ ἐστὶν ἴσον τῷ $\Delta Λ$. μέσον ἄρα καὶ τὸ $\Delta Λ$. καὶ παράκειται παρὰ ῥητὴν τὴν ΔE . ῥητὴ ἄρα ἐστὶ καὶ ἡ $MΔ$ καὶ ἀσύμμετρος τῇ ΔE μήκει. διὰ τὰ αὐτὰ δὲ καὶ ἡ MH ῥητὴ ἐστὶ καὶ ἀσύμμετρος τῇ $MΔ$, τουτέστι τῇ ΔE , μήκει· ῥητὴ ἄρα ἐστὶν ἑκάτερα τῶν $ΔΜ$, MH καὶ ἀσύμμετρος τῇ ΔE μήκει. καὶ ἐπεὶ ἀσύμμετρός ἐστὶν ἡ $ΑΓ$ τῇ $ΓΒ$ μήκει, ὡς δὲ ἡ $ΑΓ$ πρὸς τὴν $ΓΒ$, οὕτως τὸ ἀπὸ τῆς $ΑΓ$ πρὸς τὸ

Proposition 62

The square on a second binomial (straight-line) applied to a rational (straight-line) produces as breadth a third binomial (straight-line).†



Let AB be a second binomial (straight-line) having been divided into its (component) medial (straight-lines) at C , such that AC is the greater segment. And let DE be some rational (straight-line). And let the parallelogram DF , equal to the (square) on AB , have been applied to DE , producing DG as breadth. I say that DG is a third binomial (straight-line).

Let the same construction be made as that shown previously. And since AB is a second binomial (straight-line), having been divided at C , AC and CB are thus medial (straight-lines) commensurable in square only, and containing a medial (area) [Prop. 10.38]. Hence, the sum of the (squares) on AC and CB is also medial [Props. 10.15, 10.23 corr.]. And it is equal to DL . Thus, DL (is) also medial. And it is applied to the rational (straight-line) DE . MD is thus also rational, and in-

ὑπὸ τῶν ΑΓΒ, ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΑΓ τῷ ὑπὸ τῶν ΑΓΒ. ὥστε καὶ τὸ συγχείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ τῷ δις ὑπὸ τῶν ΑΓΒ ἀσύμμετρόν ἐστιν, τουτέστι τὸ ΔΑ τῷ ΜΖ· ὥστε καὶ ἡ ΔΜ τῷ ΜΗ ἀσύμμετρός ἐστιν. καὶ εἰσι ῥηταί· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. δεικτέον [δγ], ὅτι καὶ τρίτη.

Ὅμοίως δὴ τοῖς προτέροις ἐπιλογιούμεθα, ὅτι μείζων ἐστὶν ἡ ΔΜ τῆς ΜΗ, καὶ σύμμετρος ἡ ΔΚ τῇ ΚΜ. καὶ ἐστὶ τὸ ὑπὸ τῶν ΔΚΜ ἴσον τῷ ἀπὸ τῆς ΜΝ· ἡ ΔΜ ἄρα τῆς ΜΗ μείζον δύναται τῷ ἀπὸ συμέτρου ἑαυτῇ. καὶ οὐδετέρα τῶν ΔΜ, ΜΗ σύμμετρός ἐστι τῇ ΔΕ μήκει.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τρίτη· ὅπερ ἔδει δεῖξαι.

commensurable in length with DE [Prop. 10.22]. So, for the same (reasons), MG is also rational, and incommensurable in length with ML —that is to say, with DE . Thus, DM and MG are each rational, and incommensurable in length with DE . And since AC is incommensurable in length with CB , and as AC (is) to CB , so the (square) on AC (is) to the (rectangle contained) by ACB [Prop. 10.21 lem.], the (square) on AC (is) also incommensurable with the (rectangle contained) by ACB [Prop. 10.11]. And hence the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by ACB —that is to say, DL with MF [Props. 10.12, 10.13]. Hence, DM is also incommensurable (in length) with MG [Props. 6.1, 10.11]. And they are rational. DG is thus a binomial (straight-line) [Prop. 10.36]. [So] we must show that (it is) also a third (binomial straight-line).

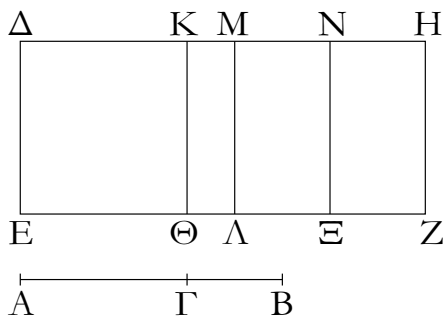
So, similarly to the previous (propositions), we can conclude that DM is greater than MG , and DK (is) commensurable (in length) with KM . And the (rectangle contained) by DKM is equal to the (square) on MN . Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) commensurable (in length) with (DM) [Prop. 10.17]. And neither of DM and MG is commensurable in length with DE .

Thus, DG is a third binomial (straight-line) [Def. 10.7]. (Which is) the very thing it was required to show.

† In other words, the square of a second binomial is a third binomial. See Prop. 10.56.

ξγ'.

Τὸ ἀπὸ τῆς μείζονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τετάρτην.

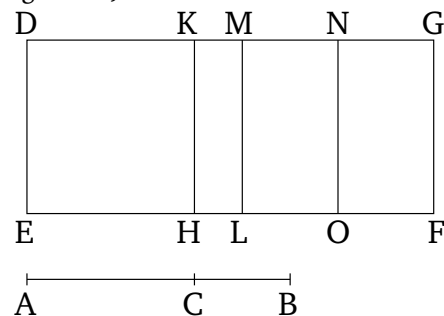


Ἐστω μείζων ἡ ΑΒ διηρημένη κατὰ τὸ Γ, ὥστε μείζονα εἶναι τὴν ΑΓ τῆς ΓΒ, ῥητὴ δὲ ἡ ΔΕ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΔΕ παραβεβλήσθω τὸ ΔΖ παραλληλόγραμμον πλάτος ποιοῦν τὴν ΔΗ· λέγω, ὅτι ἡ ΔΗ ἐκ δύο ὀνομάτων ἐστὶ τετάρτη.

Κατεσκευάσθω τὰ αὐτὰ τοῖς προδεδειγμένοις. καὶ ἐπεὶ μείζων ἐστὶν ἡ ΑΒ διηρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ δυνάμει

Proposition 63

The square on a major (straight-line) applied to a rational (straight-line) produces as breadth a fourth binomial (straight-line).[†]



Let AB be a major (straight-line) having been divided at C , such that AC is greater than CB , and (let) DE (be) a rational (straight-line). And let the parallelogram DF , equal to the (square) on AB , have been applied to DE , producing DG as breadth. I say that DG is a fourth binomial (straight-line).

Let the same construction be made as that shown pre-

εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγχείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δὲ ὑπ' αὐτῶν μέσον. ἐπεὶ οὖν ῥητόν ἐστι τὸ συγχείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΒΒ, ῥητόν ἄρα ἐστὶ τὸ ΔΑ· ῥητὴ ἄρα καὶ ἡ ΔΜ καὶ σύμμετρος τῇ ΔΕ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ δις ὑπὸ τῶν ΑΓ, ΒΒ, τουτέστι τὸ ΜΖ, καὶ παρὰ ῥητὴν ἐστὶ τὴν ΜΑ, ῥητὴ ἄρα ἐστὶ καὶ ἡ ΜΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει· ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΔΜ τῇ ΜΗ μήκει. αἱ ΔΜ, ΜΗ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. δεικτέον [δὴ], ὅτι καὶ τετάρτη.

Ὅμοιως δὴ δεῖξομεν τοῖς πρότερον, ὅτι μείζων ἐστὶν ἡ ΔΜ τῆς ΜΗ, καὶ ὅτι τὸ ὑπὸ ΔΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ. ἐπεὶ οὖν ἀσύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΓ τῷ ἀπὸ τῆς ΒΒ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ΔΘ τῷ ΚΛ· ὥστε ἀσύμμετρος καὶ ἡ ΔΚ τῇ ΚΜ ἐστίν. ἐὰν δὲ ὦσι δύο εὐθεῖαι ἄνιστοι, τῷ δὲ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ἐλάσσονος ἴσον παραλληλόγραμμον παρὰ τὴν μείζονα παραβληθῇ ἐλλείπον εἶδει τετραγώνῳ καὶ εἰς ἀσύμμετρα αὐτὴν διαιρῇ, ἡ μείζων τῆς ἐλάσσονος μείζον δυνήσεται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ μήκει· ἡ ΔΜ ἄρα τῆς ΜΗ μείζον δύνανται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ. καὶ εἰσιν αἱ ΔΜ, ΜΗ ῥηταὶ δυνάμει μόνον σύμμετροι, καὶ ἡ ΔΜ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΔΕ.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶ τετάρτη· ὅπερ ἔδει δεῖξαι.

viously. And since AB is a major (straight-line), having been divided at C , AC and CB are incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial [Prop. 10.39]. Therefore, since the sum of the (squares) on AC and CB is rational, DL is thus rational. Thus, DM (is) also rational, and commensurable in length with DE [Prop. 10.20]. Again, since twice the (rectangle contained) by AC and CB —that is to say, MF —is medial, and is (applied to) the rational (straight-line) ML , MG is thus also rational, and incommensurable in length with DE [Prop. 10.22]. DM is thus also incommensurable in length with MG [Prop. 10.13]. DM and MG are thus rational (straight-lines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. [So] we must show that (it is) also a fourth (binomial straight-line).

So, similarly to the previous (propositions), we can show that DM is greater than MG , and that the (rectangle contained) by DKM is equal to the (square) on MN . Therefore, since the (square) on AC is incommensurable with the (square) on CB , DH is also incommensurable with KL . Hence, DK is also incommensurable with KM [Props. 6.1, 10.11]. And if there are two unequal straight-lines, and a parallelogram equal to the fourth part of the (square) on the lesser, falling short by a square figure, is applied to the greater, and divides it into (parts which are) incommensurable (in length), then the square on the greater will be larger than (the square on) the lesser by the (square) on (some straight-line) incommensurable in length with the greater [Prop. 10.18]. Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) incommensurable (in length) with (DM). And DM and MG are rational (straight-lines which are) commensurable in square only. And DM is commensurable (in length) with the (previously) laid down rational (straight-line) DE .

Thus, DG is a fourth binomial (straight-line) [Def. 10.8]. (Which is) the very thing it was required to show.

† In other words, the square of a major is a fourth binomial. See Prop. 10.57.

ξδ'.

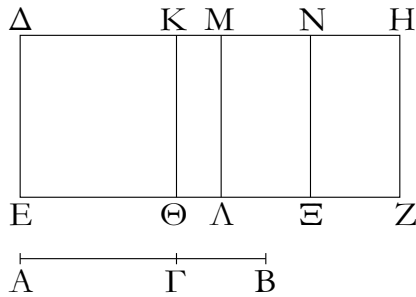
Τὸ ἀπὸ τῆς ῥητὸν καὶ μέσον δυναμένης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πέμπτην.

Ἐστω ῥητόν καὶ μέσον δυναμένη ἡ AB διηρημένη εἰς τὰς εὐθείας κατὰ τὸ Γ , ὥστε μείζονα εἶναι τὴν $ΑΓ$, καὶ ἐκκείσθω ῥητὴ ἡ $ΔΕ$, καὶ τῷ ἀπὸ τῆς AB ἴσον παρὰ τὴν $ΔΕ$ παραβεβλήσθω τὸ $ΔΖ$ πλάτος ποιοῦν τὴν $ΔΗ$ · λέγω, ὅτι ἡ $ΔΗ$ ἐκ δύο ὀνομάτων ἐστὶ πέμπτη.

Proposition 64

The square on the square-root of a rational plus a medial (area) applied to a rational (straight-line) produces as breadth a fifth binomial (straight-line).[†]

Let AB be the square-root of a rational plus a medial (area) having been divided into its (component) straight-lines at C , such that AC is greater. And let the rational (straight-line) DE be laid down. And let the (parallelogram) DF , equal to the (square) on AB , have been ap-

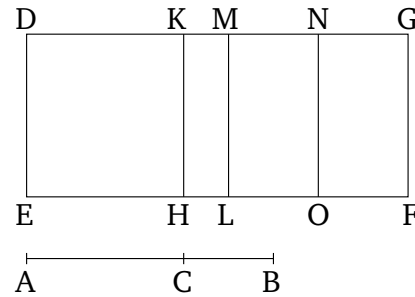


Κατεσκευάσθω τὰ αὐτὰ τοῖς πρὸ τούτου. ἐπεὶ οὖν ῥητὸν καὶ μέσον δυναμένη ἐστὶν ἡ AB διηρημένη κατὰ τὸ Γ , αἱ AG , GB ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δ' ὑπ' αὐτῶν ῥητόν. ἐπεὶ οὖν μέσον ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AG , GB , μέσον ἄρα ἐστὶ τὸ $\Delta\Lambda$. ὥστε ῥητὴ ἐστὶν ἡ ΔM καὶ μήκει ἀσύμμετρος τῇ ΔE . πάλιν, ἐπεὶ ῥητόν ἐστι τὸ δις ὑπὸ τῶν AGB , τουτέστι τὸ MZ , ῥητὴ ἄρα ἡ MH καὶ σύμμετρος τῇ ΔE . ἀσύμμετρος ἄρα ἡ ΔM τῇ MH . αἱ ΔM , MH ἄρα ῥηταὶ εἰσὶ δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔH . λέγω δὴ, ὅτι καὶ πέμπτη.

Ὅμοιως γὰρ διεχθήσεται, ὅτι τὸ ὑπὸ τῶν ΔKM ἴσον ἐστὶ τῷ ἀπὸ τῆς MN , καὶ ἀσύμμετρος ἡ ΔK τῇ KM μήκει· ἡ ΔM ἄρα τῆς MH μείζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ εἰσιν αἱ ΔM , MH [ῥηταὶ] δυνάμει μόνον σύμμετροι, καὶ ἡ ἐλάσσων ἡ MH σύμμετρος τῇ ΔE μήκει.

Ἡ ΔH ἄρα ἐκ δύο ὀνομάτων ἐστὶ πέμπτη· ὅπερ ἔδει δεῖξαι.

plied to DE , producing DG as breadth. I say that DG is a fifth binomial straight-line.



Let the same construction be made as in the (propositions) before this. Therefore, since AB is the square-root of a rational plus a medial (area), having been divided at C , AC and CB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational [Prop. 10.40]. Therefore, since the sum of the (squares) on AC and CB is medial, DL is thus medial. Hence, DM is rational and incommensurable in length with DE [Prop. 10.22]. Again, since twice the (rectangle contained) by ACB —that is to say, MF —is rational, MG (is) thus rational and commensurable (in length) with DE [Prop. 10.20]. DM (is) thus incommensurable (in length) with MG [Prop. 10.13]. Thus, DM and MG are rational (straight-lines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a fifth (binomial straight-line).

For, similarly (to the previous propositions), it can be shown that the (rectangle contained) by DKM is equal to the (square) on MN , and DK (is) incommensurable in length with KM . Thus, the square on DM is greater than (the square on) MG by the (square) on (some straight-line) incommensurable (in length) with (DM) [Prop. 10.18]. And DM and MG are [rational] (straight-lines which are) commensurable in square only, and the lesser MG is commensurable in length with DE .

Thus, DG is a fifth binomial (straight-line) [Def. 10.9]. (Which is) the very thing it was required to show.

† In other words, the square of the square-root of a rational plus medial is a fifth binomial. See Prop. 10.58.

ξε'.

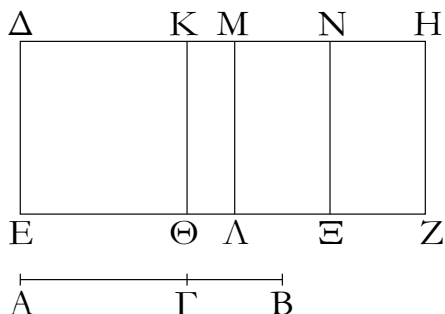
Τὸ ἀπὸ τῆς δύο μέσα δυναμένης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων ἕκτην.

Ἐστω δύο μέσα δυναμένη ἡ AB διηρημένη κατὰ τὸ Γ , ῥητὴ δὲ ἔστω ἡ ΔE , καὶ παρὰ τὴν ΔE τῷ ἀπὸ τῆς AB ἴσον παραβεβλήσθω τὸ ΔZ πλάτος ποιοῦν τὴν ΔH . λέγω, ὅτι ἡ ΔH ἐκ δύο ὀνομάτων ἐστὶν ἕκτη.

Proposition 65

The square on the square-root of (the sum of) two medial (areas) applied to a rational (straight-line) produces as breadth a sixth binomial (straight-line).[†]

Let AB be the square-root of (the sum of) two medial (areas), having been divided at C . And let DE be a rational (straight-line). And let the (parallelogram) DF , equal to the (square) on AB , have been applied to DE ,

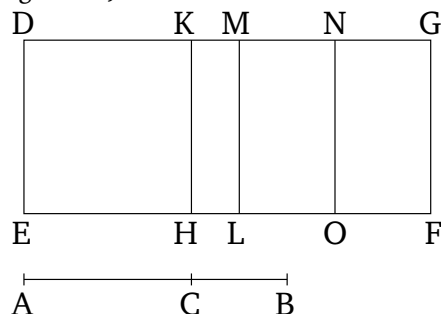


Κατεσκευάσθω γὰρ τὰ αὐτὰ τοῖς πρότερον. καὶ ἐπεὶ ἡ ΑΒ δύο μέσα δυναμένη ἐστὶ διηρημένη κατὰ τὸ Γ, αἱ ΑΓ, ΓΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τό τε συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ' αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ ἐκ τῶν ἀπ' αὐτῶν τετραγώνων συγκείμενον τῷ ὑπ' αὐτῶν ὥστε κατὰ τὰ προδεδειγμένα μέσον ἐστὶν ἑκάτερον τῶν ΔΑ, ΜΖ. καὶ παρὰ ῥητὴν τὴν ΔΕ παράκειται ῥητὴ ἄρα ἐστὶν ἑκάτερα τῶν ΔΜ, ΜΗ καὶ ἀσύμμετρος τῇ ΔΕ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ τῷ δις ὑπὸ τῶν ΑΓ, ΓΒ, ἀσύμμετρον ἄρα ἐστὶ τὸ ΔΑ τῷ ΜΖ. ἀσύμμετρος ἄρα καὶ ἡ ΔΜ τῇ ΜΗ· αἱ ΔΜ, ΜΗ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ ΔΗ. λέγω δὴ, ὅτι καὶ ἔκτῃ.

Ὅμοιως δὴ πάλιν δεῖξομεν, ὅτι τὸ ὑπὸ τῶν ΔΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ, καὶ ὅτι ἡ ΔΚ τῇ ΚΜ μήκει ἐστὶν ἀσύμμετρος· καὶ διὰ τὰ αὐτὰ δὴ ἡ ΔΜ τῇς ΜΗ μείζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει. καὶ οὐδετέρα τῶν ΔΜ, ΜΗ σύμμετρός ἐστι τῇ ἐκκειμένη ῥητῇ τῇ ΔΕ μήκει.

Ἡ ΔΗ ἄρα ἐκ δύο ὀνομάτων ἐστὶν ἔκτῃ· ὅπερ ἔδει δεῖξαι.

producing DG as breadth. I say that DG is a sixth binomial (straight-line).



For let the same construction be made as in the previous (propositions). And since AB is the square-root of (the sum of) two medial (areas), having been divided at C , AC and CB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, moreover, the sum of the squares on them incommensurable with the (rectangle contained) by them [Prop. 10.41]. Hence, according to what has been previously demonstrated, DL and MF are each medial. And they are applied to the rational (straight-line) DE . Thus, DM and MG are each rational, and incommensurable in length with DE [Prop. 10.22]. And since the sum of the (squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB , DL is thus incommensurable with MF . Thus, DM (is) also incommensurable (in length) with MG [Props. 6.1, 10.11]. DM and MG are thus rational (straight-lines which are) commensurable in square only. Thus, DG is a binomial (straight-line) [Prop. 10.36]. So, I say that (it is) also a sixth (binomial straight-line).

So, similarly (to the previous propositions), we can again show that the (rectangle contained) by DKM is equal to the (square) on MN , and that DK is incommensurable in length with KM . And so, for the same (reasons), the square on DM is greater than (the square on) MG by the (square) on (some straight-line) incommensurable in length with (DM) [Prop. 10.18]. And neither of DM and MG is commensurable in length with the (previously) laid down rational (straight-line) DE .

Thus, DG is a sixth binomial (straight-line) [Def. 10.10]. (Which is) the very thing it was required to show.

† In other words, the square of the square-root of two medials is a sixth binomial. See Prop. 10.59.

ξζ'.

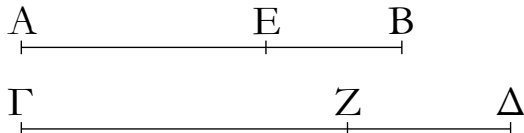
Proposition 66

Ἡ τῇ ἐκ δύο ὀνομάτων μήκει σύμμετρος καὶ αὐτὴ ἐκ δύο ὀνομάτων ἐστὶ καὶ τῇ τάξει ἡ αὐτὴ.

Ἐστω ἐκ δύο ὀνομάτων ἡ ΑΒ, καὶ τῇ ΑΒ μήκει

A (straight-line) commensurable in length with a binomial (straight-line) is itself also binomial, and the same in order.

σύμμετρος ἔστω ἡ $\Gamma\Delta$. λέγω, ὅτι ἡ $\Gamma\Delta$ ἐκ δύο ὀνομάτων ἐστὶ καὶ τῇ τάξει ἡ αὐτὴ τῇ AB .

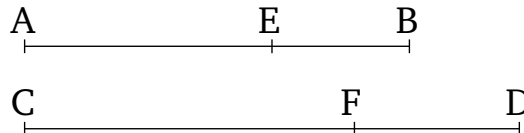


Ἐπεὶ γὰρ ἐκ δύο ὀνομάτων ἐστὶν ἡ AB , διηρήσθω εἰς τὰ ὀνόματα κατὰ τὸ E , καὶ ἔστω μείζον ὄνομα τὸ AE . αἱ AE , EB ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι. γεγονέντω ὡς ἡ AB πρὸς τὴν $\Gamma\Delta$, οὕτως ἡ AE πρὸς τὴν $\GammaΖ$. καὶ λοιπὴ ἄρα ἡ EB πρὸς λοιπὴν τὴν $Ζ\Delta$ ἐστίν, ὡς ἡ AB πρὸς τὴν $\Gamma\Delta$. σύμμετρος δὲ ἡ AB τῇ $\Gamma\Delta$ μήκει· σύμμετρος ἄρα ἐστὶ καὶ ἡ μὲν AE τῇ $\GammaΖ$, ἡ δὲ EB τῇ $Ζ\Delta$. καὶ εἰσι ῥηταὶ αἱ AE , EB . ῥηταὶ ἄρα εἰσι καὶ αἱ $\GammaΖ$, $Ζ\Delta$. καὶ ἐστίν ὡς ἡ AE πρὸς $\GammaΖ$, ἡ EB πρὸς $Ζ\Delta$. ἐναλλάξ ἄρα ἐστίν ὡς ἡ AE πρὸς EB , ἡ $\GammaΖ$ πρὸς $Ζ\Delta$. αἱ δὲ AE , EB δυνάμει μόνον [εἰσι] σύμμετροι· καὶ αἱ $\GammaΖ$, $Ζ\Delta$ ἄρα δυνάμει μόνον εἰσι σύμμετροι. καὶ εἰσι ῥηταὶ· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ $\Gamma\Delta$. λέγω δὴ, ὅτι τῇ τάξει ἐστὶν ἡ αὐτὴ τῇ AB .

Ἡ γὰρ AE τῆς EB μείζον δύναται ἥτοι τῷ ἀπὸ συμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμέτρου. εἰ μὲν οὖν ἡ AE τῆς EB μείζον δύναται τῷ ἀπὸ συμέτρου ἑαυτῇ, καὶ ἡ $\GammaΖ$ τῆς $Ζ\Delta$ μείζον δυνήσεται τῷ ἀπὸ συμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρος ἐστὶν ἡ AE τῇ ἐκκειμένῃ ῥητῇ, καὶ ἡ $\GammaΖ$ σύμμετρος αὐτῇ ἔσται, καὶ διὰ τοῦτο ἑκατέρω τῶν AB , $\Gamma\Delta$ ἐκ δύο ὀνομάτων ἐστὶ πρώτη, τουτέστι τῇ τάξει ἡ αὐτῇ. εἰ δὲ ἡ EB σύμμετρος ἐστὶ τῇ ἐκκειμένῃ ῥητῇ, καὶ ἡ $Ζ\Delta$ σύμμετρος ἐστὶν αὐτῇ, καὶ διὰ τοῦτο πάλιν τῇ τάξει ἡ αὐτῇ ἔσται τῇ AB . ἑκατέρω γὰρ αὐτῶν ἔσται ἐκ δύο ὀνομάτων δευτέρα. εἰ δὲ οὐδετέρα τῶν AE , EB σύμμετρος ἐστὶ τῇ ἐκκειμένῃ ῥητῇ, οὐδετέρα τῶν $\GammaΖ$, $Ζ\Delta$ σύμμετρος αὐτῇ ἔσται, καὶ ἐστὶν ἑκατέρα τρίτη. εἰ δὲ ἡ AE τῆς EB μείζον δύναται τῷ ἀπὸ ἀσυμέτρου ἑαυτῇ, καὶ ἡ $\GammaΖ$ τῆς $Ζ\Delta$ μείζον δύναται τῷ ἀπὸ ἀσυμέτρου ἑαυτῇ. καὶ εἰ μὲν ἡ AE σύμμετρος ἐστὶ τῇ ἐκκειμένῃ ῥητῇ, καὶ ἡ $\GammaΖ$ σύμμετρος ἐστὶν αὐτῇ, καὶ ἐστὶν ἑκατέρω τετάρτη. εἰ δὲ ἡ EB , καὶ ἡ $Ζ\Delta$, καὶ ἔσται ἑκατέρω πέμπτη. εἰ δὲ οὐδετέρα τῶν AE , EB , καὶ τῶν $\GammaΖ$, $Ζ\Delta$ οὐδετέρα σύμμετρος ἐστὶ τῇ ἐκκειμένῃ ῥητῇ, καὶ ἔσται ἑκατέρω ἕκτη.

Ὡστε ἡ τῇ ἐκ δύο ὀνομάτων μήκει σύμμετρος ἐκ δύο ὀνομάτων ἐστὶ καὶ τῇ τάξει ἡ αὐτῇ· ὅπερ ἔδει δείξαι.

Let AB be a binomial (straight-line), and let CD be commensurable in length with AB . I say that CD is a binomial (straight-line), and (is) the same in order as AB .



For since AB is a binomial (straight-line), let it have been divided into its (component) terms at E , and let AE be the greater term. AE and EB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.36]. Let it have been contrived that as AB (is) to CD , so AE (is) to CF [Prop. 6.12]. Thus, the remainder EB is also to the remainder FD , as AB (is) to CD [Props. 6.16, 5.19 corr.]. And AB (is) commensurable in length with CD . Thus, AE is also commensurable (in length) with CF , and EB with FD [Prop. 10.11]. And AE and EB are rational. Thus, CF and FD are also rational. And as AE is to CF , (so) EB (is) to FD [Prop. 5.11]. Thus, alternately, as AE is to EB , (so) CF (is) to FD [Prop. 5.16]. And AE and EB [are] commensurable in square only. Thus, CF and FD are also commensurable in square only [Prop. 10.11]. And they are rational. CD is thus a binomial (straight-line) [Prop. 10.36]. So, I say that it is the same in order as AB .

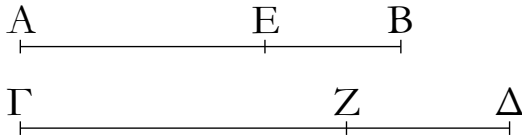
For the square on AE is greater than (the square on) EB by the (square) on (some straight-line) either commensurable or incommensurable (in length) with (AE). Therefore, if the square on AE is greater than (the square on) EB by the (square) on (some straight-line) commensurable (in length) with (AE) then the square on CF will also be greater than (the square on) FD by the (square) on (some straight-line) commensurable (in length) with (CF) [Prop. 10.14]. And if AE is commensurable (in length) with (some previously) laid down rational (straight-line) then CF will also be commensurable (in length) with it [Prop. 10.12]. And, on account of this, AB and CD are each first binomial (straight-lines) [Def. 10.5]—that is to say, the same in order. And if EB is commensurable (in length) with the (previously) laid down rational (straight-line) then FD is also commensurable (in length) with it [Prop. 10.12], and, again, on account of this, (CD) will be the same in order as AB . For each of them will be second binomial (straight-lines) [Def. 10.6]. And if neither of AE and EB is commensurable (in length) with the (previously) laid down rational (straight-line) then neither of CF and FD will be commensurable (in length) with it [Prop. 10.13], and each (of AB and CD) is a third (binomial straight-line)

[Def. 10.7]. And if the square on AE is greater than (the square on) EB by the (square) on (some straight-line) incommensurable (in length) with (AE) then the square on CF is also greater than (the square on) FD by the (square) on (some straight-line) incommensurable (in length) with (CF) [Prop. 10.14]. And if AE is commensurable (in length) with the (previously) laid down rational (straight-line) then CF is also commensurable (in length) with it [Prop. 10.12], and each (of AB and CD) is a fourth (binomial straight-line) [Def. 10.8]. And if EB (is commensurable in length with the previously laid down rational straight-line) then FD (is) also (commensurable in length with it), and each (of AB and CD) will be a fifth (binomial straight-line) [Def. 10.9]. And if neither of AE and EB (is commensurable in length with the previously laid down rational straight-line) then also neither of CF and FD is commensurable (in length) with the laid down rational (straight-line), and each (of AB and CD) will be a sixth (binomial straight-line) [Def. 10.10].

Hence, a (straight-line) commensurable in length with a binomial (straight-line) is a binomial (straight-line), and the same in order. (Which is) the very thing it was required to show.

ξξ'.

Ἡ τῇ ἐκ δύο μέσων μήκει σύμμετρος καὶ αὐτὴ ἐκ δύο μέσων ἐστὶ καὶ τῇ τάξει ἡ αὐτή.



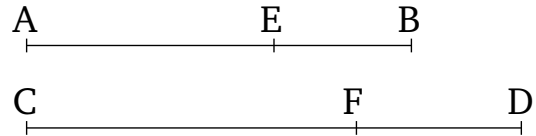
Ἐστω ἐκ δύο μέσων ἡ AB , καὶ τῇ AB σύμμετρος ἔστω μήκει ἡ $\Gamma\Delta$. λέγω, ὅτι ἡ $\Gamma\Delta$ ἐκ δύο μέσων ἐστὶ καὶ τῇ τάξει ἡ αὐτὴ τῇ AB .

Ἐπεὶ γὰρ ἐκ δύο μέσων ἐστὶν ἡ AB , διηρήσθω εἰς τὰς μέσας κατὰ τὸ E αἱ AE , EB ἅρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. καὶ γεγονέντω ὡς ἡ AB πρὸς $\Gamma\Delta$, ἡ AE πρὸς $\GammaΖ$ · καὶ λοιπὴ ἅρα ἡ EB πρὸς λοιπὴν τὴν $Z\Delta$ ἐστίν, ὡς ἡ AB πρὸς $\Gamma\Delta$. σύμμετρος δὲ ἡ AB τῇ $\Gamma\Delta$ μήκει· σύμμετρος ἅρα καὶ ἑκατέρω τῶν AE , EB ἑκατέρω τῶν $\GammaΖ$, $Z\Delta$. μέσαι δὲ αἱ AE , EB · μέσαι ἅρα καὶ αἱ $\GammaΖ$, $Z\Delta$. καὶ ἐπεὶ ἐστὶν ὡς ἡ AE πρὸς EB , ἡ $\GammaΖ$ πρὸς $Z\Delta$, αἱ δὲ AE , EB δυνάμει μόνον σύμμετροί εἰσιν, καὶ αἱ $\GammaΖ$, $Z\Delta$ [ἄρα] δυνάμει μόνον σύμμετροί εἰσιν, ἐδείχθησαν δὲ καὶ μέσαι· ἡ $\Gamma\Delta$ ἅρα ἐκ δύο μέσων ἐστίν. λέγω δὴ, ὅτι καὶ τῇ τάξει ἡ αὐτὴ ἐστὶ τῇ AB .

Ἐπεὶ γὰρ ἐστὶν ὡς ἡ AE πρὸς EB , ἡ $\GammaΖ$ πρὸς $Z\Delta$, καὶ ὡς ἅρα τὸ ἀπὸ τῆς AE πρὸς τὸ ὑπὸ τῶν AEB , οὕτως τὸ ἀπὸ τῆς $\GammaΖ$ πρὸς τὸ ὑπὸ τῶν $\GammaΖ\Delta$ · ἐναλλάξ ὡς τὸ ἀπὸ τῆς

Proposition 67

A (straight-line) commensurable in length with a binomial (straight-line) is itself also binomial, and the same in order.



Let AB be a binomial (straight-line), and let CD be commensurable in length with AB . I say that CD is binomial, and the same in order as AB .

For since AB is a binomial (straight-line), let it have been divided into its (component) medial (straight-lines) at E . Thus, AE and EB are medial (straight-lines which are) commensurable in square only [Props. 10.37, 10.38]. And let it have been contrived that as AB (is) to CD , (so) AE (is) to CF [Prop. 6.12]. And thus as the remainder EB is to the remainder FD , so AB (is) to CD [Props. 5.19 corr., 6.16]. And AB (is) commensurable in length with CD . Thus, AE and EB are also commensurable (in length) with CF and FD , respectively [Prop. 10.11]. And AE and EB (are) medial. Thus, CF and FD (are) also medial [Prop. 10.23]. And since as AE is to EB , (so) CF (is) to FD , and AE and EB are commensurable in square only, CF and FD are [thus]

ΑΕ πρὸς τὸ ἀπὸ τῆς ΓΖ, οὕτως τὸ ὑπὸ τῶν ΑΕΒ πρὸς τὸ ὑπὸ τῶν ΓΖΔ. σύμμετρον δὲ τὸ ἀπὸ τῆς ΑΕ τῷ ἀπὸ τῆς ΓΖ· σύμμετρον ἄρα καὶ τὸ ὑπὸ τῶν ΑΕΒ τῷ ὑπὸ τῶν ΓΖΔ. εἴτε οὖν ῥητόν ἐστι τὸ ὑπὸ τῶν ΑΕΒ, καὶ τὸ ὑπὸ τῶν ΓΖΔ ῥητόν ἐστιν [καὶ διὰ τοῦτό ἐστιν ἐκ δύο μέσων πρώτη]. εἴτε μέσον, μέσον, καὶ ἐστιν ἑκατέρα δευτέρα.

Καὶ διὰ τοῦτο ἔσται ἡ ΓΔ τῇ ΑΒ τῇ τάξει ἡ αὐτή· ὅπερ ἔδει δείξαι.

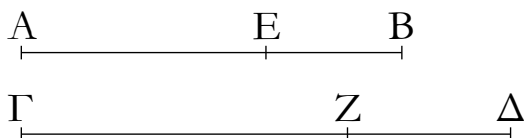
also commensurable in square only [Prop. 10.11]. And they were also shown (to be) medial. Thus, CD is a bi-medial (straight-line). So, I say that it is also the same in order as AB .

For since as AE is to EB , (so) CF (is) to FD , thus also as the (square) on AE (is) to the (rectangle contained) by AEB , so the (square) on CF (is) to the (rectangle contained) by CFD [Prop. 10.21 lem.]. Alternately, as the (square) on AE (is) to the (square) on CF , so the (rectangle contained) by AEB (is) to the (rectangle contained) by CFD [Prop. 5.16]. And the (square) on AE (is) commensurable with the (square) on CF . Thus, the (rectangle contained) by AEB (is) also commensurable with the (rectangle contained) by CFD [Prop. 10.11]. Therefore, either the (rectangle contained) by AEB is rational, and the (rectangle contained) by CFD is rational [and, on account of this, (AE and CD) are first bi-medial (straight-lines)], or (the rectangle contained by AEB is) medial, and (the rectangle contained by CFD is) medial, and (AB and CD) are each second (bi-medial straight-lines) [Props. 10.23, 10.37, 10.38].

And, on account of this, CD will be the same in order as AB . (Which is) the very thing it was required to show.

ζη'.

Ἡ τῇ μείζονι σύμμετρος καὶ αὐτὴ μείζων ἐστίν.

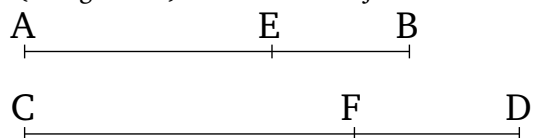


Ἐστω μείζων ἡ ΑΒ, καὶ τῇ ΑΒ σύμμετρος ἔστω ἡ ΓΔ· λέγω, ὅτι ἡ ΓΔ μείζων ἐστίν.

Διηρήσθω ἡ ΑΒ κατὰ τὸ Ε· αἱ ΑΕ, ΕΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δ' ὑπ' αὐτῶν μέσον· καὶ γεγόνετω τὰ αὐτὰ τοῖς πρότερον. καὶ ἐπεὶ ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΓΔ, οὕτως ἢ τε ΑΕ πρὸς τὴν ΓΖ καὶ ἡ ΕΒ πρὸς τὴν ΖΔ, καὶ ὡς ἄρα ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΕΒ πρὸς τὴν ΖΔ. σύμμετρος δὲ ἡ ΑΒ τῇ ΓΔ· σύμμετρος ἄρα καὶ ἑκατέρα τῶν ΑΕ, ΕΒ ἑκατέρᾳ τῶν ΓΖ, ΖΔ. καὶ ἐπεὶ ἐστὶν ὡς ἡ ΑΕ πρὸς τὴν ΓΖ, οὕτως ἡ ΕΒ πρὸς τὴν ΖΔ, καὶ ἐναλλάξ ὡς ἡ ΑΕ πρὸς ΕΒ, οὕτως ἡ ΓΖ πρὸς ΖΔ, καὶ συνθέντι ἄρα ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΓΔ πρὸς τὴν ΔΖ· καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΕ, οὕτως τὸ ἀπὸ τῆς ΓΔ πρὸς τὸ ἀπὸ τῆς ΔΖ. ὁμοίως δὲ δείξομεν, ὅτι καὶ ὡς τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΑΕ, οὕτως τὸ ἀπὸ τῆς ΓΔ πρὸς τὸ ἀπὸ τῆς ΓΖ. καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΑΒ πρὸς τὰ ἀπὸ τῶν ΑΕ, ΕΒ, οὕτως τὸ ἀπὸ τῆς ΓΔ πρὸς τὰ ἀπὸ τῶν ΓΖ, ΖΔ·

Proposition 68

A (straight-line) commensurable (in length) with a major (straight-line) is itself also major.



Let AB be a major (straight-line), and let CD be commensurable (in length) with AB . I say that CD is a major (straight-line).

Let AB have been divided (into its component terms) at E . AE and EB are thus incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial [Prop. 10.39]. And let (the) same (things) have been contrived as in the previous (propositions). And since as AB is to CD , so AE (is) to CF and EB to FD , thus also as AE (is) to CF , so EB (is) to FD [Prop. 5.11]. And AB (is) commensurable (in length) with CD . Thus, AE and EB (are) also commensurable (in length) with CF and FD , respectively [Prop. 10.11]. And since as AE is to CF , so EB (is) to FD , also, alternately, as AE (is) to EB , so CF (is) to FD [Prop. 5.16], and thus, via composition, as AB is to BE , so CD (is) to DF [Prop. 5.18]. And thus as the (square) on AB (is) to the (square) on BE , so the

καὶ ἐναλλάξ ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς $\Gamma\Delta$, οὕτως τὰ ἀπὸ τῶν AE , EB πρὸς τὰ ἀπὸ τῶν ΓZ , $Z\Delta$. σύμμετρον δὲ τὸ ἀπὸ τῆς AB τῷ ἀπὸ τῆς $\Gamma\Delta$. σύμμετρα ἄρα καὶ τὰ ἀπὸ τῶν AE , EB τοῖς ἀπὸ τῶν ΓZ , $Z\Delta$. καὶ ἐστὶ τὰ ἀπὸ τῶν AE , EB ἅμα ῥητόν, καὶ τὰ ἀπὸ τῶν ΓZ , $Z\Delta$ ἅμα ῥητόν ἐστίν. ὁμοίως δὲ καὶ τὸ δις ὑπὸ τῶν AE , EB σύμμετρόν ἐστι τῷ δις ὑπὸ τῶν ΓZ , $Z\Delta$. καὶ ἐστὶ μέσον τὸ δις ὑπὸ τῶν AE , EB . μέσον ἄρα καὶ τὸ δις ὑπὸ τῶν ΓZ , $Z\Delta$. αἱ ΓZ , $Z\Delta$ ἄρα δυνάμει ἀσύμμετροί εἰσι ποιοῦσαι τὸ μὲν συγχείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ἅμα ῥητόν, τὸ δὲ δις ὑπ' αὐτῶν μέσον. ὅλη ἄρα ἡ $\Gamma\Delta$ ἄλογός ἐστιν ἢ καλουμένη μείζων.

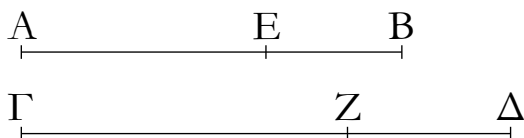
Ἡ ἄρα τῇ μείζονι σύμμετρος μείζων ἐστίν· ὅπερ ἔδει δεῖξαι.

(square) on CD (is) to the (square) on DF [Prop. 6.20]. So, similarly, we can also show that as the (square) on AB (is) to the (square) on AE , so the (square) on CD (is) to the (square) on CF . And thus as the (square) on AB (is) to (the sum of) the (squares) on AE and EB , so the (square) on CD (is) to (the sum of) the (squares) on CF and FD . And thus, alternately, as the (square) on AB is to the (square) on CD , so (the sum of) the (squares) on AE and EB (is) to (the sum of) the (squares) on CF and FD [Prop. 5.16]. And the (square) on AB (is) commensurable with the (square) on CD . Thus, (the sum of) the (squares) on AE and EB (is) also commensurable with (the sum of) the (squares) on CF and FD [Prop. 10.11]. And the (squares) on AE and EB (added) together are rational. The (squares) on CF and FD (added) together (are) thus also rational. So, similarly, twice the (rectangle contained) by AE and EB is also commensurable with twice the (rectangle contained) by CF and FD . And twice the (rectangle contained) by AE and EB is medial. Therefore, twice the (rectangle contained) by CF and FD (is) also medial [Prop. 10.23 corr.]. CF and FD are thus (straight-lines which are) incommensurable in square [Prop 10.13], simultaneously making the sum of the squares on them rational, and twice the (rectangle contained) by them medial. The whole, CD , is thus that irrational (straight-line) called major [Prop. 10.39].

Thus, a (straight-line) commensurable (in length) with a major (straight-line) is major. (Which is) the very thing it was required to show.

ξθ'.

Ἡ τῇ ῥητόν καὶ μέσον δυναμένη σύμμετρος [καὶ αὐτῇ] ῥητόν καὶ μέσον δυναμένη ἐστίν.

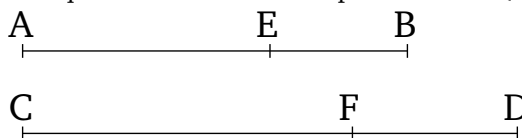


Ἐστω ῥητόν καὶ μέσον δυναμένη ἡ AB , καὶ τῇ AB σύμμετρος ἔστω ἡ $\Gamma\Delta$. δεικτέον, ὅτι καὶ ἡ $\Gamma\Delta$ ῥητόν καὶ μέσον δυναμένη ἐστίν.

Διηρήσθω ἡ AB εἰς τὰς εὐθείας κατὰ τὸ E . αἱ AE , EB ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγχείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δ' ὑπ' αὐτῶν ῥητόν· καὶ τὰ αὐτὰ κατεσκευάσθω τοῖς πρότερον. ὁμοίως δὲ δείξομεν, ὅτι καὶ αἱ ΓZ , $Z\Delta$ δυνάμει εἰσὶν ἀσύμμετροι, καὶ σύμμετρον τὸ μὲν συγχείμενον ἐκ τῶν ἀπὸ τῶν AE , EB τῷ συγχείμενῳ ἐκ τῶν ἀπὸ τῶν ΓZ , $Z\Delta$, τὸ δὲ ὑπὸ AE , EB τῷ ὑπὸ ΓZ , $Z\Delta$. ὥστε καὶ τὸ [μὲν] συγχείμενον ἐκ τῶν ἀπὸ τῶν ΓZ , $Z\Delta$ τετραγώνων ἐστὶ μέσον, τὸ δ' ὑπὸ τῶν ΓZ ,

Proposition 69

A (straight-line) commensurable (in length) with the square-root of a rational plus a medial (area) is [itself also] the square-root of a rational plus a medial (area).



Let AB be the square-root of a rational plus a medial (area), and let CD be commensurable (in length) with AB . We must show that CD is also the square-root of a rational plus a medial (area).

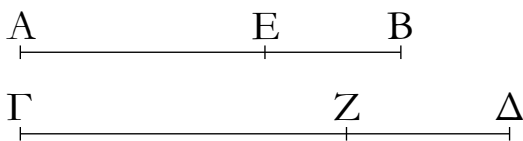
Let AB have been divided into its (component) straight-lines at E . AE and EB are thus incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them rational [Prop. 10.40]. And let the same construction have been made as in the previous (propositions). So, similarly, we can show that CF and FD are also incommensurable in square, and that the sum of the (squares) on AE and

ΖΔ ῥητόν.

Ῥητὸν ἄρα καὶ μέσον δυναμένη ἐστὶν ἡ ΓΔ· ὅπερ ἔδει δείξαι.

ο'.

Ἡ τῇ δύο μέσα δυναμένη σύμμετρος δύο μέσα δυναμένη ἐστὶν.



Ἐστω δύο μέσα δυναμένη ἡ ΑΒ, καὶ τῇ ΑΒ σύμμετρος ἡ ΓΔ· δεικτέον, ὅτι καὶ ἡ ΓΔ δύο μέσα δυναμένη ἐστὶν.

Ἐπεὶ γὰρ δύο μέσα δυναμένη ἐστὶν ἡ ΑΒ, διηρήσθω εἰς τὰς εὐθείας κατὰ τὸ Ε· αἱ ΑΕ, ΕΒ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ τε συγχείμενον ἐκ τῶν ἀπ' αὐτῶν [τετραγώνων] μέσον καὶ τὸ ὑπ' αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ συγχείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τετραγώνων τῷ ὑπὸ τῶν ΑΕ, ΕΒ· καὶ κατασκευάσθω τὰ αὐτὰ τοῖς πρότερον. ὁμοίως δὲ δείξομεν, ὅτι καὶ αἱ ΓΖ, ΖΔ δυνάμει εἰσὶν ἀσύμμετροι καὶ σύμμετρον τὸ μὲν συγχείμενον ἐκ τῶν ἀπὸ τῶν ΑΕ, ΕΒ τῷ συγχειμένῳ ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ, τὸ δὲ ὑπὸ τῶν ΑΕ, ΕΒ τῷ ὑπὸ τῶν ΓΖ, ΖΔ· ὥστε καὶ τὸ συγχείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων μέσον ἐστὶ καὶ τὸ ὑπὸ τῶν ΓΖ, ΖΔ μέσον καὶ ἔτι ἀσύμμετρον τὸ συγχείμενον ἐκ τῶν ἀπὸ τῶν ΓΖ, ΖΔ τετραγώνων τῷ ὑπὸ τῶν ΓΖ, ΖΔ.

Ἡ ἄρα ΓΔ δύο μέσα δυναμένη ἐστὶν· ὅπερ ἔδει δείξαι.

οα'.

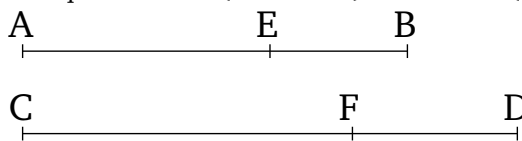
Ῥητοῦ καὶ μέσου συντιθεμένου τέσσαρες ἄλογοι γίγνονται ἥτοι ἐκ δύο ὀνομάτων ἢ ἐκ δύο μέσων πρώτη ἢ μείζων ἢ ῥητὸν καὶ μέσον δυναμένη.

EB (is) commensurable with the sum of the (squares) on CF and FD , and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD . And hence the sum of the squares on CF and FD is medial, and the (rectangle contained) by CF and FD (is) rational.

Thus, CD is the square-root of a rational plus a medial (area) [Prop. 10.40]. (Which is) the very thing it was required to show.

Proposition 70

A (straight-line) commensurable (in length) with the square-root of (the sum of) two medial (areas) is (itself also) the square-root of (the sum of) two medial (areas).



Let AB be the square-root of (the sum of) two medial (areas), and (let) CD (be) commensurable (in length) with AB . We must show that CD is also the square-root of (the sum of) two medial (areas).

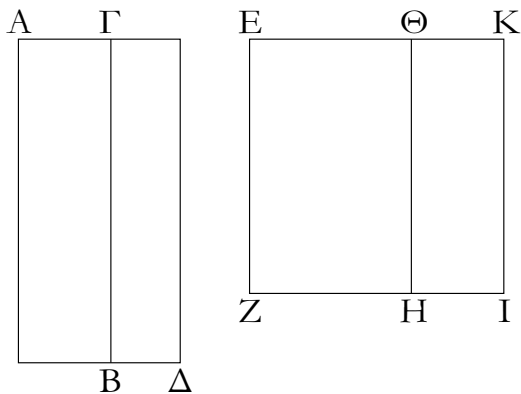
For since AB is the square-root of (the sum of) two medial (areas), let it have been divided into its (component) straight-lines at E . Thus, AE and EB are incommensurable in square, making the sum of the [squares] on them medial, and the (rectangle contained) by them medial, and, moreover, the sum of the (squares) on AE and EB incommensurable with the (rectangle) contained by AE and EB [Prop. 10.41]. And let the same construction have been made as in the previous (propositions). So, similarly, we can show that CF and FD are also incommensurable in square, and (that) the sum of the (squares) on AE and EB (is) commensurable with the sum of the (squares) on CF and FD , and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD . Hence, the sum of the squares on CF and FD is also medial, and the (rectangle contained) by CF and FD (is) medial, and, moreover, the sum of the squares on CF and FD (is) incommensurable with the (rectangle contained) by CF and FD .

Thus, CD is the square-root of (the sum of) two medial (areas) [Prop. 10.41]. (Which is) the very thing it was required to show.

Proposition 71

When a rational and a medial (area) are added together, four irrational (straight-lines) arise (as the square-roots of the total area)—either a binomial, or a first bi-

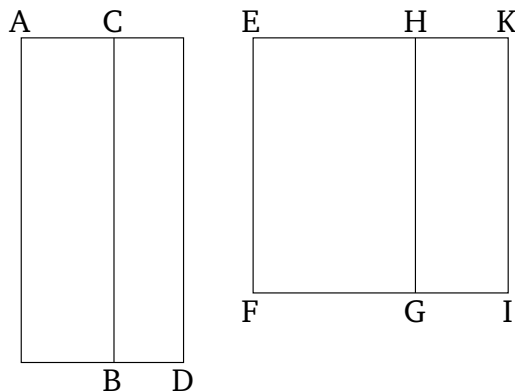
Ἐστω ῥητὸν μὲν τὸ AB , μέσον δὲ τὸ $\Gamma\Delta$. λέγω, ὅτι ἡ τὸ $A\Delta$ χωρίον δυναμένη ἤτοι ἐκ δύο ὀνομάτων ἐστὶν ἢ ἐκ δύο μέσων πρώτη ἢ μείζων ἢ ῥητὸν καὶ μέσον δυναμένη.



Τὸ γὰρ AB τοῦ $\Gamma\Delta$ ἤτοι μείζων ἐστὶν ἢ ἔλασσον. ἔστω πρότερον μείζων· καὶ ἐκκείσθω ῥητὴ ἡ EZ , καὶ παραβελήσθω παρὰ τὴν EZ τῷ AB ἴσον τὸ EH πλάτος ποιοῦν τὴν $E\Theta$. τῷ δὲ $\Delta\Gamma$ ἴσον παρὰ τὴν EZ παραβελήσθω τὸ ΘI πλάτος ποιοῦν τὴν ΘK . καὶ ἐπεὶ ῥητὸν ἐστὶ τὸ AB καὶ ἐστὶν ἴσον τῷ EH , ῥητὸν ἄρα καὶ τὸ EH . καὶ παρὰ [ῥητὴν] τὴν EZ παραβέλῃται πλάτος ποιοῦν τὴν $E\Theta$. ἡ $E\Theta$ ἄρα ῥητὴ ἐστὶ καὶ σύμμετρος τῇ EZ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ $\Gamma\Delta$ καὶ ἐστὶν ἴσον τῷ ΘI , μέσον ἄρα ἐστὶ καὶ τὸ ΘI . καὶ παρὰ ῥητὴν τὴν EZ παράκειται πλάτος ποιοῦν τὴν ΘK . ῥητὴ ἄρα ἐστὶν ἡ ΘK καὶ ἀσύμμετρος τῇ EZ μήκει. καὶ ἐπεὶ μέσον ἐστὶ τὸ $\Gamma\Delta$, ῥητὸν δὲ τὸ AB , ἀσύμμετρον ἄρα ἐστὶ τὸ AB τῷ $\Gamma\Delta$. ὥστε καὶ τὸ EH ἀσύμμετρον ἐστὶ τῷ ΘI . ὡς δὲ τὸ EH πρὸς τὸ ΘI , οὕτως ἐστὶν ἡ $E\Theta$ πρὸς τὴν ΘK . ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ $E\Theta$ τῇ ΘK μήκει. καὶ εἰσιν ἀμφοτέραι ῥηταί· αἱ $E\Theta$, ΘK ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ EK διηρημένη κατὰ τὸ Θ . καὶ ἐπεὶ μείζων ἐστὶ τὸ AB τοῦ $\Gamma\Delta$, ἴσον δὲ τὸ μὲν AB τῷ EH , τὸ δὲ $\Gamma\Delta$ τῷ ΘI , μείζων ἄρα καὶ τὸ EH τοῦ ΘI . καὶ ἡ $E\Theta$ ἄρα μείζων ἐστὶ τῆς ΘK . ἤτοι οὖν ἡ $E\Theta$ τῆς ΘK μείζων δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει ἢ τῷ ἀπὸ ἀσυμμέτρου. δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου ἑαυτῇ· καὶ ἐστὶν ἡ μείζων ἡ ΘE σύμμετρος τῇ ἐκκειμένῃ ῥητὴ τῇ EZ . ἡ ἄρα EK ἐκ δύο ὀνομάτων ἐστὶ πρώτη. ῥητὴ δὲ ἡ EZ . ἐὰν δὲ χωρίον περιέχεται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων πρώτης, ἡ τὸ χωρίον δυναμένη ἐκ δύο ὀνομάτων ἐστὶν. ἡ ἄρα τὸ EI δυναμένη ἐκ δύο ὀνομάτων ἐστὶν. ὥστε καὶ ἡ τὸ $A\Delta$ δυναμένη ἐκ δύο ὀνομάτων ἐστὶν. ἀλλὰ δὴ δυνάσθω ἡ $E\Theta$ τῆς ΘK μείζων τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ· καὶ ἐστὶν ἡ μείζων ἡ $E\Theta$ σύμμετρος τῇ ἐκκειμένῃ ῥητὴ τῇ EZ μήκει· ἡ ἄρα EK ἐκ δύο ὀνομάτων ἐστὶ τετάρτη. ῥητὴ δὲ ἡ EZ . ἐὰν δὲ χωρίον περιέχεται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο

medial, or a major, or the square-root of a rational plus a medial (area).

Let AB be a rational (area), and CD a medial (area). I say that the square-root of area AD is either binomial, or first bimedral, or major, or the square-root of a rational plus a medial (area).



For AB is either greater or less than CD . Let it, first of all, be greater. And let the rational (straight-line) EF be laid down. And let (the rectangle) EG , equal to AB , have been applied to EF , producing EH as breadth. And let (the rectangle) HI , equal to DC , have been applied to EF , producing HK as breadth. And since AB is rational, and is equal to EG , EG is thus also rational. And it has been applied to the [rational] (straight-line) EF , producing EH as breadth. EH is thus rational, and commensurable in length with EF [Prop. 10.20]. Again, since CD is medial, and is equal to HI , HI is thus also medial. And it is applied to the rational (straight-line) EF , producing HK as breadth. HK is thus rational, and incommensurable in length with EF [Prop. 10.22]. And since CD is medial, and AB rational, AB is thus incommensurable with CD . Hence, EG is also incommensurable with HI . And as EG (is) to HI , so EH is to HK [Prop. 6.1]. Thus, EH is also incommensurable in length with HK [Prop. 10.11]. And they are both rational. Thus, EH and HK are rational (straight-lines which are) commensurable in square only. EK is thus a binomial (straight-line), having been divided (into its component terms) at H [Prop. 10.36]. And since AB is greater than CD , and AB (is) equal to EG , and CD to HI , EG (is) thus also greater than HI . Thus, EH is also greater than HK [Prop. 5.14]. Therefore, the square on EH is greater than (the square on) HK either by the (square) on (some straight-line) commensurable in length with (EH) , or by the (square) on (some straight-line) incommensurable (in length with EH). Let it, first of all, be greater by the (square) on (some straight-line) commensurable (in length with EH). And the greater

ονομάτων τετάρτης, ἡ τὸ χωρίον δυναμένη ἄλογός ἐστιν ἢ καλουμένη μείζων. ἡ ἄρα τὸ EI χωρίον δυναμένη μείζων ἐστίν· ὥστε καὶ ἡ τὸ $A\Delta$ δυναμένη μείζων ἐστίν.

Ἀλλὰ δὴ ἔστω ἔλασσον τὸ AB τοῦ $\Gamma\Delta$ · καὶ τὸ EH ἄρα ἔλασσόν ἐστι τοῦ ΘI · ὥστε καὶ ἡ $E\Theta$ ἐλάσσων ἐστὶ τῆς ΘK . ἦτοι δὲ ἡ ΘK τῆς $E\Theta$ μείζων δύναται τῷ ἀπὸ συμμετρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμετρου. δυνάσθω πρότερον τῷ ἀπὸ συμμετρου ἑαυτῇ μήκει· καὶ ἐστὶν ἡ ἐλάσσων ἡ $E\Theta$ σύμμετρος τῇ ἐκκειμένη ῥητῇ τῇ EZ μήκει· ἡ ἄρα EK ἐκ δύο ὀνομάτων ἐστὶ δευτέρα. ῥητὴ δὲ ἡ EZ · ἐὰν δὲ χωρίον περιέχεται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων δευτέρας, ἡ τὸ χωρίον δυναμένη ἐκ δύο μέσων ἐστὶ πρώτη. ἡ ἄρα τὸ EI χωρίον δυναμένη ἐκ δύο μέσων ἐστὶ πρώτη· ὥστε καὶ ἡ τὸ $A\Delta$ δυναμένη ἐκ δύο μέσων ἐστὶ πρώτη. ἀλλὰ δὴ ἡ ΘK τῆς ΘE μείζων δυνάσθω τῷ ἀπὸ ἀσυμμετρου ἑαυτῇ. καὶ ἐστὶν ἡ ἐλάσσων ἡ $E\Theta$ σύμμετρος τῇ ἐκκειμένη ῥητῇ τῇ EZ · ἡ ἄρα EK ἐκ δύο ὀνομάτων ἐστὶ πέμπτη. ῥητὴ δὲ ἡ EZ · ἐὰν δὲ χωρίον περιέχεται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων πέμπτης, ἡ τὸ χωρίον δυναμένη ῥητὸν καὶ μέσον δυναμένη ἐστίν. ἡ ἄρα τὸ EI χωρίον δυναμένη ῥητὸν καὶ μέσον δυναμένη ἐστίν· ὥστε καὶ ἡ τὸ $A\Delta$ χωρίον δυναμένη ῥητὸν καὶ μέσον δυναμένη ἐστίν.

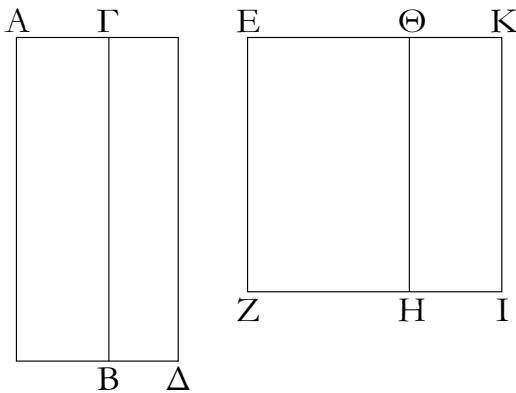
Ἐρητοῦ ἄρα καὶ μέσου συντιθεμένου τέσσαρες ἄλλοι γίνονται ἦτοι ἐκ δύο ὀνομάτων ἢ ἐκ δύο μέσων πρώτη ἢ μείζων ἢ ῥητὸν καὶ μέσον δυναμένη· ὅπερ ἔδει δεῖξαι.

(of the two components of EK) HE is commensurable (in length) with the (previously) laid down (straight-line) EF . EK is thus a first binomial (straight-line) [Def. 10.5]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a first binomial (straight-line) then the square-root of the area is a binomial (straight-line) [Prop. 10.54]. Thus, the square-root of EI is a binomial (straight-line). Hence the square-root of AD is also a binomial (straight-line). And, so, let the square on EH be greater than (the square on) HK by the (square) on (some straight-line) incommensurable (in length) with (EH). And the greater (of the two components of EK) EH is commensurable in length with the (previously) laid down rational (straight-line) EF . Thus, EK is a fourth binomial (straight-line) [Def. 10.8]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a fourth binomial (straight-line) then the square-root of the area is the irrational (straight-line) called major [Prop. 10.57]. Thus, the square-root of area EI is a major (straight-line). Hence, the square-root of AD is also major.

And so, let AB be less than CD . Thus, EG is also less than HI . Hence, EH is also less than HK [Props. 6.1, 5.14]. And the square on HK is greater than (the square on) EH either by the (square) on (some straight-line) commensurable (in length) with (HK), or by the (square) on (some straight-line) incommensurable (in length) with (HK). Let it, first of all, be greater by the square on (some straight-line) commensurable in length with (HK). And the lesser (of the two components of EK) EH is commensurable in length with the (previously) laid down rational (straight-line) EF . Thus, EK is a second binomial (straight-line) [Def. 10.6]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a second binomial (straight-line) then the square-root of the area is a first bimedral (straight-line) [Prop. 10.55]. Thus, the square-root of area EI is a first bimedral (straight-line). Hence, the square-root of AD is also a first bimedral (straight-line). And so, let the square on HK be greater than (the square on) HE by the (square) on (some straight-line) incommensurable (in length) with (HK). And the lesser (of the two components of EK) EH is commensurable (in length) with the (previously) laid down rational (straight-line) EF . Thus, EK is a fifth binomial (straight-line) [Def. 10.9]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a fifth binomial (straight-line) then the square-root of the area is the square-root of a rational plus a medial (area) [Prop. 10.58]. Thus, the square-root of area EI is the square-root of a rational plus a medial (area). Hence, the square-root of area AD is also the

ξβ'.

Δύο μέσων ἀσυμμέτρων ἀλλήλοις συντιθεμένων αἱ λοιπαὶ δύο ἄλλογοι γίνονται ἥτοι ἐκ δύο μέσων δευτέρα ἢ [ή] δύο μέσα δυναμένη.



Συγχεῖσθω γὰρ δύο μέσα ἀσύμμετρα ἀλλήλοις τὰ AB , $\Gamma\Delta$. λέγω, ὅτι ἡ τὸ $A\Delta$ χωρίον δυναμένη ἥτοι ἐκ δύο μέσων ἐστὶ δευτέρα ἢ δύο μέσα δυναμένη.

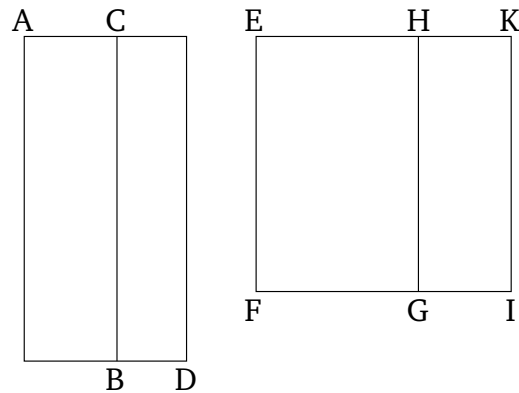
Τὸ γὰρ AB τοῦ $\Gamma\Delta$ ἥτοι μείζον ἐστὶν ἢ ἕλασσον. ἔστω, εἰ τύχον, πρότερον μείζον τὸ AB τοῦ $\Gamma\Delta$. καὶ ἐκκεῖσθω ῥητὴ ἡ EZ , καὶ τῷ μὲν AB ἴσον παρὰ τὴν EZ παραβεβλήσθω τὸ EH πλάτος ποιοῦν τὴν $E\Theta$, τῷ δὲ $\Gamma\Delta$ ἴσον τὸ ΘI πλάτος ποιοῦν τὴν ΘK . καὶ ἐπεὶ μέσον ἐστὶν ἑκάτερον τῶν AB , $\Gamma\Delta$, μέσον ἄρα καὶ ἑκάτερον τῶν EH , ΘI . καὶ παρὰ ῥητὴν τὴν ZE παράκειται πλάτος ποιοῦν τὰς $E\Theta$, ΘK . ἑκατέρω ἄρα τῶν $E\Theta$, ΘK ῥητὴ ἐστὶ καὶ ἀσύμμετρος τῇ EZ μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ AB τῷ $\Gamma\Delta$, καὶ ἐστὶν ἴσον τὸ μὲν AB τῷ EH , τὸ δὲ $\Gamma\Delta$ τῷ ΘI , ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ EH τῷ ΘI . ὥς δὲ τὸ EH πρὸς τὸ ΘI , οὕτως ἐστὶν ἡ $E\Theta$ πρὸς ΘK . ἀσύμμετρος ἄρα ἐστὶν ἡ $E\Theta$ τῇ ΘK μήκει. αἱ $E\Theta$, ΘK ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ EK . ἥτοι δὲ ἡ $E\Theta$ τῆς ΘK μείζον δύνανται τῷ ἀπὸ συμμετρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου. δυνάσθω πρότερον τῷ ἀπὸ συμμετρου ἑαυτῇ μήκει· καὶ οὐδετέρα τῶν $E\Theta$, ΘK σύμμετρός ἐστι τῇ ἐκκεκλιμένη ῥητῇ τῇ EZ μήκει· ἡ EK ἄρα ἐκ δύο ὀνομάτων ἐστὶ τρίτη. ῥητὴ δὲ ἡ EZ · ἐὰν δὲ χωρίον περιέχεται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων τρίτης, ἡ τὸ χωρίον δυναμένη ἐκ δύο μέσων ἐστὶ δευτέρα· ἢ ἄρα τὸ EI , τουτέστι τὸ $A\Delta$, δυναμένη ἐκ δύο μέσων ἐστὶ δευτέρα.

square-root of a rational plus a medial (area).

Thus, when a rational and a medial area are added together, four irrational (straight-lines) arise (as the square-roots of the total area)—either a binomial, or a first bimedral, or a major, or the square-root of a rational plus a medial (area). (Which is) the very thing it was required to show.

Proposition 72

When two medial (areas which are) incommensurable with one another are added together, the remaining two irrational (straight-lines) arise (as the square-roots of the total area)—either a second bimedral, or the square-root of (the sum of) two medial (areas).



For let the two medial (areas) AB and CD , (which are) incommensurable with one another, have been added together. I say that the square-root of area AD is either a second bimedral, or the square-root of (the sum of) two medial (areas).

For AB is either greater than or less than CD . By chance, let AB , first of all, be greater than CD . And let the rational (straight-line) EF be laid down. And let EG , equal to AB , have been applied to EF , producing EH as breadth, and HI , equal to CD , producing HK as breadth. And since AB and CD are each medial, EG and HI (are) thus also each medial. And they are applied to the rational straight-line FE , producing EH and HK (respectively) as breadth. Thus, EH and HK are each rational (straight-lines which are) incommensurable in length with EF [Prop. 10.22]. And since AB is incommensurable with CD , and AB is equal to EG , and CD to HI , EG is thus also incommensurable with HI . And as EG (is) to HI , so EH is to HK [Prop. 6.1]. EH is thus incommensurable in length with HK [Prop. 10.11]. Thus, EH and HK are rational (straight-lines which are) commensurable in square only. EK is thus a binomial (straight-line) [Prop. 10.36]. And the square on EH is greater than (the square on) HK either by the (square)

ἀλλὰ δὴ ἡ $ΕΘ$ τῆς $ΘΚ$ μείζον δυνάσθω τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ μήκει· καὶ ἀσύμμετρός ἐστιν ἑκάτερα τῶν $ΕΘ$, $ΘΚ$ τῇ $ΕΖ$ μήκει· ἡ ἄρα $ΕΚ$ ἐκ δύο ὀνομάτων ἐστὶν ἕκτη. ἐὰν δὲ χωρίον περιέχεται ὑπὸ ῥητῆς καὶ τῆς ἐκ δύο ὀνομάτων ἕκτης, ἡ τὸ χωρίον δυναμένη ἡ δύο μέσα δυναμένη ἐστίν· ὥστε καὶ ἡ τὸ $ΑΔ$ χωρίον δυναμένη ἡ δύο μέσα δυναμένη ἐστίν.

[Ὡμοίως δὴ δείξομεν, ὅτι ἂν ἔλαττον ἢ τὸ $ΑΒ$ τοῦ $ΓΔ$, ἡ τὸ $ΑΔ$ χωρίον δυναμένη ἢ ἐκ δύο μέσων δευτέρα ἐστὶν ἥτοι δύο μέσα δυναμένη].

Δύο ἄρα μέσων ἀσύμμετρων ἀλλήλοις συντιθεμένων αἱ λοιπαὶ δύο ἄλογοι γίνονται ἥτοι ἐκ δύο μέσων δευτέρα ἢ δύο μέσα δυναμένη.

Ἡ ἐκ δύο ὀνομάτων καὶ αἱ μετ' αὐτὴν ἄλογοι οὕτε τῇ μέσῃ οὕτε ἀλλήλαις εἰσὶν αἱ αὐταί. τὸ μὲν γὰρ ἀπὸ μέσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ῥητὴν καὶ ἀσύμμετρον τῇ παρ' ἣν παράκειται μήκει. τὸ δὲ ἀπὸ τῆς ἐκ δύο ὀνομάτων παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πρώτην. τὸ δὲ ἀπὸ τῆς ἐκ δύο μέσων πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων δευτέραν. τὸ δὲ ἀπὸ τῆς ἐκ δύο μέσων δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τρίτην. τὸ δὲ ἀπὸ τῆς μείζονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων τετάρτην. τὸ δὲ ἀπὸ τῆς ῥητὸν καὶ μέσον δυναμένης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων πέμπτην. τὸ δὲ ἀπὸ τῆς δύο μέσα δυναμένης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων ἕκτην. τὰ δ' εἰρημένα πλάτη διαφέρει τοῦ τε πρώτου καὶ ἀλλήλων, τοῦ μὲν πρώτου, ὅτι ῥητὴ ἐστίν, ἀλλήλων δέ, ὅτι τῇ τάξει οὐκ εἰσὶν αἱ αὐταί· ὥστε καὶ αὐταὶ αἱ ἄλογοι διαφέρουσιν ἀλλήλων.

on (some straight-line) commensurable (in length) with (EH), or by the (square) on (some straight-line) incommensurable (in length with EH). Let it, first of all, be greater by the square on (some straight-line) commensurable in length with (EH). And neither of EH or HK is commensurable in length with the (previously) laid down rational (straight-line) EF . Thus, EK is a third binomial (straight-line) [Def. 10.7]. And EF (is) rational. And if an area is contained by a rational (straight-line) and a third binomial (straight-line) then the square-root of the area is a second binomial (straight-line) [Prop. 10.56]. Thus, the square-root of EI —that is to say, of AD —is a second binomial. And so, let the square on EH be greater than (the square) on HK by the (square) on (some straight-line) incommensurable in length with (EH). And EH and HK are each incommensurable in length with EF . Thus, EK is a sixth binomial (straight-line) [Def. 10.10]. And if an area is contained by a rational (straight-line) and a sixth binomial (straight-line) then the square-root of the area is the square-root of (the sum of) two medial (areas) [Prop. 10.59]. Hence, the square-root of area AD is also the square-root of (the sum of) two medial (areas).

[So, similarly, we can show that, even if AB is less than CD , the square-root of area AD is either a second binomial or the square-root of (the sum of) two medial (areas).]

Thus, when two medial (areas which are) incommensurable with one another are added together, the remaining two irrational (straight-lines) arise (as the square-roots of the total area)—either a second binomial, or the square-root of (the sum of) two medial (areas).

A binomial (straight-line), and the (other) irrational (straight-lines) after it, are neither the same as a medial (straight-line) nor (the same) as one another. For the (square) on a medial (straight-line), applied to a rational (straight-line), produces as breadth a rational (straight-line which is) also incommensurable in length with (the straight-line) to which it is applied [Prop. 10.22]. And the (square) on a binomial (straight-line), applied to a rational (straight-line), produces as breadth a first binomial [Prop. 10.60]. And the (square) on a first binomial (straight-line), applied to a rational (straight-line), produces as breadth a second binomial [Prop. 10.61]. And the (square) on a second binomial (straight-line), applied to a rational (straight-line), produces as breadth a third binomial [Prop. 10.62]. And the (square) on a major (straight-line), applied to a rational (straight-line), produces as breadth a fourth binomial [Prop. 10.63]. And the (square) on the square-root of a rational plus a medial

ογ'.

Ἐὰν ἀπὸ ῥητῆς ῥητὴ ἀφαιρεθῇ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἀποτομή.



Ἀπὸ γὰρ ῥητῆς τῆς AB ῥητὴ ἀφηρήσθω ἡ BΓ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ· λέγω, ὅτι ἡ λοιπὴ ἡ AΓ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐπεὶ γὰρ ἀσύμμετρός ἐστιν ἡ AB τῇ BΓ μήκει, καὶ ἐστιν ὡς ἡ AB πρὸς τὴν BΓ, οὕτως τὸ ἀπὸ τῆς AB πρὸς τὸ ὑπὸ τῶν AB, BΓ, ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς AB τῷ ὑπὸ τῶν AB, BΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς AB σύμμετρά ἐστι τὰ ἀπὸ τῶν AB, BΓ τετράγωνα, τῷ δὲ ὑπὸ τῶν AB, BΓ σύμμετρόν ἐστι τὸ δις ὑπὸ τῶν AB, BΓ. καὶ ἐπειδὴ περ τὰ ἀπὸ τῶν AB, BΓ ἴσα ἐστὶ τῷ δις ὑπὸ τῶν AB, BΓ μετὰ τοῦ ἀπὸ ΓA, καὶ λοιπῷ ἄρα τῷ ἀπὸ τῆς AΓ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν AB, BΓ. ῥητὰ δὲ τὰ ἀπὸ τῶν AB, BΓ· ἄλογος ἄρα ἐστὶν ἡ AΓ· καλείσθω δὲ ἀποτομή. ὅπερ εἶδει δεῖξαι.

(area), applied to a rational (straight-line), produces as breadth a fifth binomial [Prop. 10.64]. And the (square) on the square-root of (the sum of) two medial (areas), applied to a rational (straight-line), produces as breadth a sixth binomial [Prop. 10.65]. And the aforementioned breadths differ from the first (breadth), and from one another—from the first, because it is rational—and from one another, because they are not the same in order. Hence, the (previously mentioned) irrational (straight-lines) themselves also differ from one another.

Proposition 73

If a rational (straight-line), which is commensurable in square only with the whole, is subtracted from a(nother) rational (straight-line) then the remainder is an irrational (straight-line). Let it be called an apotome.



For let the rational (straight-line) BC , which commensurable in square only with the whole, have been subtracted from the rational (straight-line) AB . I say that the remainder AC is that irrational (straight-line) called an apotome.

For since AB is incommensurable in length with BC , and as AB is to BC , so the (square) on AB (is) to the (rectangle contained) by AB and BC [Prop. 10.21 lem.], the (square) on AB is thus incommensurable with the (rectangle contained) by AB and BC [Prop. 10.11]. But, the (sum of the) squares on AB and BC is commensurable with the (square) on AB [Prop. 10.15], and twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. And, inasmuch as the (sum of the squares) on AB and BC is equal to twice the (rectangle contained) by AB and BC plus the (square) on CA [Prop. 2.7], the (sum of the squares) on AB and BC is thus also incommensurable with the remaining (square) on AC [Props. 10.13, 10.16]. And the (sum of the squares) on AB and BC is rational. AC is thus an irrational (straight-line) [Def. 10.4]. And let it be called an apotome.[†] (Which is) the very thing it was required to show.

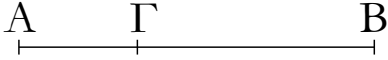
[†] See footnote to Prop. 10.36.

οδ'.

Ἐὰν ἀπὸ μέσης μέση ἀφαιρεθῇ δυνάμει μόνον σύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομή πρώτη.

Proposition 74

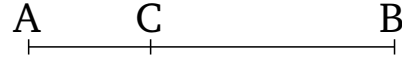
If a medial (straight-line), which is commensurable in square only with the whole, and which contains a rational (area) with the whole, is subtracted from a(nother) medial (straight-line) then the remainder is an irrational



Ἀπὸ γὰρ μέσης τῆς AB μέση ἀφηρήσθω ἡ $BΓ$ δυνάμει μόνον σύμμετρος οὕσα τῇ AB , μετὰ δὲ τῆς AB ῥητὸν ποιούσα τὸ ὑπὸ τῶν AB , $BΓ$ · λέγω, ὅτι ἡ λοιπὴ ἡ $ΑΓ$ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ πρώτη.

Ἐπεὶ γὰρ αἱ AB , $BΓ$ μέσαι εἰσὶν, μέσα ἐστὶ καὶ τὰ ἀπὸ τῶν AB , $BΓ$. ῥητὸν δὲ τὸ δις ὑπὸ τῶν AB , $BΓ$ · ἀσύμμετρα ἄρα τὰ ἀπὸ τῶν AB , $BΓ$ τῷ δις ὑπὸ τῶν AB , $BΓ$ · καὶ λοιπῶν ἄρα τῷ ἀπὸ τῆς $ΑΓ$ ἀσύμμετρόν ἐστι τὸ δις ὑπὸ τῶν AB , $BΓ$, ἐπεὶ καὶ τὸ ὅλον ἐνὶ αὐτῶν ἀσύμμετρον ἦ, καὶ τὰ ἐξ ἀρχῆς μεγέθη ἀσύμμετρα ἔσται. ῥητὸν δὲ τὸ δις ὑπὸ τῶν AB , $BΓ$ · ἄλογον ἄρα τὸ ἀπὸ τῆς $ΑΓ$ · ἄλογος ἄρα ἐστὶν ἡ $ΑΓ$ · καλείσθω δὲ μέσης ἀποτομὴ πρώτη.

(straight-line). Let it be called a first apotome of a medial (straight-line).



For let the medial (straight-line) BC , which is commensurable in square only with AB , and which makes with AB the rational (rectangle contained) by AB and BC , have been subtracted from the medial (straight-line) AB [Prop. 10.27]. I say that the remainder AC is an irrational (straight-line). Let it be called the first apotome of a medial (straight-line).

For since AB and BC are medial (straight-lines), the (sum of the squares) on AB and BC is also medial. And twice the (rectangle contained) by AB and BC (is) rational. The (sum of the squares) on AB and BC (is) thus incommensurable with twice the (rectangle contained) by AB and BC . Thus, twice the (rectangle contained) by AB and BC is also incommensurable with the remaining (square) on AC [Prop. 2.7], since if the whole is incommensurable with one of the (constituent magnitudes) then the original magnitudes will also be incommensurable (with one another) [Prop. 10.16]. And twice the (rectangle contained) by AB and BC (is) rational. Thus, the (square) on AC is irrational. Thus, AC is an irrational (straight-line) [Def. 10.4]. Let it be called a first apotome of a medial (straight-line).[†]

[†] See footnote to Prop. 10.37.

οε'.

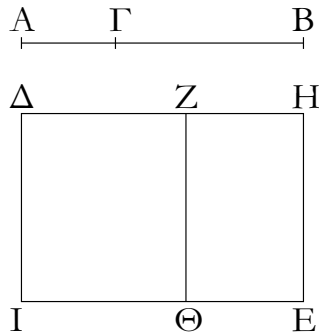
Ἐὰν ἀπὸ μέσης μέση ἀφαιρεθῇ δυνάμει μόνον σύμμετρος οὕσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ δευτέρα.

Ἀπὸ γὰρ μέσης τῆς AB μέση ἀφηρήσθω ἡ $ΓΒ$ δυνάμει μόνον σύμμετρος οὕσα τῇ ὅλῃ τῇ AB , μετὰ δὲ τῆς ὅλης τῆς AB μέσον περιέχουσα τὸ ὑπὸ τῶν AB , $BΓ$ · λέγω, ὅτι ἡ λοιπὴ ἡ $ΑΓ$ ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομὴ δευτέρα.

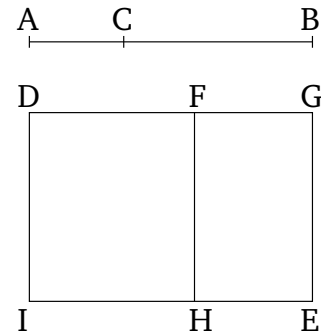
Proposition 75

If a medial (straight-line), which is commensurable in square only with the whole, and which contains a medial (area) with the whole, is subtracted from a(nother) medial (straight-line) then the remainder is an irrational (straight-line). Let it be called a second apotome of a medial (straight-line).

For let the medial (straight-line) CB , which is commensurable in square only with the whole, AB , and which contains with the whole, AB , the medial (rectangle contained) by AB and BC , have been subtracted from the medial (straight-line) AB [Prop. 10.28]. I say that the remainder AC is an irrational (straight-line). Let it be called a second apotome of a medial (straight-line).



Ἐκκείσθω γὰρ ῥητὴ ἡ ΔΙ, καὶ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ ἴσον παρὰ τὴν ΔΙ παραβεβλήσθω τὸ ΔΕ πλάτος ποιοῦν τὴν ΔΗ, τῷ δὲ δις ὑπὸ τῶν ΑΒ, ΒΓ ἴσον παρὰ τὴν ΔΙ παραβεβλήσθω τὸ ΔΘ πλάτος ποιοῦν τὴν ΔΖ· λοιπὸν ἄρα τὸ ΖΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ. καὶ ἐπεὶ μέσσα καὶ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ, μέσον ἄρα καὶ τὸ ΔΕ. καὶ παρὰ ῥητὴν τὴν ΔΙ παράκειται πλάτος ποιοῦν τὴν ΔΗ· ῥητὴ ἄρα ἐστὶν ἡ ΔΗ καὶ ἀσύμμετρος τῇ ΔΙ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ ὑπὸ τῶν ΑΒ, ΒΓ, καὶ τὸ δις ἄρα ὑπὸ τῶν ΑΒ, ΒΓ μέσον ἐστίν. καὶ ἐστὶν ἴσον τῷ ΔΘ· καὶ τὸ ΔΘ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΔΙ παραβέβληται πλάτος ποιοῦν τὴν ΔΖ· ῥητὴ ἄρα ἐστὶν ἡ ΔΖ καὶ ἀσύμμετρος τῇ ΔΙ μήκει. καὶ ἐπεὶ αἱ ΑΒ, ΒΓ δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΒ τῇ ΒΓ μήκει· ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΑΒ τετράγωνον τῷ ὑπὸ τῶν ΑΒ, ΒΓ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΒ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΒ, ΒΓ, τῷ δὲ ὑπὸ τῶν ΑΒ, ΒΓ σύμμετρόν ἐστι τὸ δις ὑπὸ τῶν ΑΒ, ΒΓ· ἀσύμμετρον ἄρα ἐστὶ τὸ δις ὑπὸ τῶν ΑΒ, ΒΓ τοῖς ἀπὸ τῶν ΑΒ, ΒΓ. ἴσον δὲ τοῖς μὲν ἀπὸ τῶν ΑΒ, ΒΓ τὸ ΔΕ, τῷ δὲ δις ὑπὸ τῶν ΑΒ, ΒΓ τὸ ΔΘ· ἀσύμμετρον ἄρα [ἐστὶ] τὸ ΔΕ τῷ ΔΘ. ὥς δὲ τὸ ΔΕ πρὸς τὸ ΔΘ, οὕτως ἡ ΗΔ πρὸς τὴν ΔΖ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΗΔ τῇ ΔΖ. καὶ εἰσιν ἀμφοτέραι ῥηταί· αἱ ἄρα ΗΔ, ΔΖ ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ ΖΗ ἄρα ἀποτομή ἐστίν. ῥητὴ δὲ ἡ ΔΙ· τὸ δὲ ὑπὸ ῥητῆς καὶ ἀλόγου περιεχόμενον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν. καὶ δύναται τὸ ΖΕ ἢ ΑΓ· ἡ ΑΓ ἄρα ἄλογός ἐστιν· καλείσθω δὲ μέσης ἀποτομῇ δευτέρα. ὅπερ ἔδει δεῖξαι.



For let the rational (straight-line) DI be laid down. And let DE , equal to the (sum of the squares) on AB and BC , have been applied to DI , producing DG as breadth. And let DH , equal to twice the (rectangle contained) by AB and BC , have been applied to DI , producing DF as breadth. The remainder FE is thus equal to the (square) on AC [Prop. 2.7]. And since the (squares) on AB and BC are medial and commensurable (with one another), DE (is) thus also medial [Props. 10.15, 10.23 corr.]. And it is applied to the rational (straight-line) DI , producing DG as breadth. Thus, DG is rational, and incommensurable in length with DI [Prop. 10.22]. Again, since the (rectangle contained) by AB and BC is medial, twice the (rectangle contained) by AB and BC is thus also medial [Prop. 10.23 corr.]. And it is equal to DH . Thus, DH is also medial. And it has been applied to the rational (straight-line) DI , producing DF as breadth. DF is thus rational, and incommensurable in length with DI [Prop. 10.22]. And since AB and BC are commensurable in square only, AB is thus incommensurable in length with BC . Thus, the square on AB (is) also incommensurable with the (rectangle contained) by AB and BC [Props. 10.21 lem., 10.11]. But, the (sum of the squares) on AB and BC is commensurable with the (square) on AB [Prop. 10.15], and twice the (rectangle contained) by AB and BC is commensurable with the (rectangle contained) by AB and BC [Prop. 10.6]. Thus, twice the (rectangle contained) by AB and BC is incommensurable with the (sum of the squares) on AB and BC [Prop. 10.13]. And DE is equal to the (sum of the squares) on AB and BC , and DH to twice the (rectangle contained) by AB and BC . Thus, DE [is] incommensurable with DH . And as DE (is) to DH , so GD (is) to DF [Prop. 6.1]. Thus, GD is incommensurable with DF [Prop. 10.11]. And they are both rational (straight-lines). Thus, GD and DF are rational (straight-lines which are) commensurable in square only. Thus, FG is an apotome [Prop. 10.73]. And DI (is) rational. And the (area) contained by a rational and an irrational (straight-line) is irrational [Prop. 10.20], and its square-root is irrational.

And AC is the square-root of FE . Thus, AC is an irrational (straight-line) [Def. 10.4]. And let it be called the second apotome of a medial (straight-line).[†] (Which is) the very thing it was required to show.

[†] See footnote to Prop. 10.38.

οστ'.

Ἐάν ἀπὸ εὐθείας εὐθεῖα ἀφαιρεθῇ δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὰ μὲν ἀπ' αὐτῶν ἅμα ῥητόν, τὸ δ' ὑπ' αὐτῶν μέσον, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἐλάσσων.



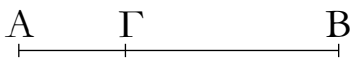
Ἀπὸ γὰρ εὐθείας τῆς AB εὐθεῖα ἀφηρήσθω ἡ $BΓ$ δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ ποιοῦσα τὰ προκείμενα. λέγω, ὅτι ἡ λοιπὴ ἡ $AΓ$ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Ἐπεὶ γὰρ τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν AB , $BΓ$ τετραγώνων ῥητόν ἐστιν, τὸ δὲ δις ὑπὸ τῶν AB , $BΓ$ μέσον, ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν AB , $BΓ$ τῷ δις ὑπὸ τῶν AB , $BΓ$ · καὶ ἀναστρέψαντι λοιπῷ τῷ ἀπὸ τῆς $AΓ$ ἀσύμμετρά ἐστὶ τὰ ἀπὸ τῶν AB , $BΓ$. ῥητὰ δὲ τὰ ἀπὸ τῶν AB , $BΓ$ · ἄλογον ἄρα τὸ ἀπὸ τῆς $AΓ$ · ἄλογος ἄρα ἡ $AΓ$ · καλείσθω δὲ ἐλάσσων. ὅπερ εἶδει δεῖξαι.

[†] See footnote to Prop. 10.39.

οζ'.

Ἐάν ἀπὸ εὐθείας εὐθεῖα ἀφαιρεθῇ δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δὲ δις ὑπ' αὐτῶν ῥητόν, ἡ λοιπὴ ἄλογός ἐστιν· καλείσθω δὲ ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα.

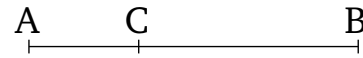


Ἀπὸ γὰρ εὐθείας τῆς AB εὐθεῖα ἀφηρήσθω ἡ $BΓ$ δυνάμει ἀσύμμετρος οὖσα τῇ AB ποιοῦσα τὰ προκείμενα· λέγω, ὅτι ἡ λοιπὴ ἡ $AΓ$ ἄλογός ἐστιν ἡ προειρημένη.

Ἐπεὶ γὰρ τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν AB , $BΓ$

Proposition 76

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the (squares) on them (added) together rational, and the (rectangle contained) by them medial, is subtracted from a(nother) straight-line then the remainder is an irrational (straight-line). Let it be called a minor (straight-line).



For let the straight-line BC , which is incommensurable in square with the whole, and fulfils the (other) prescribed (conditions), have been subtracted from the straight-line AB [Prop. 10.33]. I say that the remainder AC is that irrational (straight-line) called minor.

For since the sum of the squares on AB and BC is rational, and twice the (rectangle contained) by AB and BC (is) medial, the (sum of the squares) on AB and BC is thus incommensurable with twice the (rectangle contained) by AB and BC . And, via conversion, the (sum of the squares) on AB and BC is incommensurable with the remaining (square) on AC [Props. 2.7, 10.16]. And the (sum of the squares) on AB and BC (is) rational. The (square) on AC (is) thus irrational. Thus, AC (is) an irrational (straight-line) [Def. 10.4]. Let it be called a minor (straight-line).[†] (Which is) the very thing it was required to show.

Proposition 77

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them rational, is subtracted from a(nother) straight-line then the remainder is an irrational (straight-line). Let it be called that which makes with a rational (area) a medial whole.



For let the straight-line BC , which is incommensurable in square with AB , and fulfils the (other) prescribed (conditions), have been subtracted from the straight-line AB [Prop. 10.34]. I say that the remainder AC is the

τετραγώνων μέσον ἐστίν, τὸ δὲ δις ὑπὸ τῶν AB , $BΓ$ ῥητόν, ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν AB , $BΓ$ τῷ δις ὑπὸ τῶν AB , $BΓ$. καὶ λοιπὸν ἄρα τὸ ἀπὸ τῆς $ΑΓ$ ἀσύμμετρόν ἐστι τῷ δις ὑπὸ τῶν AB , $BΓ$. καὶ ἐστὶ τὸ δις ὑπὸ τῶν AB , $BΓ$ ῥητόν· τὸ ἄρα ἀπὸ τῆς $ΑΓ$ ἄλογόν ἐστιν· ἄλογος ἄρα ἐστὶν ἡ $ΑΓ$. καλεῖσθω δὲ ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα. ὅπερ ἔδει δεῖξαι.

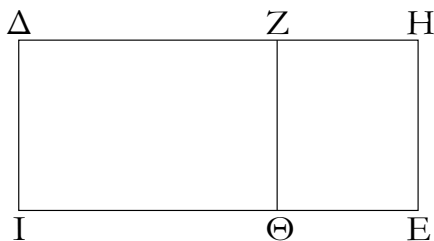
aforementioned irrational (straight-line).

For since the sum of the squares on AB and BC is medial, and twice the (rectangle contained) by AB and BC rational, the (sum of the squares) on AB and BC is thus incommensurable with twice the (rectangle contained) by AB and BC . Thus, the remaining (square) on AC is also incommensurable with twice the (rectangle contained) by AB and BC [Props. 2.7, 10.16]. And twice the (rectangle contained) by AB and BC is rational. Thus, the (square) on AC is irrational. Thus, AC is an irrational (straight-line) [Def. 10.4]. And let it be called that which makes with a rational (area) a medial whole.[†] (Which is) the very thing it was required to show.

[†] See footnote to Prop. 10.40.

ση'.

Ἐὰν ἀπὸ εὐθείας εὐθεῖα ἀφαιρεθῇ δυνάμει ἀσύμμετρος οὕσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τό τε συγχείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον τό τε δις ὑπ' αὐτῶν μέσον καὶ ἔτι τὰ ἀπ' αὐτῶν τετράγωνα ἀσύμμετρα τῷ δις ὑπ' αὐτῶν, ἡ λοιπὴ ἄλογός ἐστιν· καλεῖσθω δὲ ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα.

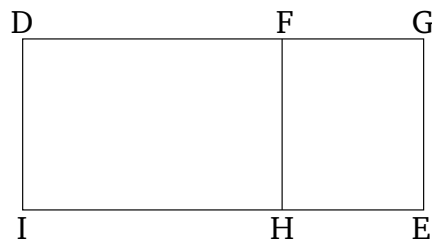


Ἀπὸ γὰρ εὐθείας τῆς AB εὐθεῖα ἀφηρήσθω ἡ $BΓ$ δυνάμει ἀσύμμετρος οὕσα τῇ AB ποιοῦσα τὰ προκείμενα· λέγω, ὅτι ἡ λοιπὴ ἡ $ΑΓ$ ἄλογός ἐστιν ἡ καλουμένη ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα.

Ἐκκείσθω γὰρ ῥητὴ ἡ $ΔΙ$, καὶ τοῖς μὲν ἀπὸ τῶν AB , $BΓ$ ἴσον παρὰ τὴν $ΔΙ$ παραβεβλήσθω τὸ $ΔΕ$ πλάτος ποιοῦν τὴν $ΔΗ$, τῷ δὲ δις ὑπὸ τῶν AB , $BΓ$ ἴσον ἀφηρήσθω τὸ $ΔΘ$ [πλάτος ποιοῦν τὴν $ΔΖ$]. λοιπὸν ἄρα τὸ $ΖΕ$ ἴσον ἐστὶ τῷ ἀπὸ τῆς $ΑΓ$. ὥστε ἡ $ΑΓ$ δύναται τὸ $ΖΕ$. καὶ ἐπεὶ τὸ συγχείμενον ἐκ τῶν ἀπὸ τῶν AB , $BΓ$ τετραγώνων μέσον ἐστὶ καὶ ἐστὶν ἴσον τῷ $ΔΕ$, μέσον ἄρα [ἐστὶ] τὸ $ΔΕ$. καὶ παρὰ ῥητὴν τὴν $ΔΙ$ παρὰκείται πλάτος ποιοῦν τὴν $ΔΗ$. ῥητὴ ἄρα ἐστὶν ἡ $ΔΗ$ καὶ ἀσύμμετρος τῇ $ΔΙ$ μήκει. πάλιν, ἐπεὶ τὸ δις ὑπὸ τῶν AB , $BΓ$ μέσον ἐστὶ καὶ ἐστὶν ἴσον τῷ $ΔΘ$, τὸ ἄρα

Proposition 78

If a straight-line, which is incommensurable in square with the whole, and with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them medial, and, moreover, the (sum of the) squares on them incommensurable with twice the (rectangle contained) by them, is subtracted from a(nother) straight-line then the remainder is an irrational (straight-line). Let it be called that which makes with a medial (area) a medial whole.



For let the straight-line BC , which is incommensurable in square AB , and fulfils the (other) prescribed (conditions), have been subtracted from the (straight-line) AB [Prop. 10.35]. I say that the remainder AC is the irrational (straight-line) called that which makes with a medial (area) a medial whole.

For let the rational (straight-line) DI be laid down. And let DE , equal to the (sum of the squares) on AB and BC , have been applied to DI , producing DG as breadth. And let DH , equal to twice the (rectangle contained) by AB and BC , have been subtracted (from DE) [producing DF as breadth]. Thus, the remainder FE is equal to the (square) on AC [Prop. 2.7]. Hence, AC is the square-root of FE . And since the sum of the squares on

$\Delta\Theta$ μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΔI παράκειται πλάτος ποιοῦν τὴν ΔZ · ῥητὴ ἄρα ἐστὶ καὶ ἡ ΔZ καὶ ἀσύμμετρος τῇ ΔI μήκει. καὶ ἐπεὶ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν AB , $B\Gamma$ τῷ δις ὑπὸ τῶν AB , $B\Gamma$, ἀσύμμετρον ἄρα καὶ τὸ ΔE τῷ $\Delta\Theta$. ὥς δὲ τὸ ΔE πρὸς τὸ $\Delta\Theta$, οὕτως ἐστὶ καὶ ἡ ΔH πρὸς τὴν ΔZ · ἀσύμμετρος ἄρα ἡ ΔH τῇ ΔZ . καὶ εἰσιν ἀμφοτέραι ῥηταί· αἱ $H\Delta$, ΔZ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἀποτομὴ ἄρα ἐστὶν ἡ ZH · ῥητὴ δὲ ἡ $Z\Theta$. τὸ δὲ ὑπὸ ῥητῆς καὶ ἀποτομῆς περιεχόμενον [ὀρθογώνιον] ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν· καὶ δύναται τὸ ZE ἢ AG · ἡ AG ἄρα ἄλογός ἐστιν· καλεῖσθω δὲ ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα. ὅπερ ἔδει δεῖξαι.

AB and BC is medial, and is equal to DE , DE [is] thus medial. And it is applied to the rational (straight-line) DI , producing DG as breadth. Thus, DG is rational, and incommensurable in length with DI [Prop 10.22]. Again, since twice the (rectangle contained) by AB and BC is medial, and is equal to DH , DH is thus medial. And it is applied to the rational (straight-line) DI , producing DF as breadth. Thus, DF is also rational, and incommensurable in length with DI [Prop. 10.22]. And since the (sum of the squares) on AB and BC is incommensurable with twice the (rectangle contained) by AB and BC , DE (is) also incommensurable with DH . And as DE (is) to DH , so DG also is to DF [Prop. 6.1]. Thus, DG (is) incommensurable (in length) with DF [Prop. 10.11]. And they are both rational. Thus, GD and DF are rational (straight-lines which are) commensurable in square only. Thus, FG is an apotome [Prop. 10.73]. And FH (is) rational. And the [rectangle] contained by a rational (straight-line) and an apotome is irrational [Prop. 10.20], and its square-root is irrational. And AC is the square-root of FE . Thus, AC is irrational. Let it be called that which makes with a medial (area) a medial whole.[†] (Which is) the very thing it was required to show.

[†] See footnote to Prop. 10.41.

οὐθ'.

Τῇ ἀποτομῇ μία [μόνον] προσαρμόζει εὐθεῖα ῥητὴ δύναμει μόνον σύμμετρος οὕσα τῇ ὅλῃ.



Ἐστω ἀποτομὴ ἡ AB , προσαρμόζουσα δὲ αὐτῇ ἡ $B\Gamma$ · αἱ AG , GB ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· λέγω, ὅτι τῇ AB ἐτέρα οὐ προσαρμόζει ῥητὴ δύναμει μόνον σύμμετρος οὕσα τῇ ὅλῃ.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ BD · καὶ αἱ AD , ΔB ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. καὶ ἐπεὶ, ὥς ὑπερέχει τὰ ἀπὸ τῶν AD , ΔB τοῦ δις ὑπὸ τῶν AD , ΔB , τούτῳ ὑπερέχει καὶ τὰ ἀπὸ τῶν AG , GB τοῦ δις ὑπὸ τῶν AG , GB · τῷ γὰρ αὐτῷ τῷ ἀπὸ τῆς AB ἀμφοτέρα ὑπερέχει· ἐναλλάξ ἄρα, ὥς ὑπερέχει τὰ ἀπὸ τῶν AD , ΔB τῶν ἀπὸ τῶν AG , GB , τούτῳ ὑπερέχει [καὶ] τὸ δις ὑπὸ τῶν AD , ΔB τοῦ δις ὑπὸ τῶν AG , GB . τὰ δὲ ἀπὸ τῶν AD , ΔB τῶν ἀπὸ τῶν AG , GB ὑπερέχει ῥητῶ· ῥητὰ γὰρ ἀμφοτέρα. καὶ τὸ δις ἄρα ὑπὸ τῶν AD , ΔB τοῦ δις ὑπὸ τῶν AG , GB ὑπερέχει ῥητῶ· ὅπερ ἐστὶν ἀδύνατον· μέσα γὰρ ἀμφοτέρα, μέσον δὲ μέσου οὐχ ὑπερέχει ῥητῶ. τῇ ἄρα AB ἐτέρα οὐ προσαρμόζει ῥητὴ δύναμει μόνον σύμμετρος οὕσα τῇ ὅλῃ.

Μία ἄρα μόνῃ τῇ ἀποτομῇ προσαρμόζει ῥητὴ δύναμει

Proposition 79

[Only] one rational straight-line, which is commensurable in square only with the whole, can be attached to an apotome.[†]



Let AB be an apotome, with BC (so) attached to it. AC and CB are thus rational (straight-lines which are) commensurable in square only [Prop. 10.73]. I say that another rational (straight-line), which is commensurable in square only with the whole, cannot be attached to AB .

For, if possible, let BD be (so) attached (to AB). Thus, AD and DB are also rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And since by whatever (area) the (sum of the squares) on AD and DB exceeds twice the (rectangle contained) by AD and DB , the (sum of the squares) on AC and CB also exceeds twice the (rectangle contained) by AC and CB by this (same area). For both exceed by the same (area)—(namely), the (square) on AB [Prop. 2.7]. Thus, alternately, by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB , twice the (rectangle contained) by AD and DB [also] exceeds twice the (rectangle contained) by AC and

μόνον σύμμετρος οὕσα τῇ ὅλῃ· ὅπερ ἔδει δεῖξαι.

CB by this (same area). And the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB by a rational (area). For both (are) rational (areas). Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.21], and a medial (area) cannot exceed a(nother) medial (area) by a rational (area) [Prop. 10.26]. Thus, another rational (straight-line), which is commensurable in square only with the whole, cannot be attached to AB .

Thus, only one rational (straight-line), which is commensurable in square only with the whole, can be attached to an apotome. (Which is) the very thing it was required to show.

† This proposition is equivalent to Prop. 10.42, with minus signs instead of plus signs.

π'.

Τῇ μέσῃ ἀποτομῇ πρώτη μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οὕσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα.



Ἐστω γὰρ μέσῃ ἀποτομῇ πρώτη ἡ AB , καὶ τῇ AB προσαρμοζέτω ἡ $BΓ$. αἱ $ΑΓ$, $ΓΒ$ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι τὸ ὑπὸ τῶν $ΑΓ$, $ΓΒ$. λέγω, ὅτι τῇ AB ἑτέρα οὐ προσαρμόζει μέση δυνάμει μόνον σύμμετρος οὕσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω καὶ ἡ $ΔΒ$. αἱ ἄρα $ΑΔ$, $ΔΒ$ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι τὸ ὑπὸ τῶν $ΑΔ$, $ΔΒ$. καὶ ἐπεὶ, ὥς ὑπερέχει τὰ ἀπὸ τῶν $ΑΔ$, $ΔΒ$ τοῦ δις ὑπὸ τῶν $ΑΔ$, $ΔΒ$, τούτῳ ὑπερέχει καὶ τὰ ἀπὸ τῶν $ΑΓ$, $ΓΒ$ τοῦ δις ὑπὸ τῶν $ΑΓ$, $ΓΒ$: τῷ γὰρ αὐτῷ [πάλιν] ὑπερέχουσι τῷ ἀπὸ τῆς AB . ἐναλλάξ ἄρα, ὥς ὑπερέχει τὰ ἀπὸ τῶν $ΑΔ$, $ΔΒ$ τῶν ἀπὸ τῶν $ΑΓ$, $ΓΒ$, τούτῳ ὑπερέχει καὶ τὸ δις ὑπὸ τῶν $ΑΔ$, $ΔΒ$ τοῦ δις ὑπὸ τῶν $ΑΓ$, $ΓΒ$. τὸ δὲ δις ὑπὸ τῶν $ΑΔ$, $ΔΒ$ τοῦ δις ὑπὸ τῶν $ΑΓ$, $ΓΒ$ ὑπερέχει ῥητῷ· ῥητὰ γὰρ ἀμφοτέρω. καὶ τὰ ἀπὸ τῶν $ΑΔ$, $ΔΒ$ ἄρα τῶν ἀπὸ τῶν $ΑΓ$, $ΓΒ$ [τετραγώνων] ὑπερέχει ῥητῷ· ὅπερ ἐστὶν ἀδύνατον· μέσα γὰρ ἐστὶν ἀμφοτέρω, μέσον δὲ μέσου οὐχ ὑπερέχει ῥητῷ.

Τῇ ἄρα μέσῃ ἀποτομῇ πρώτη μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οὕσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ῥητὸν περιέχουσα· ὅπερ ἔδει δεῖξαι.

Proposition 80

Only one medial straight-line, which is commensurable in square only with the whole, and contains a rational (area) with the whole, can be attached to a first apotome of a medial (straight-line).[†]



For let AB be a first apotome of a medial (straight-line), and let BC be (so) attached to AB . Thus, AC and CB are medial (straight-lines which are) commensurable in square only, containing a rational (area)—(namely, that contained) by AC and CB [Prop. 10.74]. I say that a(nother) medial (straight-line), which is commensurable in square only with the whole, and contains a rational (area) with the whole, cannot be attached to AB .

For, if possible, let DB also be (so) attached to AB . Thus, AD and DB are medial (straight-lines which are) commensurable in square only, containing a rational (area)—(namely, that) contained by AD and DB [Prop. 10.74]. And since by whatever (area) the (sum of the squares) on AD and DB exceeds twice the (rectangle contained) by AD and DB , the (sum of the squares) on AC and CB also exceeds twice the (rectangle contained) by AC and CB by this (same area). For [again] both exceeded by the same (area)—(namely), the (square) on AB [Prop. 2.7]. Thus, alternately, by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB , twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area). And twice the (rectangle contained) by AD and DB exceeds twice

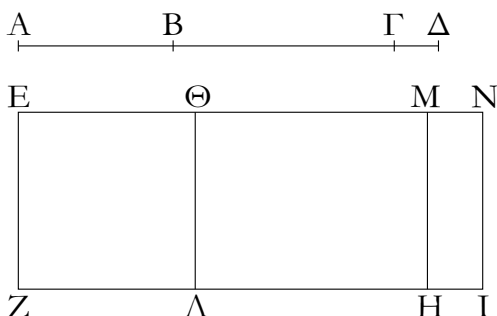
the (rectangle contained) by AC and CB by a rational (area). For both (are) rational (areas). Thus, the (sum of the squares) on AD and DB also exceeds the (sum of the) [squares] on AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Props. 10.15, 10.23 corr.], and a medial (area) cannot exceed a(nother) medial (area) by a rational (area) [Prop. 10.26].

Thus, only one medial (straight-line), which is commensurable in square only with the whole, and contains a rational (area) with the whole, can be attached to a first apotome of a medial (straight-line). (Which is) the very thing it was required to show.

† This proposition is equivalent to Prop. 10.43, with minus signs instead of plus signs.

πα'.

Τῇ μέσῃ ἀποτομῇ δευτέρᾳ μία μόνον προσαρμόζει εὐθεῖα μέση δυνάμει μόνον σύμμετρος τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα.

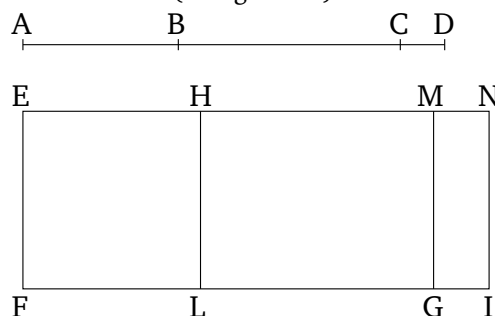


Ἐστω μέσῃ ἀποτομῇ δευτέρᾳ ἡ AB καὶ τῇ AB προσαρμόζουσα ἡ $BΓ$. αἱ ἄρα $ΑΓ$, $ΓΒ$ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι τὸ ὑπὸ τῶν $ΑΓ$, $ΓΒ$. λέγω, ὅτι τῇ AB ἐτέρα οὐ προσαρμόσει εὐθεῖα μέση δυνάμει μόνον σύμμετρος οὕσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ $BΔ$. καὶ αἱ $ΑΔ$, $ΔΒ$ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι τὸ ὑπὸ τῶν $ΑΔ$, $ΔΒ$. καὶ ἐκκείσθω ῥητὴ ἡ EZ , καὶ τοῖς μὲν ἀπὸ τῶν $ΑΓ$, $ΓΒ$ ἴσον παρὰ τὴν EZ παραβεβλήσθω τὸ EH πλάτος ποιοῦν τὴν EM . τῷ δὲ δις ὑπὸ τῶν $ΑΓ$, $ΓΒ$ ἴσον ἀφωρήσθω τὸ $ΘΗ$ πλάτος ποιοῦν τὴν $ΘΜ$. λοιπὸν ἄρα τὸ $ΕΛ$ ἴσον ἐστὶ τῷ ἀπὸ τῆς AB . ὥστε ἡ AB δύναται τὸ $ΕΛ$. πάλιν δὴ τοῖς ἀπὸ τῶν $ΑΔ$, $ΔΒ$ ἴσον παρὰ τὴν EZ παραβεβλήσθω τὸ $EΙ$ πλάτος ποιοῦν τὴν $ΕΝ$. ἔστι δὲ καὶ τὸ $ΕΛ$ ἴσον τῷ ἀπὸ τῆς AB τετραγώνῳ. λοιπὸν ἄρα τὸ $ΘΙ$ ἴσον ἐστὶ τῷ δις ὑπὸ τῶν $ΑΔ$, $ΔΒ$. καὶ ἐπεὶ μέσαι εἰσὶν αἱ $ΑΓ$, $ΓΒ$, μέσα ἄρα ἐστὶ καὶ τὰ ἀπὸ τῶν $ΑΓ$, $ΓΒ$. καὶ ἐστὶν ἴσα τῷ $ΕΗ$. μέσον ἄρα καὶ τὸ $ΕΗ$. καὶ παρὰ ῥητὴν τὴν EZ παράκειται πλάτος ποιοῦν

Proposition 81

Only one medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, can be attached to a second apotome of a medial (straight-line).†



Let AB be a second apotome of a medial (straight-line), with BC (so) attached to AB . Thus, AC and CB are medial (straight-lines which are) commensurable in square only, containing a medial (area)—(namely, that contained) by AC and CB [Prop. 10.75]. I say that a(nother) medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, cannot be attached to AB .

For, if possible, let BD be (so) attached. Thus, AD and DB are also medial (straight-lines which are) commensurable in square only, containing a medial (area)—(namely, that contained) by AD and DB [Prop. 10.75]. And let the rational (straight-line) EF be laid down. And let EG , equal to the (sum of the squares) on AC and CB , have been applied to EF , producing EM as breadth. And let HG , equal to twice the (rectangle contained) by AC and CB , have been subtracted (from EG), producing HM as breadth. The remainder EL is thus equal to the (square) on AB [Prop. 2.7]. Hence, AB is the

τὴν EM · ῥητὴ ἄρα ἐστὶν ἡ EM καὶ ἀσύμμετρος τῇ EZ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ ὑπὸ τῶν AG , GB , καὶ τὸ δις ὑπὸ τῶν AG , GB μέσον ἐστὶν. καὶ ἐστὶν ἴσον τῷ $ΘΗ$ · καὶ τὸ $ΘΗ$ ἄρα μέσον ἐστὶν. καὶ παρὰ ῥητὴν τὴν EZ παράκειται πλάτος ποιοῦν τὴν $ΘΜ$ · ῥητὴ ἄρα ἐστὶ καὶ ἡ $ΘΜ$ καὶ ἀσύμμετρος τῇ EZ μήκει. καὶ ἐπεὶ αἱ AG , GB δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα ἐστὶν ἡ AG τῇ GB μήκει. ὥς δὲ ἡ AG πρὸς τὴν GB , οὕτως ἐστὶ τὸ ἀπὸ τῆς AG πρὸς τὸ ὑπὸ τῶν AG , GB · ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς AG τῷ ὑπὸ τῶν AG , GB . ἀλλὰ τῷ μὲν ἀπὸ τῆς AG σύμμετρόν ἐστι τὰ ἀπὸ τῶν AG , GB , τῷ δὲ ὑπὸ τῶν AG , GB σύμμετρόν ἐστι τὸ δις ὑπὸ τῶν AG , GB · ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν AG , GB τῷ δις ὑπὸ τῶν AG , GB . καὶ ἐστὶ τοῖς μὲν ἀπὸ τῶν AG , GB ἴσον τὸ EH , τῷ δὲ δις ὑπὸ τῶν AG , GB ἴσον τὸ $ΗΘ$ · ἀσύμμετρον ἄρα ἐστὶ τὸ EH τῷ $ΘΗ$. ὥς δὲ τὸ EH πρὸς τὸ $ΘΗ$, οὕτως ἐστὶν ἡ EM πρὸς τὴν $ΘΜ$ · ἀσύμμετρος ἄρα ἐστὶν ἡ EM τῇ $ΜΘ$ μήκει. καὶ εἰσιν ἀμφοτέραι ῥηταί· αἱ EM , $ΜΘ$ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ $ΕΘ$, προσαρμόζουσα δὲ αὐτῇ ἡ $ΘΜ$. ὁμοίως δὲ δείξομεν, ὅτι καὶ ἡ $ΘΝ$ αὐτῇ προσαρμόζει· τῇ ἄρα ἀποτομῇ ἄλλη καὶ ἄλλη προσαρμόζει· εὐθεῖα δυνάμει μόνον σύμμετρος οὔσα τῇ ὅλῃ· ὅπερ ἐστὶν ἀδύνατον.

Τῇ ἄρα μέσης ἀποτομῇ δευτέρᾳ μία μόνον προσαρμόζει· εὐθεῖα μέση δυνάμει μόνον σύμμετρος οὔσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης μέσον περιέχουσα· ὅπερ ἔδει δεῖξαι.

square-root of EL . So, again, let EI , equal to the (sum of the squares) on AD and DB have been applied to EF , producing EN as breadth. And EL is also equal to the square on AB . Thus, the remainder HI is equal to twice the (rectangle contained) by AD and DB [Prop. 2.7]. And since AC and CB are (both) medial (straight-lines), the (sum of the squares) on AC and CB is also medial. And it is equal to EG . Thus, EG is also medial [Props. 10.15, 10.23 corr.]. And it is applied to the rational (straight-line) EF , producing EM as breadth. Thus, EM is rational, and incommensurable in length with EF [Prop. 10.22]. Again, since the (rectangle contained) by AC and CB is medial, twice the (rectangle contained) by AC and CB is also medial [Prop. 10.23 corr.]. And it is equal to HG . Thus, HG is also medial. And it is applied to the rational (straight-line) EF , producing HM as breadth. Thus, HM is also rational, and incommensurable in length with EF [Prop. 10.22]. And since AC and CB are commensurable in square only, AC is thus incommensurable in length with CB . And as AC (is) to CB , so the (square) on AC is to the (rectangle contained) by AC and CB [Prop. 10.21 corr.]. Thus, the (square) on AC is incommensurable with the (rectangle contained) by AC and CB [Prop. 10.11]. But, the (sum of the squares) on AC and CB is commensurable with the (square) on AC , and twice the (rectangle contained) by AC and CB is commensurable with the (rectangle contained) by AC and CB [Prop. 10.6]. Thus, the (sum of the squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB [Prop. 10.13]. And EG is equal to the (sum of the squares) on AC and CB . And GH is equal to twice the (rectangle contained) by AC and CB . Thus, EG is incommensurable with HG . And as EG (is) to HG , so EM is to HM [Prop. 6.1]. Thus, EM is incommensurable in length with MH [Prop. 10.11]. And they are both rational (straight-lines). Thus, EM and MH are rational (straight-lines which are) commensurable in square only. Thus, EH is an apotome [Prop. 10.73], and HM (is) attached to it. So, similarly, we can show that HN (is) also (commensurable in square only with EN and is) attached to (EH). Thus, different straight-lines, which are commensurable in square only with the whole, are attached to an apotome. The very thing is impossible [Prop. 10.79].

Thus, only one medial straight-line, which is commensurable in square only with the whole, and contains a medial (area) with the whole, can be attached to a second apotome of a medial (straight-line). (Which is) the very thing it was required to show.

† This proposition is equivalent to Prop. 10.44, with minus signs instead of plus signs.

πβ'.

Τῇ ἐλάσσονι μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ ποιοῦσα μετὰ τῆς ὅλης τὸ μὲν ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δὲ δις ὑπ' αὐτῶν μέσον.



Ἐστω ἡ ἐλάσσων ἡ AB , καὶ τῇ AB προσαρμόζουσα ἔστω ἡ $BΓ$. αἱ ἄρα $ΑΓ$, $ΓΒ$ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγχείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δὲ δις ὑπ' αὐτῶν μέσον· λέγω, ὅτι τῇ AB ἑτέρα εὐθεῖα οὐ προσαρμόσει τὰ αὐτὰ ποιοῦσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ $ΒΔ$. καὶ αἱ $ΑΔ$, $ΔΒ$ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προειρημένα. καὶ ἐπεὶ, ὥς ὑπερέχει τὰ ἀπὸ τῶν $ΑΔ$, $ΔΒ$ τῶν ἀπὸ τῶν $ΑΓ$, $ΓΒ$, τούτῳ ὑπερέχει καὶ τὸ δις ὑπὸ τῶν $ΑΔ$, $ΔΒ$ τοῦ δις ὑπὸ τῶν $ΑΓ$, $ΓΒ$, τὰ δὲ ἀπὸ τῶν $ΑΔ$, $ΔΒ$ τετραγώνων τῶν ἀπὸ τῶν $ΑΓ$, $ΓΒ$ τετραγώνων ὑπερέχει ῥητῶ· ῥητὰ γὰρ ἐστὶν ἀμφοτέρω· καὶ τὸ δις ὑπὸ τῶν $ΑΔ$, $ΔΒ$ ἄρα τοῦ δις ὑπὸ τῶν $ΑΓ$, $ΓΒ$ ὑπερέχει ῥητῶ· ὅπερ ἐστὶν ἀδύνατον· μέσα γὰρ ἐστὶν ἀμφοτέρω.

Τῇ ἄρα ἐλάσσονι μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ καὶ ποιοῦσα τὰ μὲν ἀπ' αὐτῶν τετραγώνων ἅμα ῥητόν, τὸ δὲ δις ὑπ' αὐτῶν μέσον· ὅπερ ἔδει δεῖξαι.

Proposition 82

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the (sum of the) squares on them rational, and twice the (rectangle contained) by them medial, can be attached to a minor (straight-line).



Let AB be a minor (straight-line), and let BC be (so) attached to AB . Thus, AC and CB are (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and twice the (rectangle contained) by them medial [Prop. 10.76]. I say that another another straight-line fulfilling the same (conditions) cannot be attached to AB .

For, if possible, let BD be (so) attached (to AB). Thus, AD and DB are also (straight-lines which are) incommensurable in square, fulfilling the (other) aforementioned (conditions) [Prop. 10.76]. And since by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB , twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area) [Prop. 2.7]. And the (sum of the) squares on AD and DB exceeds the (sum of the) squares on AC and CB by a rational (area). For both are rational (areas). Thus, twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.26].

Thus, only one straight-line, which is incommensurable in square with the whole, and (with the whole) makes the squares on them (added) together rational, and twice the (rectangle contained) by them medial, can be attached to a minor (straight-line). (Which is) the very thing it was required to show.

† This proposition is equivalent to Prop. 10.45, with minus signs instead of plus signs.

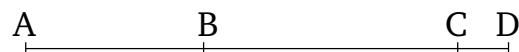
πγ'.

Τῇ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσῃ μία μόνον προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὖσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὸ μὲν συγχείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δὲ δις ὑπ' αὐτῶν ῥητόν.



Proposition 83

Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them rational, can be attached to that (straight-line) which with a rational (area) makes a medial whole.†



Ἐστω ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα ἡ AB , καὶ τῇ AB προσαρμοζέτω ἡ $BΓ$. αἱ ἄρα $ΑΓ$, $ΓΒ$ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προκείμενα· λέγω, ὅτι τῇ AB ἑτέρα οὐ προσαρμόσει τὰ αὐτὰ ποιοῦσα.

Εἰ γὰρ δυνατόν, προσαρμοζέτω ἡ $ΒΔ$. καὶ αἱ $ΑΔ$, $ΔΒ$ ἄρα εὐθεῖαι δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προκείμενα. ἐπεὶ οὖν, ὅ ὑπερέχει τὰ ἀπὸ τῶν $ΑΔ$, $ΔΒ$ τῶν ἀπὸ τῶν $ΑΓ$, $ΓΒ$, τούτῳ ὑπερέχει καὶ τὸ δις ὑπὸ τῶν $ΑΔ$, $ΔΒ$ τοῦ δις ὑπὸ τῶν $ΑΓ$, $ΓΒ$ ἀκολουθῶς τοῖς πρὸ αὐτοῦ, τὸ δὲ δις ὑπὸ τῶν $ΑΔ$, $ΔΒ$ τοῦ δις ὑπὸ τῶν $ΑΓ$, $ΓΒ$ ὑπερέχει ῥητῶ· ῥητὰ γάρ ἐστιν ἀμφοτέρω· καὶ τὰ ἀπὸ τῶν $ΑΔ$, $ΔΒ$ ἄρα τῶν ἀπὸ τῶν $ΑΓ$, $ΓΒ$ ὑπερέχει ῥητῶ· ὅπερ ἐστὶν ἀδύνατον· μέσα γάρ ἐστιν ἀμφοτέρω.

Οὐκ ἄρα τῇ AB ἑτέρα προσαρμόσει εὐθεῖα δυνάμει ἀσύμμετρος οὕσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τὰ προειρημένα· μία ἄρα μόνον προσαρμόσει· ὅπερ ἔδει δεῖξαι.

Let AB be a (straight-line) which with a rational (area) makes a medial whole, and let BC be (so) attached to AB . Thus, AC and CB are (straight-lines which are) incommensurable in square, fulfilling the (other) proscribed (conditions) [Prop. 10.77]. I say that another (straight-line) fulfilling the same (conditions) cannot be attached to AB .

For, if possible, let BD be (so) attached (to AB). Thus, AD and DB are also straight-lines (which are) incommensurable in square, fulfilling the (other) prescribed (conditions) [Prop. 10.77]. Therefore, analogously to the (propositions) before this, since by whatever (area) the (sum of the squares) on AD and DB exceeds the (sum of the squares) on AC and CB , twice the (rectangle contained) by AD and DB also exceeds twice the (rectangle contained) by AC and CB by this (same area). And twice the (rectangle contained) by AD and DB exceeds twice the (rectangle contained) by AC and CB by a rational (area). For they are (both) rational (areas). Thus, the (sum of the squares) on AD and DB also exceeds the (sum of the squares) on AC and CB by a rational (area). The very thing is impossible. For both are medial (areas) [Prop. 10.26].

Thus, another straight-line cannot be attached to AB , which is incommensurable in square with the whole, and fulfills the (other) aforementioned (conditions) with the whole. Thus, only one (such straight-line) can be (so) attached. (Which is) the very thing it was required to show.

† This proposition is equivalent to Prop. 10.46, with minus signs instead of plus signs.

πδ'.

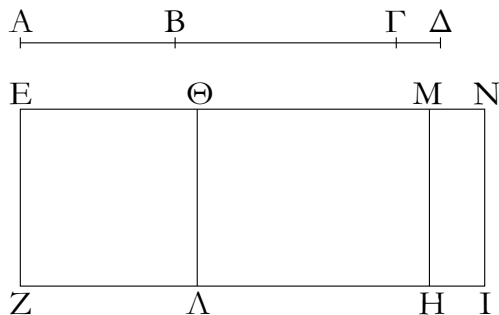
Proposition 84

Τῇ μετὰ μέσου μέσον τὸ ὅλον ποιούσῃ μία μόνῃ προσαρμόζει εὐθεῖα δυνάμει ἀσύμμετρος οὕσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιοῦσα τό τε συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον τό τε δις ὑπ' αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τῷ συγκειμένῳ ἐκ τῶν ἀπ' αὐτῶν.

Ἐστω ἡ μετὰ μέσου μέσον τὸ ὅλον ποιοῦσα ἡ AB , προσαρμόζουσα δὲ αὐτῇ ἡ $BΓ$. αἱ ἄρα $ΑΓ$, $ΓΒ$ δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὰ προειρημένα. λέγω, ὅτι τῇ AB ἑτέρα οὐ προσαρμόσει ποιοῦσα προειρημένα.

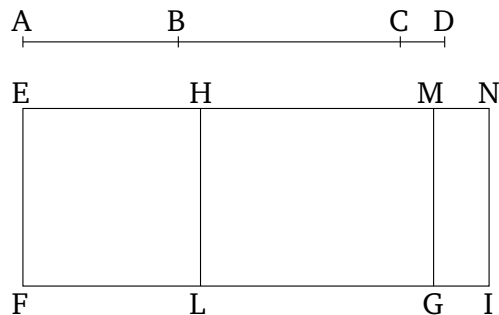
Only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the sum of the squares on them medial, and twice the (rectangle contained) by them medial, and, moreover, incommensurable with the sum of the (squares) on them, can be attached to that (straight-line) which with a medial (area) makes a medial whole.†

Let AB be a (straight-line) which with a medial (area) makes a medial whole, BC being (so) attached to it. Thus, AC and CB are incommensurable in square, fulfilling the (other) aforementioned (conditions) [Prop. 10.78]. I say that a(nother) (straight-line) fulfilling the aforementioned (conditions) cannot be attached to AB .



Εἰ γὰρ δυνατόν, προσαρμύζεται ἡ ΒΔ, ὥστε καὶ τὰς ΑΔ, ΔΒ δυνάμει ἀσύμμετρος εἶναι ποιούσας τὰ τε ἀπὸ τῶν ΑΔ, ΔΒ τετράγωνα ἅμα μέσον καὶ τὸ δις ὑπὸ τῶν ΑΔ, ΔΒ μέσον καὶ ἔτι τὰ ἀπὸ τῶν ΑΔ, ΔΒ ἀσύμμετρα τῷ δις ὑπὸ τῶν ΑΔ, ΔΒ· καὶ ἐκκείσθω ῥητὴ ἡ ΕΖ, καὶ τοῖς μὲν ἀπὸ τῶν ΑΓ, ΓΒ ἴσον παρὰ τὴν ΕΖ παραβελήσθω τὸ ΕΗ πλάτος ποιῶν τὴν ΕΜ, τῷ δὲ δις ὑπὸ τῶν ΑΓ, ΓΒ ἴσον παρὰ τὴν ΕΖ παραβελήσθω τὸ ΘΗ πλάτος ποιῶν τὴν ΘΜ· λοιπὸν ἄρα τὸ ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τῷ ΕΛ· ἡ ἄρα ΑΒ δύναται τὸ ΕΛ. πάλιν τοῖς ἀπὸ τῶν ΑΔ, ΔΒ ἴσον παρὰ τὴν ΕΖ παραβελήσθω τὸ ΕΙ πλάτος ποιῶν τὴν ΕΝ. ἔστι δὲ καὶ τὸ ἀπὸ τῆς ΑΒ ἴσον τῷ ΕΛ· λοιπὸν ἄρα τὸ δις ὑπὸ τῶν ΑΔ, ΔΒ ἴσον [ἐστὶ] τῷ ΘΙ. καὶ ἐπεὶ μέσον ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΓ, ΓΒ καὶ ἐστὶν ἴσον τῷ ΕΗ, μέσον ἄρα ἐστὶ καὶ τὸ ΕΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιῶν τὴν ΕΜ· ῥητὴ ἄρα ἐστὶν ἡ ΕΜ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ δις ὑπὸ τῶν ΑΓ, ΓΒ καὶ ἐστὶν ἴσον τῷ ΘΗ, μέσον ἄρα καὶ τὸ ΘΗ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιῶν τὴν ΘΜ· ῥητὴ ἄρα ἐστὶν ἡ ΘΜ καὶ ἀσύμμετρος τῇ ΕΖ μήκει. καὶ ἐπεὶ ἀσύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΓ, ΓΒ τῷ δις ὑπὸ τῶν ΑΓ, ΓΒ, ἀσύμμετρόν ἐστι καὶ τὸ ΕΗ τῷ ΘΗ· ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ΕΜ τῇ ΜΘ μήκει. καὶ εἰσιν ἀμφότεραι ῥηταί· αἱ ἄρα ΕΜ, ΜΘ ῥηταί· εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ ΕΘ, προσαρμύζουσα δὲ αὐτῇ ἡ ΘΜ. ὁμοίως δὲ δείξομεν, ὅτι ἡ ΕΘ πάλιν ἀποτομὴ ἐστὶν, προσαρμύζουσα δὲ αὐτῇ ἡ ΘΝ. τῇ ἄρα ἀποτομῇ ἄλλῃ καὶ ἄλλῃ προσαρμύζει ῥητὴ δύναμει μόνον σύμμετρος οὕσα τῇ ὅλῃ· ὅπερ ἐδείχθη ἀδύνατον. οὐκ ἄρα τῇ ΑΒ ἐτέρα προσαρμύσει εὐθεΐα.

Τῇ ἄρα ΑΒ μία μόνον προσαρμύζει εὐθεΐα δύναμει ἀσύμμετρος οὕσα τῇ ὅλῃ, μετὰ δὲ τῆς ὅλης ποιούσα τὰ τε ἀπ' αὐτῶν τετράγωνα ἅμα μέσον καὶ τὸ δις ὑπ' αὐτῶν μέσον καὶ ἔτι τὰ ἀπ' αὐτῶν τετράγωνα ἀσύμμετρα τῷ δις ὑπ' αὐτῶν· ὅπερ ἔδει δεῖξαι.



For, if possible, let BD be (so) attached. Hence, AD and DB are also (straight-lines which are) incommensurable in square, making the squares on AD and DB (added) together medial, and twice the (rectangle contained) by AD and DB medial, and, moreover, the (sum of the squares) on AD and DB incommensurable with twice the (rectangle contained) by AD and DB [Prop. 10.78]. And let the rational (straight-line) EF be laid down. And let EG , equal to the (sum of the squares) on AC and CB , have been applied to EF , producing EM as breadth. And let HG , equal to twice the (rectangle contained) by AC and CB , have been applied to EF , producing HM as breadth. Thus, the remaining (square) on AB is equal to EL [Prop. 2.7]. Thus, AB is the square-root of EL . Again, let EI , equal to the (sum of the squares) on AD and DB , have been applied to EF , producing EN as breadth. And the (square) on AB is also equal to EL . Thus, the remaining twice the (rectangle contained) by AD and DB [is] equal to HI [Prop. 2.7]. And since the sum of the (squares) on AC and CB is medial, and is equal to EG , EG is thus also medial. And it is applied to the rational (straight-line) EF , producing EM as breadth. EM is thus rational, and incommensurable in length with EF [Prop. 10.22]. Again, since twice the (rectangle contained) by AC and CB is medial, and is equal to HG , HG is thus also medial. And it is applied to the rational (straight-line) EF , producing HM as breadth. HM is thus rational, and incommensurable in length with EF [Prop. 10.22]. And since the (sum of the squares) on AC and CB is incommensurable with twice the (rectangle contained) by AC and CB , EG is also incommensurable with HG . Thus, EM is also incommensurable in length with MH [Props. 6.1, 10.11]. And they are both rational (straight-lines). Thus, EM and MH are rational (straight-lines which are) commensurable in square only. Thus, EH is an apotome [Prop. 10.73], with HM attached to it. So, similarly, we can show that EH is again an apotome, with HN attached to it. Thus, different rational (straight-lines), which are commensurable in square only with the whole, are attached to an apotome. The very thing was shown

(to be) impossible [Prop. 10.79]. Thus, another straight-line cannot be (so) attached to AB .

Thus, only one straight-line, which is incommensurable in square with the whole, and (together) with the whole makes the squares on them (added) together medial, and twice the (rectangle contained) by them medial, and, moreover, the (sum of the) squares on them incommensurable with the (rectangle contained) by them, can be attached to AB . (Which is) the very thing it was required to show.

† This proposition is equivalent to Prop. 10.47, with minus signs instead of plus signs.

Ὅροι τρίτοι.

ια'. Ὑποκειμένης ῥητῆς καὶ ἀποτομῆς, ἐὰν μὲν ἡ ὅλη τῆς προσαρμοζούσης μείζον δύνηται τῷ ἀπὸ συμμετρου ἑαυτῇ μήκει, καὶ ἡ ὅλη σύμμετρος ᾗ τῇ ἐκκειμένῃ ῥητῇ μήκει, καλείσθω ἀποτομή πρώτη.

ιβ'. Ἐὰν δὲ ἡ προσαρμόζουσα σύμμετρος ᾗ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ὅλη τῆς προσαρμοζούσης μείζον δύνηται τῷ ἀπὸ συμμετρου ἑαυτῇ, καλείσθω ἀποτομή δευτέρα.

ιγ'. Ἐὰν δὲ μηδετέρα σύμμετρος ᾗ τῇ ἐκκειμένῃ ῥητῇ μήκει, ἡ δὲ ὅλη τῆς προσαρμοζούσης μείζον δύνηται τῷ ἀπὸ συμμετρου ἑαυτῇ, καλείσθω ἀποτομή τρίτη.

ιδ'. Πάλιν, ἐὰν ἡ ὅλη τῆς προσαρμοζούσης μείζον δύνηται τῷ ἀπὸ ἀσυμμετρου ἑαυτῇ [μήκει], ἐὰν μὲν ἡ ὅλη σύμμετρος ᾗ τῇ ἐκκειμένῃ ῥητῇ μήκει, καλείσθω ἀποτομή τετάρτη.

ιε'. Ἐὰν δὲ ἡ προσαρμόζουσα, πέμπτη.

ις'. Ἐὰν δὲ μηδετέρα, ἕκτη.

Definitions III

11. Given a rational (straight-line) and an apotome, if the square on the whole is greater than the (square on a straight-line) attached (to the apotome) by the (square) on (some straight-line) commensurable in length with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a first apotome.

12. And if the attached (straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a second apotome.

13. And if neither of (the whole or the attached straight-line) is commensurable in length with the (previously) laid down rational (straight-line), and the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) commensurable (in length) with (the whole), then let the (apotome) be called a third apotome.

14. Again, if the square on the whole is greater than (the square on) the attached (straight-line) by the (square) on (some straight-line) incommensurable [in length] with (the whole), and the whole is commensurable in length with the (previously) laid down rational (straight-line), then let the (apotome) be called a fourth apotome.

15. And if the attached (straight-line is commensurable), a fifth (apotome).

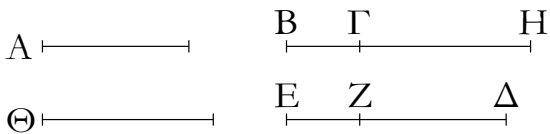
16. And if neither (the whole nor the attached straight-line is commensurable), a sixth (apotome).

πε'.

Εὑρεῖν τὴν πρώτην ἀποτομήν.

Proposition 85

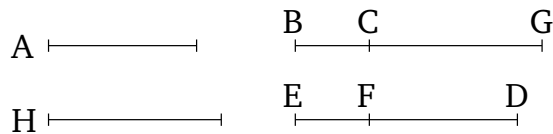
To find a first apotome.



Ἐκκείσθω ῥητὴ ἡ A , καὶ τῇ A μήκει σύμμετρος ἔστω ἡ BH . ῥητὴ ἄρα ἐστὶ καὶ ἡ BH . καὶ ἐκκείσθωσαν δύο τετράγωνοι ἀριθμοὶ οἱ ΔE , EZ , ὧν ἡ ὑπεροχὴ ὁ $Z\Delta$ μὴ ἔστω τετράγωνος· οὐδ' ἄρα ὁ $E\Delta$ πρὸς τὸν ΔZ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ πεποιήσθω ὡς ὁ $E\Delta$ πρὸς τὸν ΔZ , οὕτως τὸ ἀπὸ τῆς BH τετράγωνον πρὸς τὸ ἀπὸ τῆς $H\Gamma$ τετράγωνον· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς BH τῷ ἀπὸ τῆς $H\Gamma$. ῥητὸν δὲ τὸ ἀπὸ τῆς BH . ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς $H\Gamma$. ῥητὴ ἄρα ἐστὶ καὶ ἡ $H\Gamma$. καὶ ἐπεὶ ὁ $E\Delta$ πρὸς τὸν ΔZ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ' ἄρα τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς $H\Gamma$ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ BH τῇ $H\Gamma$ μήκει. καὶ εἰσιν ἀμφοτέραι ῥηταί· αἱ BH , $H\Gamma$ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἡ ἄρα $B\Gamma$ ἀποτομή ἐστίν. λέγω δὴ, ὅτι καὶ πρώτη.

Ὡς γὰρ μείζον ἐστὶ τὸ ἀπὸ τῆς BH τοῦ ἀπὸ τῆς $H\Gamma$, ἔστω τὸ ἀπὸ τῆς Θ . καὶ ἐπεὶ ἐστὶν ὡς ὁ $E\Delta$ πρὸς τὸν $Z\Delta$, οὕτως τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς $H\Gamma$, καὶ ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΔE πρὸς τὸν EZ , οὕτως τὸ ἀπὸ τῆς HB πρὸς τὸ ἀπὸ τῆς Θ . ὁ δὲ ΔE πρὸς τὸν EZ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἐκάτερος γὰρ τετράγωνός ἐστιν· καὶ τὸ ἀπὸ τῆς HB ἄρα πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· σύμμετρος ἄρα ἐστὶν ἡ BH τῇ Θ μήκει. καὶ δύναται ἡ BH τῆς $H\Gamma$ μείζον τῷ ἀπὸ τῆς Θ . ἡ BH ἄρα τῆς $H\Gamma$ μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καὶ ἐστὶν ἡ ὅλη ἡ BH σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ A . ἡ $B\Gamma$ ἄρα ἀποτομή ἐστὶ πρώτη.

Εὐρηται ἄρα ἡ πρώτη ἀποτομή ἡ $B\Gamma$. ὅπερ ἔδει εὐρεῖν.



Let the rational (straight-line) A be laid down. And let $B\Gamma$ be commensurable in length with A . $B\Gamma$ is thus also a rational (straight-line). And let two square numbers DE and EF be laid down, and let their difference FD be not square [Prop. 10.28 lem. I]. Thus, ED does not have to DF the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as ED (is) to DF , so the square on $B\Gamma$ (is) to the square on GC [Prop. 10.6. corr.]. Thus, the (square) on $B\Gamma$ is commensurable with the (square) on GC [Prop. 10.6]. And the (square) on $B\Gamma$ (is) rational. Thus, the (square) on GC (is) also rational. Thus, GC is also rational. And since ED does not have to DF the ratio which (some) square number (has) to (some) square number, the (square) on $B\Gamma$ thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either. Thus, $B\Gamma$ is incommensurable in length with GC [Prop. 10.9]. And they are both rational (straight-lines). Thus, $B\Gamma$ and GC are rational (straight-lines which are) commensurable in square only. Thus, BC is an apotome [Prop. 10.73]. So, I say that (it is) also a first (apotome).

Let the (square) on H be that (area) by which the (square) on $B\Gamma$ is greater than the (square) on GC [Prop. 10.13 lem.]. And since as ED is to FD , so the (square) on $B\Gamma$ (is) to the (square) on GC , thus, via conversion, as DE is to EF , so the (square) on GB (is) to the (square) on H [Prop. 5.19 corr.]. And DE has to EF the ratio which (some) square-number (has) to (some) square-number. For each is a square (number). Thus, the (square) on GB also has to the (square) on H the ratio which (some) square number (has) to (some) square number. Thus, $B\Gamma$ is commensurable in length with H [Prop. 10.9]. And the square on $B\Gamma$ is greater than (the square on) GC by the (square) on H . Thus, the square on $B\Gamma$ is greater than (the square on) GC by the (square) on (some straight-line) commensurable in length with ($B\Gamma$). And the whole, $B\Gamma$, is commensurable in length with the (previously) laid down rational (straight-line) A . Thus, BC is a first apotome [Def. 10.11].

Thus, the first apotome BC has been found. (Which is) the very thing it was required to find.

† See footnote to Prop. 10.48.

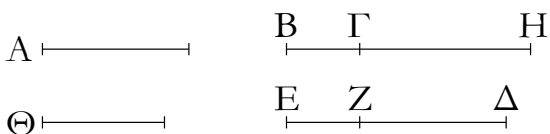
πζ'.

Εὐρεῖν τὴν δευτέραν ἀποτομήν.

Proposition 86

To find a second apotome.

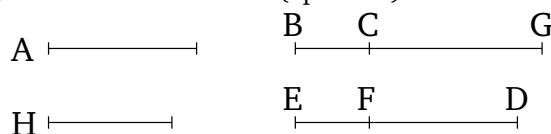
Ἐκκείσθω ῥητὴ ἡ A καὶ τῇ A σύμμετρος μήκει ἡ $HΓ$. ῥητὴ ἄρα ἐστὶν ἡ $HΓ$. καὶ ἐκκείσθωσαν δύο τετράγωνοι ἀριθμοὶ οἱ $ΔΕ$, $ΕΖ$, ὧν ἡ ὑπεροχὴ ὁ $ΔΖ$ μὴ ἔστω τετράγωνος. καὶ πεποιήσθω ὡς ὁ $ΖΔ$ πρὸς τὸν $ΔΕ$, οὕτως τὸ ἀπὸ τῆς $ΓΗ$ τετράγωνον πρὸς τὸ ἀπὸ τῆς $ΗΒ$ τετράγωνον. σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς $ΓΗ$ τετράγωνον τῷ ἀπὸ τῆς $ΗΒ$ τετράγωνῳ. ῥητὸν δὲ τὸ ἀπὸ τῆς $ΓΗ$. ῥητὸν ἄρα [ἐστὶ] καὶ τὸ ἀπὸ τῆς $ΗΒ$. ῥητὴ ἄρα ἐστὶν ἡ BH . καὶ ἐπεὶ τὸ ἀπὸ τῆς $HΓ$ τετράγωνον πρὸς τὸ ἀπὸ τῆς $ΗΒ$ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, ἀσύμμετρος ἐστὶν ἡ $ΓΗ$ τῇ $ΗΒ$ μήκει. καὶ εἰσιν ἀμφοτέραι ῥηταί· αἱ $ΓΗ$, $ΗΒ$ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἡ $BΓ$ ἄρα ἀποτομή ἐστὶν. λέγω δὴ, ὅτι καὶ δευτέρα.



Ὅτι γὰρ μεῖζόν ἐστι τὸ ἀπὸ τῆς BH τοῦ ἀπὸ τῆς $HΓ$, ἔστω τὸ ἀπὸ τῆς $Θ$. ἐπεὶ οὖν ἐστὶν ὡς τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς $HΓ$, οὕτως ὁ $ΕΔ$ ἀριθμὸς πρὸς τὸν $ΔΖ$ ἀριθμὸν, ἀναστρέψαντι ἄρα ἐστὶν ὡς τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς $Θ$, οὕτως ὁ $ΔΕ$ πρὸς τὸν $ΕΖ$. καὶ ἐστὶν ἐκάτερος τῶν $ΔΕ$, $ΕΖ$ τετράγωνος· τὸ ἄρα ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς $Θ$ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· σύμμετρος ἄρα ἐστὶν ἡ BH τῇ $Θ$ μήκει. καὶ δύναται ἡ BH τῆς $HΓ$ μεῖζον τῷ ἀπὸ τῆς $Θ$. ἡ BH ἄρα τῆς $HΓ$ μεῖζον δύναται τῷ ἀπὸ συμέτρου ἑαυτῇ μήκει. καὶ ἐστὶν ἡ προσαρμόζουσα ἡ $ΓΗ$ τῇ ἐκκειμένη ῥητῇ σύμμετρος τῇ A . ἡ $BΓ$ ἄρα ἀποτομή ἐστὶ δευτέρα.

Εὑρηται ἄρα δευτέρα ἀποτομή ἡ $BΓ$. ὅπερ ἔδει δεῖξαι.

Let the rational (straight-line) A , and GC (which is) commensurable in length with A , be laid down. Thus, GC is a rational (straight-line). And let the two square numbers DE and EF be laid down, and let their difference DF be not square [Prop. 10.28 lem. I]. And let it have been contrived that as FD (is) to DE , so the square on CG (is) to the square on GB [Prop. 10.6 corr.]. Thus, the square on CG is commensurable with the square on GB [Prop. 10.6]. And the (square) on CG (is) rational. Thus, the (square) on GB [is] also rational. Thus, BG is a rational (straight-line). And since the square on GC does not have to the (square) on GB the ratio which (some) square number (has) to (some) square number, CG is incommensurable in length with GB [Prop. 10.9]. And they are both rational (straight-lines). Thus, CG and GB are rational (straight-lines which are) commensurable in square only. Thus, BC is an apotome [Prop. 10.73]. So, I say that it is also a second (apotome).



For let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC [Prop. 10.13 lem.]. Therefore, since as the (square) on BG is to the (square) on GC , so the number ED (is) to the number DF , thus, also, via conversion, as the (square) on BG is to the (square) on H , so DE (is) to EF [Prop. 5.19 corr.]. And DE and EF are each square (numbers). Thus, the (square) on BG has to the (square) on H the ratio which (some) square number (has) to (some) square number. Thus, BG is commensurable in length with H [Prop. 10.9]. And the square on BG is greater than (the square on) GC by the (square) on H . Thus, the square on BG is greater than (the square on) GC by the (square) on (some straight-line) commensurable in length with (BG). And the attachment CG is commensurable (in length) with the (previously) laid down rational (straight-line) A . Thus, BC is a second apotome [Def. 10.12].[†]

Thus, the second apotome BC has been found. (Which is) the very thing it was required to show.

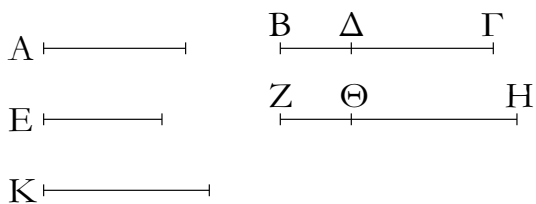
[†] See footnote to Prop. 10.49.

πζ'.

Εὑρεῖν τὴν τρίτην ἀποτομήν.

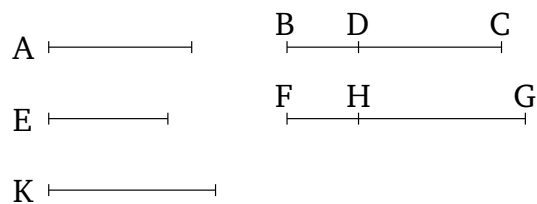
Proposition 87

To find a third apotome.



Ἐκκεῖσθω ῥητὴ ἡ A , καὶ ἐκκεῖσθωσαν τρεῖς ἀριθμοὶ οἱ E , $B\Gamma$, $\Gamma\Delta$ λόγον μὴ ἔχοντες πρὸς ἀλλήλους, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, ὃ δὲ ΓB πρὸς τὸν $B\Delta$ λόγον ἔχέτω, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, καὶ πεποιήσθω ὡς μὲν ὁ E πρὸς τὸν $B\Gamma$, οὕτως τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς ZH τετράγωνον, ὡς δὲ ὁ $B\Gamma$ πρὸς τὸν $\Gamma\Delta$, οὕτως τὸ ἀπὸ τῆς ZH τετράγωνον πρὸς τὸ ἀπὸ τῆς $H\Theta$. ἐπεὶ οὖν ἐστὶν ὡς ὁ E πρὸς τὸν $B\Gamma$, οὕτως τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς ZH τετράγωνον, σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς A τετράγωνον τῷ ἀπὸ τῆς ZH τετραγώνῳ. ῥητὸν δὲ τὸ ἀπὸ τῆς A τετράγωνον. ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ZH ῥητὴ ἄρα ἐστὶν ἡ ZH . καὶ ἐπεὶ ὁ E πρὸς τὸν $B\Gamma$ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, οὐδ' ἄρα τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς ZH [τετράγωνον] λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. ἀσύμμετρος ἄρα ἐστὶν ἡ A τῇ ZH μήκει. πάλιν, ἐπεὶ ἐστὶν ὡς ὁ $B\Gamma$ πρὸς τὸν $\Gamma\Delta$, οὕτως τὸ ἀπὸ τῆς ZH τετράγωνον πρὸς τὸ ἀπὸ τῆς $H\Theta$, σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ZH τῷ ἀπὸ τῆς $H\Theta$. ῥητὸν δὲ τὸ ἀπὸ τῆς ZH ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς $H\Theta$ ῥητὴ ἄρα ἐστὶν ἡ $H\Theta$. καὶ ἐπεὶ ὁ $B\Gamma$ πρὸς τὸν $\Gamma\Delta$ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, οὐδ' ἄρα τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς $H\Theta$ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. ἀσύμμετρος ἄρα ἐστὶν ἡ ZH τῇ $H\Theta$ μήκει. καὶ εἰσιν ἀμρότεροι ῥηταί· αἱ ZH , $H\Theta$ ἄρα ῥηταί· εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ $Z\Theta$. λέγω δὴ, ὅτι καὶ τρίτη.

Ἐπεὶ γάρ ἐστὶν ὡς μὲν ὁ E πρὸς τὸν $B\Gamma$, οὕτως τὸ ἀπὸ τῆς A τετράγωνον πρὸς τὸ ἀπὸ τῆς ZH , ὡς δὲ ὁ $B\Gamma$ πρὸς τὸν $\Gamma\Delta$, οὕτως τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς $H\Theta$, δι' ἴσου ἄρα ἐστὶν ὡς ὁ E πρὸς τὸν $\Gamma\Delta$, οὕτως τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς $H\Theta$. ὁ δὲ E πρὸς τὸν $\Gamma\Delta$ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. οὐδ' ἄρα τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς $H\Theta$ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. ἀσύμμετρος ἄρα ἡ A τῇ $H\Theta$ μήκει. οὐδετέρα ἄρα τῶν ZH , $H\Theta$ σύμμετρος ἐστὶ τῇ ἐκκεῖμένη ῥητῇ τῇ A μήκει. ὅ οὖν μείζον ἐστὶ τὸ ἀπὸ τῆς ZH τοῦ ἀπὸ τῆς $H\Theta$, ἔστω τὸ ἀπὸ τῆς K . ἐπεὶ οὖν ἐστὶν ὡς ὁ $B\Gamma$ πρὸς τὸν $\Gamma\Delta$, οὕτως τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς $H\Theta$, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ $B\Gamma$ πρὸς τὸν $B\Delta$, οὕτως τὸ ἀπὸ τῆς ZH τετράγωνον πρὸς τὸ ἀπὸ τῆς K . ὁ δὲ $B\Gamma$ πρὸς τὸν $B\Delta$ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν. καὶ τὸ ἀπὸ τῆς ZH ἄρα πρὸς τὸ ἀπὸ τῆς K λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον



Let the rational (straight-line) A be laid down. And let the three numbers, E , BC , and CD , not having to one another the ratio which (some) square number (has) to (some) square number, be laid down. And let CB have to BD the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as E (is) to BC , so the square on A (is) to the square on FG , and as BC (is) to CD , so the square on FG (is) to the (square) on GH [Prop. 10.6 corr.]. Therefore, since as E is to BC , so the square on A (is) to the square on FG , the square on A is thus commensurable with the square on FG [Prop. 10.6]. And the square on A (is) rational. Thus, the (square) on FG (is) also rational. Thus, FG is a rational (straight-line). And since E does not have to BC the ratio which (some) square number (has) to (some) square number, the square on A thus does not have to the [square] on FG the ratio which (some) square number (has) to (some) square number either. Thus, A is incommensurable in length with FG [Prop. 10.9]. Again, since as BC is to CD , so the square on FG is to the (square) on GH , the square on FG is thus commensurable with the (square) on GH [Prop. 10.6]. And the (square) on FG (is) rational. Thus, the (square) on GH (is) also rational. Thus, GH is a rational (straight-line). And since BC does not have to CD the ratio which (some) square number (has) to (some) square number, the (square) on FG thus does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. And both are rational (straight-lines). FG and GH are thus rational (straight-lines which are) commensurable in square only. Thus, FH is an apotome [Prop. 10.73]. So, I say that (it is) also a third (apotome).

For since as E is to BC , so the square on A (is) to the (square) on FG , and as BC (is) to CD , so the (square) on FG (is) to the (square) on HG , thus, via equality, as E is to CD , so the (square) on A (is) to the (square) on HG [Prop. 5.22]. And E does not have to CD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on A does not have to the (square) on GH the ratio which (some) square number (has) to (some) square number either. A (is) thus incommensurable in length with GH [Prop. 10.9]. Thus, neither of FG and GH is commensurable in length with the

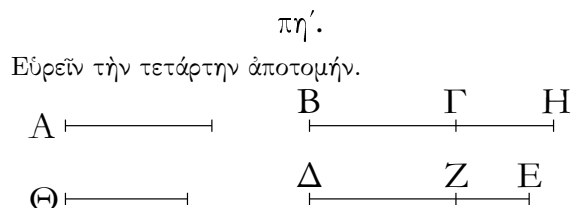
ἀριθμόν. σύμμετρος ἄρα ἐστὶν ἡ ZH τῇ K μήκει, καὶ δύναται ἡ ZH τῆς $H\Theta$ μείζον τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ οὐδετέρα τῶν ZH , $H\Theta$ σύμμετρος ἐστὶ τῇ ἐκκειμένῃ ῥητῇ τῇ A μήκει· ἡ $Z\Theta$ ἄρα ἀποτομή ἐστὶ τρίτη.

Εὕρηται ἄρα ἡ τρίτη ἀποτομή ἡ $Z\Theta$. ὅπερ ἔδει δεῖξαι.

(previously) laid down rational (straight-line) A . Therefore, let the (square) on K be that (area) by which the (square) on FG is greater than the (square) on GH [Prop. 10.13 lem.]. Therefore, since as BC is to CD , so the (square) on FG (is) to the (square) on GH , thus, via conversion, as BC is to BD , so the square on FG (is) to the square on K [Prop. 5.19 corr.]. And BC has to BD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG also has to the (square) on K the ratio which (some) square number (has) to (some) square number. FG is thus commensurable in length with K [Prop. 10.9]. And the square on FG is (thus) greater than (the square on) GH by the (square) on (some straight-line) commensurable (in length) with (FG). And neither of FG and GH is commensurable in length with the (previously) laid down rational (straight-line) A . Thus, FH is a third apotome [Def. 10.13].

Thus, the third apotome FH has been found. (Which is) very thing it was required to show.

† See footnote to Prop. 10.50.

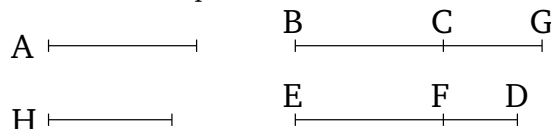


Ἐκκείσθω ῥητὴ ἡ A καὶ τῇ A μήκει σύμμετρος ἡ BH . ῥητὴ ἄρα ἐστὶ καὶ ἡ BH . καὶ ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΔZ , ZE , ὥστε τὸν ΔE ὅλον πρὸς ἑκάτερον τῶν ΔZ , EZ λόγον μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. καὶ πεποιήσθω ὡς ὁ ΔE πρὸς τὸν EZ , οὕτως τὸ ἀπὸ τῆς BH τετράγωνον πρὸς τὸ ἀπὸ τῆς $H\Gamma$. σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς BH τῷ ἀπὸ τῆς $H\Gamma$. ῥητὸν δὲ τὸ ἀπὸ τῆς BH . ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς $H\Gamma$. ῥητὴ ἄρα ἐστὶν ἡ $H\Gamma$. καὶ ἐπεὶ ὁ ΔE πρὸς τὸν EZ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ' ἄρα τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς $H\Gamma$ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν. ἀσύμμετρος ἄρα ἐστὶν ἡ BH τῇ $H\Gamma$ μήκει. καὶ εἰσὶν ἀμφοτέραι ῥηταί· αἱ BH , $H\Gamma$ ἄρα ῥηταί· εἰσι δυνάμει μόνον σύμμετροι· ἀποτομή ἄρα ἐστὶν ἡ $B\Gamma$. [λέγω δὴ, ὅτι καὶ τετάρτη.]

Ὡς οὖν μείζον ἐστὶ τὸ ἀπὸ τῆς BH τοῦ ἀπὸ τῆς $H\Gamma$, ἔστω τὸ ἀπὸ τῆς Θ . ἐπεὶ οὖν ἐστὶν ὡς ὁ ΔE πρὸς τὸν EZ , οὕτως τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς $H\Gamma$, καὶ ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ $E\Delta$ πρὸς τὸν ΔZ , οὕτως τὸ ἀπὸ τῆς $H\Theta$ πρὸς τὸ ἀπὸ τῆς Θ . ὁ δὲ $E\Delta$ πρὸς τὸν ΔZ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον

Proposition 88

To find a fourth apotome.



Let the rational (straight-line) A , and BG (which is) commensurable in length with A , be laid down. Thus, BG is also a rational (straight-line). And let the two numbers DF and FE be laid down such that the whole, DE , does not have to each of DF and EF the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as DE (is) to EF , so the square on BG (is) to the (square) on GC [Prop. 10.6 corr.]. The (square) on BG is thus commensurable with the (square) on GC [Prop. 10.6]. And the (square) on BG (is) rational. Thus, the (square) on GC (is) also rational. Thus, GC (is) a rational (straight-line). And since DE does not have to EF the ratio which (some) square number (has) to (some) square number, the (square) on BG thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with GC [Prop. 10.9]. And they are both rational (straight-lines). Thus, BG and GC are rational (straight-lines which are) commensurable in square only. Thus, BC is an apotome [Prop. 10.73]. [So, I say that (it

ἀριθμόν· οὐδ' ἄρα τὸ ἀπὸ τῆς HB πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ BH τῇ Θ μήκει. καὶ δύναται ἡ BH τῆς $H\Gamma$ μείζον τῷ ἀπὸ τῆς Θ · ἡ ἄρα BH τῆς $H\Gamma$ μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ. καὶ ἐστὶν ὅλη ἡ BH σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ A . ἡ ἄρα $B\Gamma$ ἀποτομή ἐστὶ τετάρτη.

Εὕρηται ἄρα ἡ τετάρτη ἀποτομή· ὅπερ ἔδει δεῖξαι.

is) also a fourth (apotome).]

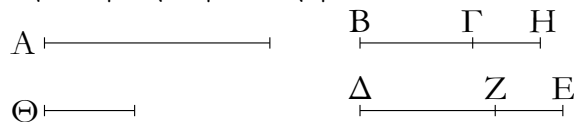
Now, let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC [Prop. 10.13 lem.]. Therefore, since as DE is to EF , so the (square) on BG (is) to the (square) on GC , thus, also, via conversion, as ED is to DF , so the (square) on GB (is) to the (square) on H [Prop. 5.19 corr.]. And ED does not have to DF the ratio which (some) square number (has) to (some) square number. Thus, the (square) on GB does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with H [Prop. 10.9]. And the square on BG is greater than (the square on) GC by the (square) on H . Thus, the square on BG is greater than (the square) on GC by the (square) on (some straight-line) incommensurable (in length) with (BG). And the whole, BG , is commensurable in length with the the (previously) laid down rational (straight-line) A . Thus, BC is a fourth apotome [Def. 10.14].[†]

Thus, a fourth apotome has been found. (Which is) the very thing it was required to show.

[†] See footnote to Prop. 10.51.

πθ'.

Εὕρεῖν τὴν πέμπτην ἀποτομήν.

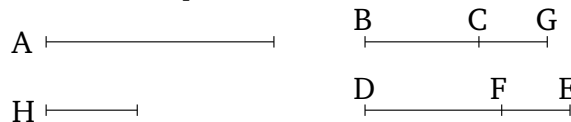


Ἐκκείσθω ῥητὴ ἡ A , καὶ τῇ A μήκει σύμμετρος ἔστω ἡ ΓH · ῥητὴ ἄρα [ἐστὶν] ἡ ΓH . καὶ ἐκκείσθωσαν δύο ἀριθμοὶ οἱ ΔZ , ZE , ὥστε τὸν ΔE πρὸς ἑκάτερον τῶν ΔZ , ZE λόγον πάλιν μὴ ἔχειν, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· καὶ πεποιήσθω ὡς ὁ ZE πρὸς τὸν $E\Delta$, οὕτως τὸ ἀπὸ τῆς ΓH πρὸς τὸ ἀπὸ τῆς HB . ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς HB · ῥητὴ ἄρα ἐστὶ καὶ ἡ BH . καὶ ἐπεὶ ἐστὶν ὡς ὁ ΔE πρὸς τὸν EZ , οὕτως τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς $H\Gamma$, ὁ δὲ ΔE πρὸς τὸν EZ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν, οὐδ' ἄρα τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς $H\Gamma$ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ BH τῇ $H\Gamma$ μήκει. καὶ εἰσὶν ἀμφοτέραι ῥηταί· αἱ BH , $H\Gamma$ ἄρα ῥηταὶ εἰσὶ δύναμει μόνον σύμμετροι· ἡ $B\Gamma$ ἄρα ἀποτομή ἐστὶν. λέγω δὴ, ὅτι καὶ πέμπτη.

Ὅτι γὰρ μείζον ἐστὶ τὸ ἀπὸ τῆς BH τοῦ ἀπὸ τῆς $H\Gamma$, ἔστω τὸ ἀπὸ τῆς Θ . ἐπεὶ οὖν ἐστὶν ὡς τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς $H\Gamma$, οὕτως ὁ ΔE πρὸς τὸν EZ , ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ $E\Delta$ πρὸς τὸν ΔZ , οὕτως τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς Θ , ὁ δὲ $E\Delta$ πρὸς τὸν ΔZ λόγον οὐκ ἔχει, ὃν

Proposition 89

To find a fifth apotome.



Let the rational (straight-line) A be laid down, and let CG be commensurable in length with A . Thus, CG [is] a rational (straight-line). And let the two numbers DF and FE be laid down such that DE again does not have to each of DF and FE the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as FE (is) to ED , so the (square) on CG (is) to the (square) on GB . Thus, the (square) on GB (is) also rational [Prop. 10.6]. Thus, BG is also rational. And since as DE is to EF , so the (square) on BG (is) to the (square) on GC . And DE does not have to EF the ratio which (some) square number (has) to (some) square number. The (square) on BG thus does not have to the (square) on GC the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with GC [Prop. 10.9]. And they are both rational (straight-lines). BG and GC are thus rational (straight-lines which are) commensurable in square only. Thus, BC is an apotome [Prop. 10.73]. So, I say that (it is) also a fifth (apotome).

τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· οὐδ' ἄρα τὸ ἀπὸ τῆς BH πρὸς τὸ ἀπὸ τῆς Θ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἀσύμμετρος ἄρα ἐστὶν ἡ BH τῇ Θ μήκει. καὶ δύναται ἡ BH τῆς $H\Gamma$ μείζον τῷ ἀπὸ τῆς Θ · ἡ HB ἄρα τῆς $H\Gamma$ μείζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει. καὶ ἐστὶν ἡ προσαρμόζουσα ἡ ΓH σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ A μήκει· ἡ ἄρα $B\Gamma$ ἀποτομή ἐστὶ πέμπτῃ.

Εὐρηται ἄρα ἡ πέμπτῃ ἀποτομή ἡ $B\Gamma$ · ὅπερ ἔδει δεῖξαι.

For, let the (square) on H be that (area) by which the (square) on BG is greater than the (square) on GC [Prop. 10.13 lem.]. Therefore, since as the (square) on BG (is) to the (square) on GC , so DE (is) to EF , thus, via conversion, as ED is to DF , so the (square) on BG (is) to the (square) on H [Prop. 5.19 corr.]. And ED does not have to DF the ratio which (some) square number (has) to (some) square number. Thus, the (square) on BG does not have to the (square) on H the ratio which (some) square number (has) to (some) square number either. Thus, BG is incommensurable in length with H [Prop. 10.9]. And the square on BG is greater than (the square on) GC by the (square) on H . Thus, the square on GB is greater than (the square on) GC by the (square) on (some straight-line) incommensurable in length with (GB). And the attachment CG is commensurable in length with the (previously) laid down rational (straight-line) A . Thus, BC is a fifth apotome [Def. 10.15].[†]

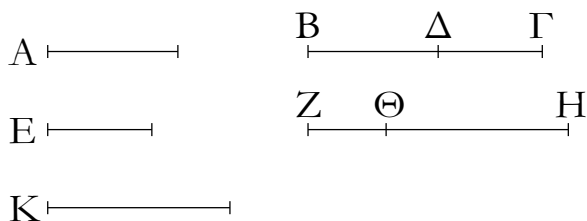
Thus, the fifth apotome BC has been found. (Which is) the very thing it was required to show.

[†] See footnote to Prop. 10.52.

ι'.

Εὐρεῖν τὴν ἕκτην ἀποτομήν.

Ἐκκείσθω ῥητὴ ἡ A καὶ τρεῖς ἀριθμοὶ οἱ E , $B\Gamma$, $\Gamma\Delta$ λόγον μὴ ἔχοντες πρὸς ἀλλήλους, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἔτι δὲ καὶ ὁ ΓB πρὸς τὸν $B\Delta$ λόγον μὴ ἔχετώ, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· καὶ πεποιήσθω ὡς μὲν ὁ E πρὸς τὸν $B\Gamma$, οὕτως τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς ZH , ὡς δὲ ὁ $B\Gamma$ πρὸς τὸν $\Gamma\Delta$, οὕτως τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς $H\Theta$.

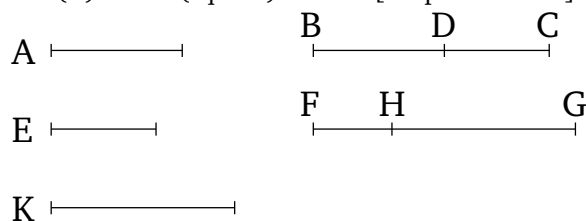


Ἐπεὶ οὖν ἐστὶν ὡς ὁ E πρὸς τὸν $B\Gamma$, οὕτως τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς ZH , σύμμετρον ἄρα τὸ ἀπὸ τῆς A τῷ ἀπὸ τῆς ZH . ῥητὸν δὲ τὸ ἀπὸ τῆς A · ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς ZH · ῥητὴ ἄρα ἐστὶ καὶ ἡ ZH . καὶ ἐπεὶ ὁ E πρὸς τὸν $B\Gamma$ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, οὐδ' ἄρα τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς ZH λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἀσύμμετρος ἄρα ἐστὶν ἡ A τῇ ZH μήκει. πάλιν, ἐπεὶ ἐστὶν ὡς ὁ $B\Gamma$ πρὸς τὸν $\Gamma\Delta$, οὕτως τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς $H\Theta$, σύμμετρον ἄρα τὸ ἀπὸ τῆς ZH τῷ ἀπὸ τῆς $H\Theta$. ῥητὸν

Proposition 90

To find a sixth apotome.

Let the rational (straight-line) A , and the three numbers E , BC , and CD , not having to one another the ratio which (some) square number (has) to (some) square number, be laid down. Furthermore, let CB also not have to BD the ratio which (some) square number (has) to (some) square number. And let it have been contrived that as E (is) to BC , so the (square) on A (is) to the (square) on FG , and as BC (is) to CD , so the (square) on FG (is) to the (square) on GH [Prop. 10.6 corr.].



Therefore, since as E is to BC , so the (square) on A (is) to the (square) on FG , the (square) on A (is) thus commensurable with the (square) on FG [Prop. 10.6]. And the (square) on A (is) rational. Thus, the (square) on FG (is) also rational. Thus, FG is also a rational (straight-line). And since E does not have to BC the ratio which (some) square number (has) to (some) square number, the (square) on A thus does not have to the (square) on FG the ratio which (some) square number (has) to (some) square number either. Thus, A is in-

δὲ τὸ ἀπὸ τῆς ZH · ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς $H\Theta$ · ῥητὴ ἄρα καὶ ἡ $H\Theta$. καὶ ἐπεὶ ὁ $B\Gamma$ πρὸς τὸν $\Gamma\Delta$ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, οὐδ' ἄρα τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς $H\Theta$ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἀσύμμετρος ἄρα ἐστὶν ἡ ZH τῇ $H\Theta$ μήκει. καὶ εἰσιν ἀμφοτέραι ῥηταί· αἱ ZH , $H\Theta$ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἡ ἄρα $Z\Theta$ ἀποτομή ἐστίν. λέγω δὴ, ὅτι καὶ ἔκτῃ.

Ἐπεὶ γάρ ἐστιν ὡς μὲν ὁ E πρὸς τὸν $B\Gamma$, οὕτως τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς ZH , ὡς δὲ ὁ $B\Gamma$ πρὸς τὸν $\Gamma\Delta$, οὕτως τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς $H\Theta$, δι' ἴσου ἄρα ἐστὶν ὡς ὁ E πρὸς τὸν $\Gamma\Delta$, οὕτως τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς $H\Theta$. ὁ δὲ E πρὸς τὸν $\Gamma\Delta$ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· οὐδ' ἄρα τὸ ἀπὸ τῆς A πρὸς τὸ ἀπὸ τῆς $H\Theta$ λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἀσύμμετρος ἄρα ἐστὶν ἡ A τῇ $H\Theta$ μήκει· οὐδετέρα ἄρα τῶν ZH , $H\Theta$ σύμμετρος ἐστὶ τῇ A ῥητῇ μήκει. ὥ οὖν μείζον ἐστὶ τὸ ἀπὸ τῆς ZH τοῦ ἀπὸ τῆς $H\Theta$, ἔστω τὸ ἀπὸ τῆς K . ἐπεὶ οὖν ἐστὶν ὡς ὁ $B\Gamma$ πρὸς τὸν $\Gamma\Delta$, οὕτως τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς $H\Theta$, ἀναστρέψαντι ἄρα ἐστὶν ὡς ὁ ΓB πρὸς τὸν $B\Delta$, οὕτως τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς K . ὁ δὲ ΓB πρὸς τὸν $B\Delta$ λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· οὐδ' ἄρα τὸ ἀπὸ τῆς ZH πρὸς τὸ ἀπὸ τῆς K λόγον ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν· ἀσύμμετρος ἄρα ἐστὶν ἡ ZH τῇ K μήκει. καὶ δύναται ἡ ZH τῆς $H\Theta$ μείζον τῷ ἀπὸ τῆς K · ἡ ZH ἄρα τῆς $H\Theta$ μείζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ μήκει. καὶ οὐδετέρα τῶν ZH , $H\Theta$ σύμμετρος ἐστὶ τῇ ἑκκαμμένῃ ῥητῇ μήκει τῇ A . ἡ ἄρα $Z\Theta$ ἀποτομή ἐστὶν ἔκτῃ.

Εὐρηταί ἄρα ἡ ἕκτῃ ἀποτομή ἡ $Z\Theta$ · ὅπερ ἔδει δεῖξαι.

commensurable in length with FG [Prop. 10.9]. Again, since as BC is to CD , so the (square) on FG (is) to the (square) on GH , the (square) on FG (is) thus commensurable with the (square) on GH [Prop. 10.6]. And the (square) on FG (is) rational. Thus, the (square) on GH (is) also rational. Thus, GH (is) also rational. And since BC does not have to CD the ratio which (some) square number (has) to (some) square number, the (square) on FG thus does not have to the (square) on GH the ratio which (some) square (number) has to (some) square (number) either. Thus, FG is incommensurable in length with GH [Prop. 10.9]. And both are rational (straight-lines). Thus, FG and GH are rational (straight-lines which are) commensurable in square only. Thus, FH is an apotome [Prop. 10.73]. So, I say that (it is) also a sixth (apotome).

For since as E is to BC , so the (square) on A (is) to the (square) on FG , and as BC (is) to CD , so the (square) on FG (is) to the (square) on GH , thus, via equality, as E is to CD , so the (square) on A (is) to the (square) on GH [Prop. 5.22]. And E does not have to CD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on A does not have to the (square) GH the ratio which (some) square number (has) to (some) square number either. A is thus incommensurable in length with GH [Prop. 10.9]. Thus, neither of FG and GH is commensurable in length with the rational (straight-line) A . Therefore, let the (square) on K be that (area) by which the (square) on FG is greater than the (square) on GH [Prop. 10.13 lem.]. Therefore, since as BC is to CD , so the (square) on FG (is) to the (square) on GH , thus, via conversion, as CB is to BD , so the (square) on FG (is) to the (square) on K [Prop. 5.19 corr.]. And CB does not have to BD the ratio which (some) square number (has) to (some) square number. Thus, the (square) on FG does not have to the (square) on K the ratio which (some) square number (has) to (some) square number either. FG is thus incommensurable in length with K [Prop. 10.9]. And the square on FG is greater than (the square on) GH by the (square) on K . Thus, the square on FG is greater than (the square on) GH by the (square) on (some straight-line) incommensurable in length with (FG). And neither of FG and GH is commensurable in length with the (previously) laid down rational (straight-line) A . Thus, FH is a sixth apotome [Def. 10.16].

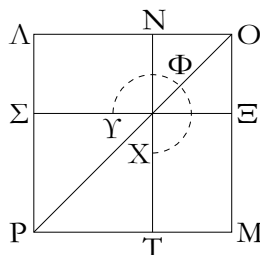
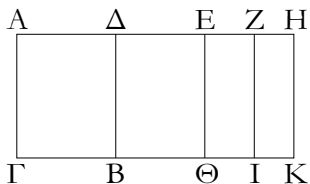
Thus, the sixth apotome FH has been found. (Which is) the very thing it was required to show.

† See footnote to Prop. 10.53.

ἡ α'.

Ἐάν χωρίον περιέχεται ὑπὸ ῥητῆς καὶ ἀποτομῆς πρώτης, ἡ τὸ χωρίον δυναμένη ἀπορομή ἐστίν.

Περιεχέσθω γὰρ χωρίον τὸ AB ὑπὸ ῥητῆς τῆς AG καὶ ἀποτομῆς πρώτης τῆς AD· λέγω, ὅτι ἡ τὸ AB χωρίον δυναμένη ἀποτομή ἐστίν.



Ἐπεὶ γὰρ ἀποτομή ἐστὶ πρώτη ἡ AD, ἔστω αὐτῇ προσαρμόζουσα ἡ ΔΗ· αἱ AH, ΗΔ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· καὶ ὅλη ἡ AH σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ AG, καὶ ἡ AH τῆς ΗΔ μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει· ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν AH παραβληθῇ ἐλλείπον εἶδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διαιρεῖ. τετμήσθω ἡ ΔΗ δίχα κατὰ τὸ Ε, καὶ τῷ ἀπὸ τῆς EH ἴσον παρὰ τὴν AH παραβεβλήσθω ἐλλείπον εἶδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν AZ, ZH· σύμμετρος ἄρα ἐστὶν ἡ AZ τῇ ZH. καὶ διὰ τῶν Ε, Ζ, Η σημείων τῇ AG παράλληλοι ἦχθωσαν αἱ ΕΘ, ΖΙ, ΗΚ.

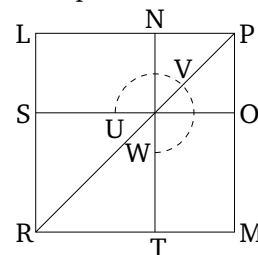
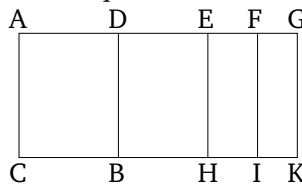
Καὶ ἐπεὶ σύμμετρός ἐστιν ἡ AZ τῇ ZH μήκει, καὶ ἡ AH ἄρα ἑκατέρᾳ τῶν AZ, ZH σύμμετρός ἐστι μήκει· ἀλλὰ ἡ AH σύμμετρός ἐστι τῇ AG· καὶ ἑκατέρα ἄρα τῶν AZ, ZH σύμμετρός ἐστι τῇ AG μήκει· καὶ ἐστὶ ῥητὴ ἡ AG· ῥητὴ ἄρα καὶ ἑκατέρα τῶν AZ, ZH· ὥστε καὶ ἑκάτερον τῶν AI, ZK ῥητόν ἐστιν· καὶ ἐπεὶ σύμμετρός ἐστιν ἡ ΔΕ τῇ EH μήκει, καὶ ἡ ΔΗ ἄρα ἑκατέρᾳ τῶν ΔΕ, EH σύμμετρός ἐστι μήκει· ῥητὴ δὲ ἡ ΔΗ καὶ ἀσύμμετρος τῇ AG μήκει· ῥητὴ ἄρα καὶ ἑκατέρα τῶν ΔΕ, EH καὶ ἀσύμμετρος τῇ AG μήκει· ἑκάτερον ἄρα τῶν ΔΘ, ΕΚ μέσον ἐστίν.

Κείσθω δὴ τῷ μὲν AI ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ZK ἴσον τετράγωνον ἀφηρήσθω κοινὴν γωνίαν ἔχον αὐτῷ τὴν ὑπὸ ΛΟΜ τὸ ΝΞ· περὶ τὴν αὐτὴν ἄρα διάμετρον ἐστὶ τὰ ΛΜ, ΝΞ τετράγωνα· ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα· ἐπεὶ οὖν ἴσον ἐστὶ τὸ ὑπὸ τῶν AZ, ZH περιεχόμενον ὀρθογώνιον τῷ ἀπὸ τῆς EH τετραγώνῳ, ἔστιν ἄρα ὡς ἡ AZ πρὸς τὴν EH, οὕτως ἡ EH πρὸς τὴν ZH· ἀλλ' ὡς μὲν ἡ AZ πρὸς τὴν EH, οὕτως τὸ AI πρὸς τὸ ΕΚ, ὡς δὲ ἡ EH πρὸς τὴν ZH, οὕτως ἐστὶ τὸ ΕΚ πρὸς τὸ ΚΖ· τῶν ἄρα AI, ΚΖ μέσον ἀνάλογόν ἐστι τὸ ΕΚ· ἐστὶ δὲ καὶ τῶν ΛΜ, ΝΞ μέσον ἀνάλογόν τὸ ΜΝ, ὡς ἐν τοῖς ἔμπροσθεν ἐδείχθη, καὶ ἐστὶ τὸ [μὲν] AI τῷ ΛΜ τετραγώνῳ ἴσον, τὸ δὲ ΚΖ τῷ ΝΞ· καὶ τὸ ΜΝ ἄρα τῷ ΕΚ ἴσον ἐστίν· ἀλλὰ τὸ μὲν ΕΚ τῷ ΔΘ ἐστὶν ἴσον, τὸ δὲ ΜΝ τῷ ΛΞ· τὸ ἄρα

Proposition 91

If an area is contained by a rational (straight-line) and a first apotome then the square-root of the area is an apotome.

For let the area AB have been contained by the rational (straight-line) AC and the first apotome AD. I say that the square-root of area AB is an apotome.



For since AD is a first apotome, let DG be its attachment. Thus, AG and DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And the whole, AG, is commensurable (in length) with the (previously) laid down rational (straight-line) AC, and the square on AG is greater than (the square on) GD by the (square) on (some straight-line) commensurable in length with (AG) [Def. 10.11]. Thus, if (an area) equal to the fourth part of the (square) on DG is applied to AG, falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17]. Let DG have been cut in half at E. And let (an area) equal to the (square) on EG have been applied to AG, falling short by a square figure. And let it be the (rectangle contained) by AF and FG. AF is thus commensurable (in length) with FG. And let EH, FI, and GK have been drawn through points E, F, and G (respectively), parallel to AC.

And since AF is commensurable in length with FG, AG is thus also commensurable in length with each of AF and FG [Prop. 10.15]. But AG is commensurable (in length) with AC. Thus, each of AF and FG is also commensurable in length with AC [Prop. 10.12]. And AC is a rational (straight-line). Thus, AF and FG (are) each also rational (straight-lines). Hence, AI and FK are also each rational (areas) [Prop. 10.19]. And since DE is commensurable in length with EG, DG is thus also commensurable in length with each of DE and EG [Prop. 10.15]. And DG (is) rational, and incommensurable in length with AC. DE and EG (are) thus each rational, and incommensurable in length with AC [Prop. 10.13]. Thus, DH and EK are each medial (areas) [Prop. 10.21].

So let the square LM, equal to AI, be laid down. And let the square NO, equal to FK, have been sub-

ΔK ἴσον ἐστὶ τῷ $\Upsilon \Phi X$ γνῶμονι καὶ τῷ $N\Xi$. ἔστι δὲ καὶ τὸ AK ἴσον τοῖς ΛM , $N\Xi$ τετραγώνοις· λοιπὸν ἄρα τὸ AB ἴσον ἐστὶ τῷ ΣT . τὸ δὲ ΣT τὸ ἀπὸ τῆς ΛN ἐστὶ τετράγωνον· τὸ ἄρα ἀπὸ τῆς ΛN τετράγωνον ἴσον ἐστὶ τῷ AB · ἡ ΛN ἄρα δύναται τὸ AB . λέγω δὴ, ὅτι ἡ ΛN ἀποτομή ἐστίν.

Ἐπεὶ γὰρ ῥητόν ἐστιν ἑκάτερον τῶν AI , ZK , καὶ ἐστὶν ἴσον τοῖς ΛM , $N\Xi$, καὶ ἑκάτερον ἄρα τῶν ΛM , $N\Xi$ ῥητόν ἐστιν, τουτέστι τὸ ἀπὸ ἑκατέρας τῶν ΛO , ON · καὶ ἑκατέρα ἄρα τῶν ΛO , ON ῥητὴ ἐστίν. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ $\Delta\Theta$ καὶ ἐστὶν ἴσον τῷ $\Lambda\Xi$, μέσον ἄρα ἐστὶ καὶ τὸ $\Lambda\Xi$. ἐπεὶ οὖν τὸ μὲν $\Lambda\Xi$ μέσον ἐστίν, τὸ δὲ $N\Xi$ ῥητόν, ἀσύμμετρον ἄρα ἐστὶ τὸ $\Lambda\Xi$ τῷ $N\Xi$ · ὥς δὲ τὸ $\Lambda\Xi$ πρὸς τὸ $N\Xi$, οὕτως ἐστὶν ἡ ΛO πρὸς τὴν ON · ἀσύμμετρος ἄρα ἐστὶν ἡ ΛO τῇ ON μήκει. καὶ εἰσιν ἀμφοτέραι ῥηταί· αἱ ΛO , ON ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἀποτομή ἄρα ἐστὶν ἡ ΛN . καὶ δύναται τὸ AB χωρίον· ἡ ἄρα τὸ AB χωρίον δυναμένη ἀποτομή ἐστίν.

Ἐὰν ἄρα χωρίον περιέχεται ὑπὸ ῥητῆς καὶ τὰ ἐξῆς.

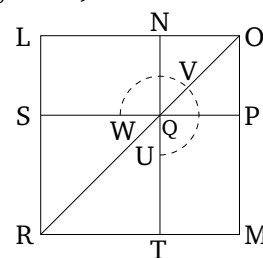
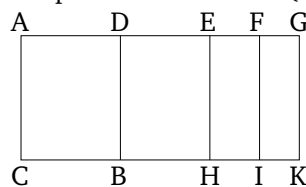
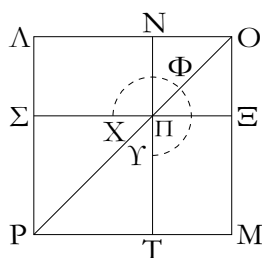
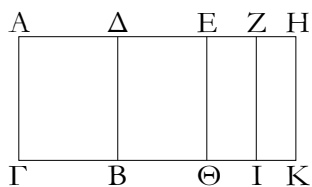
tracted (from LM), having with it the common angle LPM . Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since the rectangle contained by AF and FG is equal to the square EG , thus as AF is to EG , so EG (is) to FG [Prop. 6.17]. But, as AF (is) to EG , so AI (is) to EK , and as EG (is) to FG , so EK is to KF [Prop. 6.1]. Thus, EK is the mean proportional to AI and KF [Prop. 5.11]. And MN is also the mean proportional to LM and NO , as shown before [Prop. 10.53 lem.]. And AI is equal to the square LM , and KF to NO . Thus, MN is also equal to EK . But, EK is equal to DH , and MN to LO [Prop. 1.43]. Thus, DK is equal to the gnomon UVW and NO . And AK is also equal to (the sum of) the squares LM and NO . Thus, the remainder AB is equal to ST . And ST is the square on LN . Thus, the square on LN is equal to AB . Thus, LN is the square-root of AB . So, I say that LN is an apotome.

For since AI and FK are each rational (areas), and are equal to LM and NO (respectively), thus LM and NO —that is to say, the (squares) on each of LP and PN (respectively)—are also each rational (areas). Thus, LP and PN are also each rational (straight-lines). Again, since DH is a medial (area), and is equal to LO , LO is thus also a medial (area). Therefore, since LO is medial, and NO rational, LO is thus incommensurable with NO . And as LO (is) to NO , so LP is to PN [Prop. 6.1]. LP is thus incommensurable in length with PN [Prop. 10.11]. And they are both rational (straight-lines). Thus, LP and PN are rational (straight-lines which are) commensurable in square only. Thus, LN is an apotome [Prop. 10.73]. And it is the square-root of area AB . Thus, the square-root of area AB is an apotome.

Thus, if an area is contained by a rational (straight-line), and so on . . .

ιβ'.

Ἐὰν χωρίον περιέχεται ὑπὸ ῥητῆς καὶ ἀποτομῆς δευτέρας, ἡ τὸ χωρίον δυναμένη μέσης ἀποτομή ἐστὶ πρώτη.



Proposition 92

If an area is contained by a rational (straight-line) and a second apotome then the square-root of the area is a first apotome of a medial (straight-line).

Χωρίον γὰρ τὸ AB περιεχέσθω ὑπὸ ῥητῆς τῆς AG καὶ ἀποτομῆς δευτέρας τῆς AD . λέγω, ὅτι ἡ τὸ AB χωρίον δυναμένη μέσης ἀποτομῆ ἐστὶ πρώτη.

Ἐστω γὰρ τῇ AD προσαρμόζουσα ἡ ΔH . αἱ ἄρα AH , $H\Delta$ ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ προσαρμόζουσα ἡ ΔH σύμμετρός ἐστι τῇ ἐκκειμένη ῥητῇ τῇ AG , ἡ δὲ ὅλη ἡ AH τῆς προσαρμοζούσης τῆς $H\Delta$ μείζον δύναται τῷ ἀπὸ συμέτρου ἑαυτῇ μήκει. ἐπεὶ οὖν ἡ AH τῆς $H\Delta$ μείζον δύναται τῷ ἀπὸ συμέτρου ἑαυτῇ, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς $H\Delta$ ἴσον παρὰ τὴν AH παραβληθῇ ἐλλείπον εἶδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διαιρεῖ. τετμηθῶ οὖν ἡ ΔH δίχα κατὰ τὸ E . καὶ τῷ ἀπὸ τῆς EH ἴσον παρὰ τὴν AH παραβεβλήσθω ἐλλείπον εἶδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν AZ , ZH . σύμμετρος ἄρα ἐστὶν ἡ AZ τῇ ZH μήκει. καὶ ἡ AH ἄρα ἑκατέρᾳ τῶν AZ , ZH σύμμετρός ἐστι μήκει. ῥητὴ δὲ ἡ AH καὶ ἀσύμμετρος τῇ AG μήκει. καὶ ἑκατέρα ἄρα τῶν AZ , ZH ῥητὴ ἐστὶ καὶ ἀσύμμετρος τῇ AG μήκει. ἑκάτερον ἄρα τῶν AI , ZK μέσον ἐστίν. πάλιν, ἐπεὶ σύμμετρός ἐστὶν ἡ ΔE τῇ EH , καὶ ἡ ΔH ἄρα ἑκατέρᾳ τῶν ΔE , EH σύμμετρός ἐστίν. ἀλλ' ἡ ΔH σύμμετρός ἐστι τῇ AG μήκει [ῥητὴ ἄρα καὶ ἑκατέρα τῶν ΔE , EH καὶ σύμμετρος τῇ AG μήκει]. ἑκάτερον ἄρα τῶν $\Delta\Theta$, EK ῥητόν ἐστιν.

Συνεστάτω οὖν τῷ μὲν AI ἴσον τετράγωνον τὸ AM , τῷ δὲ ZK ἴσον ἀφηρήσθω τὸ $N\Xi$ περὶ τὴν αὐτὴν γωνίαν ὅν τῷ AM τὴν ὑπὸ τῶν LOM . περὶ τὴν αὐτὴν ἄρα ἐστὶ διάμετρον τὰ AM , $N\Xi$ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ OP , καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν τὰ AI , ZK μέσα ἐστὶ καὶ ἐστὶν ἴσα τοῖς ἀπὸ τῶν LO , ON , καὶ τὰ ἀπὸ τῶν LO , ON [ἄρα] μέσα ἐστίν. καὶ αἱ LO , ON ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. καὶ ἐπεὶ τὸ ὑπὸ τῶν AZ , ZH ἴσον ἐστὶ τῷ ἀπὸ τῆς EH , ἐστὶν ἄρα ὡς ἡ AZ πρὸς τὴν EH , οὕτως ἡ EH πρὸς τὴν ZH . ἀλλ' ὡς μὲν ἡ AZ πρὸς τὴν EH , οὕτως τὸ AI πρὸς τὸ EK . ὡς δὲ ἡ EH πρὸς τὴν ZH , οὕτως [ἐστὶ] τὸ EK πρὸς τὸ ZK . τῶν ἄρα AI , ZK μέσον ἀνάλογόν ἐστι τὸ EK . ἐστὶ δὲ καὶ τῶν AM , $N\Xi$ τετραγώνων μέσον ἀνάλογον τὸ MN . καὶ ἐστὶν ἴσον τὸ μὲν AI τῷ AM , τὸ δὲ ZK τῷ $N\Xi$. καὶ τὸ MN ἄρα ἴσον ἐστὶ τῷ EK . ἀλλὰ τῷ μὲν EK ἴσον [ἐστὶ] τὸ $\Delta\Theta$, τῷ δὲ MN ἴσον τὸ $\Lambda\Xi$. ὅλον ἄρα τὸ ΔK ἴσον ἐστὶ τῷ $\Upsilon\Phi X$ γινώμονι καὶ τῷ $N\Xi$. ἐπεὶ οὖν ὅλον τὸ AK ἴσον ἐστὶ τοῖς AM , $N\Xi$, ὣν τὸ ΔK ἴσον ἐστὶ τῷ $\Upsilon\Phi X$ γινώμονι καὶ τῷ $N\Xi$, λοιπὸν ἄρα τὸ AB ἴσον ἐστὶ τῷ $T\Sigma$. τὸ δὲ $T\Sigma$ ἐστὶ τὸ ἀπὸ τῆς ΛN . τὸ ἀπὸ τῆς ΛN ἄρα ἴσον ἐστὶ τῷ AB χωρίῳ. ἡ ΛN ἄρα δύναται τὸ AB χωρίον. λέγω [δή], ὅτι ἡ ΛN μέσης ἀποτομῆ ἐστὶ πρώτη.

Ἐπεὶ γὰρ ῥητόν ἐστι τὸ EK καὶ ἐστὶν ἴσον τῷ $\Lambda\Xi$, ῥητόν ἄρα ἐστὶ τὸ $\Lambda\Xi$, τουτέστι τὸ ὑπὸ τῶν LO , ON . μέσον δὲ ἐδείχθη τὸ $N\Xi$. ἀσύμμετρον ἄρα ἐστὶ τὸ $\Lambda\Xi$ τῷ $N\Xi$. ὡς δὲ τὸ $\Lambda\Xi$ πρὸς τὸ $N\Xi$, οὕτως ἐστὶν ἡ LO πρὸς ON . αἱ LO , ON ἄρα ἀσύμμετροί εἰσι μήκει. αἱ ἄρα LO , ON μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητόν περιέχουσιν. ἡ ΛN ἄρα

For let the area AB have been contained by the rational (straight-line) AC and the second apotome AD . I say that the square-root of area AB is the first apotome of a medial (straight-line).

For let DG be an attachment to AD . Thus, AG and GD are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and the attachment DG is commensurable (in length) with the (previously) laid down rational (straight-line) AC , and the square on the whole, AG , is greater than (the square on) the attachment, GD , by the (square) on (some straight-line) commensurable in length with (AG) [Def. 10.12]. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) commensurable (in length) with (AG) , thus if (an area) equal to the fourth part of the (square) on GD is applied to AG , falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17]. Therefore, let DG have been cut in half at E . And let (an area) equal to the (square) on EG have been applied to AG , falling short by a square figure. And let it be the (rectangle contained) by AF and FG . Thus, AF is commensurable in length with FG . AG is thus also commensurable in length with each of AF and FG [Prop. 10.15]. And AG (is) a rational (straight-line), and incommensurable in length with AC . AF and FG are thus also each rational (straight-lines), and incommensurable in length with AC [Prop. 10.13]. Thus, AI and FK are each medial (areas) [Prop. 10.21]. Again, since DE is commensurable (in length) with EG , thus DG is also commensurable (in length) with each of DE and EG [Prop. 10.15]. But, DG is commensurable in length with AC [thus, DE and EG are also each rational, and commensurable in length with AC]. Thus, DH and EK are each rational (areas) [Prop. 10.19].

Therefore, let the square LM , equal to AI , have been constructed. And let NO , equal to FK , which is about the same angle LPM as LM , have been subtracted (from LM). Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since AI and FK are medial (areas), and are equal to the (squares) on LP and PN (respectively), [thus] the (squares) on LP and PN are also medial. Thus, LP and PN are also medial (straight-lines which are) commensurable in square only.[†] And since the (rectangle contained) by AF and FG is equal to the (square) on EG , thus as AF is to EG , so EG (is) to FG [Prop. 10.17]. But, as AF (is) to EG , so AI (is) to EK . And as EG (is) to FG , so EK [is] to FK [Prop. 6.1]. Thus, EK is the mean proportional to AI

μέσης ἀποτομή ἐστὶ πρώτη καὶ δύναται τὸ AB χωρίον.

Ἡ ἄρα τὸ AB χωρίον δυναμένη μέσης ἀποτομή ἐστὶ πρώτη· ὅπερ ἔδει δεῖξαι.

and FK [Prop. 5.11]. And MN is also the mean proportional to the squares LM and NO [Prop. 10.53 lem.]. And AI is equal to LM , and FK to NO . Thus, MN is also equal to EK . But, DH [is] equal to EK , and LO equal to MN [Prop. 1.43]. Thus, the whole (of) DK is equal to the gnomon UVW and NO . Therefore, since the whole (of) AK is equal to LM and NO , of which DK is equal to the gnomon UVW and NO , the remainder AB is thus equal to TS . And TS is the (square) on LN . Thus, the (square) on LN is equal to the area AB . LN is thus the square-root of area AB . [So], I say that LN is the first apotome of a medial (straight-line).

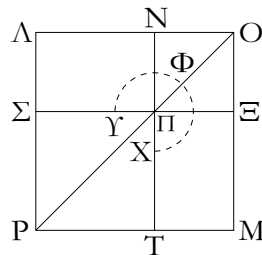
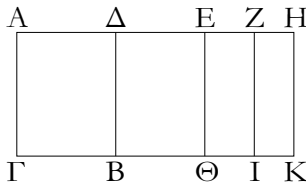
For since EK is a rational (area), and is equal to LO , LO —that is to say, the (rectangle contained) by LP and PN —is thus a rational (area). And NO was shown (to be) a medial (area). Thus, LO is incommensurable with NO . And as LO (is) to NO , so LP is to PN [Prop. 6.1]. Thus, LP and PN are incommensurable in length [Prop. 10.11]. LP and PN are thus medial (straight-lines which are) commensurable in square only, and which contain a rational (area). Thus, LN is the first apotome of a medial (straight-line) [Prop. 10.74]. And it is the square-root of area AB .

Thus, the square root of area AB is the first apotome of a medial (straight-line). (Which is) the very thing it was required to show.

† There is an error in the argument here. It should just say that LP and PN are commensurable in square, rather than in square only, since LP and PN are only shown to be incommensurable in length later on.

ιγ'.

Ἐὰν χωρίον περιέχεται ὑπὸ ῥητῆς καὶ ἀποτομῆς τρίτης, ἢ τὸ χωρίον δυναμένη μέσης ἀποτομή ἐστὶ δευτέρα.

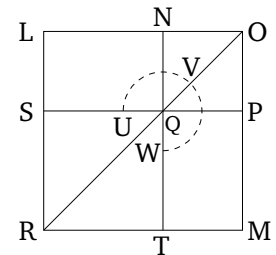
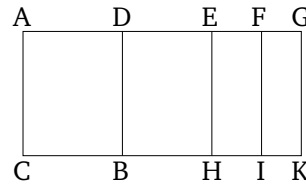


Χωρίον γὰρ τὸ AB περιεχέσθω ὑπὸ ῥητῆς τῆς $ΑΓ$ καὶ ἀποτομῆς τρίτης τῆς $ΑΔ$ · λέγω, ὅτι ἡ τὸ AB χωρίον δυναμένη μέσης ἀποτομή ἐστὶ δευτέρα.

Ἐστω γὰρ τῇ $ΑΔ$ προσαρμόζουσα ἡ $ΔΗ$ · αἱ $ΑΗ$, $ΗΔ$ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα τῶν $ΑΗ$, $ΗΔ$ σύμμετρός ἐστι μήκει τῇ ἐκκειμένη ῥητῇ τῇ $ΑΓ$, ἡ δὲ ὅλη ἡ $ΑΗ$ τῆς προσαρμόζουσας τῆς $ΔΗ$ μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ. ἐπεὶ οὖν ἡ $ΑΗ$ τῆς $ΗΔ$ μείζον

Proposition 93

If an area is contained by a rational (straight-line) and a third apotome then the square-root of the area is a second apotome of a medial (straight-line).



For let the area AB have been contained by the rational (straight-line) AC and the third apotome AD . I say that the square-root of area AB is the second apotome of a medial (straight-line).

For let DG be an attachment to AD . Thus, AG and GD are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and neither of AG and GD is commensurable in length with the (previ-

δύναται τῷ ἀπὸ συμμετρου ἐαυτῇ, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλείπον εἶδει τετραγώνῳ, εἰς σύμμετρα αὐτὴν διελεί. τετμήσθω οὖν ἡ ΔΗ δίχα κατὰ τὸ Ε, καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβελήσθω ἐλλείπον εἶδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ. καὶ ἤχθωσαν διὰ τῶν Ε, Ζ, Η σημείων τῇ ΑΓ παράλληλοι αἱ ΕΘ, ΖΙ, ΗΚ· σύμμετροι ἄρα εἰσὶν αἱ ΑΖ, ΖΗ· σύμμετρον ἄρα καὶ τὸ ΑΙ τῷ ΖΚ. καὶ ἐπεὶ αἱ ΑΖ, ΖΗ σύμμετροί εἰσι μήκει, καὶ ἡ ΑΗ ἄρα ἐκατέρᾳ τῶν ΑΖ, ΖΗ σύμμετρός ἐστι μήκει. ῥητὴ δὲ ἡ ΑΗ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ὥστε καὶ αἱ ΑΖ, ΖΗ. ἐκάτερον ἄρα τῶν ΑΙ, ΖΚ μέσον ἐστίν. πάλιν, ἐπεὶ σύμμετρός ἐστιν ἡ ΔΕ τῇ ΕΗ μήκει, καὶ ἡ ΔΗ ἄρα ἐκατέρᾳ τῶν ΔΕ, ΕΗ σύμμετρός ἐστι μήκει. ῥητὴ δὲ ἡ ΗΔ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ῥητὴ ἄρα καὶ ἐκατέρᾳ τῶν ΔΕ, ΕΗ καὶ ἀσύμμετρος τῇ ΑΓ μήκει· ἐκάτερον ἄρα τῶν ΔΘ, ΕΚ μέσον ἐστίν. καὶ ἐπεὶ αἱ ΑΗ, ΗΔ δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα ἐστὶ μήκει ἡ ΑΗ τῇ ΗΔ. ἀλλ' ἡ μὲν ΑΗ τῇ ΑΖ σύμμετρός ἐστι μήκει ἡ δὲ ΔΗ τῇ ΕΗ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΖ τῇ ΕΗ μήκει. ὥς δὲ ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἐστὶ τὸ ΑΙ πρὸς τὸ ΕΚ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΑΙ τῷ ΕΚ.

Συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΑΜ, τῷ δὲ ΖΚ ἴσον ἀφῆρήσθω τὸ ΝΞ περὶ τὴν αὐτὴν γωνίαν ὅν τῷ ΑΜ· περὶ τὴν αὐτὴν ἄρα διάμετρόν ἐστι τὰ ΑΜ, ΝΞ. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν τὸ ὑπὸ τῶν ΑΖ, ΖΗ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΗ, ἔστιν ἄρα ὡς ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἡ ΕΗ πρὸς τὴν ΖΗ. ἀλλ' ὡς μὲν ἡ ΑΖ πρὸς τὴν ΕΗ, οὕτως ἐστὶ τὸ ΑΙ πρὸς τὸ ΕΚ· ὡς δὲ ἡ ΕΗ πρὸς τὴν ΖΗ, οὕτως ἐστὶ τὸ ΕΚ πρὸς τὸ ΖΚ· καὶ ὡς ἄρα τὸ ΑΙ πρὸς τὸ ΕΚ, οὕτως τὸ ΕΚ πρὸς τὸ ΖΚ· τῶν ἄρα ΑΙ, ΖΚ μέσον ἀνάλογόν ἐστι τὸ ΕΚ. ἔστι δὲ καὶ τῶν ΑΜ, ΝΞ τετραγώνων μέσον ἀνάλογον τὸ ΜΝ· καὶ ἐστὶν ἴσον τὸ μὲν ΑΙ τῷ ΑΜ, τὸ δὲ ΖΚ τῷ ΝΞ· καὶ τὸ ΕΚ ἄρα ἴσον ἐστὶ τῷ ΜΝ. ἀλλὰ τὸ μὲν ΜΝ ἴσον ἐστὶ τῷ ΑΞ, τὸ δὲ ΕΚ ἴσον [ἐστὶ] τῷ ΔΘ· καὶ ὅλον ἄρα τὸ ΔΚ ἴσον ἐστὶ τῷ ΥΦΧ γνῶμονι καὶ τῷ ΝΞ. ἔστι δὲ καὶ τὸ ΑΚ ἴσον τοῖς ΑΜ, ΝΞ· λοιπὸν ἄρα τὸ ΑΒ ἴσον ἐστὶ τῷ ΣΤ, τουτέστι τῷ ἀπὸ τῆς ΑΝ τετραγώνῳ· ἡ ΑΝ ἄρα δύναται τὸ ΑΒ χωρίον. λέγω, ὅτι ἡ ΑΝ μέσης ἀποτομῇ ἐστὶ δευτέρα.

Ἐπεὶ γὰρ μέσα ἐδείχθη τὰ ΑΙ, ΖΚ καὶ ἐστὶν ἴσα τοῖς ἀπὸ τῶν ΑΟ, ΟΝ, μέσον ἄρα καὶ ἐκάτερον τῶν ἀπὸ τῶν ΑΟ, ΟΝ· μέση ἄρα ἐκατέρᾳ τῶν ΑΟ, ΟΝ. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ΑΙ τῷ ΖΚ, σύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΑΟ τῷ ἀπὸ τῆς ΟΝ. πάλιν, ἐπεὶ ἀσύμμετρον ἐδείχθη τὸ ΑΙ τῷ ΕΚ, ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ΑΜ τῷ ΜΝ, τουτέστι τὸ ἀπὸ τῆς ΑΟ τῷ ὑπὸ τῶν ΑΟ, ΟΝ· ὥστε καὶ ἡ ΑΟ ἀσύμμετρός ἐστι μήκει τῇ ΟΝ· αἱ ΑΟ, ΟΝ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. λέγω δὴ, ὅτι καὶ μέσον περιέχουσιν.

Ἐπεὶ γὰρ μέσον ἐδείχθη τὸ ΕΚ καὶ ἐστὶν ἴσον τῷ ὑπὸ τῶν ΑΟ, ΟΝ, μέσον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΑΟ, ΟΝ· ὥστε αἱ ΑΟ, ΟΝ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον

ously) laid down rational (straight-line) AC , and the square on the whole, AG , is greater than (the square on) the attachment, DG , by the (square) on (some straight-line) commensurable (in length) with (AG) [Def. 10.13]. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) commensurable (in length) with (AG) , thus if (an area) equal to the fourth part of the square on DG is applied to AG , falling short by a square figure, then it divides (AG) into (parts which are) commensurable (in length) [Prop. 10.17]. Therefore, let DG have been cut in half at E . And let (an area) equal to the (square) on EG have been applied to AG , falling short by a square figure. And let it be the (rectangle contained) by AF and FG . And let EH , FI , and GK have been drawn through points E , F , and G (respectively), parallel to AC . Thus, AF and FG are commensurable (in length). AI (is) thus also commensurable with FK [Props. 6.1, 10.11]. And since AF and FG are commensurable in length, AG is thus also commensurable in length with each of AF and FG [Prop. 10.15]. And AG (is) rational, and incommensurable in length with AC . Hence, AF and FG (are) also (rational, and incommensurable in length with AC) [Prop. 10.13]. Thus, AI and FK are each medial (areas) [Prop. 10.21]. Again, since DE is commensurable in length with EG , DG is also commensurable in length with each of DE and EG [Prop. 10.15]. And GD (is) rational, and incommensurable in length with AC . Thus, DE and EG (are) each also rational, and incommensurable in length with AC [Prop. 10.13]. DH and EK are thus each medial (areas) [Prop. 10.21]. And since AG and GD are commensurable in square only, AG is thus incommensurable in length with GD . But, AG is commensurable in length with AF , and DG with EG . Thus, AF is incommensurable in length with EG [Prop. 10.13]. And as AF (is) to EG , so AI is to EK [Prop. 6.1]. Thus, AI is incommensurable with EK [Prop. 10.11].

Therefore, let the square LM , equal to AI , have been constructed. And let NO , equal to FK , which is about the same angle as LM , have been subtracted (from LM). Thus, LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since the (rectangle contained) by AF and FG is equal to the (square) on EG , thus as AF is to EG , so EG (is) to FG [Prop. 6.17]. But, as AF (is) to EG , so AI is to EK [Prop. 6.1]. And as EG (is) to FG , so EK is to FK [Prop. 6.1]. And thus as AI (is) to EK , so EK (is) to FK [Prop. 5.11]. Thus, EK is the mean proportional to AI and FK . And MN is also the mean proportional to the squares LM and NO [Prop. 10.53 lem.]. And AI is

περιέχουσαι. ἡ AN ἄρα μέσης ἀποτομή ἐστὶ δευτέρα· καὶ δύναται τὸ AB χωρίον.

Ἡ ἄρα τὸ AB χωρίον δυναμένη μέσης ἀποτομή ἐστὶ δευτέρα· ὅπερ ἔδει δείξαι.

equal to LM , and FK to NO . Thus, EK is also equal to MN . But, MN is equal to LO , and EK [is] equal to DH [Prop. 1.43]. And thus the whole of DK is equal to the gnomon UVW and NO . And AK (is) also equal to LM and NO . Thus, the remainder AB is equal to ST —that is to say, to the square on LN . Thus, LN is the square-root of area AB . I say that LN is the second apotome of a medial (straight-line).

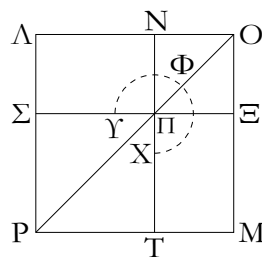
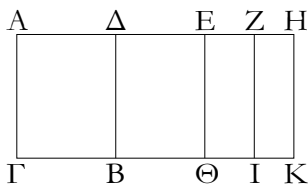
For since AI and FK were shown (to be) medial (areas), and are equal to the (squares) on LP and PN (respectively), the (squares) on each of LP and PN (are) thus also medial. Thus, LP and PN (are) each medial (straight-lines). And since AI is commensurable with FK [Props. 6.1, 10.11], the (square) on LP (is) thus also commensurable with the (square) on PN . Again, since AI was shown (to be) incommensurable with EK , LM is thus also incommensurable with MN —that is to say, the (square) on LP with the (rectangle contained) by LP and PN . Hence, LP is also incommensurable in length with PN [Props. 6.1, 10.11]. Thus, LP and PN are medial (straight-lines which are) commensurable in square only. So, I say that they also contain a medial (area).

For since EK was shown (to be) a medial (area), and is equal to the (rectangle contained) by LP and PN , the (rectangle contained) by LP and PN is thus also medial. Hence, LP and PN are medial (straight-lines which are) commensurable in square only, and which contain a medial (area). Thus, LN is the second apotome of a medial (straight-line) [Prop. 10.75]. And it is the square-root of area AB .

Thus, the square-root of area AB is the second apotome of a medial (straight-line). (Which is) the very thing it was required to show.

ιδ'.

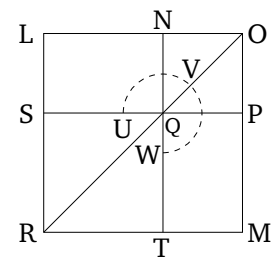
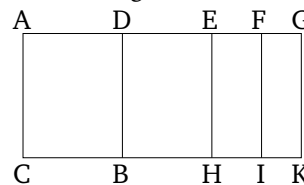
Ἐάν χωρίον περιέχεται ὑπὸ ῥητῆς καὶ ἀποτομῆς τετάρτης, ἡ τὸ χωρίον δυναμένη ἐλάσσων ἐστίν.



Χωρίον γὰρ τὸ AB περιεχέσθω ὑπὸ ῥητῆς τῆς $AΓ$ καὶ ἀποτομῆς τετάρτης τῆς $AΔ$. λέγω, ὅτι ἡ τὸ AB χωρίον δυναμένη ἐλάσσων ἐστίν.

Proposition 94

If an area is contained by a rational (straight-line) and a fourth apotome then the square-root of the area is a minor (straight-line).



For let the area AB have been contained by the rational (straight-line) AC and the fourth apotome AD . I say that the square-root of area AB is a minor (straight-

Ἐστω γάρ τῃ AD προσαρμόζουσα ἡ $ΔΗ$. αἱ ἄρα AH , $HΔ$ ῥηταί εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ AH σύμμετρος ἐστὶ τῇ ἐκκειμένη ῥητῇ τῇ AG μήκει, ἡ δὲ ὅλη ἡ AH τῆς προσαρμοζούσης τῆς $ΔΗ$ μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ μήκει. ἐπεὶ οὖν ἡ AH τῆς $HΔ$ μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ μήκει, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς $ΔΗ$ ἴσον παρὰ τὴν AH παραβληθῇ ἐλλείπον εἶδει τετραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διελεί. τετμήσθω οὖν ἡ $ΔΗ$ δίχα κατὰ τὸ E , καὶ τῷ ἀπὸ τῆς EH ἴσον παρὰ τὴν AH παραβεβλήσθω ἐλλείπον εἶδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν AZ , ZH . ἀσύμμετρος ἄρα ἐστὶ μήκει ἡ AZ τῇ ZH . ἤχθωσαν οὖν διὰ τῶν E , Z , H παράλληλοι ταῖς AG , BD αἱ $ΕΘ$, ZI , HK . ἐπεὶ οὖν ῥητὴ ἐστὶν ἡ AH καὶ σύμμετρος τῇ AG μήκει, ῥητὸν ἄρα ἐστὶν ὅλον τὸ AK . πάλιν, ἐπεὶ ἀσύμμετρος ἐστὶν ἡ $ΔΗ$ τῇ AG μήκει, καὶ εἰσὶν ἀμφοτέραι ῥηταί, μέσον ἄρα ἐστὶ τὸ $ΔK$. πάλιν, ἐπεὶ ἀσύμμετρος ἐστὶν ἡ AZ τῇ ZH μήκει, ἀσύμμετρον ἄρα καὶ τὸ AI τῷ ZK .

Συνεστάτω οὖν τῷ μὲν AI ἴσον τετράγωνον τὸ AM , τῷ δὲ ZK ἴσον ἀφηρήσθω περὶ τὴν αὐτὴν γωνίαν τὴν ὑπὸ τῶν $ΛOM$ τὸ $NΞ$. περὶ τὴν αὐτὴν ἄρα διάμετόν ἐστὶ τὰ AM , $NΞ$ τετράγωνα. ἔστω αὐτῶν διάμετος ἡ OP , καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν τὸ ὑπὸ τῶν AZ , ZH ἴσον ἐστὶ τῷ ἀπὸ τῆς EH , ἀνάλογον ἄρα ἐστὶν ὡς ἡ AZ πρὸς τὴν EH , οὕτως ἡ EH πρὸς τὴν ZH . ἀλλ' ὡς μὲν ἡ AZ πρὸς τὴν EH , οὕτως ἐστὶ τὸ AI πρὸς τὸ EK , ὡς δὲ ἡ EH πρὸς τὴν ZH , οὕτως ἐστὶ τὸ EK πρὸς τὸ ZK . τῶν ἄρα AI , ZK μέσον ἀνάλογόν ἐστὶ τὸ EK . ἐστὶ δὲ καὶ τῶν AM , $NΞ$ τετραγώνων μέσον ἀνάλογον τὸ MN , καὶ ἐστὶν ἴσον τὸ μὲν AI τῷ AM , τὸ δὲ ZK τῷ $NΞ$. καὶ τὸ EK ἄρα ἴσον ἐστὶ τῷ MN . ἀλλὰ τῷ μὲν EK ἴσον ἐστὶ τὸ $ΔΘ$, τῷ δὲ MN ἴσον ἐστὶ τὸ $ΛΞ$. ὅλον ἄρα τὸ $ΔK$ ἴσον ἐστὶ τῷ $ΥΦX$ γνῶμονι καὶ τῷ $NΞ$. ἐπεὶ οὖν ὅλον τὸ AK ἴσον ἐστὶ τοῖς AM , $NΞ$ τετραγώνοις, ὣν τὸ $ΔK$ ἴσον ἐστὶ τῷ $ΥΦX$ γνῶμονι καὶ τῷ $NΞ$ τετραγώνῳ, λοιπὸν ἄρα τὸ AB ἴσον ἐστὶ τῷ $ΣΤ$, τουτέστι τῷ ἀπὸ τῆς AN τετραγώνῳ· ἡ AN ἄρα δύναται τὸ AB χωρίον. λέγω, ὅτι ἡ AN ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Ἐπεὶ γὰρ ῥητὸν ἐστὶ τὸ AK καὶ ἐστὶν ἴσον τοῖς ἀπὸ τῶν AO , ON τετράγωνοις, τὸ ἄρα συγκείμενον ἐκ τῶν ἀπὸ τῶν AO , ON ῥητόν ἐστιν. πάλιν, ἐπεὶ τὸ $ΔK$ μέσον ἐστίν, καὶ ἐστὶν ἴσον τὸ $ΔK$ τῷ δις ὑπὸ τῶν AO , ON , τὸ ἄρα δις ὑπὸ τῶν AO , ON μέσον ἐστίν. καὶ ἐπεὶ ἀσύμμετρον ἐδείχθη τὸ AI τῷ ZK , ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς AO τετράγωνον τῷ ἀπὸ τῆς ON τετραγώνῳ. αἱ AO , ON ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητόν, τὸ δὲ δις ὑπ' αὐτῶν μέσον. ἡ AN ἄρα ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων· καὶ δύναται τὸ AB χωρίον.

Ἡ ἄρα τὸ AB χωρίον δυναμένη ἐλάσσων ἐστίν· ὅπερ ἔδει δεῖξαι.

line). For let DG be an attachment to AD . Thus, AG and DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and AG is commensurable in length with the (previously) laid down rational (straight-line) AC , and the square on the whole, AG , is greater than (the square on) the attachment, DG , by the square on (some straight-line) incommensurable in length with (AG) [Def. 10.14]. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) incommensurable in length with (AG) , thus if (some area), equal to the fourth part of the (square) on DG , is applied to AG , falling short by a square figure, then it divides (AG) into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let DG have been cut in half at E , and let (some area), equal to the (square) on EG , have been applied to AG , falling short by a square figure, and let it be the (rectangle contained) by AF and FG . Thus, AF is incommensurable in length with FG . Therefore, let EH , FI , and GK have been drawn through E , F , and G (respectively), parallel to AC and BD . Therefore, since AG is rational, and commensurable in length with AC , the whole (area) AK is thus rational [Prop. 10.19]. Again, since DG is incommensurable in length with AC , and both are rational (straight-lines), DK is thus a medial (area) [Prop. 10.21]. Again, since AF is incommensurable in length with FG , AI (is) thus also incommensurable with FK [Props. 6.1, 10.11].

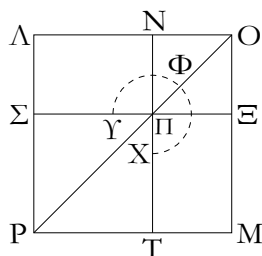
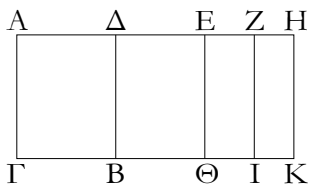
Therefore, let the square LM , equal to AI , have been constructed. And let NO , equal to FK , (and) about the same angle, LPM , have been subtracted (from LM). Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let the (rest of the) figure have been drawn. Therefore, since the (rectangle contained) by AF and FG is equal to the (square) on EG , thus, proportionally, as AF is to EG , so EG (is) to FG [Prop. 6.17]. But, as AF (is) to EG , so AI is to EK , and as EG (is) to FG , so EK is to FK [Prop. 6.1]. Thus, EK is the mean proportional to AI and FK [Prop. 5.11]. And MN is also the mean proportional to the squares LM and NO [Prop. 10.13 lem.], and AI is equal to LM , and FK to NO . EK is thus also equal to MN . But, DH is equal to EK , and LO is equal to MN [Prop. 1.43]. Thus, the whole of DK is equal to the gnomon UVW and NO . Therefore, since the whole of AK is equal to the (sum of the) squares LM and NO , of which DK is equal to the gnomon UVW and the square NO , the remainder AB is thus equal to ST —that is to say, to the square on LN . Thus, LN is the square-root of area AB . I say that LN is the irrational (straight-line which is) called minor.

For since AK is rational, and is equal to the (sum of the) squares LP and PN , the sum of the (squares) on LP and PN is thus rational. Again, since DK is medial, and DK is equal to twice the (rectangle contained) by LP and PN , thus twice the (rectangle contained) by LP and PN is medial. And since AI was shown (to be) incommensurable with FK , the square on LP (is) thus also incommensurable with the square on PN . Thus, LP and PN are (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and twice the (rectangle contained) by them medial. LN is thus the irrational (straight-line) called minor [Prop. 10.76]. And it is the square-root of area AB .

Thus, the square-root of area AB is a minor (straight-line). (Which is) the very thing it was required to show.

ἢε'.

Ἐάν χωρίον περιέχεται ὑπὸ ῥητῆς καὶ ἀποτομῆς πέμπτης, ἢ τὸ χωρίον δυναμένη [ἢ] μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσά ἐστιν.



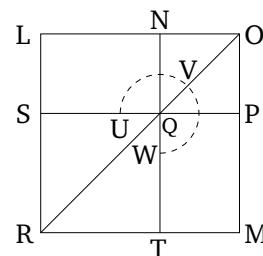
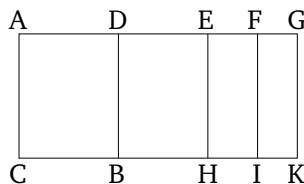
Χωρίον γὰρ τὸ AB περιεχέσθω ὑπὸ ῥητῆς τῆς AG καὶ ἀποτομῆς πέμπτης τῆς AD . λέγω, ὅτι ἢ τὸ AB χωρίον δυναμένη [ἢ] μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσά ἐστιν.

Ἐστω γὰρ τῇ AD προσαρμόζουσα ἡ $ΔΗ$. αἱ ἄρα AH , $HΔ$ ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ προσαρμόζουσα ἡ $HΔ$ σύμμετρός ἐστι μήκει τῇ ἐκκειμένῃ ῥητῇ τῇ AG , ἡ δὲ ὅλη ἡ AH τῆς προσαρμόζουσας τῆς $ΔΗ$ μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ. ἐάν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς $ΔΗ$ ἴσον παρὰ τὴν AH παραβληθῇ ἐλλείπον εἶδει τετραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διελεί. τετμήσθω οὖν ἡ $ΔΗ$ δίχα κατὰ τὸ E σημεῖον, καὶ τῷ ἀπὸ τῆς EH ἴσον παρὰ τὴν AH παραβεβλήσθω ἐλλείπον εἶδει τετραγώνῳ καὶ ἔστω τὸ ὑπὸ τῶν AZ , ZH . ἀσύμμετρος ἄρα ἐστὶν ἡ AZ τῇ ZH μήκει. καὶ ἐπεὶ ἀσύμμετρός ἐστὶν ἡ AH τῇ $ΓΑ$ μήκει, καὶ εἰσιν ἀμφοτέραι ῥηταί, μέσον ἄρα ἐστὶ τὸ AK . πάλιν, ἐπεὶ ῥητὴ ἐστὶν ἡ $ΔΗ$ καὶ σύμμετρος τῇ AG μήκει, ῥητόν ἐστι τὸ $ΔΚ$.

Συνεστάτω οὖν τῷ μὲν AI ἴσον τετράγωνον τὸ $ΛΜ$, τῷ δὲ ZK ἴσον τετράγωνον ἀφηρήσθω τὸ $ΝΞ$ περὶ τὴν αὐτὴν γωνίαν τὴν ὑπὸ $ΛΟΜ$. περὶ τὴν αὐτὴν ἄρα διάμετρον ἐστὶ τὰ $ΛΜ$, $ΝΞ$ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ $ΟΡ$, καὶ

Proposition 95

If an area is contained by a rational (straight-line) and a fifth apotome then the square-root of the area is that (straight-line) which with a rational (area) makes a medial whole.



For let the area AB have been contained by the rational (straight-line) AC and the fifth apotome AD . I say that the square-root of area AB is that (straight-line) which with a rational (area) makes a medial whole.

For let DG be an attachment to AD . Thus, AG and DG are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and the attachment GD is commensurable in length the (previously) laid down rational (straight-line) AC , and the square on the whole, AG , is greater than (the square on) the attachment, DG , by the (square) on (some straight-line) incommensurable (in length) with (AG) [Def. 10.15]. Thus, if (some area), equal to the fourth part of the (square) on DG , is applied to AG , falling short by a square figure, then it divides (AG) into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let DG have been divided in half at point E , and let (some area), equal to the (square) on EG , have been applied to AG , falling short by a square figure, and let it be the (rectangle contained) by AF and FG . Thus, AF is incommensurable in length with FG . And since AG is incommensurable

καταγεγράφθω τὸ σχῆμα. ὁμοίως δὲ δείξομεν, ὅτι ἡ ΛN δύναται τὸ AB χωρίον. λέγω, ὅτι ἡ ΛN ἢ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσά ἐστιν.

Ἐπεὶ γὰρ μέσον ἐδείχθη τὸ AK καὶ ἐστὶν ἴσον τοῖς ἀπὸ τῶν ΛO , ON , τὸ ἄρα συγκείμενον ἐκ τῶν ἀπὸ τῶν ΛO , ON μέσον ἐστίν. πάλιν, ἐπεὶ ῥητόν ἐστι τὸ ΔK καὶ ἐστὶν ἴσον τῷ δις ὑπὸ τῶν ΛO , ON , καὶ αὐτὸ ῥητόν ἐστιν. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ AI τῷ ZK , ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΛO τῷ ἀπὸ τῆς ON . αἱ ΛO , ON ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιούσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δὲ δις ὑπ' αὐτῶν ῥητόν. ἡ λοιπὴ ἄρα ἡ ΛN ἄλογός ἐστιν ἢ καλουμένη μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσα· καὶ δύναται τὸ AB χωρίον.

Ἡ τὸ AB ἄρα χωρίον δυναμένη μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσά ἐστιν· ὅπερ ἔδει δείξαι.

in length with CA , and both are rational (straight-lines), AK is thus a medial (area) [Prop. 10.21]. Again, since DG is rational, and commensurable in length with AC , DK is a rational (area) [Prop. 10.19].

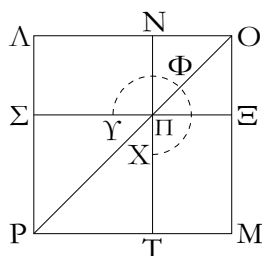
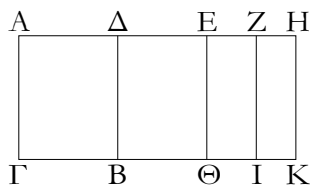
Therefore, let the square LM , equal to AI , have been constructed. And let the square NO , equal to FK , (and) about the same angle, LPM , have been subtracted (from NO). Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let (the rest of) the figure have been drawn. So, similarly (to the previous propositions), we can show that LN is the square-root of area AB . I say that LN is that (straight-line) which with a rational (area) makes a medial whole.

For since AK was shown (to be) a medial (area), and is equal to (the sum of) the squares on LP and PN , the sum of the (squares) on LP and PN is thus medial. Again, since DK is rational, and is equal to twice the (rectangle contained) by LP and PN , (the latter) is also rational. And since AI is incommensurable with FK , the (square) on LP is thus also incommensurable with the (square) on PN . Thus, LP and PN are (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by them rational. Thus, the remainder LN is the irrational (straight-line) called that which with a rational (area) makes a medial whole [Prop. 10.77]. And it is the square-root of area AB .

Thus, the square-root of area AB is that (straight-line) which with a rational (area) makes a medial whole. (Which is) the very thing it was required to show.

ιγ'.

Ἐὰν χωρίον περιέχεται ὑπὸ ῥητῆς καὶ ἀποτομῆς ἔκτης, ἢ τὸ χωρίον δυναμένη μετὰ μέσου μέσον τὸ ὅλον ποιούσά ἐστιν.

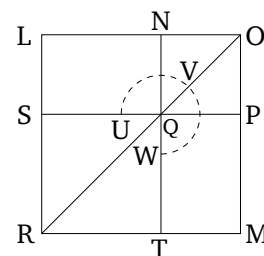
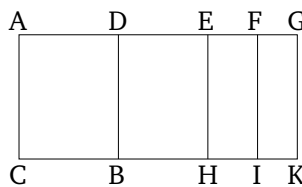


Χωρίον γὰρ τὸ AB περιεχέσθω ὑπὸ ῥητῆς τῆς AΓ καὶ ἀποτομῆς ἔκτης τῆς AΔ . λέγω, ὅτι ἡ τὸ AB χωρίον δυναμένη [ἢ] μετὰ μέσου μέσον τὸ ὅλον ποιούσά ἐστιν.

Ἐστω γὰρ τῇ AΔ προσαρμόζουσα ἡ ΔH . αἱ ἄρα AH , HΔ ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, καὶ οὐδετέρα

Proposition 96

If an area is contained by a rational (straight-line) and a sixth apotome then the square-root of the area is that (straight-line) which with a medial (area) makes a medial whole.



For let the area AB have been contained by the rational (straight-line) AC and the sixth apotome AD . I say that the square-root of area AB is that (straight-line) which with a medial (area) makes a medial whole.

For let DG be an attachment to AD . Thus, AG and

αὐτῶν σύμμετρος ἐστὶ τῇ ἐκκειμένη ρητῇ τῇ ΑΓ μήκει, ἢ δὲ ὅλη ἡ ΑΗ τῆς προσαρμοζούσης τῆς ΔΗ μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ μήκει. ἐπεὶ οὖν ἡ ΑΗ τῆς ΗΔ μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ μήκει, ἐὰν ἄρα τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΔΗ ἴσον παρὰ τὴν ΑΗ παραβληθῇ ἐλλείπον εἶδει τετραγώνῳ, εἰς ἀσύμμετρα αὐτὴν διελεί. τετμήσθω οὖν ἡ ΔΗ δίχα κατὰ τὸ Ε [σημεῖον], καὶ τῷ ἀπὸ τῆς ΕΗ ἴσον παρὰ τὴν ΑΗ παραβεβλήσθω ἐλλείπον εἶδει τετραγώνῳ, καὶ ἔστω τὸ ὑπὸ τῶν ΑΖ, ΖΗ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΖ τῇ ΖΗ μήκει. ὥς δὲ ἡ ΑΖ πρὸς τὴν ΖΗ, οὕτως ἐστὶ τὸ ΑΙ πρὸς τὸ ΖΚ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΑΙ τῷ ΖΚ. καὶ ἐπεὶ αἱ ΑΗ, ΑΓ ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, μέσον ἐστὶ τὸ ΑΚ. πάλιν, ἐπεὶ αἱ ΑΓ, ΔΗ ῥηταὶ εἰσι καὶ ἀσύμμετροι μήκει, μέσον ἐστὶ καὶ τὸ ΔΚ. ἐπεὶ οὖν αἱ ΑΗ, ΗΔ δυνάμει μόνον σύμμετροί εἰσιν, ἀσύμμετρος ἄρα ἐστὶν ἡ ΑΗ τῇ ΗΔ μήκει. ὥς δὲ ἡ ΑΗ πρὸς τὴν ΗΔ, οὕτως ἐστὶ τὸ ΑΚ πρὸς τὸ ΚΔ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΑΚ τῷ ΚΔ.

Συνεστάτω οὖν τῷ μὲν ΑΙ ἴσον τετράγωνον τὸ ΛΜ, τῷ δὲ ΖΚ ἴσον ἀφηρήσθω περὶ τὴν αὐτὴν γωνίαν τὸ ΝΞ· περὶ τὴν αὐτὴν ἄρα διάμετρον ἐστὶ τὰ ΛΜ, ΝΞ τετράγωνα. ἔστω αὐτῶν διάμετρος ἡ ΟΡ, καὶ καταγεγράφθω τὸ σχῆμα. ὁμοίως δὴ τοῖς ἐπάνω δείξομεν, ὅτι ἡ ΑΝ δύναται τὸ ΑΒ χωρίον. λέγω, ὅτι ἡ ΑΝ [ἡ] μετὰ μέσου μέσον τὸ ὅλον ποιούσά ἐστιν.

Ἐπεὶ γὰρ μέσον ἐδείχθη τὸ ΑΚ καὶ ἐστὶν ἴσον τοῖς ἀπὸ τῶν ΛΟ, ΟΝ, τὸ ἄρα συγκείμενον ἐκ τῶν ἀπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. πάλιν, ἐπεὶ μέσον ἐδείχθη τὸ ΔΚ καὶ ἐστὶν ἴσον τῷ δις ὑπὸ τῶν ΛΟ, ΟΝ, καὶ τὸ δις ὑπὸ τῶν ΛΟ, ΟΝ μέσον ἐστίν. καὶ ἐπεὶ ἀσύμμετρον ἐδείχθη τὸ ΑΚ τῷ ΔΚ, ἀσύμμετρα [ἄρα] ἐστὶ καὶ τὰ ἀπὸ τῶν ΛΟ, ΟΝ τετράγωνα τῷ δις ὑπὸ τῶν ΛΟ, ΟΝ. καὶ ἐπεὶ ἀσύμμετρον ἐστὶ τὸ ΑΙ τῷ ΖΚ, ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΛΟ τῷ ἀπὸ τῆς ΟΝ· αἱ ΛΟ, ΟΝ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιῶσαι τὸ τε συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον καὶ τὸ δις ὑπ' αὐτῶν μέσον ἔτι τε τὰ ἀπ' αὐτῶν τετράγωνα ἀσύμμετρα τῷ δις ὑπ' αὐτῶν. ἡ ἄρα ΑΝ ἄλογός ἐστιν ἢ καλουμένη μετὰ μέσου μέσον τὸ ὅλον ποιούσα· καὶ δύναται τὸ ΑΒ χωρίον.

Ἡ ἄρα τὸ χωρίον δυναμένη μετὰ μέσου μέσον τὸ ὅλον ποιούσά ἐστιν· ὅπερ ἔδει δεῖξαι.

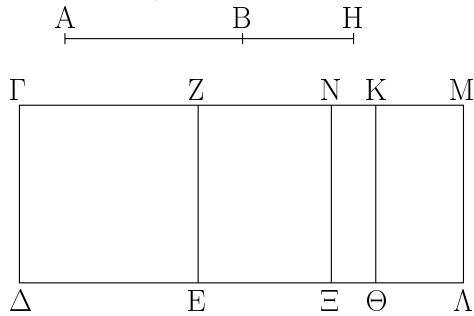
GD are rational (straight-lines which are) commensurable in square only [Prop. 10.73], and neither of them is commensurable in length with the (previously) laid down rational (straight-line) AC , and the square on the whole, AG , is greater than (the square on) the attachment, DG , by the (square) on (some straight-line) incommensurable in length with (AG) [Def. 10.16]. Therefore, since the square on AG is greater than (the square on) GD by the (square) on (some straight-line) incommensurable in length with (AG) , thus if (some area), equal to the fourth part of square on DG , is applied to AG , falling short by a square figure, then it divides (AG) into (parts which are) incommensurable (in length) [Prop. 10.18]. Therefore, let DG have been cut in half at [point] E . And let (some area), equal to the (square) on EG , have been applied to AG , falling short by a square figure. And let it be the (rectangle contained) by AF and FG . AF is thus incommensurable in length with FG . And as AF (is) to FG , so AI is to FK [Prop. 6.1]. Thus, AI is incommensurable with FK [Prop. 10.11]. And since AG and AC are rational (straight-lines which are) commensurable in square only, AK is a medial (area) [Prop. 10.21]. Again, since AC and DG are rational (straight-lines which are) incommensurable in length, DK is also a medial (area) [Prop. 10.21]. Therefore, since AG and GD are commensurable in square only, AG is thus incommensurable in length with GD . And as AG (is) to GD , so AK is to KD [Prop. 6.1]. Thus, AK is incommensurable with KD [Prop. 10.11].

Therefore, let the square LM , equal to AI , have been constructed. And let NO , equal to FK , (and) about the same angle, have been subtracted (from LM). Thus, the squares LM and NO are about the same diagonal [Prop. 6.26]. Let PR be their (common) diagonal, and let (the rest of) the figure have been drawn. So, similarly to the above, we can show that LN is the square-root of area AB . I say that LN is that (straight-line) which with a medial (area) makes a medial whole.

For since AK was shown (to be) a medial (area), and is equal to the (sum of the) squares on LP and PN , the sum of the (squares) on LP and PN is medial. Again, since DK was shown (to be) a medial (area), and is equal to twice the (rectangle contained) by LP and PN , twice the (rectangle contained) by LP and PN is also medial. And since AK was shown (to be) incommensurable with DK , [thus] the (sum of the) squares on LP and PN is also incommensurable with twice the (rectangle contained) by LP and PN . And since AI is incommensurable with FK , the (square) on LP (is) thus also incommensurable with the (square) on PN . Thus, LP and PN are (straight-lines which are) incommensu-

ιζ'.

Τὸ ἀπὸ ἀποτομῆς παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην.



Ἐστω ἀποτομή ἡ AB , ῥητὴ δὲ ἡ $\Gamma\Delta$, καὶ τῷ ἀπὸ τῆς AB ἴσον παρὰ τὴν $\Gamma\Delta$ παραβεβλήσθω τὸ $\Gamma\Theta$ πλάτος ποιοῦν τὴν $\Gamma\Xi$ · λέγω, ὅτι ἡ $\Gamma\Xi$ ἀποτομὴ ἐστὶ πρώτη.

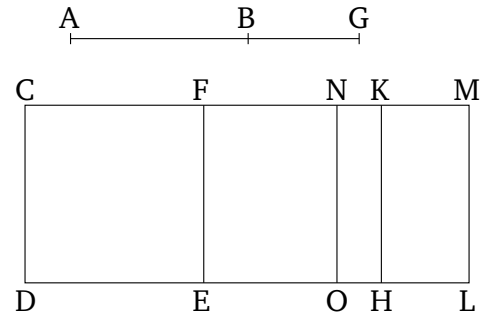
Ἐστω γάρ τῃ AB προσαρμόζουσα ἡ BH · αἱ ἄρα AH , HB ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· καὶ τῷ μὲν ἀπὸ τῆς AH ἴσον παρὰ τὴν $\Gamma\Delta$ παραβεβλήσθω τὸ $\Gamma\Theta$, τῷ δὲ ἀπὸ τῆς BH τὸ $\Theta\Lambda$. ὅλον ἄρα τὸ $\Gamma\Lambda$ ἴσον ἐστὶ τοῖς ἀπὸ τῶν AH , HB · ὧν τὸ $\Gamma\Theta$ ἴσον ἐστὶ τῷ ἀπὸ τῆς AB · λοιπὸν ἄρα τὸ $\Theta\Lambda$ ἴσον ἐστὶ τῷ δις ὑπὸ τῶν AH , HB . τετμήσθω ἡ ZM δίχα κατὰ τὸ N σημεῖον, καὶ ἄρξω διὰ τοῦ N τῇ $\Gamma\Delta$ παράλληλος ἡ $N\Xi$ · ἐκάτερον ἄρα τῶν $Z\Xi$, ΛN ἴσον ἐστὶ τῷ ὑπὸ τῶν AH , HB . καὶ ἐπεὶ τὰ ἀπὸ τῶν AH , HB ῥητὰ ἐστίν, καὶ ἐστὶ τοῖς ἀπὸ τῶν AH , HB ἴσον τὸ ΔM , ῥητὸν ἄρα ἐστὶ τὸ ΔM . καὶ παρὰ ῥητὴν τὴν $\Gamma\Delta$ παραβεβλήται πλάτος ποιοῦν τὴν ΓM · ῥητὴ ἄρα ἐστὶν ἡ ΓM καὶ σύμμετρος τῇ $\Gamma\Delta$ μήκει. πάλιν, ἐπεὶ μέσον ἐστὶ τὸ δις ὑπὸ τῶν AH , HB , καὶ τῷ δις ὑπὸ τῶν AH , HB ἴσον τὸ $Z\Lambda$, μέσον ἄρα τὸ $Z\Lambda$. καὶ παρὰ ῥητὴν τὴν $\Gamma\Delta$ παράκειται πλάτος ποιοῦν τὴν ZM · ῥητὴ ἄρα ἐστὶν ἡ ZM καὶ ἀσύμμετρος τῇ $\Gamma\Delta$ μήκει. καὶ ἐπεὶ τὰ μὲν ἀπὸ τῶν AH , HB ῥητὰ ἐστίν, τὸ δὲ δις ὑπὸ τῶν AH , HB μέσον, ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν AH , HB τῷ δις ὑπὸ τῶν AH , HB . καὶ τοῖς μὲν ἀπὸ τῶν AH , HB ἴσον ἐστὶ τὸ $\Gamma\Lambda$, τῷ δὲ δις ὑπὸ τῶν AH , HB τὸ $Z\Lambda$ · ἀσύμμετρον ἄρα ἐστὶ τὸ ΔM τῷ $Z\Lambda$. ὥς δὲ τὸ ΔM πρὸς τὸ $Z\Lambda$, οὕτως ἐστὶν ἡ ΓM πρὸς τὴν ZM . ἀσύμμετρος ἄρα ἐστὶν ἡ ΓM τῇ ZM μήκει. καὶ εἰσιν ἀμφοτέραι ῥηταί· αἱ ἄρα ΓM , MZ ῥηταὶ εἰσι

rable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by medial, and, furthermore, the (sum of the) squares on them incommensurable with twice the (rectangle contained) by them. Thus, LN is the irrational (straight-line) called that which with a medial (area) makes a medial whole [Prop. 10.78]. And it is the square-root of area AB .

Thus, the square-root of area (AB) is that (straight-line) which with a medial (area) makes a medial whole. (Which is) the very thing it was required to show.

Proposition 97

The (square) on an apotome, applied to a rational (straight-line), produces a first apotome as breadth.



Let AB be an apotome, and CD a rational (straight-line). And let CE , equal to the (square) on AB , have been applied to CD , producing CF as breadth. I say that CF is a first apotome.

For let BG be an attachment to AB . Thus, AG and GB are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let CH , equal to the (square) on AG , and KL , (equal) to the (square) on BG , have been applied to CD . Thus, the whole of CL is equal to the (sum of the squares) on AG and GB , of which CE is equal to the (square) on AB . The remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Let FM have been cut in half at point N . And let NO have been drawn through N , parallel to CD . Thus, FO and LN are each equal to the (rectangle contained) by AG and GB . And since the (sum of the squares) on AG and GB is rational, and DM is equal to the (sum of the squares) on AG and GB , DM is thus rational. And it has been applied to the rational (straight-line) CD , producing CM as breadth. Thus, CM is rational, and commensurable in length with CD [Prop. 10.20]. Again, since twice the (rectangle contained) by AG and GB is medial, and FL (is) equal to twice the (rectangle contained) by AG and GB , FL (is) thus a medial (area). And it is applied to the rational (straight-line) CD , producing FM as breadth. FM is

δυνάμει μόνον σύμμετροι· ἡ ΓΖ ἄρα ἀποτομή ἐστιν. λέγω δὴ, ὅτι καὶ πρώτη.

Ἐπεὶ γὰρ τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καὶ ἐστὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΒΗ ἴσον τὸ ΚΑ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ τὸ ΝΑ, καὶ τῶν ΓΘ, ΚΑ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΝΑ· ἐστὶν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΑ, οὕτως τὸ ΝΑ πρὸς τὸ ΚΑ. ἀλλ' ὡς μὲν τὸ ΓΘ πρὸς τὸ ΝΑ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΝΜ· ὡς δὲ τὸ ΝΑ πρὸς τὸ ΚΑ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ. καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ, σύμμετρόν [ἐστὶ] καὶ τὸ ΓΘ τῷ ΚΑ. ὡς δὲ τὸ ΓΘ πρὸς τὸ ΚΑ, οὕτως ἡ ΓΚ πρὸς τὴν ΚΜ· σύμμετρος ἄρα ἐστὶν ἡ ΓΚ τῇ ΚΜ. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ ἴσον παρὰ τὴν ΓΜ παραβέβληται ἐλλείπον εἶδει τετραγώνῳ τὸ ὑπὸ τῶν ΓΚ, ΚΜ, καὶ ἐστὶ σύμμετρος ἡ ΓΚ τῇ ΚΜ, ἡ ἄρα ΓΜ τῆς ΜΖ μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καὶ ἐστὶν ἡ ΓΜ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ μήκει· ἡ ἄρα ΓΖ ἀποτομή ἐστὶ πρώτη.

Τὸ ἄρα ἀπὸ ἀποτομῆς παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην· ὅπερ εἶδει δεῖξαι.

thus rational, and incommensurable in length with CD [Prop. 10.22]. And since the (sum of the squares) on AG and GB is rational, and twice the (rectangle contained) by AG and GB medial, the (sum of the squares) on AG and GB is thus incommensurable with twice the (rectangle contained) by AG and GB . And CL is equal to the (sum of the squares) on AG and GB , and FL to twice the (rectangle contained) by AG and GB . DM is thus incommensurable with FL . And as DM (is) to FL , so CM is to FM [Prop. 6.1]. CM is thus incommensurable in length with FM [Prop. 10.11]. And both are rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a first (apotome).

For since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and CH is equal to the (square) on AG , and KL equal to the (square) on BG , and NL to the (rectangle contained) by AG and GB , NL is thus also the mean proportional to CH and KL . Thus, as CH is to NL , so NL (is) to KL . But, as CH (is) to NL , so CK is to NM , and as NL (is) to KL , so NM is to KM [Prop. 6.1]. Thus, the (rectangle contained) by CK and KM is equal to the (square) on NM —that is to say, to the fourth part of the (square) on FM [Prop. 6.17]. And since the (square) on AG is commensurable with the (square) on GB , CH [is] also commensurable with KL . And as CH (is) to KL , so CK (is) to KM [Prop. 6.1]. CK is thus commensurable (in length) with KM [Prop. 10.11]. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM , equal to the fourth part of the (square) on FM , has been applied to CM , falling short by a square figure, and CK is commensurable (in length) with KM , the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable in length with (CM) [Prop. 10.17]. And CM is commensurable in length with the (previously) laid down rational (straight-line) CD . Thus, CF is a first apotome [Def. 10.15].

Thus, the (square) on an apotome, applied to a rational (straight-line), produces a first apotome as breadth. (Which is) the very thing it was required to show.

ιη'.

Τὸ ἀπὸ μέσης ἀποτομῆς πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν δευτέραν.

Ἐστω μέσης ἀποτομῆς πρώτη ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβελήσθω τὸ ΓΕ πλάτος

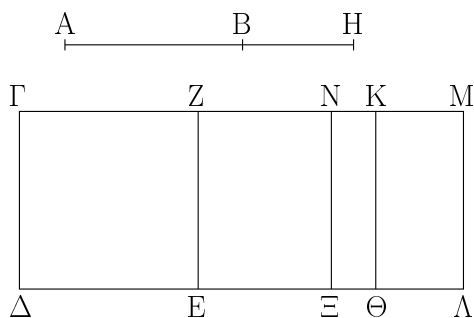
Proposition 98

The (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces a second apotome as breadth.

Let AB be a first apotome of a medial (straight-line),

ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστι δευτέρα.

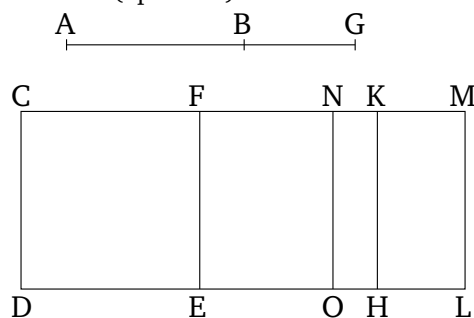
Ἐστω γὰρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι ῥητὸν περιέχουσαι. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ πλάτος ποιοῦν τὴν ΚΜ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ· μέσον ἄρα καὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράκειται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ὧν τὸ ἀπὸ τῆς ΑΒ ἴσον ἐστὶ τῷ ΓΕ, λοιπὸν ἄρα τὸ δις ὑπὸ τῶν ΑΗ, ΗΒ ἴσον ἐστὶ τῷ ΖΛ. ῥητὸν δέ [ἐστὶ] τὸ δις ὑπὸ τῶν ΑΗ, ΗΒ· ῥητὸν ἄρα τὸ ΖΛ. καὶ παρὰ ῥητὴν τὴν ΖΕ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶ καὶ ἡ ΖΜ καὶ σύμμετρος τῇ ΓΔ μήκει. ἐπεὶ οὖν τὰ μὲν ἀπὸ τῶν ΑΗ, ΗΒ, τουτέστι τὸ ΓΛ, μέσον ἐστίν, τὸ δὲ δις ὑπὸ τῶν ΑΗ, ΗΒ, τουτέστι τὸ ΖΛ, ῥητὸν ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΛ τῷ ΖΛ. ὥς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΖΜ· ἀσύμμετρος ἄρα ἡ ΓΜ τῇ ΖΜ μήκει. καὶ εἰσὶν ἀμφοτέραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταὶ εἰσὶ δυνάμει μόνον σύμμετροι· ἡ ΓΖ ἄρα ἀποτομή ἐστίν. λέγω δὴ, ὅτι καὶ δευτέρα.



Τετμήσθω γὰρ ἡ ΖΜ δίχα κατὰ τὸ Ν, καὶ ἤχθω διὰ τοῦ Ν τῇ ΓΔ παράλληλος ἡ ΝΞ· ἑκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ τῶν ἀπὸ τῶν ΑΗ, ΗΒ τετραγώνων μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καὶ ἐστὶν ἴσον τὸ μὲν ἀπὸ τῆς ΑΗ τῷ ΓΘ, τὸ δὲ ὑπὸ τῶν ΑΗ, ΗΒ τῷ ΝΛ, τὸ δὲ ἀπὸ τῆς ΒΗ τῷ ΚΛ, καὶ τῶν ΓΘ, ΚΛ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΝΛ· ἐστὶν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΛ, οὕτως τὸ ΝΛ πρὸς τὸ ΚΛ. ἀλλ' ὡς μὲν τὸ ΓΘ πρὸς

and CD a rational (straight-line). And let CE , equal to the (square) on AB , have been applied to CD , producing CF as breadth. I say that CF is a second apotome.

For let BG be an attachment to AB . Thus, AG and GB are medial (straight-lines which are) commensurable in square only, containing a rational (area) [Prop. 10.74]. And let CH , equal to the (square) on AG , have been applied to CD , producing CK as breadth, and KL , equal to the (square) on GB , producing KM as breadth. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB . Thus, CL (is) also a medial (area) [Props. 10.15, 10.23 corr.]. And it is applied to the rational (straight-line) CD , producing CM as breadth. CM is thus rational, and incommensurable in length with CD [Prop. 10.22]. And since CL is equal to the (sum of the squares) on AG and GB , of which the (square) on AB is equal to CE , the remainder, twice the (rectangle contained) by AG and GB , is thus equal to FL [Prop. 2.7]. And twice the (rectangle contained) by AG and GB [is] rational. Thus, FL (is) rational. And it is applied to the rational (straight-line) FE , producing FM as breadth. FM is thus also rational, and commensurable in length with CD [Prop. 10.20]. Therefore, since the (sum of the squares) on AG and GB —that is to say, CL —is medial, and twice the (rectangle contained) by AG and GB —that is to say, FL —(is) rational, CL is thus incommensurable with FL . And as CL (is) to FL , so CM is to FM [Prop. 6.1]. Thus, CM (is) incommensurable in length with FM [Prop. 10.11]. And they are both rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a second (apotome).



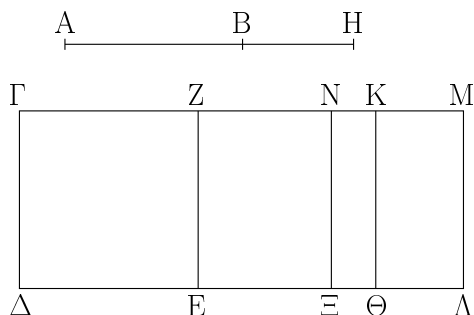
For let FM have been cut in half at N . And let NO have been drawn through (point) N , parallel to CD . Thus, FO and NL are each equal to the (rectangle contained) by AG and GB . And since the (rectangle contained) by AG and GB is the mean proportional to the squares on AG and GB [Prop. 10.21 lem.], and the (square) on AG is equal to CH , and the (rectangle contained) by AG and GB to NL , and the (square) on

τὸ ΝΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΝΜ, ὥς δὲ τὸ ΝΛ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΜΚ· ὥς ἄρα ἡ ΓΚ πρὸς τὴν ΝΜ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ [καὶ ἐπεὶ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΒΗ, σύμμετρόν ἐστι καὶ τὸ ΓΘ τῷ ΚΛ, τουτέστιν ἡ ΓΚ τῇ ΚΜ]. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΜΖ ἴσον παρὰ τὴν μείζονα τὴν ΓΜ παραβέβληται ἐλλείπον εἶδει τετραγώνῳ τὸ ὑπὸ τῶν ΓΚ, ΚΜ καὶ εἰς σύμμετρα αὐτὴν διαιρεῖ, ἡ ἄρα ΓΜ τῆς ΜΖ μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ μήκει. καὶ ἐστὶν ἡ προσαρμόζουσα ἡ ΖΜ σύμμετρος μήκει τῇ ἐκκειμένη ρητῇ τῇ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστὶ δευτέρα.

Τὸ ἄρα ἀπὸ μέσης ἀποτομῆς πρώτης παρὰ ρητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν δευτέραν· ὅπερ ἔδει δεῖξαι.

ιθ'.

Τὸ ἀπὸ μέσης ἀποτομῆς δευτέρας παρὰ ρητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τρίτην.



Ἐστω μέσης ἀποτομή δευτέρα ἡ ΑΒ, ρητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβελήσθω τὸ ΓΕ πλάτος ποιοῦν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστὶ τρίτη.

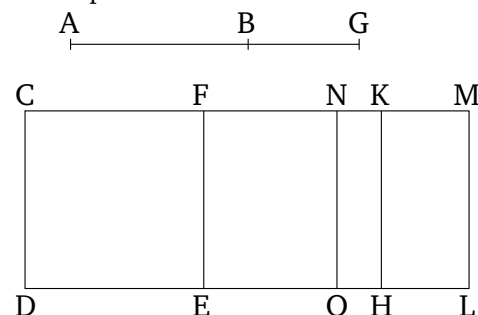
Ἐστω γάρ τῇ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ μέσαι εἰσὶ δυνάμει μόνον σύμμετροι μέσον περιέχουσαι. καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβελήσθω τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΒΗ ἴσον παρὰ τὴν ΚΘ παραβελήσθω τὸ ΚΛ πλάτος ποιοῦν τὴν ΚΜ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ [καὶ ἐστὶ μέσα τὰ ἀπὸ τῶν ΑΗ, ΗΒ]· μέσον ἄρα καὶ τὸ ΓΛ. καὶ παρὰ ρητὴν τὴν

BG to KL , NL is thus also the mean proportional to CH and KL . Thus, as CH is to NL , so NL (is) to KL [Prop. 5.11]. But, as CH (is) to NL , so CK is to NM , and as NL (is) to KL , so NM is to MK [Prop. 6.1]. Thus, as CK (is) to NM , so NM is to KM [Prop. 5.11]. The (rectangle contained) by CK and KM is thus equal to the (square) on NM [Prop. 6.17]—that is to say, to the fourth part of the (square) on FM [and since the (square) on AG is commensurable with the (square) on BG , CH is also commensurable with KL —that is to say, CK with KM]. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM , equal to the fourth part of the (square) on MF , has been applied to the greater CM , falling short by a square figure, and divides it into commensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable in length with (CM) [Prop. 10.17]. The attachment FM is also commensurable in length with the (previously) laid down rational (straight-line) CD . CF is thus a second apotome [Def. 10.16].

Thus, the (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces a second apotome as breadth. (Which is) the very thing it was required to show.

Proposition 99

The (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces a third apotome as breadth.



Let AB be the second apotome of a medial (straight-line), and CD a rational (straight-line). And let CE , equal to the (square) on AB , have been applied to CD , producing CF as breadth. I say that CF is a third apotome.

For let BG be an attachment to AB . Thus, AG and GB are medial (straight-lines which are) commensurable in square only, containing a medial (area) [Prop. 10.75]. And let CH , equal to the (square) on AG , have been applied to CD , producing CK as breadth. And let KL ,

ΓΔ παραβέβληται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ ὅλον τὸ ΓΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ὡς τὸ ΓΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ, λοιπὸν ἄρα τὸ ΑΖ ἴσον ἐστὶ τῷ δις ὑπὸ τῶν ΑΗ, ΗΒ. τετμήσθω οὖν ἡ ΖΜ δίχα κατὰ τὸ Ν σημεῖον, καὶ τῇ ΓΔ παράλληλος ἦχθω ἡ ΝΞ· ἐκάτερον ἄρα τῶν ΖΞ, ΝΑ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. μέσον δὲ τὸ ὑπὸ τῶν ΑΗ, ΗΒ· μέσον ἄρα ἐστὶ καὶ τὸ ΖΑ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα καὶ ἡ ΖΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ αἱ ΑΗ, ΗΒ δυνάμει μόνον εἰσὶ σύμμετροι, ἀσύμμετρος ἄρα [ἐστὶ] μήκει ἡ ΑΗ τῇ ΗΒ· ἀσύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΑΗ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΗ σύμμετρά ἐστι τὰ ἀπὸ τῶν ΑΗ, ΗΒ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ τὸ δις ὑπὸ τῶν ΑΗ, ΗΒ· ἀσύμμετρα ἄρα ἐστὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ τῷ δις ὑπὸ τῶν ΑΗ, ΗΒ. ἀλλὰ τοῖς μὲν ἀπὸ τῶν ΑΗ, ΗΒ ἴσον ἐστὶ τὸ ΓΑ, τῷ δὲ δις ὑπὸ τῶν ΑΗ, ΗΒ ἴσον ἐστὶ τὸ ΖΑ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΑ τῷ ΖΑ. ὥς δὲ τὸ ΓΑ πρὸς τὸ ΖΑ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΖΜ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ ΖΜ μήκει. καὶ εἰσιν ἀμφοτέραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἀποτομή ἄρα ἐστὶν ἡ ΓΖ. λέγω δὴ, ὅτι καὶ τρίτη.

Ἐπεὶ γὰρ σύμμετρόν ἐστι τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ, σύμμετρον ἄρα καὶ τὸ ΓΘ τῷ ΚΑ· ὥστε καὶ ἡ ΓΚ τῇ ΚΜ. καὶ ἐπεὶ τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καὶ ἐστὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΑ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΝΑ, καὶ τῶν ΓΘ, ΚΑ ἄρα μέσον ἀνάλογόν ἐστι τὸ ΝΑ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΑ, οὕτως τὸ ΝΑ πρὸς τὸ ΚΑ. ἀλλ' ὡς μὲν τὸ ΓΘ πρὸς τὸ ΝΑ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΝΜ, ὥς δὲ τὸ ΝΑ πρὸς τὸ ΚΑ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· ὡς ἄρα ἡ ΓΚ πρὸς τὴν ΜΝ, οὕτως ἐστὶν ἡ ΜΝ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ [ἀπὸ τῆς ΜΝ, τουτέστι τῷ] τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ ἴσον παρὰ τὴν ΓΜ παραβέβληται ἐλλείπον εἶδει τετραγώνῳ καὶ εἰς σύμμετρα αὐτὴν διαιρεῖ, ἡ ΓΜ ἄρα τῆς ΜΖ μείζον δύναται τῷ ἀπὸ συμέτρου ἑαυτῇ. καὶ οὐδετέρω τῶν ΓΜ, ΜΖ σύμμετρός ἐστι μήκει τῇ ἐκκειμένη ῥητῇ τῇ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστὶ τρίτη.

Τὸ ἄρα ἀπὸ μέσης ἀποτομῆς δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τρίτην· ὅπερ ἔδει δεῖξαι.

equal to the (square) on BG , have been applied to KH , producing KM as breadth. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB [and the (sum of the squares) on AG and GB is medial]. CL (is) thus also medial [Props. 10.15, 10.23 corr.]. And it has been applied to the rational (straight-line) CD , producing CM as breadth. Thus, CM is rational, and incommensurable in length with CD [Prop. 10.22]. And since the whole of CL is equal to the (sum of the squares) on AG and GB , of which CE is equal to the (square) on AB , the remainder LF is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Therefore, let FM have been cut in half at point N . And let NO have been drawn parallel to CD . Thus, FO and NL are each equal to the (rectangle contained) by AG and GB . And the (rectangle contained) by AG and GB (is) medial. Thus, FL is also medial. And it is applied to the rational (straight-line) EF , producing FM as breadth. FM is thus rational, and incommensurable in length with CD [Prop. 10.22]. And since AG and GB are commensurable in square only, AG [is] thus incommensurable in length with GB . Thus, the (square) on AG is also incommensurable with the (rectangle contained) by AG and GB [Props. 6.1, 10.11]. But, the (sum of the squares) on AG and GB is commensurable with the (square) on AG , and twice the (rectangle contained) by AG and GB with the (rectangle contained) by AG and GB . The (sum of the squares) on AG and GB is thus incommensurable with twice the (rectangle contained) by AG and GB [Prop. 10.13]. But, CL is equal to the (sum of the squares) on AG and GB , and FL is equal to twice the (rectangle contained) by AG and GB . Thus, CL is incommensurable with FL . And as CL (is) to FL , so CM is to FM [Prop. 6.1]. CM is thus incommensurable in length with FM [Prop. 10.11]. And they are both rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a third (apotome).

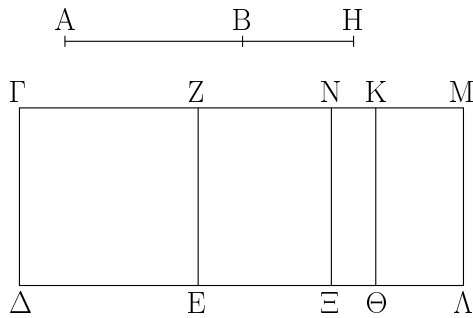
For since the (square) on AG is commensurable with the (square) on GB , CH (is) thus also commensurable with KL . Hence, CK (is) also (commensurable in length) with KM [Props. 6.1, 10.11]. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and CH is equal to the (square) on AG , and KL equal to the (square) on GB , and NL equal to the (rectangle contained) by AG and GB , NL is thus also the mean proportional to CH and KL . Thus, as CH is to NL , so NL (is) to KL . But, as CH (is) to NL , so CK (is) to NM , and as NL (is) to KL , so NM (is) to KM [Prop. 6.1].

Thus, as CK (is) to MN , so MN is to KM [Prop. 5.11]. Thus, the (rectangle contained) by CK and KM is equal to the [(square) on MN —that is to say, to the] fourth part of the (square) on FM [Prop. 6.17]. Therefore, since CM and MF are two unequal straight-lines, and (some area), equal to the fourth part of the (square) on FM , has been applied to CM , falling short by a square figure, and divides it into commensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) commensurable (in length) with (CM) [Prop. 10.17]. And neither of CM and MF is commensurable in length with the (previously) laid down rational (straight-line) CD . CF is thus a third apotome [Def. 10.13].

Thus, the (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces a third apotome as breadth. (Which is) the very thing it was required to show.

ρ'.

Τὸ ἀπὸ ἐλάσσονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τετάρτην.

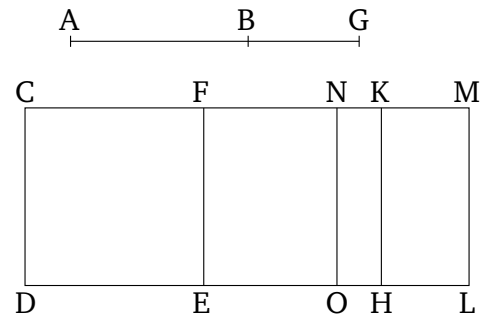


Ἐστω ἐλάσσων ἡ AB , ῥητὴ δὲ ἡ $ΓΔ$, καὶ τῷ ἀπὸ τῆς AB ἴσον παρὰ ῥητὴν τὴν $ΓΔ$ παραβεβλήσθω τὸ $ΓΕ$ πλάτος ποιοῦν τὴν $ΓΖ$ · λέγω, ὅτι ἡ $ΓΖ$ ἀποτομή ἐστὶ τετάρτη.

Ἐστω γὰρ τῇ AB προσαρμόζουσα ἡ BH · αἱ ἄρα AH , HB δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν AH , HB τετραγώνων ῥητόν, τὸ δὲ δις ὑπὸ τῶν AH , HB μέσον. καὶ τῷ μὲν ἀπὸ τῆς AH ἴσον παρὰ τὴν $ΓΔ$ παραβεβλήσθω τὸ $ΓΘ$ πλάτος ποιοῦν τὴν $ΓΚ$, τῷ δὲ ἀπὸ τῆς BH ἴσον τὸ $ΚΛ$ πλάτος ποιοῦν τὴν $ΚΜ$ · ὅλον ἄρα τὸ $ΓΛ$ ἴσον ἐστὶ τοῖς ἀπὸ τῶν AH , HB . καὶ ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AH , HB ῥητόν· ῥητόν ἄρα ἐστὶ καὶ τὸ $ΓΛ$. καὶ παρὰ ῥητὴν τὴν $ΓΔ$ παράκειται πλάτος ποιοῦν τὴν $ΓΜ$ · ῥητὴ ἄρα καὶ ἡ $ΓΜ$ καὶ σύμμετρος τῇ $ΓΔ$ μήκει. καὶ ἐπεὶ ὅλον τὸ $ΓΛ$ ἴσον ἐστὶ τοῖς ἀπὸ τῶν AH , HB , ὧν τὸ $ΓΕ$ ἴσον ἐστὶ τῷ ἀπὸ τῆς AB , λοιπὸν ἄρα τὸ $ΖΛ$ ἴσον ἐστὶ τῷ δις ὑπὸ τῶν AH , HB . τετμήσθω οὖν ἡ $ΖΜ$ δίχα κατὰ τὸ N σημεῖον, καὶ ἦχθω διὰ τοῦ N ὁποτέρῃ τῶν $ΓΔ$,

Proposition 100

The (square) on a minor (straight-line), applied to a rational (straight-line), produces a fourth apotome as breadth.



Let AB be a minor (straight-line), and CD a rational (straight-line). And let CE , equal to the (square) on AB , have been applied to the rational (straight-line) CD , producing CF as breadth. I say that CF is a fourth apotome.

For let BG be an attachment to AB . Thus, AG and GB are incommensurable in square, making the sum of the squares on AG and GB rational, and twice the (rectangle contained) by AG and GB medial [Prop. 10.76]. And let CH , equal to the (square) on AG , have been applied to CD , producing CK as breadth, and KL , equal to the (square) on BG , producing KM as breadth. Thus, the whole of CL is equal to the (sum of the squares) on AG and GB . And the sum of the (squares) on AG and GB is rational. CL is thus also rational. And it is applied to the rational (straight-line) CD , producing CM as breadth. Thus, CM (is) also rational, and commensurable in length with CD [Prop. 10.20]. And since the

ΜΑ παράλληλος ἡ ΝΞ· ἐκάτερον ἄρα τῶν ΖΞ, ΝΑ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ τὸ δις ὑπὸ τῶν ΑΗ, ΗΒ μέσον ἐστὶ καὶ ἐστὶν ἴσον τῷ ΖΑ, καὶ τὸ ΖΑ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΖΕ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΑΗ, ΗΒ ῥητόν ἐστιν, τὸ δὲ δις ὑπὸ τῶν ΑΗ, ΗΒ μέσον, ἀσύμμετρα [ἄρα] ἐστὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ τῷ δις ὑπὸ τῶν ΑΗ, ΗΒ. ἴσον δέ [ἐστὶ] τὸ ΓΑ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, τῷ δὲ δις ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΖΑ· ἀσύμμετρον ἄρα [ἐστὶ] τὸ ΓΑ τῷ ΖΑ. ὥς δὲ τὸ ΓΑ πρὸς τὸ ΖΑ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΜΖ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ ΜΖ μήκει. καὶ εἰσιν ἀμφοτέραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἀποτομή ἄρα ἐστὶν ἡ ΓΖ. λέγω [δὴ], ὅτι καὶ τετάρτη.

Ἐπεὶ γὰρ αἱ ΑΗ, ΗΒ δυνάμει εἰσὶν ἀσύμμετροι, ἀσύμμετρον ἄρα καὶ τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ. καὶ ἐστὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΑ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΘ τῷ ΚΑ. ὥς δὲ τὸ ΓΘ πρὸς τὸ ΚΑ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΚΜ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΚ τῇ ΚΜ μήκει. καὶ ἐπεὶ τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καὶ ἐστὶν ἴσον τὸ μὲν ἀπὸ τῆς ΑΗ τῷ ΓΘ, τὸ δὲ ἀπὸ τῆς ΗΒ τῷ ΚΑ, τὸ δὲ ὑπὸ τῶν ΑΗ, ΗΒ τῷ ΝΑ, τῶν ἄρα ΓΘ, ΚΑ μέσον ἀνάλογόν ἐστι τὸ ΝΑ· ἔστιν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΑ, οὕτως τὸ ΝΑ πρὸς τὸ ΚΑ. ἀλλ' ὡς μὲν τὸ ΓΘ πρὸς τὸ ΝΑ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΝΜ, ὥς δὲ τὸ ΝΑ πρὸς τὸ ΚΑ, οὕτως ἐστὶν ἡ ΝΜ πρὸς τὴν ΚΜ· ὡς ἄρα ἡ ΓΚ πρὸς τὴν ΜΝ, οὕτως ἐστὶν ἡ ΜΝ πρὸς τὴν ΚΜ· τὸ ἄρα ὑπὸ τῶν ΓΚ, ΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΜΝ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΜΖ ἴσον παρὰ τὴν ΓΜ παραβέβληται ἐλλείπον εἶδει τετραγώνῳ τὸ ὑπὸ τῶν ΓΚ, ΚΜ καὶ εἰς ἀσύμμετρα αὐτὴν διαιρεῖ, ἡ ἄρα ΓΜ τῆς ΜΖ μείζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ ἐστὶν ὅλη ἡ ΓΜ σύμμετρος μήκει τῇ ἐκκειμένη ῥητῇ τῇ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστὶ τετάρτη.

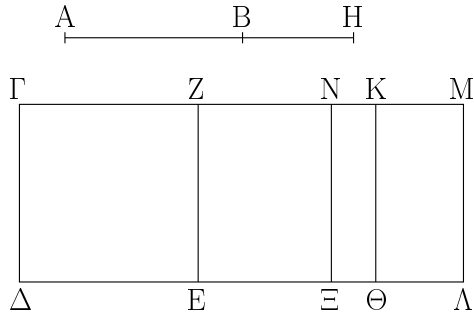
Τὸ ἄρα ἀπὸ ἐλάσσονος καὶ τὰ ἐξῆς.

whole of CL is equal to the (sum of the squares) on AG and GB , of which CE is equal to the (square) on AB , the remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Therefore, let FM have been cut in half at point N . And let NO have been drawn through N , parallel to either of CD or ML . Thus, FO and NL are each equal to the (rectangle contained) by AG and GB . And since twice the (rectangle contained) by AG and GB is medial, and is equal to FL , FL is thus also medial. And it is applied to the rational (straight-line) FE , producing FM as breadth. Thus, FM is rational, and incommensurable in length with CD [Prop. 10.22]. And since the sum of the (squares) on AG and GB is rational, and twice the (rectangle contained) by AG and GB medial, the (sum of the squares) on AG and GB is [thus] incommensurable with twice the (rectangle contained) by AG and GB . And CL (is) equal to the (sum of the squares) on AG and GB , and FL equal to twice the (rectangle contained) by AG and GB . CL [is] thus incommensurable with FL . And as CL (is) to FL , so CM is to MF [Prop. 6.1]. CM is thus incommensurable in length with MF [Prop. 10.11]. And both are rational (straight-lines). Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. [So], I say that (it is) also a fourth (apotome).

For since AG and GB are incommensurable in square, the (square) on AG (is) thus also incommensurable with the (square) on GB . And CH is equal to the (square) on AG , and KL equal to the (square) on GB . Thus, CH is incommensurable with KL . And as CH (is) to KL , so CK is to KM [Prop. 6.1]. CK is thus incommensurable in length with KM [Prop. 10.11]. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and the (square) on AG is equal to CH , and the (square) on GB to KL , and the (rectangle contained) by AG and GB to NL , NL is thus the mean proportional to CH and KL . Thus, as CH is to NL , so NL (is) to KL . But, as CH (is) to NL , so CK is to NM , and as NL (is) to KL , so NM is to KM [Prop. 6.1]. Thus, as CK (is) to MN , so MN is to KM [Prop. 5.11]. The (rectangle contained) by CK and KM is thus equal to the (square) on MN —that is to say, to the fourth part of the (square) on FM [Prop. 6.17]. Therefore, since CM and MF are two unequal straight-lines, and the (rectangle contained) by CK and KM , equal to the fourth part of the (square) on MF , has been applied to CM , falling short by a square figure, and divides it into incommensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) incommensurable

ρα'.

Τὸ ἀπὸ τῆς μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πέμπτην.



Ἐστω ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσα ἡ ΑΒ, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς ΑΒ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΕ πλάτος ποιούν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστὶ πέμπτη.

Ἐστω γάρ τῃ ΑΒ προσαρμόζουσα ἡ ΒΗ· αἱ ἄρα ΑΗ, ΗΒ εὐθεῖαι δυνάμει εἰσὶν ἀσύμμετροι ποιούσαι τὸ μὲν συγχείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον, τὸ δὲ δις ὑπ' αὐτῶν ῥητόν, καὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ· ὅλον ἄρα τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ. τὸ δὲ συγχείμενον ἐκ τῶν ἀπὸ τῶν ΑΗ, ΗΒ ἅμα μέσον ἐστίν· μέσον ἄρα ἐστὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράκειται πλάτος ποιούν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ ἀσύμμετρος τῇ ΓΔ. καὶ ἐπεὶ ὅλον τὸ ΓΛ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ὧν τὸ ΓΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΒ, λοιπὸν ἄρα τὸ ΖΛ ἴσον ἐστὶ τῷ δις ὑπὸ τῶν ΑΗ, ΗΒ. τετμήσθω οὖν ἡ ΖΜ δίχα κατὰ τὸ Ν, καὶ ῥηθὼ διὰ τοῦ Ν ὁποτέρῃ τῶν ΓΔ, ΜΛ παράλληλος ἡ ΝΞ· ἐκάτερον ἄρα τῶν ΖΞ, ΝΛ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ, καὶ ἐπεὶ τὸ δις ὑπὸ τῶν ΑΗ, ΗΒ ῥητόν ἐστὶ καὶ [ἐστίν] ἴσον τῷ ΖΛ, ῥητόν ἄρα ἐστὶ τὸ ΖΛ. καὶ παρὰ ῥητὴν τὴν ΕΖ παράκειται πλάτος ποιούν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ σύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὸ μὲν ΓΛ μέσον ἐστίν, τὸ δὲ ΖΛ ῥητόν, ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΛ τῷ ΖΛ. ὥς δὲ τὸ ΓΛ πρὸς τὸ ΖΛ, οὕτως ἡ ΓΜ πρὸς τὴν ΜΖ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ ΜΖ μήκει. καὶ εἰσὶν ἀμφοτέραι ῥηταί· αἱ ἄρα ΓΜ, ΜΖ ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἀποτομή ἄρα ἐστὶν ἡ ΓΖ. λέγω δὴ, ὅτι καὶ πέμπτη.

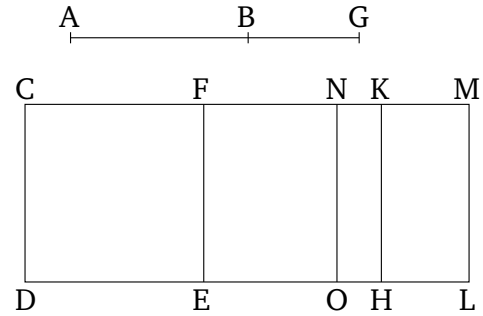
Ὅμοιως γὰρ δεῖξομεν, ὅτι τὸ ὑπὸ τῶν ΓΚΜ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΝΜ, τουτέστι τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς

(in length) with (CM) [Prop. 10.18]. And the whole of CM is commensurable in length with the (previously) laid down rational (straight-line) CD. Thus, CF is a fourth apotome [Def. 10.14].

Thus, the (square) on a minor, and so on ...

Proposition 101

The (square) on that (straight-line) which with a rational (area) makes a medial whole, applied to a rational (straight-line), produces a fifth apotome as breadth.



Let AB be that (straight-line) which with a rational (area) makes a medial whole, and CD a rational (straight-line). And let CE, equal to the (square) on AB, have been applied to CD, producing CF as breadth. I say that CF is a fifth apotome.

Let BG be an attachment to AB. Thus, the straight-lines AG and GB are incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle contained) by them rational [Prop. 10.77]. And let CH, equal to the (square) on AG, have been applied to CD, and KL, equal to the (square) on GB. The whole of CL is thus equal to the (sum of the squares) on AG and GB. And the sum of the (squares) on AG and GB together is medial. Thus, CL is medial. And it has been applied to the rational (straight-line) CD, producing CM as breadth. CM is thus rational, and incommensurable (in length) with CD [Prop. 10.22]. And since the whole of CL is equal to the (sum of the squares) on AG and GB, of which CE is equal to the (square) on AB, the remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. Therefore, let FM have been cut in half at N. And let NO have been drawn through N, parallel to either of CD or ML. Thus, FO and NL are each equal to the (rectangle contained) by AG and GB. And since twice the (rectangle contained) by AG and GB is rational, and [is] equal to FL, FL is thus rational. And it is applied to the rational (straight-line) EF, producing FM as breadth. Thus, FM is rational, and commensurable in length with CD [Prop. 10.20]. And since CL is medial, and FL rational,

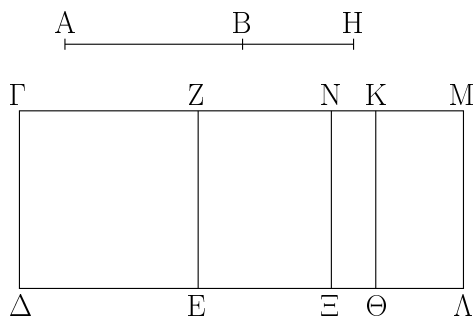
ZM. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ ἀπὸ τῆς AH τῷ ἀπὸ τῆς HB, ἴσον δὲ τὸ μὲν ἀπὸ τῆς AH τῷ ΓΘ, τὸ δὲ ἀπὸ τῆς HB τῷ ΚΑ, ἀσύμμετρον ἄρα τὸ ΓΘ τῷ ΚΑ. ὥς δὲ τὸ ΓΘ πρὸς τὸ ΚΑ, οὕτως ἡ ΓΚ πρὸς τὴν ΚΜ· ἀσύμμετρος ἄρα ἡ ΓΚ τῇ ΚΜ μήκει. ἐπεὶ οὖν δύο εὐθεῖαι ἄνισοί εἰσιν αἱ ΓΜ, ΜΖ, καὶ τῷ τετάρτῳ μέρει τοῦ ἀπὸ τῆς ΖΜ ἴσον παρὰ τὴν ΓΜ παραβέβληται ἐλλείπον εἶδει τετραγώνῳ καὶ εἰς ἀσύμμετρα αὐτὴν διαιρεῖ, ἡ ἄρα ΓΜ τῆς ΜΖ μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ. καὶ ἐστὶν ἡ προσαρμόζουσα ἡ ΖΜ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ· ἡ ἄρα ΓΖ ἀποτομή ἐστὶ πέμπτη· ὅπερ ἔδει δεῖξαι.

CL is thus incommensurable with FL . And as CL (is) to FL , so CM (is) to MF [Prop. 6.1]. CM is thus incommensurable in length with MF [Prop. 10.11]. And both are rational. Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a fifth (apotome).

For, similarly (to the previous propositions), we can show that the (rectangle contained) by CKM is equal to the (square) on NM —that is to say, to the fourth part of the (square) on FM . And since the (square) on AG is incommensurable with the (square) on GB , and the (square) on AG (is) equal to CH , and the (square) on GB to KL , CH (is) thus incommensurable with KL . And as CH (is) to KL , so CK (is) to KM [Prop. 6.1]. Thus, CK (is) incommensurable in length with KM [Prop. 10.11]. Therefore, since CM and MF are two unequal straight-lines, and (some area), equal to the fourth part of the (square) on FM , has been applied to CM , falling short by a square figure, and divides it into incommensurable (parts), the square on CM is thus greater than (the square on) MF by the (square) on (some straight-line) incommensurable (in length) with (CM) [Prop. 10.18]. And the attachment FM is commensurable with the (previously) laid down rational (straight-line) CD . Thus, CF is a fifth apotome [Def. 10.15]. (Which is) the very thing it was required to show.

ρβ'.

Τὸ ἀπὸ τῆς μετὰ μέσου μέσον τὸ ὅλον ποιούσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν ἕκτην.

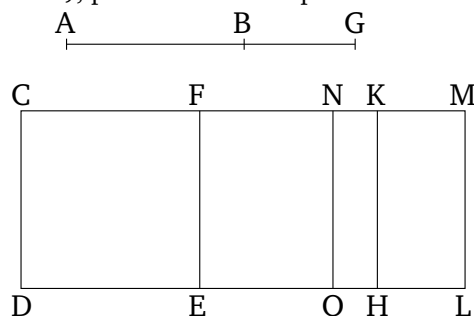


Ἐστω ἡ μετὰ μέσου μέσον τὸ ὅλον ποιούσα ἡ AB, ῥητὴ δὲ ἡ ΓΔ, καὶ τῷ ἀπὸ τῆς AB ἴσον παρὰ τὴν ΓΔ παραβελήσθω τὸ ΓΕ πλάτος ποιῶν τὴν ΓΖ· λέγω, ὅτι ἡ ΓΖ ἀποτομή ἐστὶν ἕκτη.

Ἐστω γὰρ τῇ AB προσαρμόζουσα ἡ BH· αἱ ἄρα AH, HB δυνάμει εἰσὶν ἀσύμμετροι ποιούσαι τό τε συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον καὶ τὸ δις ὑπὸ τῶν AH, HB μέσον καὶ ἀσύμμετρον τὰ ἀπὸ τῶν AH, HB τῷ

Proposition 102

The (square) on that (straight-line) which with a medial (area) makes a medial whole, applied to a rational (straight-line), produces a sixth apotome as breadth.



Let AB be that (straight-line) which with a medial (area) makes a medial whole, and CD a rational (straight-line). And let CE , equal to the (square) on AB , have been applied to CD , producing CF as breadth. I say that CF is a sixth apotome.

For let BG be an attachment to AB . Thus, AG and GB are incommensurable in square, making the sum of the squares on them medial, and twice the (rectangle

δις ὑπὸ τῶν ΑΗ, ΗΒ. παραβεβλήσθω οὖν παρὰ τὴν ΓΔ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ πλάτος ποιοῦν τὴν ΓΚ, τῷ δὲ ἀπὸ τῆς ΒΗ τὸ ΚΛ· ὅλον ἄρα τὸ ΓΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ· μέσον ἄρα [ἐστὶ] καὶ τὸ ΓΛ. καὶ παρὰ ῥητὴν τὴν ΓΔ παράκειται πλάτος ποιοῦν τὴν ΓΜ· ῥητὴ ἄρα ἐστὶν ἡ ΓΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. ἐπεὶ οὖν τὸ ΓΑ ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΑΗ, ΗΒ, ὡς τὸ ΓΕ ἴσον τῷ ἀπὸ τῆς ΑΒ, λοιπὸν ἄρα τὸ ΖΑ ἴσον ἐστὶ τῷ δις ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐστὶ τὸ δις ὑπὸ τῶν ΑΗ, ΗΒ μέσον· καὶ τὸ ΖΑ ἄρα μέσον ἐστίν. καὶ παρὰ ῥητὴν τὴν ΖΕ παράκειται πλάτος ποιοῦν τὴν ΖΜ· ῥητὴ ἄρα ἐστὶν ἡ ΖΜ καὶ ἀσύμμετρος τῇ ΓΔ μήκει. καὶ ἐπεὶ τὰ ἀπὸ τῶν ΑΗ, ΗΒ ἀσύμμετρά ἐστι τῷ δις ὑπὸ τῶν ΑΗ, ΗΒ, καὶ ἐστὶ τοῖς μὲν ἀπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΓΑ, τῷ δὲ δις ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΖΑ, ἀσύμμετρος ἄρα [ἐστὶ] τὸ ΓΑ τῷ ΖΑ. ὡς δὲ τὸ ΓΑ πρὸς τὸ ΖΑ, οὕτως ἐστὶν ἡ ΓΜ πρὸς τὴν ΜΖ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΜ τῇ ΜΖ μήκει. καὶ εἰσιν ἀμφοτέραι ῥηταί. αἱ ΓΜ, ΜΖ ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι· ἀποτομή ἄρα ἐστὶν ἡ ΓΖ. λέγω δὴ, ὅτι καὶ ἕκτη.

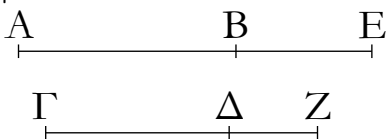
Ἐπεὶ γάρ τὸ ΖΑ ἴσον ἐστὶ τῷ δις ὑπὸ τῶν ΑΗ, ΗΒ, τετμήσθω δίχα ἡ ΖΜ κατὰ τὸ Ν, καὶ ἵχθω διὰ τοῦ Ν τῇ ΓΔ παράλληλος ἡ ΝΞ· ἐκάτερον ἄρα τῶν ΖΞ, ΝΑ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΑΗ, ΗΒ. καὶ ἐπεὶ αἱ ΑΗ, ΗΒ δυνάμει εἰσιν ἀσύμμετροι, ἀσύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΑΗ τῷ ἀπὸ τῆς ΗΒ. ἀλλὰ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον ἐστὶ τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον ἐστὶ τὸ ΚΛ· ἀσύμμετρον ἄρα ἐστὶ τὸ ΓΘ τῷ ΚΛ. ὡς δὲ τὸ ΓΘ πρὸς τὸ ΚΛ, οὕτως ἐστὶν ἡ ΓΚ πρὸς τὴν ΚΜ· ἀσύμμετρος ἄρα ἐστὶν ἡ ΓΚ τῇ ΚΜ. καὶ ἐπεὶ τῶν ἀπὸ τῶν ΑΗ, ΗΒ μέσον ἀνάλογόν ἐστι τὸ ὑπὸ τῶν ΑΗ, ΗΒ, καὶ ἐστὶ τῷ μὲν ἀπὸ τῆς ΑΗ ἴσον τὸ ΓΘ, τῷ δὲ ἀπὸ τῆς ΗΒ ἴσον τὸ ΚΛ, τῷ δὲ ὑπὸ τῶν ΑΗ, ΗΒ ἴσον τὸ ΝΑ, καὶ τῶν ἄρα ΓΘ, ΚΛ μέσον ἀνάλογόν ἐστι τὸ ΝΑ· ἐστὶν ἄρα ὡς τὸ ΓΘ πρὸς τὸ ΝΑ, οὕτως τὸ ΝΑ πρὸς τὸ ΚΛ. καὶ διὰ τὰ αὐτὰ ἡ ΓΜ τῆς ΜΖ μείζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ οὐδετέρα αὐτῶν σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ τῇ ΓΔ· ἡ ΓΖ ἄρα ἀποτομή ἐστὶν ἕκτη· ὅπερ εἶδει δεῖξαι.

contained) by AG and GB medial, and the (sum of the squares) on AG and GB incommensurable with twice the (rectangle contained) by AG and GB [Prop. 10.78]. Therefore, let CH , equal to the (square) on AG , have been applied to CD , producing CK as breadth, and KL , equal to the (square) on BG . Thus, the whole of CL is equal to the (sum of the squares) on AG and GB . CL [is] thus also medial. And it is applied to the rational (straight-line) CD , producing CM as breadth. Thus, CM is rational, and incommensurable in length with CD [Prop. 10.22]. Therefore, since CL is equal to the (sum of the squares) on AG and GB , of which CE (is) equal to the (square) on AB , the remainder FL is thus equal to twice the (rectangle contained) by AG and GB [Prop. 2.7]. And twice the (rectangle contained) by AG and GB (is) medial. Thus, FL is also medial. And it is applied to the rational (straight-line) FE , producing FM as breadth. FM is thus rational, and incommensurable in length with CD [Prop. 10.22]. And since the (sum of the squares) on AG and GB is incommensurable with twice the (rectangle contained) by AG and GB , and CL equal to the (sum of the squares) on AG and GB , and FL equal to twice the (rectangle contained) by AG and GB , CL [is] thus incommensurable with FL . And as CL (is) to FL , so CM is to MF [Prop. 6.1]. Thus, CM is incommensurable in length with MF [Prop. 10.11]. And they are both rational. Thus, CM and MF are rational (straight-lines which are) commensurable in square only. CF is thus an apotome [Prop. 10.73]. So, I say that (it is) also a sixth (apotome).

For since FL is equal to twice the (rectangle contained) by AG and GB , let FM have been cut in half at N , and let NO have been drawn through N , parallel to CD . Thus, FO and NL are each equal to the (rectangle contained) by AG and GB . And since AG and GB are incommensurable in square, the (square) on AG is thus incommensurable with the (square) on GB . But, CH is equal to the (square) on AG , and KL is equal to the (square) on GB . Thus, CH is incommensurable with KL . And as CH (is) to KL , so CK is to KM [Prop. 6.1]. Thus, CK is incommensurable (in length) with KM [Prop. 10.11]. And since the (rectangle contained) by AG and GB is the mean proportional to the (squares) on AG and GB [Prop. 10.21 lem.], and CH is equal to the (square) on AG , and KL equal to the (square) on GB , and NL equal to the (rectangle contained) by AG and GB , NL is thus also the mean proportional to CH and KL . Thus, as CH is to NL , so NL (is) to KL . And for the same (reasons as the preceding propositions), the square on CM is greater than (the square on) MF by the (square on) (some straight-line)

ργ'.

Ἡ τῇ ἀποτομῇ μήκει σύμμετρος ἀποτομή ἐστι καὶ τῇ τάξει ἡ αὐτή.



Ἐστω ἀποτομή ἡ AB , καὶ τῇ AB μήκει σύμμετρος ἔστω ἡ $\Gamma\Delta$. λέγω, ὅτι καὶ ἡ $\Gamma\Delta$ ἀποτομή ἐστι καὶ τῇ τάξει ἡ αὐτὴ τῇ AB .

Ἐπεὶ γὰρ ἀποτομή ἐστὶν ἡ AB , ἔστω αὐτῇ προσαρμόζουσα ἡ BE . αἱ AE , EB ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι. καὶ τῷ τῆς AB πρὸς τὴν $\Gamma\Delta$ λόγῳ ὁ αὐτὸς γεγονέτω ὁ τῆς BE πρὸς τὴν ΔZ . καὶ ὡς ἐν ἄρα πρὸς ἐν, πάντα [ἐστὶ] πρὸς πάντα. ἔστιν ἄρα καὶ ὡς ὅλη ἡ AE πρὸς ὅλην τὴν ΓZ , οὕτως ἡ AB πρὸς τὴν $\Gamma\Delta$. σύμμετρος δὲ ἡ AB τῇ $\Gamma\Delta$ μήκει· σύμμετρος ἄρα καὶ ἡ AE μὲν τῇ ΓZ , ἡ δὲ BE τῇ ΔZ . καὶ αἱ AE , EB ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· καὶ αἱ ΓZ , ΔZ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι [ἀποτομὴ ἄρα ἐστὶν ἡ $\Gamma\Delta$. λέγω δὴ, ὅτι καὶ τῇ τάξει ἡ αὐτὴ τῇ AB].

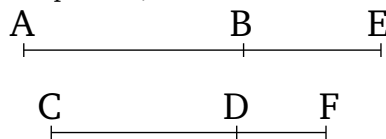
Ἐπεὶ οὖν ἐστὶν ὡς ἡ AE πρὸς τὴν ΓZ , οὕτως ἡ BE πρὸς τὴν ΔZ , ἐναλλάξ ἄρα ἐστὶν ὡς ἡ AE πρὸς τὴν EB , οὕτως ἡ ΓZ πρὸς τὴν ΔZ . ἥτοι δὴ ἡ AE τῆς EB μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυνμέτρου. εἰ μὲν οὖν ἡ AE τῆς EB μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ ἡ ΓZ τῆς ΔZ μείζον δύνησεται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστιν ἡ AE τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΓZ , εἰ δὲ ἡ BE , καὶ ἡ ΔZ , εἰ δὲ οὐδετέρα τῶν AE , EB , καὶ οὐδετέρα τῶν ΓZ , ΔZ . εἰ δὲ ἡ AE [τῆς EB] μείζον δύναται τῷ ἀπὸ ἀσυνμέτρου ἑαυτῇ, καὶ ἡ ΓZ τῆς ΔZ μείζον δύνησεται τῷ ἀπὸ ἀσυνμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρός ἐστὶν ἡ AE τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ΓZ , εἰ δὲ ἡ BE , καὶ ἡ ΔZ , εἰ δὲ οὐδετέρα τῶν AE , EB , οὐδετέρα τῶν ΓZ , ΔZ .

Ἀποτομὴ ἄρα ἐστὶν ἡ $\Gamma\Delta$ καὶ τῇ τάξει ἡ αὐτὴ τῇ AB . ὅπερ ἔδει δεῖξαι.

incommensurable (in length) with (CM) [Prop. 10.18]. And neither of them is commensurable with the (previously) laid down rational (straight-line) CD . Thus, CF is a sixth apotome [Def. 10.16]. (Which is) the very thing it was required to show.

Proposition 103

A (straight-line) commensurable in length with an apotome is an apotome, and (is) the same in order.



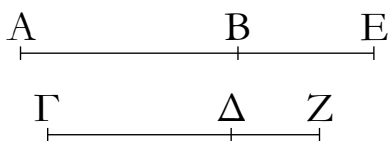
Let AB be an apotome, and let CD be commensurable in length with AB . I say that CD is also an apotome, and (is) the same in order as AB .

For since AB is an apotome, let BE be an attachment to it. Thus, AE and EB are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let it have been contrived that the (ratio) of BE to DF is the same as the ratio of AB to CD [Prop. 6.12]. Thus, also, as one is to one, (so) all [are] to all [Prop. 5.12]. And thus as the whole AE is to the whole CF , so AB (is) to CD . And AB (is) commensurable in length with CD . AE (is) thus also commensurable (in length) with CF , and BE with DF [Prop. 10.11]. And AE and BE are rational (straight-lines which are) commensurable in square only. Thus, CF and FD are also rational (straight-lines which are) commensurable in square only [Prop. 10.13]. [CD is thus an apotome. So, I say that (it is) also the same in order as AB .]

Therefore, since as AE is to CF , so BE (is) to DF , thus, alternately, as AE is to EB , so CF (is) to FD [Prop. 5.16]. So, the square on AE is greater than (the square on) EB either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (AE) . Therefore, if the (square) on AE is greater than (the square on) EB by the (square) on (some straight-line) commensurable (in length) with (AE) then the square on CF will also be greater than (the square on) FD by the (square) on (some straight-line) commensurable (in length) with (CF) [Prop. 10.14]. And if AE is commensurable in length with a (previously) laid down rational (straight-line) then so (is) CF [Prop. 10.12], and if BE (is commensurable), so (is) DF , and if neither of AE or EB (are commensurable), neither (are) either of CF or FD [Prop. 10.13]. And if the (square) on AE is greater [than (the square on) EB] by the (square) on (some straight-line) incommensurable (in

ρδ'.

Ἡ τῇ μέσῃ ἀποτομῇ σύμμετρος μέσῃ ἀποτομῇ ἐστὶ καὶ τῇ τάξει ἡ αὐτῇ.



Ἐστω μέσῃ ἀποτομῇ ἡ AB , καὶ τῇ AB μήκει σύμμετρος ἔστω ἡ $\Gamma\Delta$. λέγω, ὅτι καὶ ἡ $\Gamma\Delta$ μέσῃ ἀποτομῇ ἐστὶ καὶ τῇ τάξει ἡ αὐτῇ τῇ AB .

Ἐπεὶ γὰρ μέσῃ ἀποτομῇ ἐστὶν ἡ AB , ἔστω αὐτῇ προσαρμόζουσα ἡ EB . αἱ AE , EB ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι. καὶ γεγονέντω ὡς ἡ AB πρὸς τὴν $\Gamma\Delta$, οὕτως ἡ BE πρὸς τὴν $\Delta Ζ$. σύμμετρος ἄρα [ἐστὶ] καὶ ἡ AE τῇ $\Gamma Ζ$, ἡ δὲ BE τῇ $\Delta Ζ$. αἱ δὲ AE , EB μέσαι εἰσὶ δυνάμει μόνον σύμμετροι· καὶ αἱ $\Gamma Ζ$, $\Delta Ζ$ ἄρα μέσαι εἰσὶ δυνάμει μόνον σύμμετροι· μέσῃ ἄρα ἀποτομῇ ἐστὶν ἡ $\Gamma\Delta$. λέγω δὲ, ὅτι καὶ τῇ τάξει ἐστὶν ἡ αὐτῇ τῇ AB .

Ἐπεὶ [γὰρ] ἐστὶν ὡς ἡ AE πρὸς τὴν EB , οὕτως ἡ $\Gamma Ζ$ πρὸς τὴν $\Delta Ζ$ [ἀλλ' ὡς μὲν ἡ AE πρὸς τὴν EB , οὕτως τὸ ἀπὸ τῆς AE πρὸς τὸ ὑπὸ τῶν AE , EB , ὡς δὲ ἡ $\Gamma Ζ$ πρὸς τὴν $\Delta Ζ$, οὕτως τὸ ἀπὸ τῆς $\Gamma Ζ$ πρὸς τὸ ὑπὸ τῶν $\Gamma Ζ$, $\Delta Ζ$], ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς AE πρὸς τὸ ὑπὸ τῶν AE , EB , οὕτως τὸ ἀπὸ τῆς $\Gamma Ζ$ πρὸς τὸ ὑπὸ τῶν $\Gamma Ζ$, $\Delta Ζ$ [καὶ ἐναλλάξ ὡς τὸ ἀπὸ τῆς AE πρὸς τὸ ἀπὸ τῆς $\Gamma Ζ$, οὕτως τὸ ὑπὸ τῶν AE , EB πρὸς τὸ ὑπὸ τῶν $\Gamma Ζ$, $\Delta Ζ$]. σύμμετρον δὲ τὸ ἀπὸ τῆς AE τῷ ἀπὸ τῆς $\Gamma Ζ$ · σύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν AE , EB τῷ ὑπὸ τῶν $\Gamma Ζ$, $\Delta Ζ$. εἴτε οὖν ῥητόν ἐστι τὸ ὑπὸ τῶν AE , EB , ῥητόν ἐστὶ καὶ τὸ ὑπὸ τῶν $\Gamma Ζ$, $\Delta Ζ$, εἴτε μέσον [ἐστὶ] τὸ ὑπὸ τῶν AE , EB , μέσον [ἐστὶ] καὶ τὸ ὑπὸ τῶν $\Gamma Ζ$, $\Delta Ζ$.

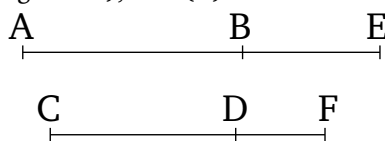
Μέσῃ ἄρα ἀποτομῇ ἐστὶν ἡ $\Gamma\Delta$ καὶ τῇ τάξει ἡ αὐτῇ τῇ AB . ὁπερ εἶδει δεῖξαι.

length) with (AE) then the (square) on CF will also be greater than (the square on) FD by the (square) on (some straight-line) incommensurable (in length) with (CF) [Prop. 10.14]. And if AE is commensurable in length with a (previously) laid down rational (straight-line), so (is) CF [Prop. 10.12], and if BE (is commensurable), so (is) DF , and if neither of AE or EB (are commensurable), neither (are) either of CF or FD [Prop. 10.13].

Thus, CD is an apotome, and (is) the same in order as AB [Defs. 10.11—10.16]. (Which is) the very thing it was required to show.

Proposition 104

A (straight-line) commensurable (in length) with an apotome of a medial (straight-line) is an apotome of a medial (straight-line), and (is) the same in order.



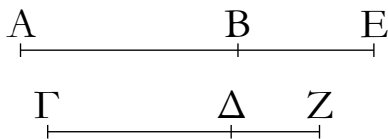
Let AB be an apotome of a medial (straight-line), and let CD be commensurable in length with AB . I say that CD is also an apotome of a medial (straight-line), and (is) the same in order as AB .

For since AB is an apotome of a medial (straight-line), let EB be an attachment to it. Thus, AE and EB are medial (straight-lines which are) commensurable in square only [Props. 10.74, 10.75]. And let it have been contrived that as AB is to CD , so BE (is) to DF [Prop. 6.12]. Thus, AE [is] also commensurable (in length) with CF , and BE with DF [Props. 5.12, 10.11]. And AE and EB are medial (straight-lines which are) commensurable in square only. CF and FD are thus also medial (straight-lines which are) commensurable in square only [Props. 10.23, 10.13]. Thus, CD is an apotome of a medial (straight-line) [Props. 10.74, 10.75]. So, I say that it is also the same in order as AB .

[For] since as AE is to EB , so CF (is) to FD [Props. 5.12, 5.16] [but as AE (is) to EB , so the (square) on AE (is) to the (rectangle contained) by AE and EB , and as CF (is) to FD , so the (square) on CF (is) to the (rectangle contained) by CF and FD], thus as the (square) on AE is to the (rectangle contained) by AE and EB , so the (square) on CF also (is) to the (rectangle contained) by CF and FD [Prop. 10.21 lem.] [and, alternately, as the (square) on AE (is) to the (square) on CF , so the (rectangle contained) by AE and EB (is) to the (rectangle contained) by CF and FD]. And the (square) on AE (is) commensurable with the (square)

ρε'.

Ἡ τῇ ἐλάσσονι σύμμετρος ἐλάσσων ἐστίν.



Ἐστω γὰρ ἐλάσσων ἡ AB καὶ τῇ AB σύμμετρος ἡ $\Gamma\Delta$. λέγω, ὅτι καὶ ἡ $\Gamma\Delta$ ἐλάσσων ἐστίν.

Γεγονέντω γὰρ τὰ αὐτά· καὶ ἐπεὶ αἱ AE , EB δυνάμει εἰσὶν ἀσύμμετροι, καὶ αἱ ΓZ , $Z\Delta$ ἄρα δυνάμει εἰσὶν ἀσύμμετροι. ἐπεὶ οὖν ἐστὶν ὡς ἡ AE πρὸς τὴν EB , οὕτως ἡ ΓZ πρὸς τὴν $Z\Delta$, ἔστιν ἄρα καὶ ὡς τὸ ἀπὸ τῆς AE πρὸς τὸ ἀπὸ τῆς EB , οὕτως τὸ ἀπὸ τῆς ΓZ πρὸς τὸ ἀπὸ τῆς $Z\Delta$. συνθέντι ἄρα ἐστὶν ὡς τὰ ἀπὸ τῶν AE , EB πρὸς τὸ ἀπὸ τῆς EB , οὕτως τὰ ἀπὸ τῶν ΓZ , $Z\Delta$ πρὸς τὸ ἀπὸ τῆς $Z\Delta$ [καὶ ἐναλλάξ]· σύμμετρον δὲ ἐστὶ τὸ ἀπὸ τῆς BE τῷ ἀπὸ τῆς ΔZ · σύμμετρον ἄρα καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AE , EB τετραγώνων τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΓZ , $Z\Delta$ τετραγώνων. ῥητὸν δὲ ἐστὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AE , EB τετραγώνων· ῥητὸν ἄρα ἐστὶ καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓZ , $Z\Delta$ τετραγώνων. πάλιν, ἐπεὶ ἐστὶν ὡς τὸ ἀπὸ τῆς AE πρὸς τὸ ὑπὸ τῶν AE , EB , οὕτως τὸ ἀπὸ τῆς ΓZ πρὸς τὸ ὑπὸ τῶν ΓZ , $Z\Delta$, σύμμετρον δὲ τὸ ἀπὸ τῆς AE τετράγωνον τῷ ἀπὸ τῆς ΓZ τετραγώνῳ, σύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν AE , EB τῷ ὑπὸ τῶν ΓZ , $Z\Delta$. μέσον δὲ τὸ ὑπὸ τῶν AE , EB · μέσον ἄρα καὶ τὸ ὑπὸ τῶν ΓZ , $Z\Delta$ · αἱ ΓZ , $Z\Delta$ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιοῦσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων ῥητὸν, τὸ δ' ὑπ' αὐτῶν μέσον.

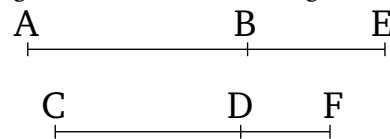
Ἐλάσσων ἄρα ἐστὶν ἡ $\Gamma\Delta$ · ὅπερ ἔδει δεῖξαι.

on CF . Thus, the (rectangle contained) by AE and EB is also commensurable with the (rectangle contained) by CF and FD [Props. 5.16, 10.11]. Therefore, either the (rectangle contained) by AE and EB is rational, and the (rectangle contained) by CF and FD will also be rational [Def. 10.4], or the (rectangle contained) by AE and EB [is] medial, and the (rectangle contained) by CF and FD [is] also medial [Prop. 10.23 corr.].

Therefore, CD is the apotome of a medial (straight-line), and is the same in order as AB [Props. 10.74, 10.75]. (Which is) the very thing it was required to show.

Proposition 105

A (straight-line) commensurable (in length) with a minor (straight-line) is a minor (straight-line).

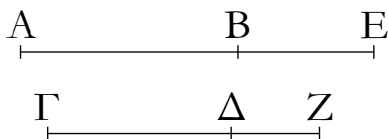


For let AB be a minor (straight-line), and (let) CD (be) commensurable (in length) with AB . I say that CD is also a minor (straight-line).

For let the same things have been contrived (as in the former proposition). And since AE and EB are (straight-lines which are) incommensurable in square [Prop. 10.76], CF and FD are thus also (straight-lines which are) incommensurable in square [Prop. 10.13]. Therefore, since as AE is to EB , so CF (is) to FD [Props. 5.12, 5.16], thus also as the (square) on AE is to the (square) on EB , so the (square) on CF (is) to the (square) on FD [Prop. 6.22]. Thus, via composition, as the (sum of the squares) on AE and EB is to the (square) on EB , so the (sum of the squares) on CF and FD (is) to the (square) on FD [Prop. 5.18], [also alternately]. And the (square) on BE is commensurable with the (square) on DF [Prop. 10.104]. The sum of the squares on AE and EB (is) thus also commensurable with the sum of the squares on CF and FD [Prop. 5.16, 10.11]. And the sum of the (squares) on AE and EB is rational [Prop. 10.76]. Thus, the sum of the (squares) on CF and FD is also rational [Def. 10.4]. Again, since as the (square) on AE is to the (rectangle contained) by AE and EB , so the (square) on CF (is) to the (rectangle contained) by CF and FD [Prop. 10.21 lem.], and the square on AE (is) commensurable with the square on CF , the (rectangle contained) by AE and EB is thus also commensurable with the (rectangle contained) by CF and FD . And the (rectangle contained) by AE and EB (is) medial [Prop. 10.76]. Thus, the (rectangle contained) by CF and FD (is) also medial [Prop. 10.23 corr.]. CF and

ρτ'.

Ἡ τῇ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσῃ σύμμετρος μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσά ἐστιν.



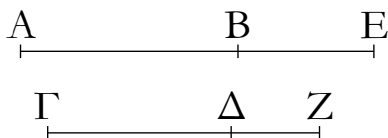
Ἐστω μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσα ἡ AB καὶ τῇ AB σύμμετρος ἡ $\Gamma\Delta$. λέγω, ὅτι καὶ ἡ $\Gamma\Delta$ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσά ἐστιν.

Ἐστω γὰρ τῇ AB προσαρμόζουσα ἡ BE . αἱ AE , EB ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιούσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν AE , EB τετραγώνων μέσον, τὸ δ' ὑπ' αὐτῶν ῥητόν. καὶ τὰ αὐτὰ κατεσκευάσθω. ὁμοίως δὲ δείξομεν τοῖς πρότερον, ὅτι αἱ ΓZ , $Z\Delta$ ἐν τῷ αὐτῷ λόγῳ εἰσὶ ταῖς AE , EB , καὶ σύμμετρόν ἐστι τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓZ , $Z\Delta$ τετραγώνων τῷ συγκειμένῳ ἐκ τῶν ἀπὸ τῶν ΓZ , $Z\Delta$ τετραγώνων, τὸ δὲ ὑπὸ τῶν AE , EB τῷ ὑπὸ τῶν ΓZ , $Z\Delta$. ὥστε καὶ αἱ ΓZ , $Z\Delta$ δυνάμει εἰσὶν ἀσύμμετροι ποιούσαι τὸ μὲν συγκείμενον ἐκ τῶν ἀπὸ τῶν ΓZ , $Z\Delta$ τετραγώνων μέσον, τὸ δ' ὑπ' αὐτῶν ῥητόν.

Ἡ $\Gamma\Delta$ ἄρα μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσά ἐστιν ὅπερ ἔδει δείξαι.

ρζ'.

Ἡ τῇ μετὰ μέσου μέσον τὸ ὅλον ποιούσῃ σύμμετρος καὶ αὐτὴ μετὰ μέσου μέσον τὸ ὅλον ποιούσά ἐστιν.



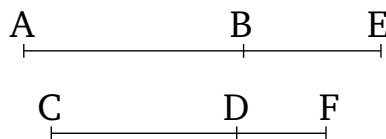
Ἐστω μετὰ μέσου μέσον τὸ ὅλον ποιούσα ἡ AB , καὶ τῇ

FD are thus (straight-lines which are) incommensurable in square, making the sum of the squares on them rational, and the (rectangle contained) by them medial.

Thus, CD is a minor (straight-line) [Prop. 10.76]. (Which is) the very thing it was required to show.

Proposition 106

A (straight-line) commensurable (in length) with a (straight-line) which with a rational (area) makes a medial whole is a (straight-line) which with a rational (area) makes a medial whole.



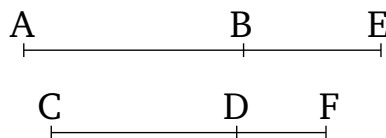
Let AB be a (straight-line) which with a rational (area) makes a medial whole, and (let) CD (be) commensurable (in length) with AB . I say that CD is also a (straight-line) which with a rational (area) makes a medial (whole).

For let BE be an attachment to AB . Thus, AE and EB are (straight-lines which are) incommensurable in square, making the sum of the squares on AE and EB medial, and the (rectangle contained) by them rational [Prop. 10.77]. And let the same construction have been made (as in the previous propositions). So, similarly to the previous (propositions), we can show that CF and FD are in the same ratio as AE and EB , and the sum of the squares on AE and EB is commensurable with the sum of the squares on CF and FD , and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD . Hence, CF and FD are also (straight-lines which are) incommensurable in square, making the sum of the squares on CF and FD medial, and the (rectangle contained) by them rational.

CD is thus a (straight-line) which with a rational (area) makes a medial whole [Prop. 10.77]. (Which is) the very thing it was required to show.

Proposition 107

A (straight-line) commensurable (in length) with a (straight-line) which with a medial (area) makes a medial whole is itself also a (straight-line) which with a medial (area) makes a medial whole.



Let AB be a (straight-line) which with a medial (area)

AB ἔστω σύμμετρος ἡ $\Gamma\Delta$ · λέγω, ὅτι καὶ ἡ $\Gamma\Delta$ μετὰ μέσου μέσον τὸ ὅλον ποιούσά ἐστιν.

Ἐστω γὰρ τῇ AB προσαρμόζουσα ἡ BE , καὶ τὰ αὐτὰ κατεσκευάσθω· αἱ AE , EB ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιῶσαι τὸ τε συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ' αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων τῷ ὑπ' αὐτῶν. καὶ εἰσιν, ὡς ἐδείχθη, αἱ AE , EB σύμμετροι ταῖς ΓZ , $Z\Delta$, καὶ τὸ συγκείμενον ἐκ τῶν ἀπὸ τῶν AE , EB τετραγώνων τῷ συγκείμενῳ ἐκ τῶν ἀπὸ τῶν ΓZ , $Z\Delta$, τὸ δὲ ὑπὸ τῶν AE , EB τῷ ὑπὸ τῶν ΓZ , $Z\Delta$ · καὶ αἱ ΓZ , $Z\Delta$ ἄρα δυνάμει εἰσὶν ἀσύμμετροι ποιῶσαι τὸ τε συγκείμενον ἐκ τῶν ἀπ' αὐτῶν τετραγώνων μέσον καὶ τὸ ὑπ' αὐτῶν μέσον καὶ ἔτι ἀσύμμετρον τὸ συγκείμενον ἐκ τῶν ἀπ' αὐτῶν [τετραγώνων] τῷ ὑπ' αὐτῶν.

Ἡ $\Gamma\Delta$ ἄρα μετὰ μέσου μέσον τὸ ὅλον ποιούσά ἐστιν· ὅπερ ἔδει δεῖξαι.

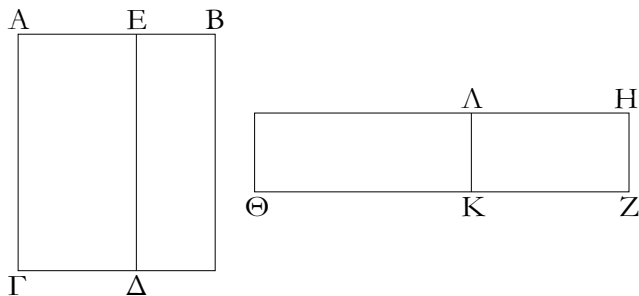
makes a medial whole, and let CD be commensurable (in length) with AB . I say that CD is also a (straight-line) which with a medial (area) makes a medial whole.

For let BE be an attachment to AB . And let the same construction have been made (as in the previous propositions). Thus, AE and EB are (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, further, the sum of the squares on them incommensurable with the (rectangle contained) by them [Prop. 10.78]. And, as was shown (previously), AE and EB are commensurable (in length) with CF and FD (respectively), and the sum of the squares on AE and EB with the sum of the squares on CF and FD , and the (rectangle contained) by AE and EB with the (rectangle contained) by CF and FD . Thus, CF and FD are also (straight-lines which are) incommensurable in square, making the sum of the squares on them medial, and the (rectangle contained) by them medial, and, further, the sum of the [squares] on them incommensurable with the (rectangle contained) by them.

Thus, CD is a (straight-line) which with a medial (area) makes a medial whole [Prop. 10.78]. (Which is) the very thing it was required to show.

ρη'.

Ἀπὸ ῥητοῦ μέσου ἀφαιρουμένου ἡ τὸ λοιπὸν χωρίον δυναμένη μία δύο ἀλόγων γίνεται ἥτοι ἀποτομή ἢ ἐλάσσων.

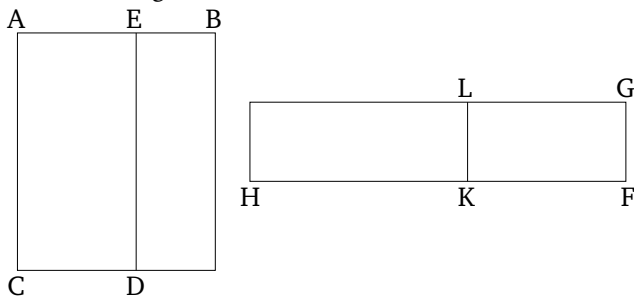


Ἀπὸ γὰρ ῥητοῦ τοῦ $B\Gamma$ μέσον ἀφηρήσθω τὸ $B\Delta$ · λέγω, ὅτι ἡ τὸ λοιπὸν δυναμένη τὸ EG μία δύο ἀλόγων γίνεται ἥτοι ἀποτομή ἢ ἐλάσσων.

Ἐκκείσθω γὰρ ῥητὴ ἡ ZH , καὶ τῷ μὲν $B\Gamma$ ἴσον παρὰ τὴν ZH παραβελήσθω ὀρθογώνιον παραλληλόγραμμον τὸ $\Theta\Lambda$, τῷ δὲ ΔB ἴσον ἀφηρήσθω τὸ HK · λοιπὸν ἄρα τὸ EG ἴσον ἐστὶ τῷ $\Lambda\Theta$. ἐπεὶ οὖν ῥητὸν μὲν ἐστὶ τὸ $B\Gamma$, μέσον δὲ τὸ $B\Delta$, ἴσον δὲ τὸ μὲν $B\Gamma$ τῷ $\Theta\Lambda$, τὸ δὲ $B\Delta$ τῷ HK , ῥητὸν μὲν ἄρα ἐστὶ τὸ $\Theta\Lambda$, μέσον δὲ τὸ HK . καὶ παρὰ ῥητὴν τὴν ZH παράκειται· ῥητὴ μὲν ἄρα ἡ $Z\Theta$ καὶ σύμμετρος τῇ ZH μήκει, ῥητὴ δὲ ἡ ZK καὶ ἀσύμμετρος τῇ ZH μήκει· ἀσύμμετρος ἄρα

Proposition 108

A medial (area) being subtracted from a rational (area), one of two irrational (straight-lines) arise (as) the square-root of the remaining area—either an apotome, or a minor (straight-line).



For let the medial (area) BD have been subtracted from the rational (area) BC . I say that one of two irrational (straight-lines) arise (as) the square-root of the remaining (area), EC —either an apotome, or a minor (straight-line).

For let the rational (straight-line) FG have been laid out, and let the right-angled parallelogram GH , equal to BC , have been applied to FG , and let GK , equal to DB , have been subtracted (from GH). Thus, the remainder EC is equal to LH . Therefore, since BC is a rational (area), and BD a medial (area), and BC (is) equal to

ἐστὶν ἡ $Z\Theta$ τῇ ZK μήκει. αἱ $Z\Theta$, ZK ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ $K\Theta$, προσαρμόζουσα δὲ αὐτῇ ἡ KZ . ἥτοι δὴ ἡ ΘZ τῆς ZK μείζον δύναται τῷ ἀπὸ συμμέτρου ἢ οὐ.

Δυνάσθω πρότερον τῷ ἀπὸ συμμέτρου. καὶ ἐστὶν ὅλη ἡ ΘZ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ZH · ἀποτομὴ ἄρα πρώτη ἐστὶν ἡ $K\Theta$. τὸ δ' ὑπὸ ῥητῆς καὶ ἀποτομῆς πρώτης περιεχόμενον ἡ δυναμένη ἀποτομὴ ἐστὶν. ἡ ἄρα τὸ $\Lambda\Theta$, τουτέστι τὸ $E\Gamma$, δυναμένη ἀποτομὴ ἐστὶν.

Εἰ δὲ ἡ ΘZ τῆς ZK μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ, καὶ ἐστὶν ὅλη ἡ $Z\Theta$ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ZH , ἀποτομὴ τετάρτη ἐστὶν ἡ $K\Theta$. τὸ δ' ὑπὸ ῥητῆς καὶ ἀποτομῆς τετάρτης περιεχόμενον ἡ δυναμένη ἐλάσσων ἐστὶν· ὅπερ ἔδει δεῖξαι.

GH , and BD to GK , GH is thus a rational (area), and GK a medial (area). And they are applied to the rational (straight-line) FG . Thus, FH (is) rational, and commensurable in length with FG [Prop. 10.20], and FK (is) also rational, and incommensurable in length with FG [Prop. 10.22]. Thus, FH is incommensurable in length with FK [Prop. 10.13]. FH and FK are thus rational (straight-lines which are) commensurable in square only. Thus, KH is an apotome [Prop. 10.73], and KF an attachment to it. So, the square on HF is greater than (the square on) FK by the (square) on (some straight-line which is) either commensurable, or not (commensurable), (in length with HF).

First, let the square (on it) be (greater) by the (square) on (some straight-line which is) commensurable (in length with HF). And the whole of HF is commensurable in length with the (previously) laid down rational (straight-line) FG . Thus, KH is a first apotome [Def. 10.1]. And the square-root of an (area) contained by a rational (straight-line) and a first apotome is an apotome [Prop. 10.91]. Thus, the square-root of LH —that is to say, (of) EC —is an apotome.

And if the square on HF is greater than (the square on) FK by the (square) on (some straight-line which is) incommensurable (in length) with (HF), and (since) the whole of FH is commensurable in length with the (previously) laid down rational (straight-line) FG , KH is a fourth apotome [Prop. 10.14]. And the square-root of an (area) contained by a rational (straight-line) and a fourth apotome is a minor (straight-line) [Prop. 10.94]. (Which is) the very thing it was required to show.

ρθ'.

Ἀπὸ μέσου ῥητοῦ ἀφαιρουμένου ἄλλαι δύο ἄλλοι γίνονται ἥτοι μέσης ἀποτομὴ πρώτη ἢ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα.

Ἀπὸ γὰρ μέσου τοῦ $B\Gamma$ ῥητὸν ἀφηρήσθω τὸ $B\Delta$. λέγω, ὅτι ἡ τὸ λοιπὸν τὸ $E\Gamma$ δυναμένη μία δύο ἀλόγων γίνεται ἥτοι μέσης ἀποτομὴ πρώτη ἢ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσα.

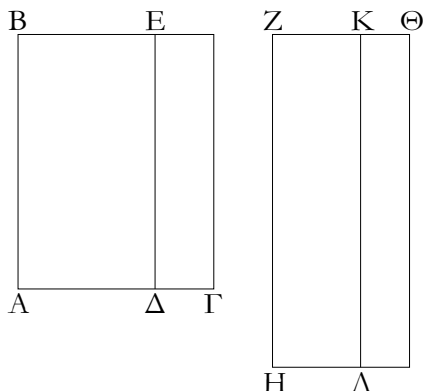
Ἐκκείσθω γὰρ ῥητὴ ἡ ZH , καὶ παραβεβλήσθω ὁμοίως τὰ χωρία. ἔστι δὴ ἀκολουθῶς ῥητὴ μὲν ἡ $Z\Theta$ καὶ ἀσύμμετρος τῇ ZH μήκει, ῥητὴ δὲ ἡ KZ καὶ σύμμετρος τῇ ZH μήκει· αἱ $Z\Theta$, ZK ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ $K\Theta$, προσαρμόζουσα δὲ ταύτῃ ἡ ZK . ἥτοι δὴ ἡ ΘZ τῆς ZK μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσύμμετρου.

Proposition 109

A rational (area) being subtracted from a medial (area), two other irrational (straight-lines) arise (as the square-root of the remaining area)—either a first apotome of a medial (straight-line), or that (straight-line) which with a rational (area) makes a medial whole.

For let the rational (area) BD have been subtracted from the medial (area) BC . I say that one of two irrational (straight-lines) arise (as) the square-root of the remaining (area), EC —either a first apotome of a medial (straight-line), or that (straight-line) which with a rational (area) makes a medial whole.

For let the rational (straight-line) FG be laid down, and let similar areas (to the preceding proposition) have been applied (to it). So, accordingly, FH is rational, and incommensurable in length with FG , and KF (is) also rational, and commensurable in length with FG . Thus, FH and FK are rational (straight-lines which are) com-



Εἰ μὲν οὖν ἡ ΘΖ τῆς ΖΚ μείζον δύναται τῷ ἀπὸ συμμετρου ἑαυτῇ, καὶ ἐστὶν ἡ προσαρμόζουσα ἡ ΖΚ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΖΗ, ἀποτομὴ δευτέρα ἐστὶν ἡ ΚΘ. ῥητὴ δὲ ἡ ΖΗ· ὥστε ἡ τὸ ΛΘ, τουτέστι τὸ ΕΓ, δυναμένη μέσης ἀποτομῇ πρώτη ἐστίν.

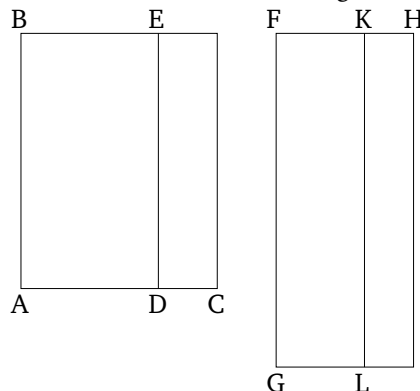
Εἰ δὲ ἡ ΘΖ τῆς ΖΚ μείζον δύναται τῷ ἀπὸ ἀσυμμέτρου, καὶ ἐστὶν ἡ προσαρμόζουσα ἡ ΖΚ σύμμετρος τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ ΖΗ, ἀποτομὴ πέμπτη ἐστὶν ἡ ΚΘ· ὥστε ἡ τὸ ΕΓ δυναμένη μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσά ἐστιν· ὅπερ ἔδει δεῖξαι.

ρι'.

Ἀπὸ μέσου μέσου ἀφαιρουμένου ἀσυμμέτρου τῷ ὅλῳ αἱ λοιπαὶ δύο ἄλλοι γίνονται ἥτοι μέσης ἀποτομῇ δευτέρα ἢ μετὰ μέσου μέσον τὸ ὅλον ποιούσα.

Ἀφηρήσθω γὰρ ὡς ἐπὶ τῶν προκειμένων καταγραφῶν ἀπὸ μέσου τοῦ ΒΓ μέσον τὸ ΒΔ ἀσύμμετρον τῷ ὅλῳ· λέγω, ὅτι ἡ τὸ ΕΓ δυναμένη μία ἐστὶ δύο ἀλόγων ἥτοι μέσης ἀποτομῇ δευτέρα ἢ μετὰ μέσου μέσον τὸ ὅλον ποιούσα.

measurable in square only [Prop. 10.13]. KH is thus an apotome [Prop. 10.73], and FK an attachment to it. So, the square on HF is greater than (the square on) FK either by the (square) on (some straight-line) commensurable (in length) with (HF) , or by the (square) on (some straight-line) incommensurable (in length with HF).



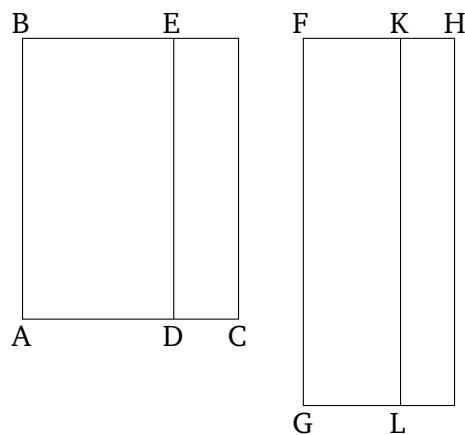
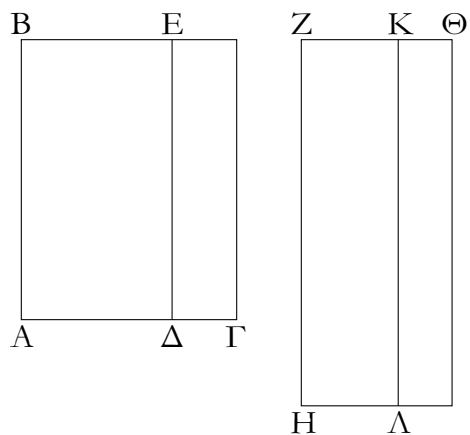
Therefore, if the square on HF is greater than (the square on) FK by the (square) on (some straight-line) commensurable (in length) with (HF) , and (since) the attachment FK is commensurable in length with the (previously) laid down rational (straight-line) FG , KH is a second apotome [Def. 10.12]. And FG (is) rational. Hence, the square-root of LH —that is to say, (of) EC —is a first apotome of a medial (straight-line) [Prop. 10.92].

And if the square on HF is greater than (the square on) FK by the (square) on (some straight-line) incommensurable (in length with HF), and (since) the attachment FK is commensurable in length with the (previously) laid down rational (straight-line) FG , KH is a fifth apotome [Def. 10.15]. Hence, the square-root of EC is that (straight-line) which with a rational (area) makes a medial whole [Prop. 10.95]. (Which is) the very thing it was required to show.

Proposition 110

A medial (area), incommensurable with the whole, being subtracted from a medial (area), the two remaining irrational (straight-lines) arise (as) the (square-root of the area)—either a second apotome of a medial (straight-line), or that (straight-line) which with a medial (area) makes a medial whole.

For, as in the previous figures, let the medial (area) BD , incommensurable with the whole, have been subtracted from the medial (area) BC . I say that the square-root of EC is one of two irrational (straight-lines)—either a second apotome of a medial (straight-line), or that (straight-line) which with a medial (area) makes a medial whole.



Ἐπεὶ γὰρ μέσον ἐστὶν ἑκάτερον τῶν $B\Gamma$, $B\Delta$, καὶ ἀσύμμετρον τὸ $B\Gamma$ τῷ $B\Delta$, ἔσται ἀκολούθως ῥητὴ ἑκατέρα τῶν $Z\Theta$, ZK καὶ ἀσύμμετρος τῇ ZH μήκει. καὶ ἐπεὶ ἀσύμμετρόν ἐστι τὸ $B\Gamma$ τῷ $B\Delta$, τουτέστι τὸ $H\Theta$ τῷ HK , ἀσύμμετρος καὶ ἡ ΘZ τῇ ZK . αἱ $Z\Theta$, ZK ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ $K\Theta$ [προσαρμόζουσα δὲ ἡ ZK . ἥτοι δὴ ἡ $Z\Theta$ τῆς ZK μείζον δύνανται τῷ ἀπὸ συμμέτρου ἢ τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ].

Εἰ μὲν δὴ ἡ $Z\Theta$ τῆς ZK μείζον δύνανται τῷ ἀπὸ συμμέτρου ἑαυτῇ, καὶ οὐθετέρα τῶν $Z\Theta$, ZK σύμμετρος ἐστὶ τῇ ἐκκεκλιμένῃ ῥητῇ μήκει τῇ ZH , ἀποτομὴ τρίτη ἐστὶν ἡ $K\Theta$. ῥητὴ δὲ ἡ $K\Lambda$, τὸ δ' ὑπὸ ῥητῆς καὶ ἀποτομῆς τρίτης περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλεῖται δὲ μέσης ἀποτομὴ δευτέρα· ὥστε ἡ τὸ $\Lambda\Theta$, τουτέστι τὸ $E\Gamma$, δυναμένη μέσης ἀποτομῆ ἐστὶ δευτέρα.

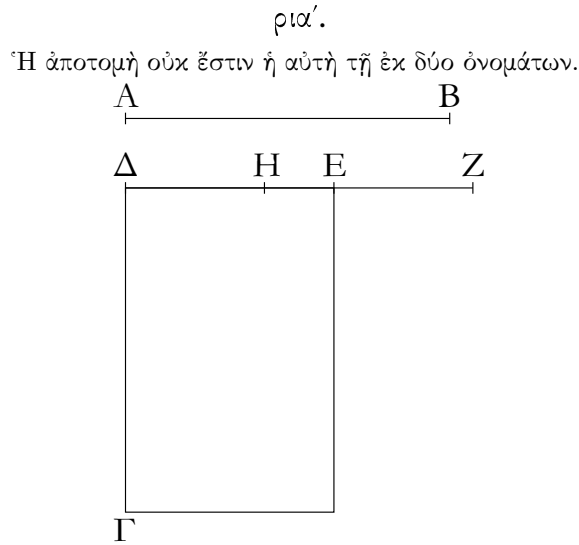
Εἰ δὲ ἡ $Z\Theta$ τῆς ZK μείζον δύνανται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ [μήκει], καὶ οὐθετέρα τῶν ΘZ , ZK σύμμετρος ἐστὶ τῇ ZH μήκει, ἀποτομὴ ἕκτη ἐστὶν ἡ $K\Theta$. τὸ δ' ὑπὸ ῥητῆς καὶ ἀποτομῆς ἕκτης ἡ δυναμένη ἐστὶ μετὰ μέσου μέσον τὸ ὅλον ποιούσα. ἡ τὸ $\Lambda\Theta$ ἄρα, τουτέστι τὸ $E\Gamma$, δυναμένη μετὰ μέσου μέσον τὸ ὅλον ποιούσα ἐστὶν· ὅπερ ἔδει δεῖξαι.

For since BC and BD are each medial (areas), and BC (is) incommensurable with BD , accordingly, FH and FK will each be rational (straight-lines), and incommensurable in length with FG [Prop. 10.22]. And since BC is incommensurable with BD —that is to say, GH with GK — HF (is) also incommensurable (in length) with FK [Props. 6.1, 10.11]. Thus, FH and FK are rational (straight-lines which are) commensurable in square only. KH is thus as apotome [Prop. 10.73], [and FK an attachment (to it)]. So, the square on FH is greater than (the square on) FK either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (FH) .]

So, if the square on FH is greater than (the square on) FK by the (square) on (some straight-line) commensurable (in length) with (FH) , and (since) neither of FH and FK is commensurable in length with the (previously) laid down rational (straight-line) FG , KH is a third apotome [Def. 10.3]. And KL (is) rational. And the rectangle contained by a rational (straight-line) and a third apotome is irrational, and the square-root of it is that irrational (straight-line) called a second apotome of a medial (straight-line) [Prop. 10.93]. Hence, the square-root of LH —that is to say, (of) EC —is a second apotome of a medial (straight-line).

And if the square on FH is greater than (the square on) FK by the (square) on (some straight-line) incommensurable [in length] with (FH) , and (since) neither of HF and FK is commensurable in length with FG , KH is a sixth apotome [Def. 10.16]. And the square-root of the (rectangle contained) by a rational (straight-line) and a sixth apotome is that (straight-line) which with a medial (area) makes a medial whole [Prop. 10.96]. Thus, the square-root of LH —that is to say, (of) EC —is that (straight-line) which with a medial (area) makes a medial whole. (Which is) the very thing it was required to

show.



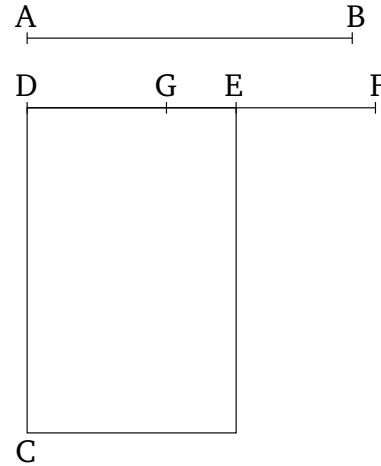
Ἐστω ἀποτομή ἡ AB · λέγω, ὅτι ἡ AB οὐκ ἔστιν ἡ αὐτὴ τῇ ἐκ δύο ὀνομάτων.

Εἰ γὰρ δυνατόν, ἔστω· καὶ ἐκκείσθω ῥητὴ ἡ $\Delta\Gamma$, καὶ τῷ ἀπὸ τῆς AB ἴσον παρὰ τὴν $\Gamma\Delta$ παραβεβλήσθω ὀρθογώνιον τὸ ΓE πλάτος ποιῶν τὴν ΔE . ἐπεὶ οὖν ἀποτομή ἐστὶν ἡ AB , ἀποτομή πρώτη ἐστὶν ἡ ΔE . ἔστω αὐτῇ προσαρμόζουσα ἡ EZ · αἱ ΔZ , ZE ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΔZ τῆς ZE μείζον δύναται τῷ ἀπὸ συμέτρου ἑαυτῆς, καὶ ἡ ΔZ σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ $\Delta\Gamma$. πάλιν, ἐπεὶ ἐκ δύο ὀνομάτων ἐστὶν ἡ AB , ἐκ δύο ἄρα ὀνομάτων πρώτη ἐστὶν ἡ ΔE . διηρήσθω εἰς τὰ ὀνόματα κατὰ τὸ H , καὶ ἔστω μείζον ὄνομα τὸ ΔH · αἱ ΔH , HE ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι, καὶ ἡ ΔH τῆς HE μείζον δύναται τῷ ἀπὸ συμέτρου ἑαυτῆς, καὶ τὸ μείζον ἡ ΔH σύμμετρός ἐστι τῇ ἐκκειμένῃ ῥητῇ μήκει τῇ $\Delta\Gamma$. καὶ ἡ ΔZ ἄρα τῇ ΔH σύμμετρός ἐστι μήκει· καὶ λοιπὴ ἄρα ἡ HZ σύμμετρός ἐστι τῇ ΔZ μήκει. [ἐπεὶ οὖν σύμμετρός ἐστιν ἡ ΔZ τῇ HZ , ῥητὴ δὲ ἐστὶν ἡ ΔZ , ῥητὴ ἄρα ἐστὶ καὶ ἡ HZ . ἐπεὶ οὖν σύμμετρός ἐστὶν ἡ ΔZ τῇ HZ μήκει] ἀσύμμετρος δὲ ἡ ΔZ τῇ EZ μήκει. ἀσύμμετρος ἄρα ἐστὶ καὶ ἡ ZH τῇ EZ μήκει. αἱ HZ , ZE ἄρα ῥηταὶ [εἰσι] δυνάμει μόνον σύμμετροι· ἀποτομή ἄρα ἐστὶν ἡ EH . ἀλλὰ καὶ ῥητὴ· ὅπερ ἐστὶν ἀδύνατον.

Ἡ ἄρα ἀποτομή οὐκ ἔστιν ἡ αὐτὴ τῇ ἐκ δύο ὀνομάτων· ὅπερ ἔδει δεῖξαι.

Proposition 111

An apotome is not the same as a binomial.



Let AB be an apotome. I say that AB is not the same as a binomial.

For, if possible, let it be (the same). And let a rational (straight-line) DC be laid down. And let the rectangle CE , equal to the (square) on AB , have been applied to CD , producing DE as breadth. Therefore, since AB is an apotome, DE is a first apotome [Prop. 10.97]. Let EF be an attachment to it. Thus, DF and FE are rational (straight-lines which are) commensurable in square only, and the square on DF is greater than (the square on) FE by the (square) on (some straight-line) commensurable (in length) with (DF), and DF is commensurable in length with the (previously) laid down rational (straight-line) DC [Def. 10.10]. Again, since AB is a binomial, DE is thus a first binomial [Prop. 10.60]. Let (DE) have been divided into its (component) terms at G , and let DG be the greater term. Thus, DG and GE are rational (straight-lines which are) commensurable in square only, and the square on DG is greater than (the square on) GE by the (square) on (some straight-line) commensurable (in length) with (DG), and the greater (term) DG is commensurable in length with the (previously) laid down rational (straight-line) DC [Def. 10.5]. Thus, DF is also commensurable in length with DG [Prop. 10.12]. The remainder GF is thus commensurable in length with DF [Prop. 10.15]. [Therefore, since DF is commensurable with GF , and DF is rational, GF is thus also rational. Therefore, since DF is commensurable in length with GF ,] DF (is) incommensurable in length with EF . Thus, FG is also incommensurable in length with EF [Prop. 10.13]. GF and FE [are] thus rational (straight-lines which are) commensurable in square only. Thus,

EG is an apotome [Prop. 10.73]. But, (it is) also rational. The very thing is impossible.

Thus, an apotome is not the same as a binomial. (Which is) the very thing it was required to show.

[Πόρισμα.]

Ἡ ἀποτομή καὶ αἱ μετ' αὐτὴν ἄλλοι οὐτε τῇ μέσῃ οὐτε ἀλλήλαις εἰσὶν αἱ αὐταί.

Τὸ μὲν γὰρ ἀπὸ μέσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ῥητὴν καὶ ἀσύμμετρον τῇ, παρ' ἣν παράκειται, μήκει, τὸ δὲ ἀπὸ ἀποτομῆς παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην, τὸ δὲ ἀπὸ μέσης ἀποτομῆς πρώτης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν δευτέραν, τὸ δὲ ἀπὸ μέσης ἀποτομῆς δευτέρας παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τρίτην, τὸ δὲ ἀπὸ ἐλάσσονος παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν τετάρτην, τὸ δὲ ἀπὸ τῆς μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πέμπτην, τὸ δὲ ἀπὸ τῆς μετὰ μέσου μέσον τὸ ὅλον ποιούσης παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν ἕκτην. ἐπεὶ οὖν τὰ εἰρημένα πλάτη διαφέρει τοῦ τε πρώτου καὶ ἀλλήλων, τοῦ μὲν πρώτου, ὅτι ῥητὴ ἐστίν, ἀλλήλων δὲ, ἐπεὶ τῇ τάξει οὐκ εἰσὶν αἱ αὐταί, δῆλον, ὥς καὶ αὐταί αἱ ἄλλοι διαφέρουσιν ἀλλήλων. καὶ ἐπεὶ δέδεικται ἡ ἀποτομή οὐκ οὔσα ἡ αὐτὴ τῇ ἐκ δύο ὀνομάτων, ποιούσι δὲ πλάτη παρὰ ῥητὴν παραβαλλόμενα αἱ μετὰ τὴν ἀποτομὴν ἀποτομὰς ἀκολουθῶς ἐκάστη τῇ τάξει τῇ καθ' αὐτήν, αἱ δὲ μετὰ τὴν ἐκ δύο ὀνομάτων τὰς ἐκ δύο ὀνομάτων καὶ αὐταί τῇ τάξει ἀκολουθῶς, ἕτεροι ἄρα εἰσὶν αἱ μετὰ τὴν ἀποτομὴν καὶ ἕτεροι αἱ μετὰ τὴν ἐκ δύο ὀνομάτων, ὥς εἶναι τῇ τάξει πάσας ἀλόγους $\overline{\Gamma}$,

[Corollary]

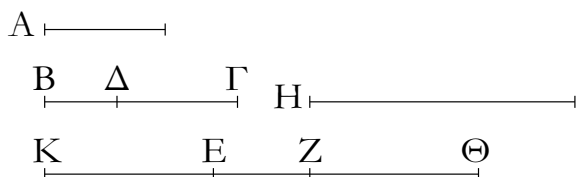
The apotome and the irrational (straight-lines) after it are neither the same as a medial (straight-line) nor (the same) as one another.

For the (square) on a medial (straight-line), applied to a rational (straight-line), produces as breadth a rational (straight-line which is) incommensurable in length with the (straight-line) to which (the area) is applied [Prop. 10.22]. And the (square) on an apotome, applied to a rational (straight-line), produces as breadth a first apotome [Prop. 10.97]. And the (square) on a first apotome of a medial (straight-line), applied to a rational (straight-line), produces as breadth a second apotome [Prop. 10.98]. And the (square) on a second apotome of a medial (straight-line), applied to a rational (straight-line), produces as breadth a third apotome [Prop. 10.99]. And (square) on a minor (straight-line), applied to a rational (straight-line), produces as breadth a fourth apotome [Prop. 10.100]. And (square) on that (straight-line) which with a rational (area) produces a medial whole, applied to a rational (straight-line), produces as breadth a fifth apotome [Prop. 10.101]. And (square) on that (straight-line) which with a medial (area) produces a medial whole, applied to a rational (straight-line), produces as breadth a sixth apotome [Prop. 10.102]. Therefore, since the aforementioned breadths differ from the first (breadth), and from one another—from the first, because it is rational, and from one another since they are not the same in order—clearly, the irrational (straight-lines) themselves also differ from one another. And since it has been shown that an apotome is not the same as a binomial [Prop. 10.111], and (that) the (irrational straight-lines) after the apotome, being applied to a rational (straight-line), produce as breadth, each according to its own (order), apotomes, and (that) the (irrational straight-lines) after the binomial themselves also (produce as breadth), according (to their) order, binomials, the (irrational straight-lines) after the apotome are thus different, and the (irrational straight-lines) after the binomial (are also) different, so that there are, in order, 13 irrational (straight-lines) in all:

Μέσην,
 Ἐκ δύο ὀνομάτων,
 Ἐκ δύο μέσων πρώτην,
 Ἐκ δύο μέσων δευτέραν,
 Μείζονα,
 Ῥητὸν καὶ μέσον δυναμένην,
 Δύο μέσα δυναμένην,
 Ἀποτομήν,
 Μέσης ἀποτομὴν πρώτην,
 Μέσης ἀποτομὴν δευτέραν,
 Ἐλάσσονα,
 Μετὰ ῥητοῦ μέσον τὸ ὅλον ποιοῦσαν,
 Μετὰ μέσου μέσον τὸ ὅλον ποιοῦσαν.

ριβ'.

Τὸ ἀπὸ ῥητῆς παρὰ τὴν ἐκ δύο ὀνομάτων παραβαλλόμενον πλάτος ποιεῖ ἀποτομήν, ἥς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ἐκ δύο ὀνομάτων ὀνόμασι καὶ ἔτι ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ γινομένη ἀποτομή τὴν αὐτὴν ἔξει τάξιν τῇ ἐκ δύο ὀνομάτων.



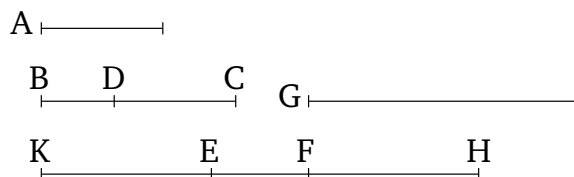
Ἐστω ῥητὴ μὲν ἡ Α, ἐκ δύο ὀνομάτων δὲ ἡ ΒΓ, ἥς μείζον ὄνομα ἔστω ἡ ΔΓ, καὶ τῷ ἀπὸ τῆς Α ἴσον ἔστω τὸ ὑπὸ τῶν ΒΓ, ΕΖ· λέγω, ὅτι ἡ ΕΖ ἀποτομή ἐστίν, ἥς τὰ ὀνόματα σύμμετρά ἐστι τοῖς ΓΔ, ΔΒ, καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ ΕΖ τὴν αὐτὴν ἔξει τάξιν τῇ ΒΓ.

Ἐστω γὰρ πάλιν τῷ ἀπὸ τῆς Α ἴσον τὸ ὑπὸ τῶν ΒΔ, Η. ἐπεὶ οὖν τὸ ὑπὸ τῶν ΒΓ, ΕΖ ἴσον ἐστὶ τῷ ὑπὸ τῶν ΒΔ, Η, ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν ΒΔ, οὕτως ἡ Η πρὸς τὴν ΕΖ. μείζων δὲ ἡ ΓΒ τῆς ΒΔ· μείζων ἄρα ἐστὶ καὶ ἡ Η τῆς ΕΖ. ἔστω τῇ Η ἴση ἡ ΕΘ· ἔστιν ἄρα ὡς ἡ ΓΒ πρὸς τὴν ΒΔ, οὕτως ἡ ΘΕ πρὸς τὴν ΕΖ· διελόντι ἄρα ἐστὶν ὡς ἡ ΓΔ πρὸς τὴν ΒΔ, οὕτως ἡ ΘΖ πρὸς τὴν ΖΕ. γεγονέντω ὡς ἡ ΘΖ πρὸς τὴν ΖΕ, οὕτως ἡ ΖΕ πρὸς τὴν ΚΕ· καὶ ὅλη ἄρα ἡ ΘΚ πρὸς ὅλην τὴν ΚΖ ἐστίν, ὡς ἡ ΖΚ πρὸς ΚΕ· ὡς γὰρ ἐν τῶν ἡγουμένων πρὸς ἐν τῶν ἐπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἐπόμενα. ὡς δὲ ἡ ΖΚ πρὸς ΚΕ, οὕτως ἐστὶν ἡ ΓΔ πρὸς τὴν ΔΒ· καὶ ὡς ἄρα ἡ ΘΚ πρὸς ΚΖ, οὕτως ἡ ΓΔ πρὸς τὴν ΔΒ. σύμμετρον δὲ τὸ ἀπὸ τῆς ΓΔ τῷ ἀπὸ τῆς ΔΒ· σύμμετρον ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς ΘΚ τῷ

Medial,
 Binomial,
 First bimedral,
 Second bimedral,
 Major,
 Square-root of a rational plus a medial (area),
 Square-root of (the sum of) two medial (areas),
 Apotome,
 First apotome of a medial,
 Second apotome of a medial,
 Minor,
 That which with a rational (area) produces a medial whole,
 That which with a medial (area) produces a medial whole.

Proposition 112[†]

The (square) on a rational (straight-line), applied to a binomial (straight-line), produces as breadth an apotome whose terms are commensurable (in length) with the terms of the binomial, and, furthermore, in the same ratio. Moreover, the created apotome will have the same order as the binomial.



Let A be a rational (straight-line), and BC a binomial (straight-line), of which let DC be the greater term. And let the (rectangle contained) by BC and EF be equal to the (square) on A . I say that EF is an apotome whose terms are commensurable (in length) with CD and DB , and in the same ratio, and, moreover, that EF will have the same order as BC .

For, again, let the (rectangle contained) by BD and G be equal to the (square) on A . Therefore, since the (rectangle contained) by BC and EF is equal to the (rectangle contained) by BD and G , thus as CB is to BD , so G (is) to EF [Prop. 6.16]. And CB (is) greater than BD . Thus, G is also greater than EF [Props. 5.16, 5.14]. Let EH be equal to G . Thus, as CB is to BD , so HE (is) to EF . Thus, via separation, as CD is to BD , so HF (is) to FE [Prop. 5.17]. Let it have been contrived that as HF (is) to FE , so FK (is) to KE . And, thus, the whole HK is to the whole KF , as FK (is) to KE . For as one of the leading (proportional magnitudes is) to one of the

ἀπὸ τῆς KZ . καὶ ἐστὶν ὡς τὸ ἀπὸ τῆς ΘK πρὸς τὸ ἀπὸ τῆς KZ , οὕτως ἡ ΘK πρὸς τὴν KE , ἐπεὶ αἱ τρεῖς αἱ ΘK , KZ , KE ἀνάλογόν εἰσιν. σύμμετρος ἄρα ἡ ΘK τῇ KE μήκει. ὥστε καὶ ἡ ΘE τῇ EK σύμμετρος ἐστὶ μήκει. καὶ ἐπεὶ τὸ ἀπὸ τῆς A ἴσον ἐστὶ τῷ ὑπὸ τῶν $E\Theta$, $B\Delta$, ῥητὸν δέ ἐστὶ τὸ ἀπὸ τῆς A , ῥητὸν ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν $E\Theta$, $B\Delta$. καὶ παρὰ ῥητὴν τὴν $B\Delta$ παράκειται· ῥητὴ ἄρα ἐστὶν ἡ $E\Theta$ καὶ σύμμετρος τῇ $B\Delta$ μήκει· ὥστε καὶ ἡ σύμμετρος αὐτῇ ἡ EK ῥητὴ ἐστὶ καὶ σύμμετρος τῇ $B\Delta$ μήκει. ἐπεὶ οὖν ἐστὶν ὡς ἡ $\Gamma\Delta$ πρὸς ΔB , οὕτως ἡ ZK πρὸς KE , αἱ δὲ $\Gamma\Delta$, ΔB δυνάμει μόνον εἰσὶ σύμμετροι, καὶ αἱ ZK , KE δυνάμει μόνον εἰσὶ σύμμετροι. ῥητὴ δέ ἐστὶν ἡ KE · ῥητὴ ἄρα ἐστὶ καὶ ἡ ZK . αἱ ZK , KE ἄρα ῥηταὶ δυνάμει μόνον εἰσὶ σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ EZ .

Ἦτοι δὲ ἡ $\Gamma\Delta$ τῆς ΔB μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῇ ἢ τῷ ἀπὸ ἀσυμμέτρου.

Εἰ μὲν οὖν ἡ $\Gamma\Delta$ τῆς ΔB μείζον δύναται τῷ ἀπὸ συμμέτρου [ἑαυτῇ], καὶ ἡ ZK τῆς KE μείζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῇ. καὶ εἰ μὲν σύμμετρος ἐστὶν ἡ $\Gamma\Delta$ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ZK · εἰ δὲ ἡ $B\Delta$, καὶ ἡ KE · εἰ δὲ οὐδετέρω τῶν $\Gamma\Delta$, ΔB , καὶ οὐδετέρω τῶν ZK , KE .

Εἰ δὲ ἡ $\Gamma\Delta$ τῆς ΔB μείζον δύναται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ, καὶ ἡ ZK τῆς KE μείζον δυνήσεται τῷ ἀπὸ ἀσυμμέτρου ἑαυτῇ. καὶ εἰ μὲν ἡ $\Gamma\Delta$ σύμμετρος ἐστὶ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ ZK · εἰ δὲ ἡ $B\Delta$, καὶ ἡ KE · εἰ δὲ οὐδετέρω τῶν $\Gamma\Delta$, ΔB , καὶ οὐδετέρω τῶν ZK , KE · ὥστε ἀποτομὴ ἐστὶν ἡ ZE , ἥς τὰ ὀνόματα τὰ ZK , KE σύμμετρά ἐστὶ τοῖς τῆς ἐκ δύο ὀνομάτων ὀνόμασι τοῖς $\Gamma\Delta$, ΔB καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ τὴν αὐτὴν τάξιν ἔχει τῇ $B\Gamma$ · ὅπερ ἔδει δείξαι.

following, so all of the leading (magnitudes) are to all of the following [Prop. 5.12]. And as FK (is) to KE , so CD is to DB [Prop. 5.11]. And, thus, as HK (is) to KF , so CD is to DB [Prop. 5.11]. And the (square) on CD (is) commensurable with the (square) on DB [Prop. 10.36]. The (square) on HK is thus also commensurable with the (square) on KF [Props. 6.22, 10.11]. And as the (square) on HK is to the (square) on KF , so HK (is) to KE , since the three (straight-lines) HK , KF , and KE are proportional [Def. 5.9]. HK is thus commensurable in length with KE [Prop. 10.11]. Hence, HE is also commensurable in length with EK [Prop. 10.15]. And since the (square) on A is equal to the (rectangle contained) by EH and BD , and the (square) on A is rational, the (rectangle contained) by EH and BD is thus also rational. And it is applied to the rational (straight-line) BD . Thus, EH is rational, and commensurable in length with BD [Prop. 10.20]. And, hence, the (straight-line) commensurable (in length) with it, EK , is also rational [Def. 10.3], and commensurable in length with BD [Prop. 10.12]. Therefore, since as CD is to DB , so FK (is) to KE , and CD and DB are (straight-lines which are) commensurable in square only, FK and KE are also commensurable in square only [Prop. 10.11]. And KE is rational. Thus, FK is also rational. FK and KE are thus rational (straight-lines which are) commensurable in square only. Thus, EF is an apotome [Prop. 10.73].

And the square on CD is greater than (the square on) DB either by the (square) on (some straight-line) commensurable, or by the (square) on (some straight-line) incommensurable, (in length) with (CD).

Therefore, if the square on CD is greater than (the square on) DB by the (square) on (some straight-line) commensurable (in length) with [CD] then the square on FK will also be greater than (the square on) KE by the (square) on (some straight-line) commensurable (in length) with (FK) [Prop. 10.14]. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FK [Props. 10.11, 10.12]. And if BD (is commensurable), (so) also (is) KE [Prop. 10.12]. And if neither of CD or DB (is commensurable), neither also (are) either of FK or KE .

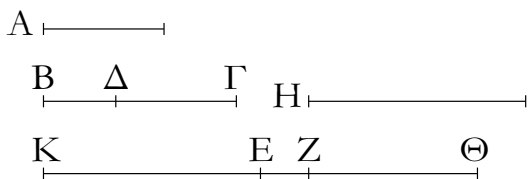
And if the square on CD is greater than (the square on) DB by the (square) on (some straight-line) incommensurable (in length) with (CD) then the square on FK will also be greater than (the square on) KE by the (square) on (some straight-line) incommensurable (in length) with (FK) [Prop. 10.14]. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FK [Props. 10.11, 10.12]. And if BD (is commensurable), (so) also (is) KE

[Prop. 10.12]. And if neither of CD or DB (is commensurable), neither also (are) either of FK or KE . Hence, FE is an apotome whose terms, FK and KE , are commensurable (in length) with the terms, CD and DB , of the binomial, and in the same ratio. And (FE) has the same order as BC [Defs. 10.5—10.10]. (Which is) the very thing it was required to show.

† Heiberg considers this proposition, and the succeeding ones, to be relatively early interpolations into the original text.

ριγ'.

Τὸ ἀπὸ ῥητῆς παρὰ ἀποτομὴν παραβαλλόμενον πλάτος ποιεῖ τὴν ἐκ δύο ὀνομάτων, ἥς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ἀποτομῆς ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, ἔτι δὲ ἡ γινομένη ἐκ δύο ὀνομάτων τὴν αὐτὴν τάξιν ἔχει τῇ ἀποτομῇ.

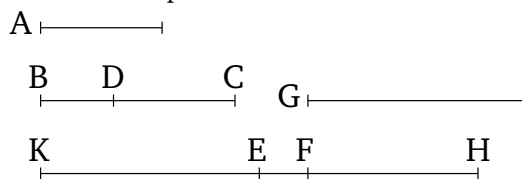


Ἐστω ῥητὴ μὲν ἡ A , ἀποτομὴ δὲ ἡ $BΔ$, καὶ τῷ ἀπὸ τῆς A ἴσον ἔστω τὸ ὑπὸ τῶν $BΔ$, $KΘ$, ὥστε τὸ ἀπὸ τῆς A ῥητῆς παρὰ τὴν $BΔ$ ἀποτομὴν παραβαλλόμενον πλάτος ποιεῖ τὴν $KΘ$. λέγω, ὅτι ἐκ δύο ὀνομάτων ἐστὶν ἡ $KΘ$, ἥς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς $BΔ$ ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ $KΘ$ τὴν αὐτὴν ἔχει τάξιν τῇ $BΔ$.

Ἐστω γὰρ τῇ $BΔ$ προσαρμόζουσα ἡ $ΔΓ$. αἱ $BΓ$, $ΓΔ$ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι. καὶ τῷ ἀπὸ τῆς A ἴσον ἔστω καὶ τὸ ὑπὸ τῶν $BΓ$, H . ῥητὸν δὲ τὸ ἀπὸ τῆς A ῥητὸν ἄρα καὶ τὸ ὑπὸ τῶν $BΓ$, H . καὶ παρὰ ῥητὴν τὴν $BΓ$ παραβέβληται ῥητὴ ἄρα ἐστὶν ἡ H καὶ σύμμετρος τῇ $BΓ$ μήκει. ἐπεὶ οὖν τὸ ὑπὸ τῶν $BΓ$, H ἴσον ἐστὶ τῷ ὑπὸ τῶν $BΔ$, $KΘ$, ἀνάλογον ἄρα ἐστὶν ὡς ἡ $ΓB$ πρὸς $BΔ$, οὕτως ἡ $KΘ$ πρὸς H . μείζων δὲ ἡ $BΓ$ τῆς $BΔ$. μείζων ἄρα καὶ ἡ $KΘ$ τῆς H . κείσθω τῇ H ἴση ἡ KE . σύμμετρος ἄρα ἐστὶν ἡ KE τῇ $BΓ$ μήκει. καὶ ἐπεὶ ἐστὶν ὡς ἡ $ΓB$ πρὸς $BΔ$, οὕτως ἡ $ΘK$ πρὸς KE , ἀναστρέψαντι ἄρα ἐστὶν ὡς ἡ $BΓ$ πρὸς τὴν $ΓΔ$, οὕτως ἡ $KΘ$ πρὸς $ΘE$. γεγονέτω ὡς ἡ $KΘ$ πρὸς $ΘE$, οὕτως ἡ $ΘZ$ πρὸς ZE . καὶ λοιπὴ ἄρα ἡ KZ πρὸς $ZΘ$ ἐστὶν, ὡς ἡ $KΘ$ πρὸς $ΘE$, τουτέστιν [ὡς] ἡ $BΓ$ πρὸς $ΓΔ$. αἱ δὲ $BΓ$, $ΓΔ$ δυνάμει μόνον [εἰσὶ] σύμμετροι. καὶ αἱ KZ , $ZΘ$ ἄρα δυνάμει μόνον εἰσὶ σύμμετροι. καὶ ἐπεὶ ἐστὶν ὡς ἡ $KΘ$ πρὸς $ΘE$, ἡ KZ πρὸς $ZΘ$, ἀλλ' ὡς ἡ $KΘ$ πρὸς $ΘE$, ἡ $ΘZ$ πρὸς ZE , καὶ ὡς ἄρα ἡ KZ πρὸς $ZΘ$, ἡ $ΘZ$ πρὸς ZE . ὥστε καὶ ὡς ἡ πρώτη πρὸς τὴν τρίτην, τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας. καὶ ὡς ἄρα ἡ KZ πρὸς ZE , οὕτως τὸ ἀπὸ τῆς KZ πρὸς τὸ ἀπὸ τῆς $ZΘ$. σύμμετρον δὲ ἐστὶ τὸ ἀπὸ τῆς KZ τῷ ἀπὸ τῆς $ZΘ$. αἱ γὰρ KZ , $ZΘ$ δυνάμει εἰσὶ σύμμετροι. σύμμετρος ἄρα ἐστὶ καὶ ἡ KZ τῇ ZE μήκει. ὥστε ἡ KZ καὶ

Proposition 113

The (square) on a rational (straight-line), applied to an apotome, produces as breadth a binomial whose terms are commensurable with the terms of the apotome, and in the same ratio. Moreover, the created binomial has the same order as the apotome.



Let A be a rational (straight-line), and BD an apotome. And let the (rectangle contained) by BD and KH be equal to the (square) on A , such that the square on the rational (straight-line) A , applied to the apotome BD , produces KH as breadth. I say that KH is a binomial whose terms are commensurable with the terms of BD , and in the same ratio, and, moreover, that KH has the same order as BD .

For let DC be an attachment to BD . Thus, BC and CD are rational (straight-lines which are) commensurable in square only [Prop. 10.73]. And let the (rectangle contained) by BC and G also be equal to the (square) on A . And the (square) on A (is) rational. The (rectangle contained) by BC and G (is) thus also rational. And it has been applied to the rational (straight-line) BC . Thus, G is rational, and commensurable in length with BC [Prop. 10.20]. Therefore, since the (rectangle contained) by BC and G is equal to the (rectangle contained) by BD and KH , thus, proportionally, as CB is to BD , so KH (is) to G [Prop. 6.16]. And BC (is) greater than BD . Thus, KH (is) also greater than G [Prop. 5.16, 5.14]. Let KE be made equal to G . KE is thus commensurable in length with BC . And since as CB is to BD , so HK (is) to KE , thus, via conversion, as BC (is) to CD , so KH (is) to HE [Prop. 5.19 corr.]. Let it have been contrived that as KH (is) to HE , so HF (is) to FE . And thus the remainder KF is to FH , as KH (is) to HE —that is to say, [as] BC (is) to CD [Prop. 5.19]. And BC and CD [are] commensurable in square only.

τῇ KE σύμμετρος [ἐστὶ] μήκει. ῥητὴ δὲ ἐστὶν ἡ KE καὶ σύμμετρος τῇ $BΓ$ μήκει. ῥητὴ ἄρα καὶ ἡ KZ καὶ σύμμετρος τῇ $BΓ$ μήκει. καὶ ἐπεὶ ἐστὶν ὡς ἡ $BΓ$ πρὸς $ΓΔ$, οὕτως ἡ KZ πρὸς $ZΘ$, ἐναλλάξ ὡς ἡ $BΓ$ πρὸς KZ , οὕτως ἡ $ΔΓ$ πρὸς $ZΘ$. σύμμετρος δὲ ἡ $BΓ$ τῇ KZ . σύμμετρος ἄρα καὶ ἡ $ZΘ$ τῇ $ΓΔ$ μήκει. αἱ $BΓ$, $ΓΔ$ δὲ ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· καὶ αἱ KZ , $ZΘ$ ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἐκ δύο ὀνομάτων ἐστὶν ἄρα ἡ $KΘ$.

Εἰ μὲν οὖν ἡ $BΓ$ τῆς $ΓΔ$ μείζον δύναται τῷ ἀπὸ συμμέτρου ἑαυτῆς, καὶ ἡ KZ τῆς $ZΘ$ μείζον δυνήσεται τῷ ἀπὸ συμμέτρου ἑαυτῆς. καὶ εἰ μὲν σύμμετρος ἐστὶν ἡ $BΓ$ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ KZ , εἰ δὲ ἡ $ΓΔ$ σύμμετρος ἐστὶ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ $ZΘ$, εἰ δὲ οὐδετέρα τῶν $BΓ$, $ΓΔ$, οὐδετέρα τῶν KZ , $ZΘ$.

Εἰ δὲ ἡ $BΓ$ τῆς $ΓΔ$ μείζον δύναται τῷ ἀπὸ ἀσυνμέτρου ἑαυτῆς, καὶ ἡ KZ τῆς $ZΘ$ μείζον δυνήσεται τῷ ἀπὸ ἀσυνμέτρου ἑαυτῆς. καὶ εἰ μὲν σύμμετρος ἐστὶν ἡ $BΓ$ τῇ ἐκκειμένῃ ῥητῇ μήκει, καὶ ἡ KZ , εἰ δὲ ἡ $ΓΔ$, καὶ ἡ $ZΘ$, εἰ δὲ οὐδετέρα τῶν $BΓ$, $ΓΔ$, οὐδετέρα τῶν KZ , $ZΘ$.

Ἐκ δύο ἄρα ὀνομάτων ἐστὶν ἡ $KΘ$, ἥς τὰ ὀνόματα τὰ KZ , $ZΘ$ σύμμετρα [ἐστὶ] τοῖς τῆς ἀποτομῆς ὀνόμασι τοῖς $BΓ$, $ΓΔ$ καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔτι ἡ $KΘ$ τῇ $BΓ$ τὴν αὐτὴν ἕξει τάξιν· ὅπερ ἔδει δεῖξαι.

KF and FH are thus also commensurable in square only [Prop. 10.11]. And since as KH is to HE , (so) KF (is) to FH , but as KH (is) to HE , (so) HF (is) to FE , thus, also as KF (is) to FH , (so) HF (is) to FE [Prop. 5.11]. And hence as the first (is) to the third, so the (square) on the first (is) to the (square) on the second [Def. 5.9]. And thus as KF (is) to FE , so the (square) on KF (is) to the (square) on FH . And the (square) on KF is commensurable with the (square) on FH . For KF and FH are commensurable in square. Thus, KF is also commensurable in length with FE [Prop. 10.11]. Hence, KF [is] also commensurable in length with KE [Prop. 10.15]. And KE is rational, and commensurable in length with BC . Thus, KF (is) also rational, and commensurable in length with BC [Prop. 10.12]. And since as BC is to CD , (so) KF (is) to FH , alternately, as BC (is) to KF , so DC (is) to FH [Prop. 5.16]. And BC (is) commensurable (in length) with KF . Thus, FH (is) also commensurable in length with CD [Prop. 10.11]. And BC and CD are rational (straight-lines which are) commensurable in square only. KF and FH are thus also rational (straight-lines which are) commensurable in square only [Def. 10.3, Prop. 10.13]. Thus, KH is a binomial [Prop. 10.36].

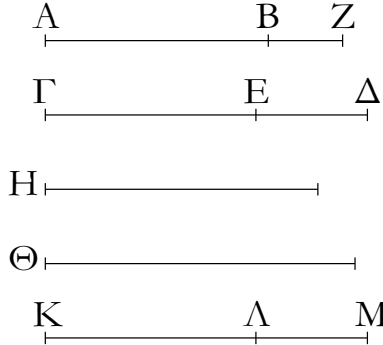
Therefore, if the square on BC is greater than (the square on) CD by the (square) on (some straight-line) commensurable (in length) with (BC) , then the square on KF will also be greater than (the square on) FH by the (square) on (some straight-line) commensurable (in length) with (KF) [Prop. 10.14]. And if BC is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) KF [Prop. 10.12]. And if CD is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) FH [Prop. 10.12]. And if neither of BC or CD (are commensurable), neither also (are) either of KF or FH [Prop. 10.13].

And if the square on BC is greater than (the square on) CD by the (square) on (some straight-line) incommensurable (in length) with (BC) then the square on KF will also be greater than (the square on) FH by the (square) on (some straight-line) incommensurable (in length) with (KF) [Prop. 10.14]. And if BC is commensurable in length with a (previously) laid down rational (straight-line), (so) also (is) KF [Prop. 10.12]. And if CD is commensurable, (so) also (is) FH [Prop. 10.12]. And if neither of BC or CD (are commensurable), neither also (are) either of KF or FH [Prop. 10.13].

KH is thus a binomial whose terms, KF and FH , [are] commensurable (in length) with the terms, BC and CD , of the apotome, and in the same ratio. Moreover,

ριδ'.

Ἐάν χωρίον περιέχεται ὑπὸ ἀποτομῆς καὶ τῆς ἐκ δύο ὀνομάτων, ἥς τὰ ὀνόματα σύμμετρά τέ ἐστι τοῖς τῆς ἀποτομῆς ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, ἡ τὸ χωρίον δυναμένη ῥητὴ ἐστίν.



Περιεχέσθω γὰρ χωρίον τὸ ὑπὸ τῶν AB, ΓΔ ὑπὸ ἀποτομῆς τῆς AB καὶ τῆς ἐκ δύο ὀνομάτων τῆς ΓΔ, ἥς μείζον ὄνομα ἔστω τὸ ΓΕ, καὶ ἔστω τὰ ὀνόματα τῆς ἐκ δύο ὀνομάτων τὰ ΓΕ, ΕΔ σύμμετρά τε τοῖς τῆς ἀποτομῆς ὀνόμασι τοῖς ΑΖ, ΖΒ καὶ ἐν τῷ αὐτῷ λόγῳ, καὶ ἔστω ἡ τὸ ὑπὸ τῶν AB, ΓΔ δυναμένη ἡ Η· λέγω, ὅτι ῥητὴ ἐστίν ἡ Η.

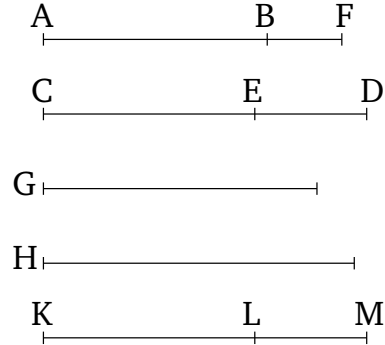
Ἐκκείσθω γὰρ ῥητὴ ἡ Θ, καὶ τῷ ἀπὸ τῆς Θ ἴσον παρὰ τὴν ΓΔ παραβεβλήσθω πλάτος ποιοῦν τὴν ΚΛ· ἀποτομὴ ἄρα ἐστὶν ἡ ΚΛ, ἥς τὰ ὀνόματα ἔστω τὰ ΚΜ, ΜΛ σύμμετρα τοῖς τῆς ἐκ δύο ὀνομάτων ὀνόμασι τοῖς ΓΕ, ΕΔ καὶ ἐν τῷ αὐτῷ λόγῳ. ἀλλὰ καὶ αἱ ΓΕ, ΕΔ σύμμετροί τε εἰσι ταῖς ΑΖ, ΖΒ καὶ ἐν τῷ αὐτῷ λόγῳ· ἔστιν ἄρα ὡς ἡ ΑΖ πρὸς τὴν ΖΒ, οὕτως ἡ ΚΜ πρὸς τὴν ΜΛ. ἐναλλάξ ἄρα ἐστὶν ὡς ἡ ΑΖ πρὸς τὴν ΚΜ, οὕτως ἡ ΒΖ πρὸς τὴν ΜΛ· καὶ λοιπὴ ἄρα ἡ ΑΒ πρὸς λοιπὴν τὴν ΚΛ ἐστὶν ὡς ἡ ΑΖ πρὸς ΚΜ. σύμμετρος δὲ ἡ ΑΖ τῇ ΚΜ· σύμμετρος ἄρα ἐστὶ καὶ ἡ ΑΒ τῇ ΚΛ. καὶ ἐστὶν ὡς ἡ ΑΒ πρὸς ΚΛ, οὕτως τὸ ὑπὸ τῶν ΓΔ, ΑΒ πρὸς τὸ ὑπὸ τῶν ΓΔ, ΚΛ· σύμμετρον ἄρα ἐστὶ καὶ τὸ ὑπὸ τῶν ΓΔ, ΑΒ τῷ ἀπὸ τῆς Θ· σύμμετρον ἄρα ἐστὶ τὸ ὑπὸ τῶν ΓΔ, ΑΒ τῷ ἀπὸ τῆς Θ. τῷ δὲ ὑπὸ τῶν ΓΔ, ΑΒ ἴσον ἐστὶ τὸ ἀπὸ τῆς Η· σύμμετρον ἄρα ἐστὶ τὸ ἀπὸ τῆς Η τῷ ἀπὸ τῆς Θ. ῥητὸν δὲ τὸ ἀπὸ τῆς Θ· ῥητὸν ἄρα ἐστὶ καὶ τὸ ἀπὸ τῆς Η· ῥητὴ ἄρα ἐστὶν ἡ Η. καὶ δύναται τὸ ὑπὸ τῶν ΓΔ, ΑΒ.

Ἐάν ἄρα χωρίον περιέχεται ὑπὸ ἀποτομῆς καὶ τῆς ἐκ δύο ὀνομάτων, ἥς τὰ ὀνόματα σύμμετρά ἐστι τοῖς τῆς ἀποτομῆς ὀνόμασι καὶ ἐν τῷ αὐτῷ λόγῳ, ἡ τὸ χωρίον δυναμένη ῥητὴ ἐστίν.

KH will have the same order as BC [Defs. 10.5—10.10]. (Which is) the very thing it was required to show.

Proposition 114

If an area is contained by an apotome, and a binomial whose terms are commensurable with, and in the same ratio as, the terms of the apotome then the square-root of the area is a rational (straight-line).



For let an area, the (rectangle contained) by AB and CD , have been contained by the apotome AB , and the binomial CD , of which let the greater term be CE . And let the terms of the binomial, CE and ED , be commensurable with the terms of the apotome, AF and FB (respectively), and in the same ratio. And let the square-root of the (rectangle contained) by AB and CD be G . I say that G is a rational (straight-line).

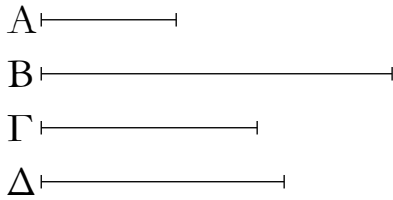
For let the rational (straight-line) H be laid down. And let (some rectangle), equal to the (square) on H , have been applied to CD , producing KL as breadth. Thus, KL is an apotome, of which let the terms, KM and ML , be commensurable with the terms of the binomial, CE and ED (respectively), and in the same ratio [Prop. 10.112]. But, CE and ED are also commensurable with AF and FB (respectively), and in the same ratio. Thus, as AF is to FB , so KM (is) to ML . Thus, alternately, as AF is to KM , so BF (is) to LM [Prop. 5.16]. Thus, the remainder AB is also to the remainder KL as AF (is) to KM [Prop. 5.19]. And AF (is) commensurable with KM [Prop. 10.12]. AB is thus also commensurable with KL [Prop. 10.11]. And as AB is to KL , so the (rectangle contained) by CD and AB (is) to the (rectangle contained) by CD and KL [Prop. 6.1]. Thus, the (rectangle contained) by CD and AB is also commensurable with the (rectangle contained) by CD and KL [Prop. 10.11]. And the (rectangle contained) by CD and KL (is) equal to the (square) on H . Thus, the (rectangle contained) by CD and AB is commensurable with the (square) on H . And the (square) on G is equal to the (rectangle contained) by CD and AB . The (square) on G

Πόρισμα.

Καὶ γέγονεν ἡμῖν καὶ διὰ τούτου φανερόν, ὅτι δυνατόν ἐστι ῥητὸν χωρίον ὑπὸ ἀλόγων εὐθειῶν περιέχεσθαι. ὅπερ ἔδει δεῖξαι.

ριε'.

Ἀπὸ μέσης ἄπειροι ἄλογοι γίνονται, καὶ οὐδεμία οὐδεμιᾶ τῶν πρότερον ἢ αὐτή.



Ἐστω μέση ἡ A . λέγω, ὅτι ἀπὸ τῆς A ἄπειροι ἄλογοι γίνονται, καὶ οὐδεμία οὐδεμιᾶ τῶν πρότερον ἢ αὐτή.

Ἐκκείσθω ῥητὴ ἡ B , καὶ τῷ ὑπὸ τῶν B, A ἴσον ἔστω τὸ ἀπὸ τῆς Γ . ἄλογος ἄρα ἐστὶν ἡ Γ . τὸ γὰρ ὑπὸ ἀλόγου καὶ ῥητῆς ἀλογόν ἐστιν. καὶ οὐδεμιᾶ τῶν πρότερον ἢ αὐτή· τὸ γὰρ ἀπ' οὐδεμιᾶς τῶν πρότερον παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ μέσην. πάλιν δὴ τῷ ὑπὸ τῶν B, Γ ἴσον ἔστω τὸ ἀπὸ τῆς Δ . ἄλογον ἄρα ἐστὶ τὸ ἀπὸ τῆς Δ . ἄλογος ἄρα ἐστὶν ἡ Δ . καὶ οὐδεμιᾶ τῶν πρότερον ἢ αὐτή· τὸ γὰρ ἀπ' οὐδεμιᾶς τῶν πρότερον παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ τὴν Γ . ὁμοίως δὴ τῆς τοιαύτης τάξεως ἐπ' ἄπειρον προβαίνουσιν φανερόν, ὅτι ἀπὸ τῆς μέσης ἄπειροι ἄλογοι γίνονται, καὶ οὐδεμία οὐδεμιᾶ τῶν πρότερον ἢ αὐτή· ὅπερ ἔδει δεῖξαι.

is thus commensurable with the (square) on H . And the (square) on H (is) rational. Thus, the (square) on G is also rational. G is thus rational. And it is the square-root of the (rectangle contained) by CD and AB .

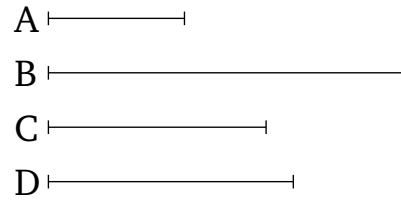
Thus, if an area is contained by an apotome, and a binomial whose terms are commensurable with, and in the same ratio as, the terms of the apotome, then the square-root of the area is a rational (straight-line).

Corollary

And it has also been made clear to us, through this, that it is possible for a rational area to be contained by irrational straight-lines. (Which is) the very thing it was required to show.

Proposition 115

An infinite (series) of irrational (straight-lines) can be created from a medial (straight-line), and none of them is the same as any of the preceding (straight-lines).



Let A be a medial (straight-line). I say that an infinite (series) of irrational (straight-lines) can be created from A , and that none of them is the same as any of the preceding (straight-lines).

Let the rational (straight-line) B be laid down. And let the (square) on C be equal to the (rectangle contained) by B and A . Thus, C is irrational [Def. 10.4]. For an (area contained) by an irrational and a rational (straight-line) is irrational [Prop. 10.20]. And (C is) not the same as any of the preceding (straight-lines). For the (square) on none of the preceding (straight-lines), applied to a rational (straight-line), produces a medial (straight-line) as breadth. So, again, let the (square) on D be equal to the (rectangle contained) by B and C . Thus, the (square) on D is irrational [Prop. 10.20]. D is thus irrational [Def. 10.4]. And (D is) not the same as any of the preceding (straight-lines). For the (square) on none of the preceding (straight-lines), applied to a rational (straight-line), produces C as breadth. So, similarly, this arrangement being advanced to infinity, it is clear that an infinite (series) of irrational (straight-lines) can be created from a medial (straight-line), and that none of them is the same as any of the preceding (straight-lines). (Which is) the very thing it was required to show.

ELEMENTS BOOK 11

Elementary Stereometry

Ὅροι.

α'. Στερεόν ἐστὶ τὸ μήκος καὶ πλάτος καὶ βάθος ἔχον.

β'. Στερεοῦ δὲ πέρας ἐπιφάνεια.

γ'. Εὐθεία πρὸς ἐπίπεδον ὀρθή ἐστίν, ὅταν πρὸς πάσας τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ [ὑποκειμένῳ] ἐπιπέδῳ ὀρθὰς ποιῇ γωνίας.

δ'. Ἐπίπεδον πρὸς ἐπίπεδον ὀρθόν ἐστίν, ὅταν αἱ τῇ κοινῇ τομῇ τῶν ἐπιπέδων πρὸς ὀρθὰς ἀγόμεναι εὐθεῖαι ἐν ἐνὶ τῶν ἐπιπέδων τῷ λοιπῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν.

ε'. Εὐθείας πρὸς ἐπίπεδον κλίσις ἐστίν, ὅταν ἀπὸ τοῦ μετεώρου πέρατος τῆς εὐθείας ἐπὶ τὸ ἐπίπεδον κάθετος ἀχθῇ, καὶ ἀπὸ τοῦ γενομένου σημείου ἐπὶ τὸ ἐν τῷ ἐπιπέδῳ πέρας τῆς εὐθείας εὐθεῖα ἐπιζευχθῇ, ἡ περιεχομένη γωνία ὑπὸ τῆς ἀχθείσης καὶ τῆς ἐφεστώσης.

ς'. Ἐπίπεδον πρὸς ἐπίπεδον κλίσις ἐστὶν ἡ περιεχομένη ὀξεῖα γωνία ὑπὸ τῶν πρὸς ὀρθὰς τῇ κοινῇ τομῇ ἀγομένων πρὸς τῷ αὐτῷ σημείῳ ἐν ἑκατέρῳ τῶν ἐπιπέδων.

ζ'. Ἐπίπεδον πρὸς ἐπίπεδον ὁμοίως κεκλίσθαι λέγεται καὶ ἕτερον πρὸς ἕτερον, ὅταν αἱ εἰρημέναι τῶν κλίσεων γωνίαι ἴσαι ἀλλήλαις ὦσιν.

η'. Παράλληλα ἐπίπεδα ἐστὶ τὰ ἀσύμπτωτα.

θ'. Ὅμοια στερεὰ σχήματ' ἐστὶ τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τὸ πλήθος.

ι'. Ἵσα δὲ καὶ ὅμοια στερεὰ σχήματ' ἐστὶ τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τῷ πλήθει καὶ τῷ μεγέθει.

ια'. Στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γραμμῶν ἀπτομένων ἀλλήλων καὶ μὴ ἐν τῇ αὐτῇ ἐπιφανείᾳ οὐσῶν πρὸς πάσαις ταῖς γραμμαῖς κλίσις. ἄλλως· στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γωνιῶν ἐπιπέδων περιεχομένη μὴ οὐσῶν ἐν τῷ αὐτῷ ἐπιπέδῳ πρὸς ἐνὶ σημείῳ συνισταμένων.

ιβ'. Πυραμὶς ἐστὶ σχῆμα στερεὸν ἐπιπέδοις περιχόμενον ἀπὸ ἐνὸς ἐπιπέδου πρὸς ἐνὶ σημείῳ συνεστῶς.

ιγ'. Πρίσμα ἐστὶ σχῆμα στερεὸν ἐπιπέδοις περιχόμενον, ὧν δύο τὰ ἀπεναντίον ἴσα τε καὶ ὁμοιά ἐστὶ καὶ παράλληλα, τὰ δὲ λοιπὰ παραλληλόγραμμα.

ιδ'. Σφαῖρά ἐστίν, ὅταν ἡμικυκλίου μενούσης τῆς διαμέτρου περιενεχθῇ τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.

ιε'. Ἄξων δὲ τῆς σφαίρας ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ ἡμικύκλιον στρέφεται.

ις'. Κέντρον δὲ τῆς σφαίρας ἐστὶ τὸ αὐτό, ὃ καὶ τοῦ ἡμικυκλίου.

ιζ'. Διάμετρος δὲ τῆς σφαίρας ἐστὶν εὐθεῖα τις διὰ τοῦ κέντρου ἡγμένη καὶ περατουμένη ἐφ' ἑκάτερα τὰ μέρη ὑπὸ τῆς ἐπιφανείας τῆς σφαίρας.

ιη'. Κῶνός ἐστιν, ὅταν ὀρθογωνίου τριγώνου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιενεχθῇ τὸ τρίγωνον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο

Definitions

1. A solid is a (figure) having length and breadth and depth.

2. The extremity of a solid (is) a surface.

3. A straight-line is at right-angles to a plane when it makes right-angles with all of the straight-lines joined to it which are also in the plane.

4. A plane is at right-angles to a(nother) plane when (all of) the straight-lines drawn in one of the planes, at right-angles to the common section of the planes, are at right-angles to the remaining plane.

5. The inclination of a straight-line to a plane is the angle contained by the drawn and standing (straight-lines), when a perpendicular is lead to the plane from the end of the (standing) straight-line raised (out of the plane), and a straight-line is (then) joined from the point (so) generated to the end of the (standing) straight-line (lying) in the plane.

6. The inclination of a plane to a(nother) plane is the acute angle contained by the (straight-lines), (one) in each of the planes, drawn at right-angles to the common segment (of the planes), at the same point.

7. A plane is said to have been similarly inclined to a plane, as another to another, when the aforementioned angles of inclination are equal to one another.

8. Parallel planes are those which do not meet (one another).

9. Similar solid figures are those contained by equal numbers of similar planes (which are similarly arranged).

10. But equal and similar solid figures are those contained by similar planes equal in number and in magnitude (which are similarly arranged).

11. A solid angle is the inclination (constituted) by more than two lines joining one another (at the same point), and not being in the same surface, to all of the lines. Otherwise, a solid angle is that contained by more than two plane angles, not being in the same plane, and constructed at one point.

12. A pyramid is a solid figure, contained by planes, (which is) constructed from one plane to one point.

13. A prism is a solid figure, contained by planes, of which the two opposite (planes) are equal, similar, and parallel, and the remaining (planes are) parallelograms.

14. A sphere is the figure enclosed when, the diameter of a semicircle remaining (fixed), the semicircle is carried around, and again established at the same (position) from which it began to be moved.

15. And the axis of the sphere is the fixed straight-line about which the semicircle is turned.

φέρεισθαι, τὸ περιληφθὲν σχῆμα. καὶ μὲν ἡ μένουσα εὐθεῖα ἴση ἢ τῇ λοιπῇ [τῇ] περὶ τὴν ὀρθὴν περιφερομένη, ὀρθογώνιος ἔσται ὁ κῶνος, ἐὰν δὲ ἐλάττων, ἀμβλυγώνιος, ἐὰν δὲ μείζων, ὀξυγώνιος.

ιθ'. Ἀξων δὲ τοῦ κώνου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ τρίγωνον στρέφεται.

κ'. Βάσις δὲ ὁ κύκλος ὁ ὑπὸ τῆς περιφερομένης εὐθείας γραφόμενος.

κα'. Κύλινδρος ἐστίν, ὅταν ὀρθογωνίου παραλληλογράμμου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιεχθὲν τὸ παραλληλόγραμμον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.

κβ'. Ἀξων δὲ τοῦ κυλίνδρου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ παραλληλόγραμμον στρέφεται.

κγ'. Βάσεις δὲ οἱ κύκλοι οἱ ὑπὸ τῶν ἀπεναντίον περιεχόμενων δύο πλευρῶν γραφόμενοι.

κδ'. Ὅμοιοι κῶνοι καὶ κύλινδροι εἰσιν, ὧν οἱ τε ἄξονες καὶ αἱ διαμέτροι τῶν βάσεων ἀνάλογόν εἰσιν.

κε'. Κύβος ἐστὶ σχῆμα στερεὸν ὑπὸ ἑξ τετραγώνων ἴσων περιεχόμενον.

κς'. Ὀκτάεδρον ἐστὶ σχῆμα στερεὸν ὑπὸ ὀκτῶ τριγώνων ἴσων καὶ ἰσοπλευρῶν περιεχόμενον.

κζ'. Εἰκοσάεδρον ἐστὶ σχῆμα στερεὸν ὑπὸ εἴκοσι τριγώνων ἴσων καὶ ἰσοπλευρῶν περιεχόμενον.

κη'. Δωδεκάεδρον ἐστὶ σχῆμα στερεὸν ὑπὸ δώδεκα πενταγώνων ἴσων καὶ ἰσοπλευρῶν καὶ ἰσογωνίων περιεχόμενον.

16. And the center of the sphere is the same as that of the semicircle.

17. And the diameter of the sphere is any straight-line which is drawn through the center and terminated in both directions by the surface of the sphere.

18. A cone is the figure enclosed when, one of the sides of a right-angled triangle about the right-angle remaining (fixed), the triangle is carried around, and again established at the same (position) from which it began to be moved. And if the fixed straight-line is equal to the remaining (straight-line) about the right-angle, (which is) carried around, then the cone will be right-angled, and if less, obtuse-angled, and if greater, acute-angled.

19. And the axis of the cone is the fixed straight-line about which the triangle is turned.

20. And the base (of the cone is) the circle described by the (remaining) straight-line (about the right-angle which is) carried around (the axis).

21. A cylinder is the figure enclosed when, one of the sides of a right-angled parallelogram about the right-angle remaining (fixed), the parallelogram is carried around, and again established at the same (position) from which it began to be moved.

22. And the axis of the cylinder is the stationary straight-line about which the parallelogram is turned.

23. And the bases (of the cylinder are) the circles described by the two opposite sides (which are) carried around.

24. Similar cones and cylinders are those for which the axes and the diameters of the bases are proportional.

25. A cube is a solid figure contained by six equal squares.

26. An octahedron is a solid figure contained by eight equal and equilateral triangles.

27. An icosahedron is a solid figure contained by twenty equal and equilateral triangles.

28. A dodecahedron is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

α'.

Proposition 1[†]

Εὐθείας γραμμῆς μέρος μὲν τι οὐκ ἔστιν ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, μέρος δὲ τι ἐν μετεωροτέρῳ.

Εἰ γὰρ δυνατόν, εὐθείας γραμμῆς τῆς $AB\Gamma$ μέρος μὲν τι τὸ AB ἔστω ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, μέρος δὲ τι τὸ $B\Gamma$ ἐν μετεωροτέρῳ.

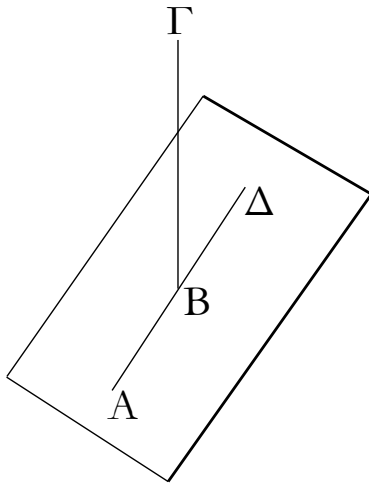
Ἔσται δὴ τις τῇ AB συνεχῆς εὐθεῖα ἐπ' εὐθείας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ. ἔστω ἡ $B\Delta$. δύο ἄρα εὐθειῶν τῶν $AB\Gamma$, $AB\Delta$ κοινὸν τμήμα ἐστὶν ἡ AB . ὅπερ ἐστὶν ἀδύνατον, ἐπειδὴ περ ἐὰν κέντρῳ τῷ B καὶ διαστήματι τῷ AB κύκλον γράψωμεν, αἱ διαμέτροι ἀνίσους ἀπολήφονται τοῦ κύκλου

Some part of a straight-line cannot be in a reference plane, and some part in a more elevated (plane).

For, if possible, let some part, AB , of the straight-line ABC be in a reference plane, and some part, BC , in a more elevated (plane).

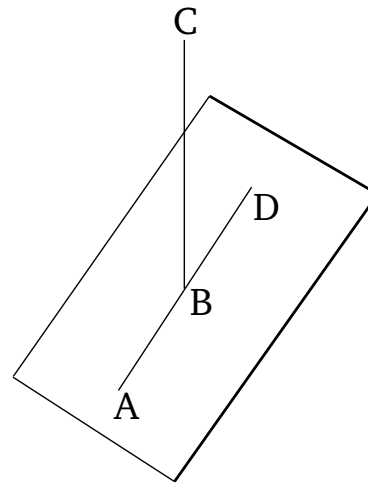
In the reference plane, there will be some straight-line continuous with, and straight-on to, AB .[‡] Let it be BD . Thus, AB is a common segment of the two (different) straight-lines ABC and ABD . The very thing is impossible, inasmuch as if we draw a circle with center B and

περιφερείας.



Εὐθείας ἄρα γραμμῆς μέρος μὲν τι οὐκ ἔστιν ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ ἐν μετεωροτέρῳ· ὅπερ ἔδει δείξαι.

radius AB then the diameters (ABD and ABC) will cut off unequal circumferences of the circle.



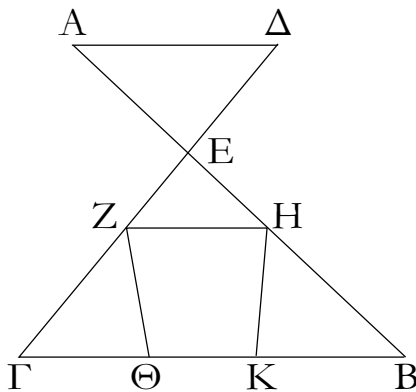
Thus, some part of a straight-line cannot be in a reference plane, and (some part) in a more elevated (plane). (Which is) the very thing it was required to show.

† The proofs of the first three propositions in this book are not at all rigorous. Hence, these three propositions should properly be regarded as additional axioms.

‡ This assumption essentially presupposes the validity of the proposition under discussion.

β'.

Ἐάν δύο εὐθεῖαι τέμνωσιν ἀλλήλας, ἐν ἐνί εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἐνί ἐστιν ἐπιπέδῳ.

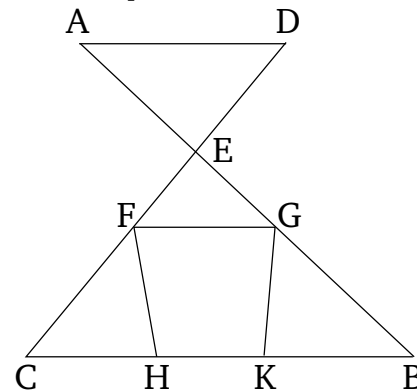


Δύο γὰρ εὐθεῖαι αἱ AB , $\Gamma\Delta$ τεμνέτωσαν ἀλλήλας κατὰ τὸ E σημεῖον. λέγω, ὅτι αἱ AB , $\Gamma\Delta$ ἐν ἐνί εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἐνί ἐστιν ἐπιπέδῳ.

Εἰλήφθω γὰρ ἐπὶ τῶν $ΕΓ$, $ΕΒ$ τυχόντα σημεῖα τὰ Z , H , καὶ ἐπεζεύχθωσαν αἱ ΓB , ZH , καὶ διήχθωσαν αἱ $Z\Theta$, HK . λέγω πρῶτον, ὅτι τὸ $ΕΓΒ$ τρίγωνον ἐν ἐνί ἐστιν ἐπιπέδῳ. εἰ γὰρ ἐστὶ τοῦ $ΕΓΒ$ τριγώνου μέρος ἥτοι τὸ $Z\Theta\Gamma$ ἢ τὸ HBK ἐν τῷ ὑποκειμένῳ [ἐπιπέδῳ], τὸ δὲ λοιπὸν ἐν ἄλλῳ, ἔσται καὶ μιᾶς τῶν $ΕΓ$, $ΕΒ$ εὐθειῶν μέρος μὲν τι ἐν τῷ ὑποκειμένῳ

Proposition 2

If two straight-lines cut one another then they are in one plane, and every triangle (formed using segments of both lines) is in one plane.



For let the two straight-lines AB and CD have cut one another at point E . I say that AB and CD are in one plane, and that every triangle (formed using segments of both lines) is in one plane.

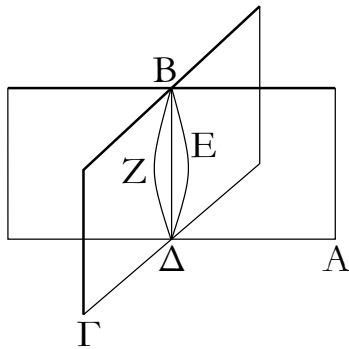
For let the random points F and G have been taken on EC and EB (respectively). And let CB and FG have been joined, and let FH and GK have been drawn across. I say, first of all, that triangle ECB is in one (reference) plane. For if part of triangle ECB , either FHC

ἐπιπέδῳ, τὸ δὲ ἐν ἄλλῳ. εἰ δὲ τοῦ ΕΓΒ τριγώνου τὸ ΖΓΒΗ μέρος ἢ ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ λοιπὸν ἐν ἄλλῳ, ἔσται καὶ ἀμφοτέρων τῶν ΕΓ, ΕΒ εὐθειῶν μέρος μὲν τι ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, τὸ δὲ ἐν ἄλλῳ· ὅπερ ἄτοπον ἐδείχθη. τὸ ἄρα ΕΓΒ τρίγωνον ἐν ἐνὶ ἐστὶν ἐπιπέδῳ. ἐν ᾧ δὲ ἐστὶ τὸ ΕΓΒ τρίγωνον, ἐν τούτῳ καὶ ἑκατέρω τῶν ΕΓ, ΕΒ, ἐν ᾧ δὲ ἑκατέρω τῶν ΕΓ, ΕΒ, ἐν τούτῳ καὶ αἱ ΑΒ, ΓΔ. αἱ ΑΒ, ΓΔ ἄρα εὐθεῖαι ἐν ἐνὶ εἰσιν ἐπιπέδῳ, καὶ πᾶν τρίγωνον ἐν ἐνὶ ἐστὶν ἐπιπέδῳ· ὅπερ ἔδει δεῖξαι.

or GBK , is in the reference [plane], and the remainder in a different (plane) then a part of one the straight-lines EC and EB will also be in the reference plane, and (a part) in a different (plane). And if the part $FCBG$ of triangle ECB is in the reference plane, and the remainder in a different (plane) then parts of both of the straight-lines EC and EB will also be in the reference plane, and (parts) in a different (plane). The very thing was shown to be absurd [Prop. 11.1]. Thus, triangle ECB is in one plane. And in whichever (plane) triangle ECB is (found), in that (plane) EC and EB (will) each also (be found). And in whichever (plane) EC and EB (are) each (found), in that (plane) AB and CD (will) also (be found) [Prop. 11.1]. Thus, the straight-lines AB and CD are in one plane, and every triangle (formed using segments of both lines) is in one plane. (Which is) the very thing it was required to show.

Υ'.

Ἐάν δύο ἐπίπεδα τεμνῇ ἄλληλα, ἡ κοινὴ αὐτῶν τομὴ εὐθεῖα ἐστίν.



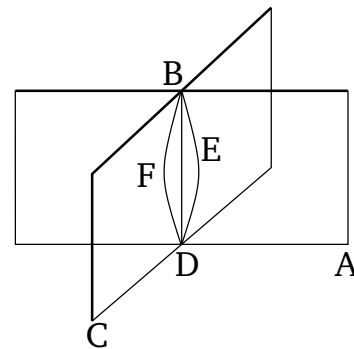
Δύο γὰρ ἐπίπεδα τὰ ΑΒ, ΒΓ τεμνέτω ἄλληλα, κοινὴ δὲ αὐτῶν τομὴ ἔστω ἡ ΔΒ γραμμὴ· λέγω, ὅτι ἡ ΔΒ γραμμὴ εὐθεῖα ἐστίν.

Εἰ γὰρ μή, ἐπεζεύχθω ἀπὸ τοῦ Δ ἐπὶ τὸ Β ἐν μὲν τῷ ΑΒ ἐπιπέδῳ εὐθεῖα ἡ ΔΕΒ, ἐν δὲ τῷ ΒΓ ἐπιπέδῳ εὐθεῖα ἡ ΔΖΒ. ἔσται δὴ δύο εὐθειῶν τῶν ΔΕΒ, ΔΖΒ τὰ αὐτὰ πέρατα, καὶ περιέξουσιν δηλαδὴ χωρίον· ὅπερ ἄτοπον. οὐκ ἄρα αἱ ΔΕΒ, ΔΖΒ εὐθεῖαι εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι οὐδὲ ἄλλη τις ἀπὸ τοῦ Δ ἐπὶ τὸ Β ἐπιζευγνυμένη εὐθεῖα ἔσται πλὴν τῆς ΔΒ κοινῆς τομῆς τῶν ΑΒ, ΒΓ ἐπιπέδων.

Ἐάν ἄρα δύο ἐπίπεδα τέμνη ἄλληλα, ἡ κοινὴ αὐτῶν τομὴ εὐθεῖα ἐστίν· ὅπερ ἔδει δεῖξαι.

Proposition 3

If two planes cut one another then their common section is a straight-line.



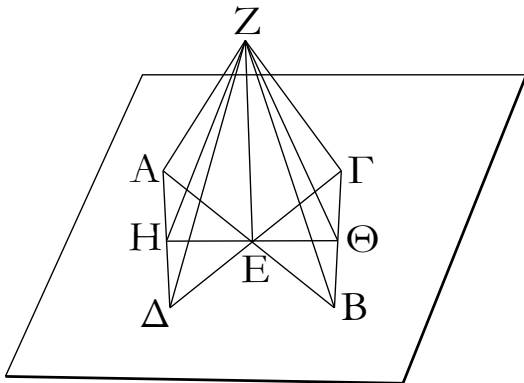
For let the two planes AB and BC cut one another, and let their common section be the line DB . I say that the line DB is straight.

For, if not, let the straight-line DEB have been joined from D to B in the plane AB , and the straight-line DFB in the plane BC . So two straight-lines, DEB and DFB , will have the same ends, and they will clearly enclose an area. The very thing (is) absurd. Thus, DEB and DFB are not straight-lines. So, similarly, we can show that no other straight-line can be joined from D to B except DB , the common section of the planes AB and BC .

Thus, if two planes cut one another then their common section is a straight-line. (Which is) the very thing it was required to show.

δ'.

Ἐάν εὐθεῖα δύο εὐθείαις τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῇ, καὶ τῷ δι' αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.



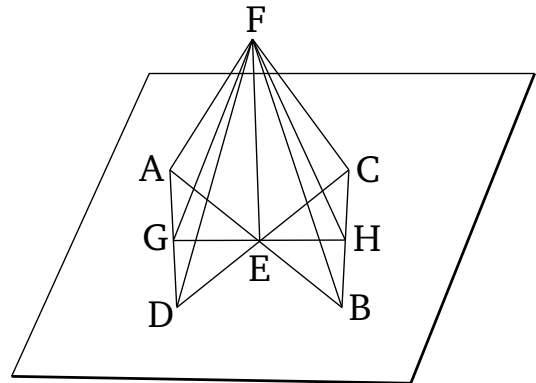
Εὐθεῖα γάρ τις ἡ EZ δύο εὐθείαις ταῖς AB, ΓΔ τεμνούσαις ἀλλήλας κατὰ τὸ E σημείον ἀπὸ τοῦ E πρὸς ὀρθὰς ἐφεστάτω· λέγω, ὅτι ἡ EZ καὶ τῷ διὰ τῶν AB, ΓΔ ἐπιπέδῳ πρὸς ὀρθὰς ἔστιν.

Ἀπειλήφθωσαν γὰρ αἱ AE, EB, ΓE, ED ἴσαι ἀλλήλαις, καὶ διήχθω τις διὰ τοῦ E, ὡς ἔτυχεν, ἡ HEΘ, καὶ ἐπεζεύχθωσαν αἱ AΔ, ΓB, καὶ ἔτι ἀπὸ τυχόντος τοῦ Z ἐπεζεύχθωσαν αἱ ZA, ZH, ZΔ, ZΓ, ZΘ, ZB.

Καὶ ἐπεὶ δύο αἱ AE, ED δυοὶ ταῖς ΓE, EB ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ AΔ βάσει τῇ ΓB ἴση ἐστίν, καὶ τὸ AED τρίγωνον τῷ ΓEB τριγώνῳ ἴσον ἔσται· ὥστε καὶ γωνία ἡ ὑπὸ ΔAE γωνία τῇ ὑπὸ EBG ἴση [ἐστίν]. ἔστι δὲ καὶ ἡ ὑπὸ AEH γωνία τῇ ὑπὸ BEΘ ἴση. δύο δὲ τριγώνῳ ἐστὶ τὰ AHE, BEΘ τὰς δύο γωνίας δυοὶ γωνίας ἴσας ἔχοντα ἑκατέραν ἑκατέρᾳ καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν πρὸς ταῖς ἴσαις γωνίαις τὴν AE τῇ EB· καὶ τὰς λοιπὰς ἄρα πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξουσιν. ἴση ἄρα ἡ μὲν HE τῇ ΕΘ, ἡ δὲ AH τῇ ΒΘ. καὶ ἐπεὶ ἴση ἐστὶν ἡ AE τῇ EB, κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ ZE, βάσις ἄρα ἡ ZA βάσει τῇ ZB ἔστιν ἴση. διὰ τὰ αὐτὰ δὲ καὶ ἡ ZΓ τῇ ZΔ ἔστιν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ AΔ τῇ ΓB, ἔστι δὲ καὶ ἡ ZA τῇ ZB ἴση, δύο δὲ αἱ ZA, AΔ δυοὶ ταῖς ZB, BΓ ἴσαι εἰσὶν ἑκατέρᾳ ἑκατέρᾳ· καὶ βάσις ἡ ZΔ βάσει τῇ ZΓ ἐδείχθη ἴση· καὶ γωνία ἄρα ἡ ὑπὸ ZAΔ γωνία τῇ ὑπὸ ZBG ἴση ἐστίν. καὶ ἐπεὶ πάλιν ἐδείχθη ἡ AH τῇ ΒΘ ἴση, ἀλλὰ μὴν καὶ ἡ ZA τῇ ZB ἴση, δύο δὲ αἱ ZA, AH δυοὶ ταῖς ZB, BΘ ἴσαι εἰσὶν. καὶ γωνία ἡ ὑπὸ ZAH ἐδείχθη ἴση τῇ ὑπὸ ZBΘ· βάσις ἄρα ἡ ZH βάσει τῇ ZΘ ἔστιν ἴση. καὶ ἐπεὶ πάλιν ἴση ἐδείχθη ἡ HE τῇ ΕΘ, κοινὴ δὲ ἡ EZ, δύο δὲ αἱ HE, EZ δυοὶ ταῖς ΘE, EZ ἴσαι εἰσὶν· καὶ βάσις ἡ ZH βάσει τῇ ZΘ ἴση· γωνία ἄρα ἡ ὑπὸ HEZ γωνία τῇ ὑπὸ ΘEZ ἴση ἐστίν. ὀρθὴ ἄρα ἑκατέρα τῶν ὑπὸ HEZ, ΘEZ γωνιῶν. ἡ ZE ἄρα πρὸς τὴν ΗΘ τυχόντως διὰ τοῦ E ἀχθεῖσαν ὀρθὴ ἐστίν. ὁμοίως δὲ δείξομεν, ὅτι ἡ ZE καὶ

Proposition 4

If a straight-line is set up at right-angles to two straight-lines cutting one another, at the common point of section, then it will also be at right-angles to the plane (passing) through them (both).



For let some straight-line EF have (been) set up at right-angles to two straight-lines, AB and CD , cutting one another at point E , at E . I say that EF is also at right-angles to the plane (passing) through AB and CD .

For let AE , EB , CE and ED have been cut off from (the two straight-lines so as to be) equal to one another. And let GEH have been drawn, at random, through E (in the plane passing through AB and CD). And let AD and CB have been joined. And, furthermore, let FA , FG , FD , FC , FH , and FB have been joined from the random (point) F (on EF).

For since the two (straight-lines) AE and ED are equal to the two (straight-lines) CE and EB , and they enclose equal angles [Prop. 1.15], the base AD is thus equal to the base CB , and triangle AED will be equal to triangle CEB [Prop. 1.4]. Hence, the angle DAE [is] equal to the angle EBC . And the angle AEG (is) also equal to the angle BEH [Prop. 1.15]. So AGE and BEH are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), those by the equal angles, AE and EB . Thus, they will also have the remaining sides equal to the remaining sides [Prop. 1.26]. Thus, GE (is) equal to EH , and AG to BH . And since AE is equal to EB , and FE is common and at right-angles, the base FA is thus equal to the base FB [Prop. 1.4]. So, for the same (reasons), FC is also equal to FD . And since AD is equal to CB , and FA is also equal to FB , the two (straight-lines) FA and AD are equal to the two (straight-lines) FB and BC , respectively. And the base FD was shown (to be) equal to the base FC . Thus, the angle FAD is also equal to the angle FBC [Prop. 1.8]. And, again, since AG was shown (to be) equal to BH , but FA (is) also equal to

πρὸς πάσας τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. εὐθεῖα δὲ πρὸς ἐπίπεδον ὀρθή ἐστιν, ὅταν πρὸς πάσας τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ αὐτῷ ἐπιπέδῳ ὀρθὰς ποιῇ γωνίας· ἡ ZE ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἐστιν. τὸ δὲ ὑποκείμενον ἐπίπεδόν ἐστι τὸ διὰ τῶν AB , $\Gamma\Delta$ εὐθειῶν. ἡ ZE ἄρα πρὸς ὀρθὰς ἐστι τῷ διὰ τῶν AB , $\Gamma\Delta$ ἐπιπέδῳ.

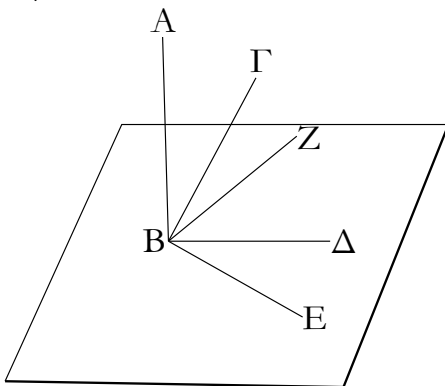
Ἐὰν ἄρα εὐθεῖα δύο εὐθείαις τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῇ, καὶ τῷ δι' αὐτῶν ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

FB , the two (straight-lines) FA and AG are equal to the two (straight-lines) FB and BH (respectively). And the angle FAG was shown (to be) equal to the angle FBH . Thus, the base FG is equal to the base FH [Prop. 1.4]. And, again, since GE was shown (to be) equal to EH , and EF (is) common, the two (straight-lines) GE and EF are equal to the two (straight-lines) HE and EF (respectively). And the base FG (is) equal to the base FH . Thus, the angle GEF is equal to the angle HEF [Prop. 1.8]. Each of the angles GEF and HEF (are) thus right-angles [Def. 1.10]. Thus, FE is at right-angles to GH , which was drawn at random through E (in the reference plane passing through AB and AC). So, similarly, we can show that FE will make right-angles with all straight-lines joined to it which are in the reference plane. And a straight-line is at right-angles to a plane when it makes right-angles with all straight-lines joined to it which are in the plane [Def. 11.3]. Thus, FE is at right-angles to the reference plane. And the reference plane is that (passing) through the straight-lines AB and CD . Thus, FE is at right-angles to the plane (passing) through AB and CD .

Thus, if a straight-line is set up at right-angles to two straight-lines cutting one another, at the common point of section, then it will also be at right-angles to the plane (passing) through them (both). (Which is) the very thing it was required to show.

ε'.

Ἐὰν εὐθεῖα τρισὶν εὐθείαις ἀπτομέναις ἀλλήλων πρὸς ὀρθὰς ἐπὶ τῆς κοινῆς τομῆς ἐπισταθῇ, αἱ τρεῖς εὐθεῖαι ἐν ἐνὶ εἰσιν ἐπιπέδῳ.

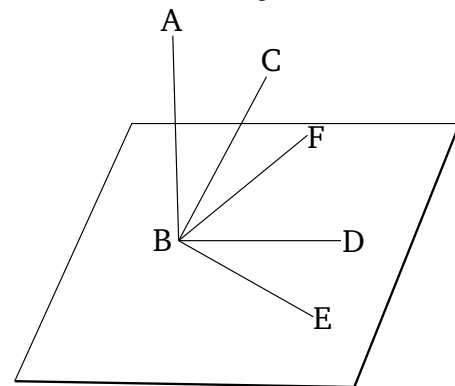


Εὐθεῖα γάρ τις ἡ AB τρισὶν εὐθείαις ταῖς $B\Gamma$, $B\Delta$, BE πρὸς ὀρθὰς ἐπὶ τῆς κατὰ τὸ B ἀφῆς ἐφεστάτω· λέγω, ὅτι αἱ $B\Gamma$, $B\Delta$, BE ἐν ἐνὶ εἰσιν ἐπιπέδῳ.

Μὴ γάρ, ἀλλ' εἰ δυνατόν, ἔστωσαν αἱ μὲν $B\Delta$, BE ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ, ἡ δὲ $B\Gamma$ ἐν μετεωροτέρῳ, καὶ ἐκβεβλήσθω τὸ διὰ τῶν AB , $B\Gamma$ ἐπίπεδον· κοινὴν δὲ τομὴν

Proposition 5

If a straight-line is set up at right-angles to three straight-lines cutting one another, at the common point of section, then the three straight-lines are in one plane.



For let some straight-line AB have been set up at right-angles to three straight-lines BC , BD , and BE , at the (common) point of section B . I say that BC , BD , and BE are in one plane.

For (if) not, and if possible, let BD and BE be in the reference plane, and BC in a more elevated (plane).

ποιήσει ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεΐαν. ποιείτω τὴν BZ. ἐν ἐνὶ ἅρᾳ εἰσὶν ἐπιπέδῳ τῷ διηγμένῳ διὰ τῶν AB, BG αἱ τρεῖς εὐθεΐαι αἱ AB, BG, BZ. καὶ ἐπεὶ ἡ AB ὀρθὴ ἐστὶ πρὸς ἑκατέραν τῶν BΔ, BE, καὶ τῷ διὰ τῶν BΔ, BE ἅρᾳ ἐπιπέδῳ ὀρθὴ ἐστὶν ἡ AB. τὸ δὲ διὰ τῶν BΔ, BE ἐπίπεδον τὸ ὑποκείμενόν ἐστιν· ἡ AB ἅρᾳ ὀρθὴ ἐστὶ πρὸς τὸ ὑποκείμενον ἐπίπεδον. ὥστε καὶ πρὸς πάσας τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας ἡ AB. ἄπτεται δὲ αὐτῆς ἡ BZ οὐσα ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ· ἡ ἅρᾳ ὑπὸ ABZ γωνία ὀρθὴ ἐστὶν. ὑπόκειται δὲ καὶ ἡ ὑπὸ ABΓ ὀρθὴ· ἴση ἅρᾳ ἡ ὑπὸ ABZ γωνία τῇ ὑπὸ ABΓ. καὶ εἰσὶν ἐν ἐνὶ ἐπιπέδῳ· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἅρᾳ ἡ BG εὐθεΐα ἐν μετεωροτέρῳ ἐστὶν ἐπιπέδῳ· αἱ τρεῖς ἅρᾳ εὐθεΐαι αἱ BG, BΔ, BE ἐν ἐνὶ εἰσὶν ἐπιπέδῳ.

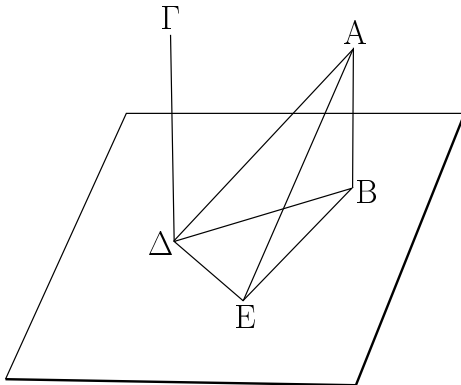
Ἐὰν ἅρᾳ εὐθεΐα τρισὶν εὐθείαις ἀπτομέναις ἀλλήλων ἐπὶ τῆς ἀφῆς πρὸς ὀρθὰς ἐπισταθῇ, αἱ τρεῖς εὐθεΐαι ἐν ἐνὶ εἰσὶν ἐπιπέδῳ· ὅπερ εἶδει δεῖξαι.

And let the plane through AB and BC have been produced. So it will make a straight-line as a common section with the reference plane [Def. 11.3]. Let it make BF . Thus, the three straight-lines AB , BC , and BF are in one plane—(namely), that drawn through AB and BC . And since AB is at right-angles to each of BD and BE , AB is thus also at right-angles to the plane (passing) through BD and BE [Prop. 11.4]. And the plane (passing) through BD and BE is the reference plane. Thus, AB is at right-angles to the reference plane. Hence, AB will also make right-angles with all straight-lines joined to it which are also in the reference plane [Def. 11.3]. And BF , which is in the reference plane, is joined to it. Thus, the angle ABF is a right-angle. And ABC was also assumed to be a right-angle. Thus, angle ABF (is) equal to ABC . And they are in one plane. The very thing is impossible. Thus, BC is not in a more elevated plane. Thus, the three straight-lines BC , BD , and BE are in one plane.

Thus, if a straight-line is set up at right-angles to three straight-lines cutting one another, at the (common) point of section, then the three straight-lines are in one plane. (Which is) the very thing it was required to show.

τ'.

Ἐὰν δύο εὐθεΐαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν, παράλληλοι ἔσονται αἱ εὐθεΐαι.



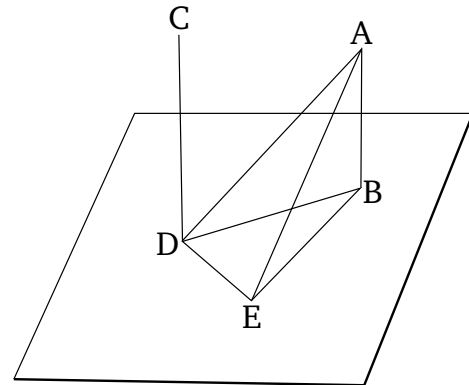
Δύο γὰρ εὐθεΐαι αἱ AB, ΓΔ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστωσαν· λέγω, ὅτι παράλληλός ἐστιν ἡ AB τῇ ΓΔ.

Συμβαλλέτωσαν γὰρ τῷ ὑποκειμένῳ ἐπιπέδῳ κατὰ τὰ B, Δ σημεῖα, καὶ ἐπεζεύχθω ἡ BΔ εὐθεΐα, καὶ ἤχθω τῇ BΔ πρὸς ὀρθὰς ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ἡ ΔΕ, καὶ κείσθω τῇ AB ἴση ἡ ΔΕ, καὶ ἐπεζεύχθωσαν αἱ BE, AE, AD.

Καὶ ἐπεὶ ἡ AB ὀρθὴ ἐστὶ πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ πρὸς πάσας [ἅρᾳ] τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἄπτεται δὲ τῆς AB ἑκατέρα τῶν BΔ, BE οὐσα ἐν τῷ ὑπο-

Proposition 6

If two straight-lines are at right-angles to the same plane then the straight-lines will be parallel.[†]



For let the two straight-lines AB and CD be at right-angles to a reference plane. I say that AB is parallel to CD .

For let them meet the reference plane at points B and D (respectively). And let the straight-line BD have been joined. And let DE have been drawn at right-angles to BD in the reference plane. And let DE be made equal to AB . And let BE , AE , and AD have been joined.

And since AB is at right-angles to the reference plane, it will [thus] also make right-angles with all straight-lines joined to it which are in the reference plane [Def. 11.3].

κειμένω ἐπιπέδω· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ $AB\Delta$, ABE γωνιῶν. διὰ τὰ αὐτὰ δὴ καὶ ἑκατέρα τῶν ὑπὸ $\Gamma\Delta B$, $\Gamma\Delta E$ ὀρθαὶ ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ AB τῇ ΔE , κοινὴ δὲ ἡ $B\Delta$, δύο δὴ αἱ AB , $B\Delta$ δυοὶ ταῖς $E\Delta$, ΔB ἴσαι εἰσὶν· καὶ γωνίας ὀρθὰς περιέχουσιν· βάσις ἄρα ἡ $A\Delta$ βάσει τῇ BE ἐστὶν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ AB τῇ ΔE , ἀλλὰ καὶ ἡ $A\Delta$ τῇ BE , δύο δὴ αἱ AB , BE δυοὶ ταῖς $E\Delta$, ΔA ἴσαι εἰσὶν· καὶ βάσις αὐτῶν κοινὴ ἡ AE · γωνία ἄρα ἡ ὑπὸ ABE γωνία τῇ ὑπὸ $E\Delta A$ ἐστὶν ἴση. ὀρθὴ δὲ ἡ ὑπὸ ABE · ὀρθὴ ἄρα καὶ ἡ ὑπὸ $E\Delta A$ · ἡ $E\Delta$ ἄρα πρὸς τὴν ΔA ὀρθὴ ἐστίν. ἔστι δὲ καὶ πρὸς ἑκατέραν τῶν $B\Delta$, $\Delta\Gamma$ ὀρθή. ἡ $E\Delta$ ἄρα τρισὶν εὐθείαις ταῖς $B\Delta$, ΔA , $\Delta\Gamma$ πρὸς ὀρθὰς ἐπὶ τῆς ἀφ᾽ ἧς ἐφέστηκεν· αἱ τρεῖς ἄρα εὐθεῖαι αἱ $B\Delta$, ΔA , $\Delta\Gamma$ ἐν ἐνὶ εἰσὶν ἐπιπέδω. ἐν ᾧ δὲ αἱ ΔB , ΔA , ἐν τούτῳ καὶ ἡ AB · πᾶν γὰρ τρίγωνον ἐν ἐνὶ ἐστὶν ἐπιπέδω· αἱ ἄρα AB , $B\Delta$, $\Delta\Gamma$ εὐθεῖαι ἐν ἐνὶ εἰσὶν ἐπιπέδω. καὶ ἐστὶν ὀρθὴ ἑκατέρα τῶν ὑπὸ $AB\Delta$, $B\Delta\Gamma$ γωνιῶν· παράλληλος ἄρα ἐστὶν ἡ AB τῇ $\Gamma\Delta$.

Ἐὰν ἄρα δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ᾧσιν, παράλληλοι ἔσονται αἱ εὐθεῖαι· ὅπερ ἔδει δεῖξαι.

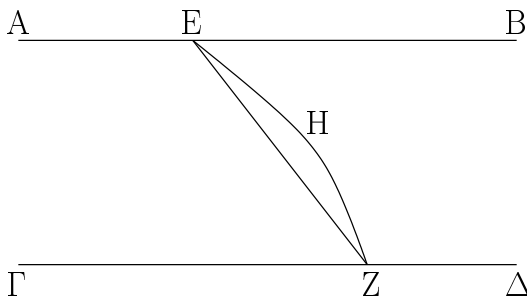
And BD and BE , which are in the reference plane, are each joined to AB . Thus, each of the angles ABD and ABE are right-angles. So, for the same (reasons), each of the angles CDB and CDE are also right-angles. And since AB is equal to DE , and BD (is) common, the two (straight-lines) AB and BD are equal to the two (straight-lines) ED and DB (respectively). And they contain right-angles. Thus, the base AD is equal to the base BE [Prop. 1.4]. And since AB is equal to DE , and AD (is) also (equal) to BE , the two (straight-lines) AB and BE are thus equal to the two (straight-lines) ED and DA (respectively). And their base AE (is) common. Thus, angle ABE is equal to angle EDA [Prop. 1.8]. And ABE (is) a right-angle. Thus, EDA (is) also a right-angle. ED is thus at right-angles to DA . And it is also at right-angles to each of BD and DC . Thus, ED is standing at right-angles to the three straight-lines BD , DA , and DC at the (common) point of section. Thus, the three straight-lines BD , DA , and DC are in one plane [Prop. 11.5]. And in which(ever) plane DB and DA (are found), in that (plane) AB (will) also (be found). For every triangle is in one plane [Prop. 11.2]. And each of the angles ABD and BDC is a right-angle. Thus, AB is parallel to CD [Prop. 1.28].

Thus, if two straight-lines are at right-angles to the same plane then the straight-lines will be parallel. (Which is) the very thing it was required to show.

† In other words, the two straight-lines lie in the same plane, and never meet when produced in either direction.

ζ'.

Ἐὰν ᾧσι δύο εὐθεῖαι παράλληλοι, ληφθῇ δὲ ἐφ' ἑκατέρας αὐτῶν τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις.

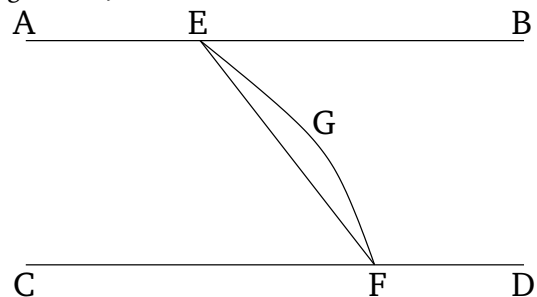


Ἐστωσαν δύο εὐθεῖαι παράλληλοι αἱ AB , $\Gamma\Delta$, καὶ εἰληφθῶ ἐφ' ἑκατέρας αὐτῶν τυχόντα σημεῖα τὰ E , Z · λέγω, ὅτι ἡ ἐπὶ τὰ E , Z σημεῖα ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις.

Μὴ γάρ, ἀλλ' εἰ δυνατόν, ἔστω ἐν μετεωροτέρῳ ὡς ἡ EHZ , καὶ διήχθῳ διὰ τῆς EHZ ἐπίπεδον· τομὴν δὴ ποιήσει

Proposition 7

If there are two parallel straight-lines, and random points are taken on each of them, then the straight-line joining the two points is in the same plane as the parallel (straight-lines).



Let AB and CD be two parallel straight-lines, and let the random points E and F have been taken on each of them (respectively). I say that the straight-line joining points E and F is in the same (reference) plane as the parallel (straight-lines).

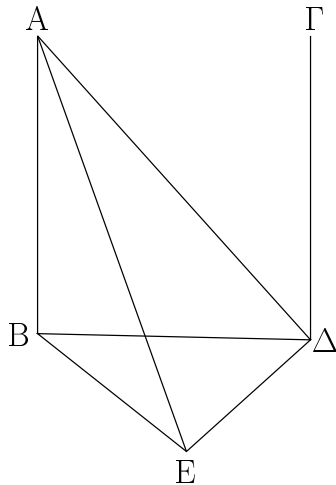
For (if) not, and if possible, let it be in a more elevated

ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεΐαν. ποιείτω ὡς τὴν EZ · δύο ἄρα εὐθεΐαι αἱ EHZ , EZ χωρίον περιέξουσιν· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἡ ἀπὸ τοῦ E ἐπὶ τὸ Z ἐπιζευγνυμένη εὐθεΐα ἐν μετεωροτέρῳ ἐστὶν ἐπιπέδῳ· ἐν τῷ διὰ τῶν AB , ΓB ἄρα παραλλήλων ἐστὶν ἐπιπέδῳ ἡ ἀπὸ τοῦ E ἐπὶ τὸ Z ἐπιζευγνυμένη εὐθεΐα.

Ἐὰν ἄρα ὧσι δύο εὐθεΐαι παράλληλοι, ληφθῇ δὲ ἐφ' ἑκατέρας αὐτῶν τυχόντα σημεῖα, ἡ ἐπὶ τὰ σημεῖα ἐπιζευγνυμένη εὐθεΐα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις· ὅπερ ἔδει δεῖξαι.

η'.

Ἐὰν ὧσι δύο εὐθεΐαι παράλληλοι, ἡ δὲ ἑτέρα αὐτῶν ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾗ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.



Ἐστωσαν δύο εὐθεΐαι παράλληλοι αἱ AB , $\Gamma\Delta$, ἡ δὲ ἑτέρα αὐτῶν ἡ AB τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω· λέγω, ὅτι καὶ ἡ λοιπὴ ἡ $\Gamma\Delta$ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Συμβαλλέτωσαν γὰρ αἱ AB , $\Gamma\Delta$ τῷ ὑποκειμένῳ ἐπιπέδῳ κατὰ τὰ B , Δ σημεῖα, καὶ ἐπεζεύχθω ἡ BD · αἱ AB , $\Gamma\Delta$, BD ἄρα ἐν ἐνὶ εἰσιν ἐπιπέδῳ. ἤχθω τῇ BA πρὸς ὀρθὰς ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ἡ ΔE , καὶ κείσθω τῇ AB ἴση ἡ ΔE , καὶ ἐπεζεύχθωσαν αἱ BE , AE , AD .

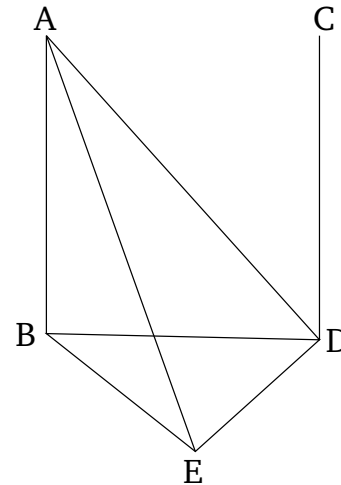
Καὶ ἐπεὶ ἡ AB ὀρθὴ ἐστὶ πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἐστὶν ἡ AB · ὀρθὴ ἄρα [ἐστὶν] ἑκάτερα τῶν ὑπὸ $AB\Delta$, ABE γωνιῶν. καὶ ἐπεὶ εἰς παραλλήλους τὰς AB , $\Gamma\Delta$ εὐθεΐα ἐμπέπτωκεν ἡ BD , αἱ ἄρα ὑπὸ $AB\Delta$, $\Gamma\Delta B$ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσίν. ὀρθὴ δὲ ἡ ὑπὸ $AB\Delta$ · ὀρθὴ ἄρα καὶ ἡ ὑπὸ $\Gamma\Delta B$ · ἡ $\Gamma\Delta$ ἄρα πρὸς τὴν BD ὀρθὴ ἐστὶν. καὶ ἐπεὶ ἴση ἐστὶν ἡ AB τῇ ΔE , κοινὴ δὲ ἡ BD ,

(plane), such as EGF . And let a plane have been drawn through EGF . So it will make a straight cutting in the reference plane [Prop. 11.3]. Let it make EF . Thus, two straight-lines (with the same end-points), EGF and EF , will enclose an area. The very thing is impossible. Thus, the straight-line joining E to F is not in a more elevated plane. The straight-line joining E to F is thus in the plane through the parallel (straight-lines) AB and CD .

Thus, if there are two parallel straight-lines, and random points are taken on each of them, then the straight-line joining the two points is in the same plane as the parallel (straight-lines). (Which is) the very thing it was required to show.

Proposition 8

If two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (one) will also be at right-angles to the same plane.



Let AB and CD be two parallel straight-lines, and let one of them, AB , be at right-angles to a reference plane. I say that the remaining (one), CD , will also be at right-angles to the same plane.

For let AB and CD meet the reference plane at points B and D (respectively). And let BD have been joined. AB , CD , and BD are thus in one plane [Prop. 11.7]. Let DE have been drawn at right-angles to BD in the reference plane, and let DE be made equal to AB , and let BE , AE , and AD have been joined.

And since AB is at right-angles to the reference plane, AB is thus also at right-angles to all of the straight-lines joined to it which are in the reference plane [Def. 11.3]. Thus, the angles ABD and ABE [are] each right-angles. And since the straight-line BD has met the parallel (straight-lines) AB and CD , the (sum of the) angles ABD and CDB is thus equal to two right-angles

δύο δὴ αἱ AB , BD δυσὶ ταῖς ED , ΔB ἴσαι εἰσὶν· καὶ γωνία ἡ ὑπὸ $AB\Delta$ γωνία τῇ ὑπὸ $E\Delta B$ ἴση· ὀρθὴ γὰρ ἑκατέρα· βάσις ἄρα ἡ $A\Delta$ βάσει τῇ BE ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν AB τῇ ΔE , ἡ δὲ BE τῇ $A\Delta$, δύο δὴ αἱ AB , BE δυσὶ ταῖς ED , ΔA ἴσαι εἰσὶν ἑκατέρα ἑκατέρα. καὶ βάσις αὐτῶν κοινὴ ἡ AE · γωνία ἄρα ἡ ὑπὸ ABE γωνία τῇ ὑπὸ $E\Delta A$ ἐστὶν ἴση. ὀρθὴ δὲ ἡ ὑπὸ ABE · ὀρθὴ ἄρα καὶ ἡ ὑπὸ $E\Delta A$ · ἡ $E\Delta$ ἄρα πρὸς τὴν $A\Delta$ ὀρθὴ ἐστὶν. ἔστι δὲ καὶ πρὸς τὴν ΔB ὀρθὴ· ἡ $E\Delta$ ἄρα καὶ τῷ διὰ τῶν $B\Delta$, ΔA ἐπιπέδῳ ὀρθὴ ἐστὶν. καὶ πρὸς πάσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ διὰ τῶν $B\Delta A$ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας ἡ $E\Delta$. ἐν δὲ τῷ διὰ τῶν $B\Delta A$ ἐπιπέδῳ ἐστὶν αἱ AB , $B\Delta$, ἐν ᾧ δὲ αἱ AB , $B\Delta$, ἐν τούτῳ ἐστὶ καὶ ἡ $\Delta\Gamma$. ἡ $E\Delta$ ἄρα τῇ $\Delta\Gamma$ πρὸς ὀρθὰς ἐστὶν· ὥστε καὶ ἡ $\Gamma\Delta$ τῇ ΔE πρὸς ὀρθὰς ἐστὶν. ἔστι δὲ καὶ ἡ $\Gamma\Delta$ τῇ $B\Delta$ πρὸς ὀρθὰς. ἡ $\Gamma\Delta$ ἄρα δύο εὐθείαις τεμνούσαις ἀλλήλας ταῖς ΔE , ΔB ἀπὸ τῆς κατὰ τὸ Δ τομῆς πρὸς ὀρθὰς ἐφέστηκεν· ὥστε ἡ $\Gamma\Delta$ καὶ τῷ διὰ τῶν ΔE , ΔB ἐπιπέδῳ πρὸς ὀρθὰς ἐστὶν. τὸ δὲ διὰ τῶν ΔE , ΔB ἐπίπεδον τὸ ὑποκείμενόν ἐστιν· ἡ $\Gamma\Delta$ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἐστὶν.

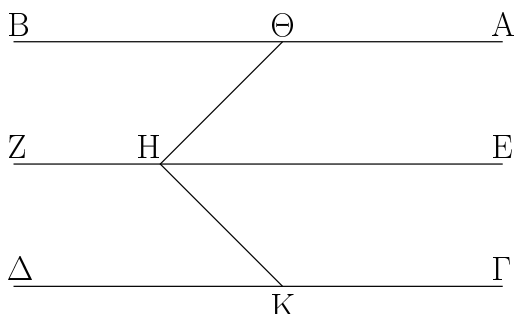
Ἐὰν ἄρα ὥσι δύο εὐθεῖαι παράλληλοι, ἡ δὲ μία αὐτῶν ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾗ, καὶ ἡ λοιπὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

[Prop. 1.29]. And ABD (is) a right-angle. Thus, CDB (is) also a right-angle. CD is thus at right-angles to BD . And since AB is equal to DE , and BD (is) common, the two (straight-lines) AB and BD are equal to the two (straight-lines) ED and DB (respectively). And angle ABD (is) equal to angle EDB . For each (is) a right-angle. Thus, the base AD (is) equal to the base BE [Prop. 1.4]. And since AB is equal to DE , and BE to AD , the two (sides) AB , BE are equal to the two (sides) ED , DA , respectively. And their base AE is common. Thus, angle ABE is equal to angle EDA [Prop. 1.8]. And ABE (is) a right-angle. EDA (is) thus also a right-angle. Thus, ED is at right-angles to AD . And it is also at right-angles to DB . Thus, ED is also at right-angles to the plane through BD and DA [Prop. 11.4]. And ED will thus make right-angles with all of the straight-lines joined to it which are also in the plane through BDA . And DC is in the plane through BDA , inasmuch as AB and BD are in the plane through BDA [Prop. 11.2], and in which (ever plane) AB and BD (are found), DC is also (found). Thus, ED is at right-angles to DC . Hence, CD is also at right-angles to DE . And CD is also at right-angles to BD . Thus, CD is standing at right-angles to two straight-lines, DE and DB , which meet one another, at the (point) of section, D . Hence, CD is also at right-angles to the plane through DE and DB [Prop. 11.4]. And the plane through DE and DB is the reference (plane). CD is thus at right-angles to the reference plane.

Thus, if two straight-lines are parallel, and one of them is at right-angles to some plane, then the remaining (one) will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

θ'.

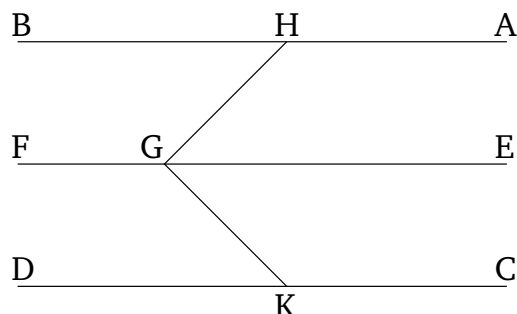
Αἱ τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ μὴ οὐσαι αὐτῇ ἐν τῷ αὐτῷ ἐπιπέδῳ καὶ ἀλλήλαις εἰσὶ παράλληλοι.



Ἐστω γὰρ ἑκατέρα τῶν AB , $\Gamma\Delta$ τῇ EZ παράλληλος μὴ οὐσαι αὐτῇ ἐν τῷ αὐτῷ ἐπιπέδῳ· λέγω, ὅτι παράλληλός

Proposition 9

(Straight-lines) parallel to the same straight-line, and which are not in the same plane as it, are also parallel to one another.



For let AB and CD each be parallel to EF , not being in the same plane as it. I say that AB is parallel to CD .

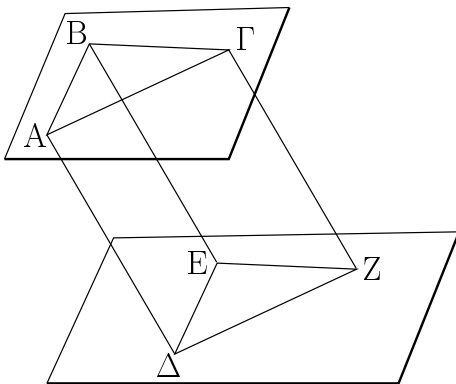
ἐστὶν ἡ AB τῇ $\Gamma\Delta$.

Εἰλήφθω γὰρ ἐπὶ τῆς EZ τυχὸν σημεῖον τὸ H , καὶ ἀπ' αὐτοῦ τῇ EZ ἐν μὲν τῷ διὰ τῶν EZ , AB ἐπιπέδῳ πρὸς ὀρθὰς ἤχθω ἡ $H\Theta$, ἐν δὲ τῷ διὰ τῶν ZE , $\Gamma\Delta$ τῇ EZ πάλιν πρὸς ὀρθὰς ἤχθω ἡ HK .

Καὶ ἐπεὶ ἡ EZ πρὸς ἑκατέραν τῶν $H\Theta$, HK ὀρθὴ ἐστίν, ἡ EZ ἄρα καὶ τῷ διὰ τῶν $H\Theta$, HK ἐπιπέδῳ πρὸς ὀρθὰς ἐστίν. καὶ ἐστὶν ἡ EZ τῇ AB παράλληλος· καὶ ἡ AB ἄρα τῷ διὰ τῶν ΘHK ἐπιπέδῳ πρὸς ὀρθὰς ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ἡ $\Gamma\Delta$ τῷ διὰ τῶν ΘHK ἐπιπέδῳ πρὸς ὀρθὰς ἐστίν· ἑκατέρα ἄρα τῶν AB , $\Gamma\Delta$ τῷ διὰ τῶν ΘHK ἐπιπέδῳ πρὸς ὀρθὰς ἐστίν. ἐὰν δὲ δύο εὐθεῖαι τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν, παράλληλοί εἰσιν αἱ εὐθεῖαι· παράλληλος ἄρα ἐστὶν ἡ AB τῇ $\Gamma\Delta$. ὅπερ εἶδει δεῖξαι.

ι'.

Ἐὰν δύο εὐθεῖαι ἀπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἀπτομένας ἀλλήλων ὥσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, ἴσας γωνίας περιέξουσιν.



Δύο γὰρ εὐθεῖαι αἱ AB , $B\Gamma$ ἀπτόμεναι ἀλλήλων παρὰ δύο εὐθείας τὰς ΔE , EZ ἀπτομένας ἀλλήλων ἔστωσαν μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ· λέγω, ὅτι ἴση ἐστὶν ἡ ὑπὸ $AB\Gamma$ γωνία τῇ ὑπὸ ΔEZ .

Ἀπειλήφθωσαν γὰρ αἱ BA , $B\Gamma$, $E\Delta$, EZ ἴσαι ἀλλήλαις, καὶ ἐπεξεύχθωσαν αἱ $A\Delta$, ΓZ , BE , $A\Gamma$, ΔZ .

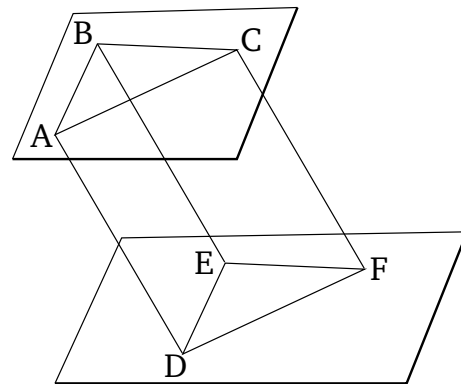
Καὶ ἐπεὶ ἡ BA τῇ $E\Delta$ ἴση ἐστὶ καὶ παράλληλος, καὶ ἡ $A\Delta$ ἄρα τῇ BE ἴση ἐστὶ καὶ παράλληλος. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΓZ τῇ BE ἴση ἐστὶ καὶ παράλληλος· ἑκατέρα ἄρα τῶν $A\Delta$, ΓZ τῇ BE ἴση ἐστὶ καὶ παράλληλος. αἱ δὲ τῇ αὐτῇ εὐθείᾳ παράλληλοι καὶ μὴ οὖσαι αὐτῇ ἐν τῷ αὐτῷ ἐπιπέδῳ καὶ ἀλλήλαις εἰσὶ παράλληλοι· παράλληλος ἄρα ἐστὶν ἡ $A\Delta$ τῇ ΓZ καὶ ἴση. καὶ ἐπιζευγνύουσιν αὐτὰς αἱ $A\Gamma$, ΔZ · καὶ ἡ $A\Gamma$ ἄρα τῇ ΔZ ἴση ἐστὶ καὶ παράλληλος. καὶ ἐπεὶ δύο αἱ AB , $B\Gamma$ δυσὶ ταῖς ΔE , EZ ἴσαι εἰσὶν, καὶ βάσεις ἡ $A\Gamma$ βάσει τῇ ΔZ ἴση, γωνία ἄρα ἡ ὑπὸ $AB\Gamma$ γωνία τῇ ὑπὸ ΔEZ ἐστίν

For let some point G have been taken at random on EF . And from it let GH have been drawn at right-angles to EF in the plane through EF and AB . And let GK have been drawn, again at right-angles to EF , in the plane through FE and CD .

And since EF is at right-angles to each of GH and GK , EF is thus also at right-angles to the plane through GH and GK [Prop. 11.4]. And EF is parallel to AB . Thus, AB is also at right-angles to the plane through HGK [Prop. 11.8]. So, for the same (reasons), CD is also at right-angles to the plane through HGK . Thus, AB and CD are each at right-angles to the plane through HGK . And if two straight-lines are at right-angles to the same plane then the straight-lines are parallel [Prop. 11.6]. Thus, AB is parallel to CD . (Which is) the very thing it was required to show.

Proposition 10

If two straight-lines joined to one another are (respectively) parallel to two straight-lines joined to one another, (but are) not in the same plane, then they will contain equal angles.



For let the two straight-lines joined to one another, AB and BC , be (respectively) parallel to the two straight-lines joined to one another, DE and EF , (but) not in the same plane. I say that angle ABC is equal to (angle) DEF .

For let BA , BC , ED , and EF have been cut off (so as to be, respectively) equal to one another. And let AD , CF , BE , AC , and DF have been joined.

And since BA is equal and parallel to ED , AD is thus also equal and parallel to BE [Prop. 1.33]. So, for the same reasons, CF is also equal and parallel to BE . Thus, AD and CF are each equal and parallel to BE . And straight-lines parallel to the same straight-line, and which are not in the same plane as it, are also parallel to one another [Prop. 11.9]. Thus, AD is parallel and equal to CF . And AC and DF join them. Thus, AC is also equal and

δὲ ἡ AZ καὶ πρὸς τὴν ΔE ὀρθή· ἡ AZ ἄρα πρὸς ἑκατέραν τῶν $H\Theta$, ΔE ὀρθή ἐστίν. ἐὰν δὲ εὐθεῖα δυσὶν εὐθείαις τεμνούσαις ἀλλήλας ἐπὶ τῆς τομῆς πρὸς ὀρθάς ἐπισταθῇ, καὶ τῷ δι' αὐτῶν ἐπιπέδῳ πρὸς ὀρθάς ἔσται· ἡ ZA ἄρα τῷ διὰ τῶν $E\Delta$, $H\Theta$ ἐπιπέδῳ πρὸς ὀρθάς ἐστίν. τὸ δὲ διὰ τῶν $E\Delta$, $H\Theta$ ἐπίπεδόν ἐστι τὸ ὑποκείμενον· ἡ AZ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστίν.

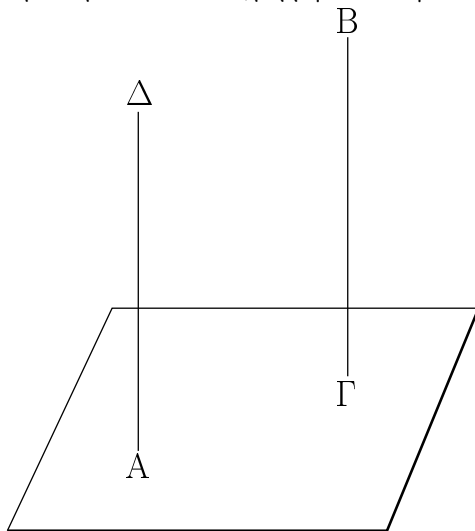
Ἀπὸ τοῦ ἄρα δοθέντος σημείου μετέωρου τοῦ A ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετος εὐθεῖα γραμμὴ ῥηται ἡ AZ · ὅπερ ἔδει ποιῆσαι.

ED and DA . And GH is thus at right-angles to all of the straight-lines joined to it which are also in the plane through ED and AD [Def. 11.3]. And AF , which is in the plane through ED and DA , is joined to it. Thus, GH is at right-angles to FA . Hence, FA is also at right-angles to HG . And AF is also at right-angles to DE . Thus, AF is at right-angles to each of GH and DE . And if a straight-line is set up at right-angles to two straight-lines cutting one another, at the point of section, then it will also be at right-angles to the plane through them [Prop. 11.4]. Thus, FA is at right-angles to the plane through ED and GH . And the plane through ED and GH is the reference (plane). Thus, AF is at right-angles to the reference plane.

Thus, the straight-line AF has been drawn from the given raised point A perpendicular to the reference plane. (Which is) the very thing it was required to do.

ιβ'.

Τῷ δοθέντι ἐπιπέδῳ ἀπὸ τοῦ πρὸς αὐτῷ δοθέντος σημείου πρὸς ὀρθάς εὐθεῖαν γραμμὴν ἀναστήσαι.



Ἐστω τὸ μὲν δοθὲν ἐπίπεδον τὸ ὑποκείμενον, τὸ δὲ πρὸς αὐτῷ σημεῖον τὸ A · δεῖ δὲ ἀπὸ τοῦ A σημείου τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς εὐθεῖαν γραμμὴν ἀναστήσαι.

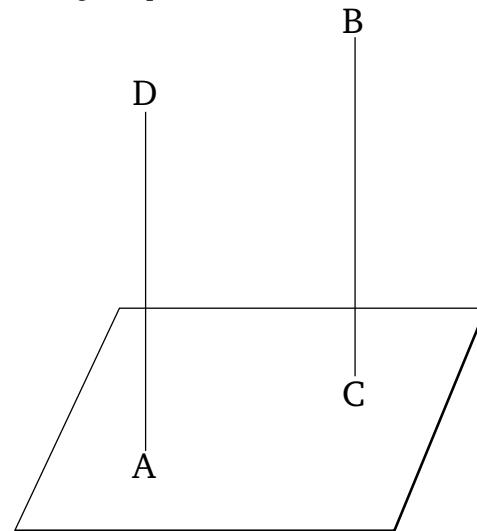
Νενοήσθω τι σημεῖον μετέωρον τὸ B , καὶ ἀπὸ τοῦ B ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετος ῥηθῶ ἡ BF , καὶ διὰ τοῦ A σημείου τῇ BF παράλληλος ῥηθῶ ἡ AD .

Ἐπεὶ οὖν δύο εὐθεῖαι παράλληλοι εἰσιν αἱ AD , FB , ἡ δὲ μία αὐτῶν ἡ BF τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστίν, καὶ ἡ λοιπὴ ἄρα ἡ AD τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθάς ἐστίν.

Τῷ ἄρα δοθέντι ἐπιπέδῳ ἀπὸ τοῦ πρὸς αὐτῷ σημείου τοῦ A πρὸς ὀρθάς ἀνέσταται ἡ AD · ὅπερ ἔδει ποιῆσαι.

Proposition 12

To set up a straight-line at right-angles to a given plane from a given point in it.



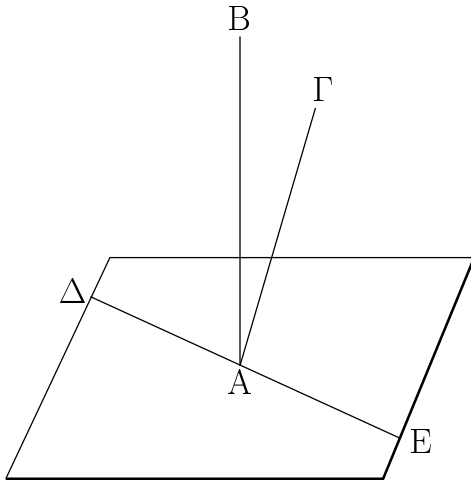
Let the given plane be the reference (plane), and A a point in it. So, it is required to set up a straight-line at right-angles to the reference plane at point A .

Let some raised point B have been assumed, and let the perpendicular (straight-line) BC have been drawn from B to the reference plane [Prop. 11.11]. And let AD have been drawn from point A parallel to BC [Prop. 1.31].

Therefore, since AD and CB are two parallel straight-lines, and one of them, BC , is at right-angles to the reference plane, the remaining (one) AD is thus also at right-angles to the reference plane [Prop. 11.8].

ιγ'.

Ἀπὸ τοῦ αὐτοῦ σημείου τῷ αὐτῷ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς οὐκ ἀναστήσονται ἐπὶ τὰ αὐτὰ μέρη.



Εἰ γὰρ δυνατόν, ἀπὸ τοῦ αὐτοῦ σημείου τοῦ Α τῷ ὑποκειμένῳ ἐπιπέδῳ δύο εὐθεῖαι αἱ ΑΒ, ΒΓ πρὸς ὀρθὰς ἀνεστάτωσαν ἐπὶ τὰ αὐτὰ μέρη, καὶ διήχθω τὸ διὰ τῶν ΒΑ, ΑΓ ἐπίπεδον· τομὴν δὴ ποιήσει διὰ τοῦ Α ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ εὐθεῖαν· ποιεῖτω τὴν ΔΑΕ· αἱ ἄρα ΑΒ, ΑΓ, ΔΑΕ εὐθεῖαι ἐν ἐνὶ εἰσιν ἐπιπέδῳ. καὶ ἐπεὶ ἡ ΓΑ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἐστίν, καὶ πρὸς πάσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἄπτεται δὲ αὐτῆς ἡ ΔΑΕ οὐσα ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ· ἡ ἄρα ὑπὸ ΓΑΕ γωνία ὀρθὴ ἐστίν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ ΒΑΕ ὀρθὴ ἐστίν· ἴση ἄρα ἡ ὑπὸ ΓΑΕ τῇ ὑπὸ ΒΑΕ καὶ εἰσιν ἐν ἐνὶ ἐπιπέδῳ· ὅπερ ἐστὶν ἀδύνατον.

Οὐκ ἄρα ἀπὸ τοῦ αὐτοῦ σημείου τῷ αὐτῷ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς ἀνασταθήσονται ἐπὶ τὰ αὐτὰ μέρη· ὅπερ ἔδει δεῖξαι.

ιδ'.

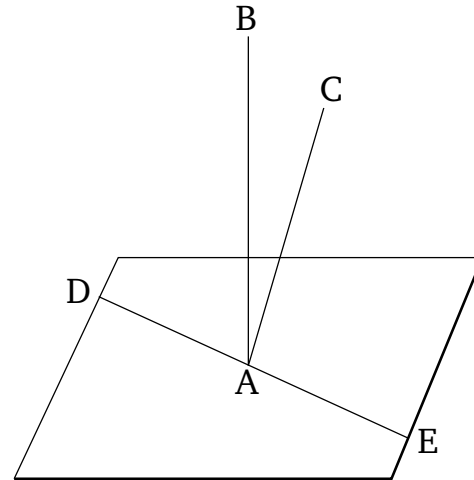
Πρὸς ἃ ἐπίπεδα ἡ αὐτὴ εὐθεῖα ὀρθὴ ἐστίν, παράλληλα ἔσται τὰ ἐπίπεδα.

Εὐθεῖα γάρ τις ἡ ΑΒ πρὸς ἐκάτερον τῶν ΓΔ, ΕΖ ἐπιπέδων πρὸς ὀρθὰς ἔστω· λέγω, ὅτι παράλληλά ἐστι τὰ ἐπίπεδα.

Thus, AD has been set up at right-angles to the given plane, from the point in it, A . (Which is) the very thing it was required to do.

Proposition 13

Two (different) straight-lines cannot be set up at the same point at right-angles to the same plane, on the same side.



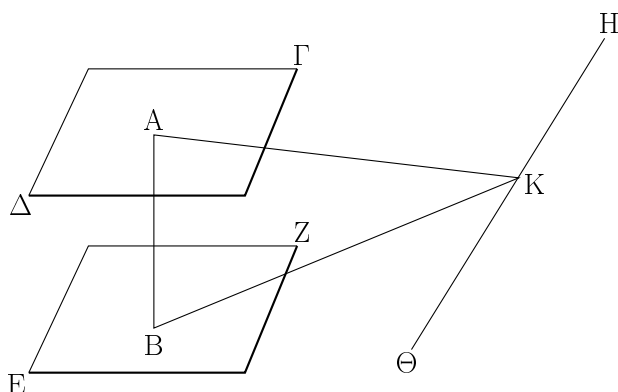
For, if possible, let the two straight-lines AB and AC have been set up at the same point A at right-angles to the reference plane, on the same side. And let the plane through BA and AC have been drawn. So it will make a straight cutting (passing) through (point) A in the reference plane [Prop. 11.3]. Let it have made DAE . Thus, AB , AC , and DAE are straight-lines in one plane. And since CA is at right-angles to the reference plane, it will thus also make right-angles with all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. And DAE , which is in the reference plane, is joined to it. Thus, angle CAE is a right-angle. So, for the same (reasons), BAE is also a right-angle. Thus, CAE (is) equal to BAE . And they are in one plane. The very thing is impossible.

Thus, two (different) straight-lines cannot be set up at the same point at right-angles to the same plane, on the same side. (Which is) the very thing it was required to show.

Proposition 14

Planes to which the same straight-line is at right-angles will be parallel planes.

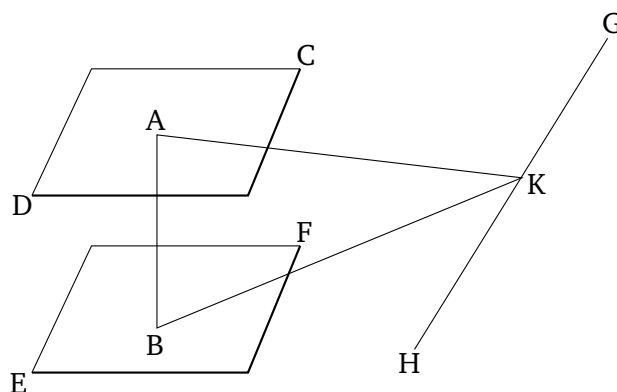
For let some straight-line AB be at right-angles to each of the planes CD and EF . I say that the planes are parallel.



Εἰ γὰρ μή, ἐκβαλλόμενα συμπεσοῦνται. συμπιπτέ-
ωσαν ποιήσουσι δὴ κοινὴν τομὴν εὐθεῖαν. ποιείτωσαν τὴν
ΗΘ, καὶ εἰλήφθω ἐπὶ τῆς ΗΘ τυχὸν σημεῖον τὸ Κ, καὶ
ἐπεζεύχθωσαν αἱ ΑΚ, ΒΚ.

Καὶ ἐπεὶ ἡ AB ὀρθὴ ἐστὶ πρὸς τὸ EZ ἐπίπεδον, καὶ πρὸς τὴν BK ἄρα εὐθεΐαν οὖσαν ἐν τῷ EZ ἐκβληθέντι ἐπιπέδῳ ὀρθὴ ἐστὶν ἡ AB · ἡ ἄρα ὑπὸ ABK γωνία ὀρθὴ ἐστὶν. διὰ τὰ αὐτὰ δὴ καὶ ἡ ὑπὸ BAK ὀρθὴ ἐστὶν. τριγώνου δὴ τοῦ ABK αἱ δύο γωνίαι αἱ ὑπὸ ABK , BAK δυσὶν ὀρθαῖς εἰσιν ἴσαι· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τὰ $\Gamma\Delta$, EZ ἐπίπεδα ἐκβαλλόμενα συμπεσοῦνται· παράλληλα ἄρα ἐστὶ τὰ $\Gamma\Delta$, EZ ἐπίπεδα.

Πρὸς ἃ ἐπίπεδα ἄρα ἡ αὐτὴ εὐθεΐα ὀρθή ἐστιν, παράλληλά ἐστι τὰ ἐπίπεδα· ὅπερ ἔδει δεῖξαι.



For, if not, being produced, they will meet. Let them have met. So they will make a straight-line as a common section [Prop. 11.3]. Let them have made GH . And let some random point K have been taken on GH . And let AK and BK have been joined.

And since AB is at right-angles to the plane EF , AB is thus also at right-angles to BK , which is a straight-line in the produced plane EF [Def. 11.3]. Thus, angle ABK is a right-angle. So, for the same (reasons), BAK is also a right-angle. So the (sum of the) two angles ABK and BAK in the triangle ABK is equal to two right-angles. The very thing is impossible [Prop. 1.17]. Thus, planes CD and EF , being produced, will not meet. Planes CD and EF are thus parallel [Def. 11.8].

Thus, planes to which the same straight-line is at right-angles are parallel planes. (Which is) the very thing it was required to show.

13.

Ἐὰν δύο εὐθεῖαι ἀπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἀπτομένας ἀλλήλων ὥσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ οὔσαι, παράλληλά ἐστι τὰ δι' αὐτῶν ἐπίπεδα.

Δύο γὰρ εὐθεῖαι ἀπτόμεναι ἀλλήλων αἱ AB, BG παρὰ δύο εὐθείας ἀπτομένας ἀλλήλων τὰς ΔΕ, ΕΖ ἔστωσαν μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι· λέγω, ὅτι ἐκβαλλόμενα τὰ διὰ τῶν AB, BG, ΔΕ, ΕΖ ἐπίπεδα οὐ συμπεσεῖται ἀλλήλοις.

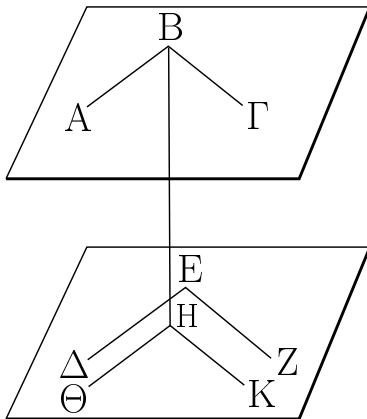
Ἦχθω γὰρ ἀπὸ τοῦ Β σημείου ἐπὶ τὸ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδον κάθετος ἡ ΒΗ καὶ συμβαλλέτω τῷ ἐπιπέδῳ κατὰ τὸ Η σημεῖον, καὶ διὰ τοῦ Η τῇ μὲν ΕΔ παράλληλος ἦχθω ἡ ΗΘ, τῇ δὲ ΕΖ ἡ ΗΚ.

Proposition 15

If two straight-lines joined to one another are parallel (respectively) to two straight-lines joined to one another, which are not in the same plane, then the planes through them are parallel (to one another).

For let the two straight-lines joined to one another, AB and BC , be parallel to the two straight-lines joined to one another, DE and EF (respectively), not being in the same plane. I say that the planes through AB , BC and DE , EF will not meet one another (when) produced.

For let BG have been drawn from point B perpendicular to the plane through DE and EF [Prop. 11.11], and let it meet the plane at point G . And let GH have been drawn through G parallel to ED , and GK (parallel) to EF [Prop. 1.31].



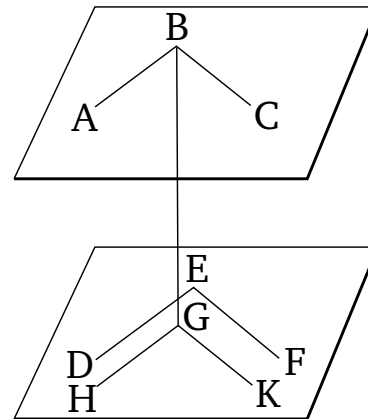
Καὶ ἐπεὶ ἡ BH ὀρθὴ ἐστὶ πρὸς τὸ διὰ τῶν ΔΕ, ΕΖ ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὖσας ἐν τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἄπτεται δὲ αὐτῆς ἑκατέρα τῶν ΗΘ, ΗΚ οὖσα ἐν τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδῳ· ὀρθὴ ἄρα ἐστὶν ἑκατέρα τῶν ὑπὸ BHΘ, BHK γωνιῶν. καὶ ἐπεὶ παράλληλός ἐστιν ἡ BA τῇ ΗΘ, αἱ ἄρα ὑπὸ HBA, BHΘ γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσὶν. ὀρθὴ δὲ ἡ ὑπὸ BHΘ· ὀρθὴ ἄρα καὶ ἡ ὑπὸ HBA· ἡ HB ἄρα τῇ BA πρὸς ὀρθὰς ἐστὶν. διὰ τὰ αὐτὰ δὴ ἡ HB καὶ τῇ BΓ ἐστὶ πρὸς ὀρθὰς. ἐπεὶ οὖν εὐθεῖα ἡ HB δυσὶν εὐθείαις ταῖς BA, BΓ τεμνούσαις ἀλλήλας πρὸς ὀρθὰς ἐφέστηκεν, ἡ HB ἄρα καὶ τῷ διὰ τῶν BA, BΓ ἐπιπέδῳ πρὸς ὀρθὰς ἐστὶν. [διὰ τὰ αὐτὰ δὴ ἡ BH καὶ τῷ διὰ τῶν ΗΘ, ΗΚ ἐπιπέδῳ πρὸς ὀρθὰς ἐστὶν. τὸ δὲ διὰ τῶν ΗΘ, ΗΚ ἐπίπεδόν ἐστὶ τὸ διὰ τῶν ΔΕ, ΕΖ· ἡ BH ἄρα τῷ διὰ τῶν ΔΕ, ΕΖ ἐπιπέδῳ ἐστὶ πρὸς ὀρθὰς. ἐδείχθη δὲ ἡ HB καὶ τῷ διὰ τῶν AB, BΓ ἐπιπέδῳ πρὸς ὀρθὰς]. πρὸς ἃ δὲ ἐπίπεδα ἡ αὐτὴ εὐθεῖα ὀρθὴ ἐστὶν, παράλληλά ἐστι τὰ ἐπίπεδα· παράλληλον ἄρα ἐστὶ τὸ διὰ τῶν AB, BΓ ἐπίπεδον τῷ διὰ τῶν ΔΕ, ΕΖ.

Ἐὰν ἄρα δύο εὐθεῖαι ἀπτόμεναι ἀλλήλων παρὰ δύο εὐθείας ἀπτομένας ἀλλήλων ὥσι μὴ ἐν τῷ αὐτῷ ἐπιπέδῳ, παράλληλά ἐστι τὰ δι' αὐτῶν ἐπίπεδα· ὅπερ ἔδει δεῖξαι.

ις'.

Ἐὰν δύο ἐπίπεδα παράλληλα ὑπὸ ἐπιπέδου τινὸς τέμνηται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν.

Δύο γὰρ ἐπίπεδα παράλληλα τὰ AB, ΓΔ ὑπὸ ἐπιπέδου τοῦ EZHΘ τεμνέσθω, κοιναὶ δὲ αὐτῶν τομαὶ ἔστωσαν αἱ EZ, ΗΘ· λέγω, ὅτι παράλληλός ἐστιν ἡ EZ τῇ ΗΘ.



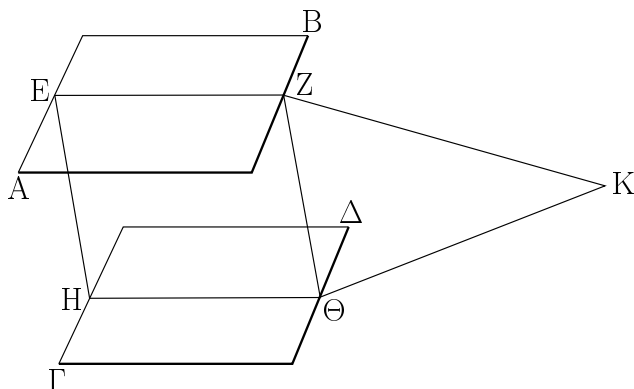
And since BH is at right-angles to the plane through DE and EF , it will thus also make right-angles with all of the straight-lines joined to it, which are also in the plane through DE and EF [Def. 11.3]. And each of GH and GK , which are in the plane through DE and EF , are joined to it. Thus, each of the angles BGH and BGK are right-angles. And since BA is parallel to GH [Prop. 11.9], the (sum of the) angles GBA and BGH is equal to two right-angles [Prop. 1.29]. And BGH (is) a right-angle. GBA (is) thus also a right-angle. Thus, GB is at right-angles to BA . So, for the same (reasons), GB is also at right-angles to BC . Therefore, since the straight-line GB has been set up at right-angles to two straight-lines, BA and BC , cutting one another, GB is thus at right-angles to the plane through BA and BC [Prop. 11.4]. [So, for the same (reasons), BG is also at right-angles to the plane through GH and GK . And the plane through GH and GK is the (plane) through DE and EF . And it was also shown that GB is at right-angles to the plane through AB and BC .] And planes to which the same straight-line is at right-angles are parallel planes [Prop. 11.14]. Thus, the plane through AB and BC is parallel to the (plane) through DE and EF .

Thus, if two straight-lines joined to one another are parallel (respectively) to two straight-lines joined to one another, which are not in the same plane, then the planes through them are parallel (to one another). (Which is) the very thing it was required to show.

Proposition 16

If two parallel planes are cut by some plane then their common sections are parallel.

For let the two parallel planes AB and CD have been cut by the plane $EFGH$. And let EF and GH be their common sections. I say that EF is parallel to GH .



Εἰ γὰρ μή, ἐκβαλλόμεναι αἱ EZ, HΘ ἤτοι ἐπὶ τὰ Z, Θ μέρη ἢ ἐπὶ τὰ E, H συμπεσοῦνται. ἐκβεβλήσθωσαν ὥς ἐπὶ τὰ Z, Θ μέρη καὶ συμπίπτεωσαν πρότερον κατὰ τὸ K. καὶ ἐπεὶ ἡ EZK ἐν τῷ AB ἐστὶν ἐπιπέδῳ, καὶ πάντα ἄρα τὰ ἐπὶ τῆς EZK σημεία ἐν τῷ AB ἐστὶν ἐπιπέδῳ. ἐν δὲ τῶν ἐπὶ τῆς EZK εὐθείας σημείων ἐστὶ τὸ K· τὸ K ἄρα ἐν τῷ AB ἐστὶν ἐπιπέδῳ. διὰ τὰ αὐτὰ δὴ τὸ K καὶ ἐν τῷ ΓΔ ἐστὶν ἐπιπέδῳ· τὰ AB, ΓΔ ἄρα ἐπίπεδα ἐκβαλλόμενα συμπεσοῦνται. οὐ συμπίπτουσι δὲ διὰ τὸ παράλληλα ὑποκεῖσθαι· οὐκ ἄρα αἱ EZ, HΘ εὐθεῖαι ἐκβαλλόμεναι ἐπὶ τὰ Z, Θ μέρη συμπεσοῦνται. ὁμοίως δὲ δεῖξομεν, ὅτι αἱ EZ, HΘ εὐθεῖαι οὐδὲ ἐπὶ τὰ E, H μέρη ἐκβαλλόμεναι συμπεσοῦνται. αἱ δὲ ἐπὶ μηδέτερα τὰ μέρη συμπίπτουσαι παράλληλοί εἰσιν. παράλληλος ἄρα ἐστὶν ἡ EZ τῇ HΘ.

Ἐάν ἄρα δύο ἐπίπεδα παράλληλα ὑπὸ ἐπιπέδου τινὸς τέμνηται, αἱ κοινὰ αὐτῶν τομαὶ παράλληλοί εἰσιν· ὅπερ ἔδει δεῖξαι.

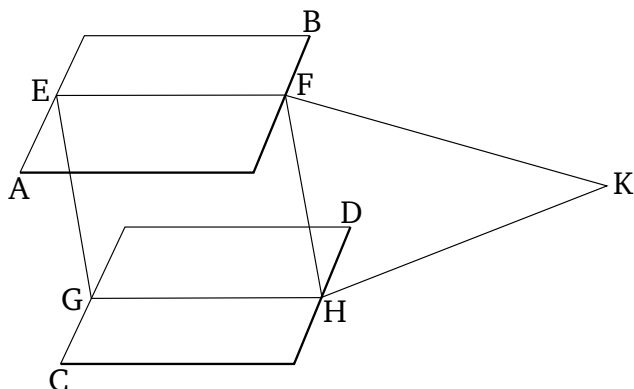
ιζ'.

Ἐάν δύο εὐθεῖαι ὑπὸ παραλλήλων ἐπιπέδων τέμνωνται, εἰς τοὺς αὐτοὺς λόγους τμηθήσονται.

Δύο γὰρ εὐθεῖαι αἱ AB, ΓΔ ὑπὸ παραλλήλων ἐπιπέδων τῶν HΘ, ΚΛ, ΜΝ τεμνέσθωσαν κατὰ τὰ A, E, B, Γ, Z, Δ σημεία· λέγω, ὅτι ἐστὶν ὡς ἡ AE εὐθεῖα πρὸς τὴν EB, οὕτως ἡ ΓZ πρὸς τὴν ZΔ.

Ἐπεξεύχθωσαν γὰρ αἱ ΑΓ, ΒΔ, ΑΔ, καὶ συμβαλλέτω ἡ ΑΔ τῷ ΚΛ ἐπιπέδῳ κατὰ τὸ Ξ σημεῖον, καὶ ἐπεξεύχθωσαν αἱ ΕΞ, ΕΖ.

Καὶ ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΚΛ, ΜΝ ὑπὸ ἐπιπέδου τοῦ ΕΒΔΞ τέμνεται, αἱ κοινὰ αὐτῶν τομαὶ αἱ ΕΞ, ΒΔ παράλληλοί εἰσιν. διὰ τὰ αὐτὰ δὴ ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ ΗΘ, ΚΛ ὑπὸ ἐπιπέδου τοῦ ΑΞΖΓ τέμνεται, αἱ κοινὰ αὐτῶν τομαὶ αἱ ΑΓ, ΕΖ παράλληλοί εἰσιν. καὶ ἐπεὶ τριγώνου τοῦ ΑΒΔ παρὰ μίαν τῶν πλευρῶν τὴν ΒΔ εὐθεῖα ἤχται ἡ ΕΞ, ἀνάλογον ἄρα ἐστὶν ὡς ἡ AE πρὸς EB, οὕτως



For, if not, being produced, EF and GH will meet either in the direction of F, H , or of E, G . Let them be produced, as in the direction of F, H , and let them, first of all, have met at K . And since EFK is in the plane AB , all of the points on EFK are thus also in the plane AB [Prop. 11.1]. And K is one of the points on EFK . Thus, K is in the plane AB . So, for the same (reasons), K is also in the plane CD . Thus, the planes AB and CD , being produced, will meet. But they do not meet, on account of being (initially) assumed (to be mutually) parallel. Thus, the straight-lines EF and GH , being produced in the direction of F, H , will not meet. So, similarly, we can show that the straight-lines EF and GH , being produced in the direction of E, G , will not meet either. And (straight-lines in one plane which), being produced, do not meet in either direction are parallel [Def. 1.23]. EF is thus parallel to GH .

Thus, if two parallel planes are cut by some plane then their common sections are parallel. (Which is) the very thing it was required to show.

Proposition 17

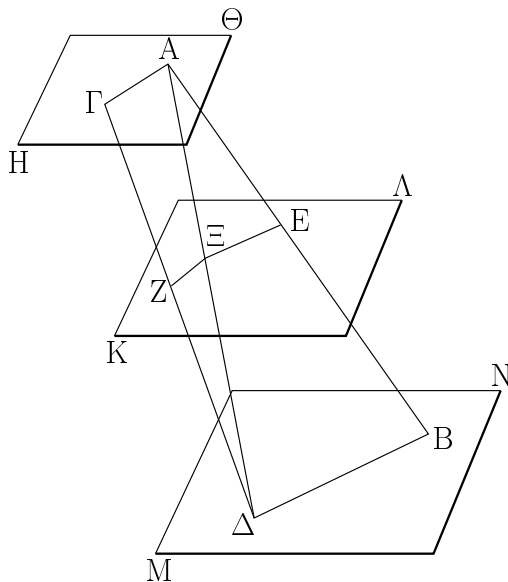
If two straight-lines are cut by parallel planes then they will be cut in the same ratios.

For let the two straight-lines AB and CD be cut by the parallel planes GH, KL , and MN at the points A, E, B , and C, F, D (respectively). I say that as the straight-line AE is to EB , so CF (is) to FD .

For let AC, BD , and AD have been joined, and let AD meet the plane KL at point O , and let EO and OF have been joined.

And since two parallel planes KL and MN are cut by the plane $EBDO$, their common sections EO and BD are parallel [Prop. 11.16]. So, for the same (reasons), since two parallel planes GH and KL are cut by the plane $AOFC$, their common sections AC and OF are parallel [Prop. 11.16]. And since the straight-line EO has been drawn parallel to one of the sides BD of trian-

ἡ $ΑΞ$ πρὸς $ΞΔ$. πάλιν ἐπεὶ τριγώνου τοῦ $ΑΔΓ$ παρὰ μίαν τῶν πλευρῶν τὴν $ΑΓ$ εὐθεΐα ῥηται ἡ $ΞΖ$, ἀνάλογόν ἐστιν ὡς ἡ $ΑΞ$ πρὸς $ΞΔ$, οὕτως ἡ $ΓΖ$ πρὸς $ΖΔ$. ἐδείχθη δὲ καὶ ὡς ἡ $ΑΞ$ πρὸς $ΞΔ$, οὕτως ἡ $ΑΕ$ πρὸς $ΕΒ$ · καὶ ὡς ἄρα ἡ $ΑΕ$ πρὸς $ΕΒ$, οὕτως ἡ $ΓΖ$ πρὸς $ΖΔ$.



Ἐὰν ἄρα δύο εὐθεΐαι ὑπὸ παραλλήλων ἐπιπέδων τέμνονται, εἰς τοὺς αὐτοὺς λόγους τμηθήσονται· ὅπερ ἔδει δεῖξαι.

ιη'.

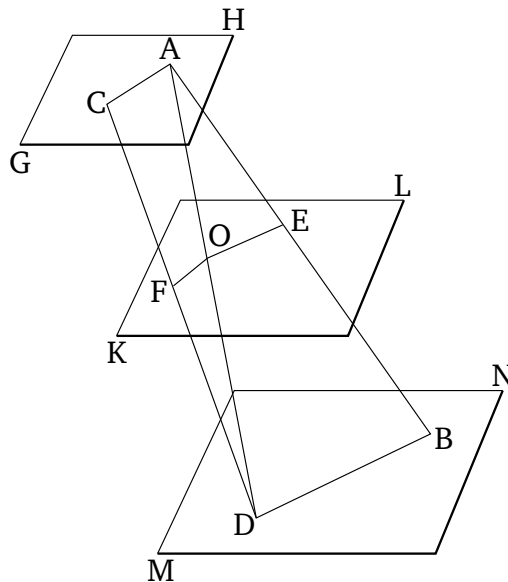
Ἐὰν εὐθεΐα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾖ, καὶ πάντα τὰ δι' αὐτῆς ἐπίπεδα τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.

Εὐθεΐα γάρ τις ἡ $ΑΒ$ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω· λέγω, ὅτι καὶ πάντα τὰ διὰ τῆς $ΑΒ$ ἐπίπεδα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστιν.

Ἐκβεβλήσθω γὰρ διὰ τῆς $ΑΒ$ ἐπίπεδον τὸ $ΔΕ$, καὶ ἔστω κοινὴ τομὴ τοῦ $ΔΕ$ ἐπιπέδου καὶ τοῦ ὑποκειμένου ἡ $ΓΕ$, καὶ εἰλήφθω ἐπὶ τῆς $ΓΕ$ τυχὸν σημεῖον τὸ $Ζ$, καὶ ἀπὸ τοῦ $Ζ$ τῇ $ΓΕ$ πρὸς ὀρθὰς ῥηθῶ ἐν τῷ $ΔΕ$ ἐπιπέδῳ ἡ $ΖΗ$.

Καὶ ἐπεὶ ἡ $ΑΒ$ πρὸς τὸ ὑποκείμενον ἐπίπεδον ὀρθὴ ἐστίν, καὶ πρὸς πάσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ ὑποκειμένῳ ἐπιπέδῳ ὀρθὴ ἐστίν ἡ $ΑΒ$ · ὥστε καὶ πρὸς τὴν $ΓΕ$ ὀρθὴ ἐστίν· ἡ ἄρα ὑπὸ $ΑΒΖ$ γωνία ὀρθὴ ἐστίν. ἔστι δὲ καὶ ἡ ὑπὸ $ΗΖΒ$ ὀρθή· παράλληλος ἄρα ἐστίν ἡ $ΑΒ$ τῇ $ΖΗ$. ἡ δὲ $ΑΒ$ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἐστίν· καὶ ἡ $ΖΗ$ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἐστίν. καὶ ἐπίπεδον πρὸς ἐπίπεδον ὀρθόν ἐστίν, ὅταν αἱ τῇ κοινῇ τομῇ τῶν ἐπιπέδων πρὸς ὀρθὰς ἀγόμεναι εὐθεΐαι ἐν ἐνὶ τῶν ἐπιπέδων τῷ λοιπῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν. καὶ τῇ κοινῇ τομῇ τῶν ἐπιπέδων τῇ $ΓΕ$ ἐν ἐνὶ τῶν ἐπιπέδων

gle $ΑΒΔ$, thus, proportionally, as $ΑΕ$ is to $ΕΒ$, so $ΑΟ$ (is) to $ΟΔ$ [Prop. 6.2]. Again, since the straight-line $ΟΓ$ has been drawn parallel to one of the sides $ΑΓ$ of triangle $ΑΔΓ$, proportionally, as $ΑΟ$ is to $ΟΔ$, so $ΟΓ$ (is) to $ΓΔ$ [Prop. 6.2]. And it was also shown that as $ΑΟ$ (is) to $ΟΔ$, so $ΑΕ$ (is) to $ΕΒ$. And thus as $ΑΕ$ (is) to $ΕΒ$, so $ΟΓ$ (is) to $ΓΔ$ [Prop. 5.11].



Thus, if two straight-lines are cut by parallel planes then they will be cut in the same ratios. (Which is) the very thing it was required to show.

Proposition 18

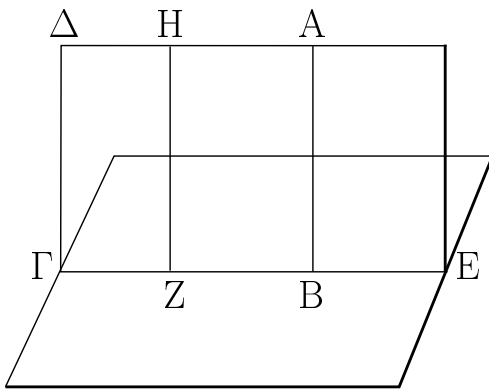
If a straight-line is at right-angles to some plane then all of the planes (passing) through it will also be at right-angles to the same plane.

For let some straight-line $ΑΒ$ be at right-angles to a reference plane. I say that all of the planes (passing) through $ΑΒ$ are also at right-angles to the reference plane.

For let the plane $ΔΕ$ have been produced through $ΑΒ$. And let $ΓΕ$ be the common section of the plane $ΔΕ$ and the reference (plane). And let some random point $Ζ$ have been taken on $ΓΕ$. And let $ΖΗ$ have been drawn from $Ζ$, at right-angles to $ΓΕ$, in the plane $ΔΕ$ [Prop. 1.11].

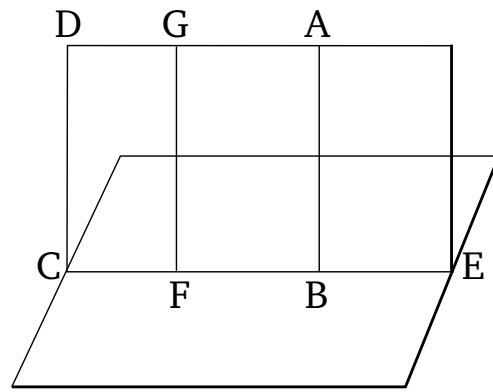
And since $ΑΒ$ is at right-angles to the reference plane, $ΑΒ$ is thus also at right-angles to all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. Hence, it is also at right-angles to $ΓΕ$. Thus, angle $ΑΒΖ$ is a right-angle. And $ΗΒΖ$ is also a right-angle. Thus, $ΑΒ$ is parallel to $ΖΗ$ [Prop. 1.28]. And $ΑΒ$ is at right-angles to the reference plane. Thus, $ΖΗ$ is also

τῷ ΔE πρὸς ὀρθὰς ἀχθεῖσα ἡ ZH ἐδείχθη τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς· τὸ ἄρα ΔE ἐπίπεδον ὀρθόν ἐστι πρὸς τὸ ὑποκείμενον. ὁμοίως δὲ δειχθήσεται καὶ πάντα τὰ διὰ τῆς AB ἐπίπεδα ὀρθὰ τυγχάνοντα πρὸς τὸ ὑποκείμενον ἐπίπεδον.



Ἐάν ἄρα εὐθεῖα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾗ, καὶ πάντα τὰ δι' αὐτῆς ἐπίπεδα τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δεῖξαι.

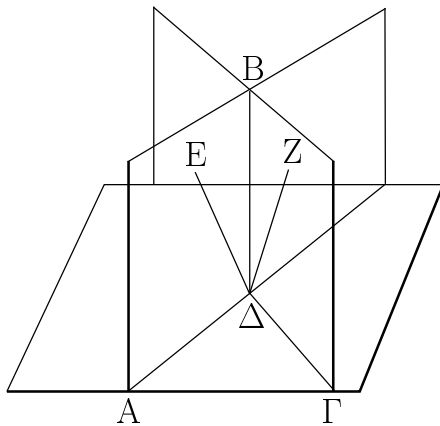
at right-angles to the reference plane [Prop. 11.8]. And a plane is at right-angles to a(nother) plane when the straight-lines drawn at right-angles to the common section of the planes, (and lying) in one of the planes, are at right-angles to the remaining plane [Def. 11.4]. And FG , (which was) drawn at right-angles to the common section of the planes, CE , in one of the planes, DE , was shown to be at right-angles to the reference plane. Thus, plane DE is at right-angles to the reference (plane). So, similarly, it can be shown that all of the planes (passing) at random through AB (are) at right-angles to the reference plane.



Thus, if a straight-line is at right-angles to some plane then all of the planes (passing) through it will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

ιθ'.

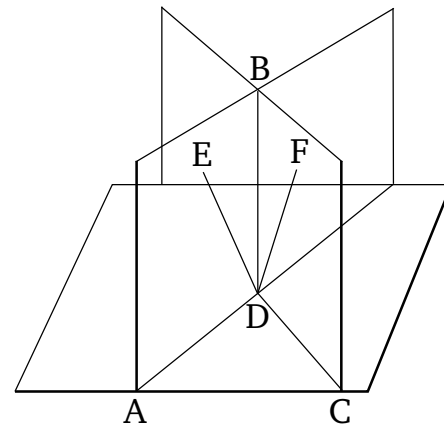
Ἐάν δύο ἐπίπεδα τέμνοντα ἄλληλα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾗ, καὶ ἡ κοινὴ αὐτῶν τομὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται.



Δύο γὰρ ἐπίπεδα τὰ AB , $B\Gamma$ τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἔστω, κοινὴ δὲ αὐτῶν τομὴ ἔστω ἡ BD · λέγω, ὅτι ἡ BD τῷ ὑποκειμένῳ ἐπιπέδῳ πρὸς ὀρθὰς ἐστίν.

Proposition 19

If two planes cutting one another are at right-angles to some plane then their common section will also be at right-angles to the same plane.



For let the two planes AB and BC be at right-angles to a reference plane, and let their common section be BD . I say that BD is at right-angles to the reference

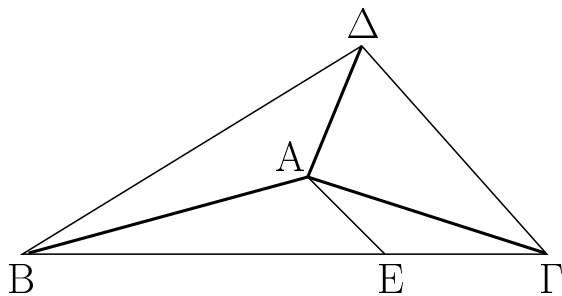
Μὴ γάρ, καὶ ἤχθωσαν ἀπὸ τοῦ Δ σημείου ἐν μὲν τῷ AB ἐπιπέδῳ τῇ AD εὐθείᾳ πρὸς ὀρθὰς ἢ ΔE , ἐν δὲ τῷ $B\Gamma$ ἐπιπέδῳ τῇ $\Gamma\Delta$ πρὸς ὀρθὰς ἢ ΔZ .

Καὶ ἐπεὶ τὸ AB ἐπίπεδον ὀρθὸν ἐστὶ πρὸς τὸ ὑποκείμενον, καὶ τῇ κοινῇ αὐτῶν τομῇ τῇ AD πρὸς ὀρθὰς ἐν τῷ AB ἐπιπέδῳ ἤκται ἢ ΔE , ἢ ΔE ἄρα ὀρθὴ ἐστὶ πρὸς τὸ ὑποκείμενον ἐπίπεδον. ὁμοίως δὲ δείξομεν, ὅτι καὶ ἢ ΔZ ὀρθὴ ἐστὶ πρὸς τὸ ὑποκείμενον ἐπίπεδον. ἀπὸ τοῦ αὐτοῦ ἄρα σημείου τοῦ Δ τῷ ὑποκειμένῳ ἐπιπέδῳ δύο εὐθεῖαι πρὸς ὀρθὰς ἀνεσταμέναι εἰσὶν ἐπὶ τὰ αὐτὰ μέρη· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα τῷ ὑποκειμένῳ ἐπιπέδῳ ἀπὸ τοῦ Δ σημείου ἀνασταθήσεται πρὸς ὀρθὰς πλὴν τῆς ΔB κοινῆς τομῆς τῶν AB , $B\Gamma$ ἐπιπέδων.

Ἐὰν ἄρα δύο ἐπίπεδα τέμνοντα ἀλλήλα ἐπιπέδῳ τινὶ πρὸς ὀρθὰς ᾗ, καὶ ἡ κοινὴ αὐτῶν τομὴ τῷ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἔσται· ὅπερ ἔδει δείξαι.

κ'.

Ἐὰν στερεὰ γωνία ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχεται, δύο ὁποιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβάνομεναι.



Στερεὰ γάρ γωνία ἢ πρὸς τῷ A ὑπὸ τριῶν γωνιῶν ἐπιπέδων τῶν ὑπὸ BAG , $\Gamma\Delta\Delta$, ΔAB περιεχέσθω· λέγω, ὅτι τῶν ὑπὸ BAG , $\Gamma\Delta\Delta$, ΔAB γωνιῶν δύο ὁποιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβάνομεναι.

Εἰ μὲν οὖν αἱ ὑπὸ BAG , $\Gamma\Delta\Delta$, ΔAB γωνίαι ἴσαι ἀλλήλαις εἰσίν, φανερόν, ὅτι δύο ὁποιοῦν τῆς λοιπῆς μείζονές εἰσιν. εἰ δὲ οὐ, ἔστω μείζων ἢ ὑπὸ BAG , καὶ συνεστάτω πρὸς τῇ AB εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ A τῇ ὑπὸ ΔAB γωνίᾳ ἐν τῷ διὰ τῶν BAG ἐπιπέδῳ ἴση ἢ ὑπὸ BAE , καὶ κείσθω τῇ AD ἴση ἢ AE , καὶ διὰ τοῦ E σημείου διαχθεῖσα ἢ $BE\Gamma$ τεμνέτω τὰς AB , $A\Gamma$ εὐθείας κατὰ τὰ B , Γ σημεία, καὶ ἐπεζεύχθωσαν αἱ ΔB , $\Delta\Gamma$.

Καὶ ἐπεὶ ἴση ἐστὶν ἢ ΔA τῇ AE , κοινὴ δὲ ἢ AB , δύο δυσὶν ἴσαι· καὶ γωνία ἢ ὑπὸ ΔAB γωνία τῇ ὑπὸ BAE ἴση· βάσεις ἄρα ἢ ΔB βάσει τῇ BE ἐστὶν ἴση. καὶ ἐπεὶ δύο αἱ ΔB , $\Delta\Gamma$ τῆς $B\Gamma$ μείζονές εἰσιν, ὧν ἢ ΔB τῇ BE ἐδείχθη ἴση,

plane.

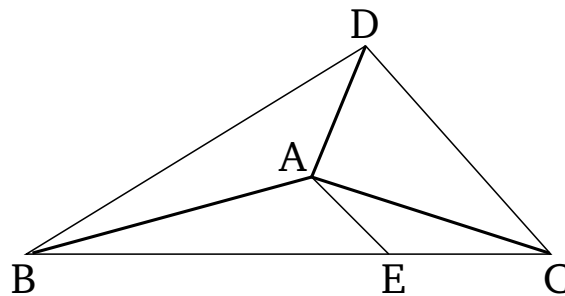
For (if) not, let DE also have been drawn from point D , in the plane AB , at right-angles to the straight-line AD , and DF , in the plane BC , at right-angles to CD .

And since the plane AB is at right-angles to the reference (plane), and DE has been drawn at right-angles to their common section AD , in the plane AB , DE is thus at right-angles to the reference plane [Def. 11.4]. So, similarly, we can show that DF is also at right-angles to the reference plane. Thus, two (different) straight-lines are set up, at the same point D , at right-angles to the reference plane, on the same side. The very thing is impossible [Prop. 11.13]. Thus, no (other straight-line) except the common section DB of the planes AB and BC can be set up at point D , at right-angles to the reference plane.

Thus, if two planes cutting one another are at right-angles to some plane then their common section will also be at right-angles to the same plane. (Which is) the very thing it was required to show.

Proposition 20

If a solid angle is contained by three plane angles then (the sum of) any two (angles) is greater than the remaining (one), (the angles) being taken up in any (possible way).



For let the solid angle A have been contained by the three plane angles BAC , CAD , and DAB . I say that (the sum of) any two of the angles BAC , CAD , and DAB is greater than the remaining (one), (the angles) being taken up in any (possible way).

For if the angles BAC , CAD , and DAB are equal to one another then (it is) clear that (the sum of) any two is greater than the remaining (one). But, if not, let BAC be greater (than CAD or DAB). And let (angle) BAE , equal to the angle DAB , have been constructed in the plane through BAC , on the straight-line AB , at the point A on it. And let AE be made equal to AD . And BEC being drawn across through point E , let it cut the straight-lines AB and AC at points B and C (respectively). And let DB and DC have been joined.

And since DA is equal to AE , and AB (is) common,

λοιπή ἄρα ἡ $\Delta\Gamma$ λοιπῆς τῆς $ΕΓ$ μείζων ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ $\Delta Α$ τῇ $ΑΕ$, κοινὴ δὲ ἡ $ΑΓ$, καὶ βάσεις ἡ $\Delta\Gamma$ βάσεως τῆς $ΕΓ$ μείζων ἐστίν, γωνία ἄρα ἡ ὑπὸ $\Delta ΑΓ$ γωνία τῆς ὑπὸ $Ε ΑΓ$ μείζων ἐστίν. ἐδείχθη δὲ καὶ ἡ ὑπὸ $\Delta ΑΒ$ τῇ ὑπὸ $Β ΑΕ$ ἴση· αἱ ἄρα ὑπὸ $\Delta ΑΒ$, $\Delta ΑΓ$ τῆς ὑπὸ $Β ΑΓ$ μείζονές εἰσιν. ὁμοίως δὲ δείξομεν, ὅτι καὶ αἱ λοιπαὶ σύνδυο λαμβανόμεναι τῆς λοιπῆς μείζονές εἰσιν.

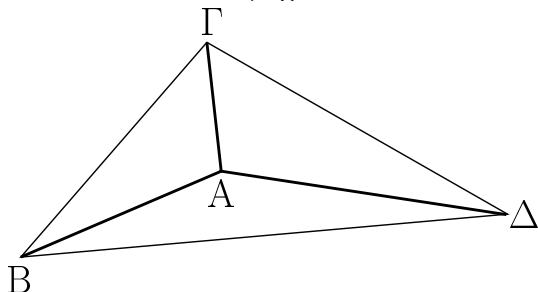
Ἐάν ἄρα στερεὰ γωνία ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχεται, δύο ὁποιοῦν τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι· ὅπερ ἔδει δεῖξαι.

the two (straight-lines AD and AB are) equal to the two (straight-lines EA and AB , respectively). And angle DAB (is) equal to angle BAE . Thus, the base DB is equal to the base BE [Prop. 1.4]. And since the (sum of the) two (straight-lines) BD and DC is greater than BC [Prop. 1.20], of which DB was shown (to be) equal to BE , the remainder DC is thus greater than the remainder EC . And since DA is equal to AE , but AC (is) common, and the base DC is greater than the base EC , the angle DAC is thus greater than the angle EAC [Prop. 1.25]. And DAB was also shown (to be) equal to BAE . Thus, (the sum of) DAB and DAC is greater than BAC . So, similarly, we can also show that the remaining (angles), being taken in pairs, are greater than the remaining (one).

Thus, if a solid angle is contained by three plane angles then (the sum of) any two (angles) is greater than the remaining (one), (the angles) being taken up in any (possible way). (Which is) the very thing it was required to show.

κα'.

Ἐπασα στερεὰ γωνία ὑπὸ ἐλασσόνων [ῆ] τεσσάρων ὀρθῶν γωνιῶν ἐπιπέδων περιέχεται.

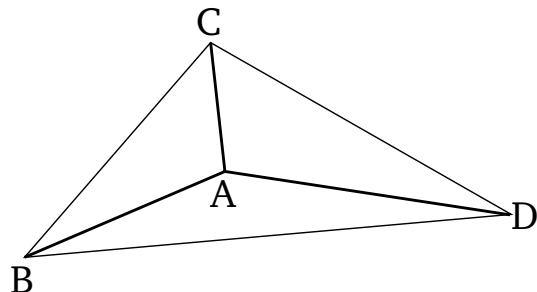


Ἐστω στερεὰ γωνία ἡ πρὸς τῷ A περιεχομένη ὑπὸ ἐπιπέδων γωνιῶν τῶν ὑπὸ $Β ΑΓ$, $Γ ΑΔ$, $\Delta ΑΒ$ · λέγω, ὅτι αἱ ὑπὸ $Β ΑΓ$, $Γ ΑΔ$, $\Delta ΑΒ$ τεσσάρων ὀρθῶν ἐλάσσονές εἰσιν.

Εἰλήφθω γὰρ ἐφ' ἐκάστης τῶν $ΑΒ$, $ΑΓ$, $ΑΔ$ τυχόντα σημεῖα τὰ $Β$, $Γ$, Δ , καὶ ἐπεζεύχθωσαν αἱ $ΒΓ$, $ΓΔ$, $\Delta Β$. καὶ ἐπεὶ στερεὰ γωνία ἡ πρὸς τῷ B ὑπὸ τριῶν γωνιῶν ἐπιπέδων περιέχεται τῶν ὑπὸ $Γ Β Α$, $Α Β Δ$, $Γ Β Δ$, δύο ὁποιοῦν τῆς λοιπῆς μείζονές εἰσιν· αἱ ἄρα ὑπὸ $Γ Β Α$, $Α Β Δ$ τῆς ὑπὸ $Γ Β Δ$ μείζονές εἰσιν. διὰ τὰ αὐτὰ δὲ καὶ αἱ μὲν ὑπὸ $Β Γ Α$, $Α Γ Δ$ τῆς ὑπὸ $Β Γ Δ$ μείζονές εἰσιν, αἱ δὲ ὑπὸ $Γ Δ Α$, $Α Δ Β$ τῆς ὑπὸ $Γ Δ Β$ μείζονές εἰσιν· αἱ ἔξ ἄρα γωνίαι αἱ ὑπὸ $Γ Β Α$, $Α Β Δ$, $Β Γ Α$, $Α Γ Δ$, $Γ Δ Α$, $Α Δ Β$ τριῶν τῶν ὑπὸ $Γ Β Δ$, $Β Γ Α$, $Γ Δ Β$ μείζονές εἰσιν. ἀλλὰ αἱ τρεῖς αἱ ὑπὸ $Γ Β Δ$, $Β Δ Γ$, $Β Γ Δ$ δυσὶν ὀρθαῖς ἴσαι εἰσιν· αἱ ἔξ ἄρα αἱ ὑπὸ $Γ Β Α$, $Α Β Δ$, $Β Γ Α$, $Α Γ Δ$, $Γ Δ Α$, $Α Δ Β$ δύο ὀρθῶν μείζονές εἰσιν. καὶ ἐπεὶ ἐκάστου τῶν $ΑΒΓ$, $ΑΓΔ$, $ΑΔΒ$ τριγώνων αἱ τρεῖς γωνίαι δυσὶν ὀρθαῖς ἴσαι εἰσιν, αἱ ἄρα τῶν τριῶν τριγώνων ἑννέα γωνίαι αἱ ὑπὸ

Proposition 21

Any solid angle is contained by plane angles (whose sum is) less [than] four right-angles.[†]



Let the solid angle A be contained by the plane angles BAC , CAD , and DAB . I say that (the sum of) BAC , CAD , and DAB is less than four right-angles.

For let the random points B , C , and D have been taken on each of (the straight-lines) AB , AC , and AD (respectively). And let BC , CD , and DB have been joined. And since the solid angle at B is contained by the three plane angles CBA , ABD , and CBD , (the sum of) any two is greater than the remaining (one) [Prop. 11.20]. Thus, (the sum of) CBA and ABD is greater than CBD . So, for the same (reasons), (the sum of) BCA and ACD is also greater than BCD , and (the sum of) CDA and ADB is greater than CDB . Thus, the (sum of the) six angles CBA , ABD , BCA , ACD , CDA , and ADB is greater than the (sum of the) three (angles) CBD , BCD , and CDB . But, the (sum of the) three (angles) CBD , BDC , and BCD is equal to two

ΓΒΑ, ΑΓΒ, ΒΑΓ, ΑΓΔ, ΓΔΑ, ΓΑΔ, ΑΔΒ, ΔΒΑ, ΒΑΔ ἔξ ὀρθαῖς ἴσαι εἰσίν, ὧν αἱ ὑπὸ ΑΒΓ, ΒΓΑ, ΑΓΔ, ΓΔΑ, ΑΔΒ, ΔΒΑ ἔξ γωνίαι δύο ὀρθῶν εἰσι μείζονες· λοιπαὶ ἄρα αἱ ὑπὸ ΒΑΓ, ΓΑΔ, ΔΑΒ τρεῖς [γωνίαι] περιέχουσιν τὴν στερεὰν γωνίαν τεσσάρων ὀρθῶν ἐλάσσονες εἰσιν.

Ἄπαντα ἄρα στερεὰ γωνία ὑπὸ ἐλασσόνων [ῥ] τεσσάρων ὀρθῶν γωνιῶν ἐπιπέδων περιέχεται· ὅπερ ἔδει δεῖξαι.

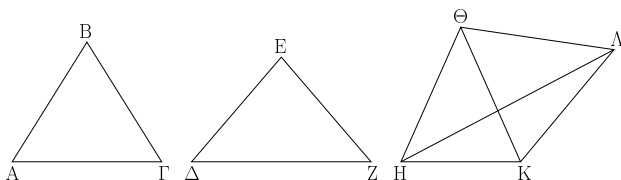
right-angles [Prop. 1.32]. Thus, the (sum of the) six angles CBA , ABD , BCA , ACD , CDA , and ADB is greater than two right-angles. And since the (sum of the) three angles of each of the triangles ABC , ACD , and ADB is equal to two right-angles, the (sum of the) nine angles CBA , ACB , BAC , ACD , CDA , CAD , ADB , DBA , and BAD of the three triangles is equal to six right-angles, of which the (sum of the) six angles ABC , BCA , ACD , CDA , ADB , and DBA is greater than two right-angles. Thus, the (sum of the) remaining three [angles] BAC , CAD , and DAB , containing the solid angle, is less than four right-angles.

Thus, any solid angle is contained by plane angles (whose sum is) less [than] four right-angles. (Which is) the very thing it was required to show.

† This proposition is only proved for the case of a solid angle contained by three plane angles. However, the generalization to a solid angle contained by more than three plane angles is straightforward.

κβ'.

Ἐὰν ὧσι τρεῖς γωνίαι ἐπίπεδοι, ὧν αἱ δύο τῆς λοιπῆς μείζονες εἰσι πάντῃ μεταλαμβάνομεναι, περιέχουσι δὲ αὐτὰς ἴσαι εὐθεῖαι, δυνατόν ἐστιν ἐκ τῶν ἐπιζευγνυουσῶν τὰς ἴσας εὐθείας τρίγωνον συστήσασθαι.

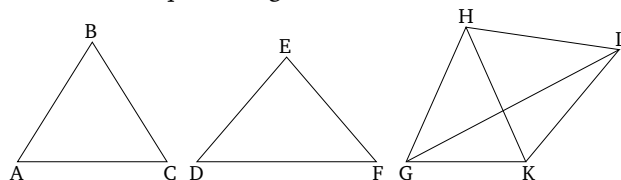


Ἐστωσαν τρεῖς γωνίαι ἐπίπεδοι αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ, ὧν αἱ δύο τῆς λοιπῆς μείζονες εἰσι πάντῃ μεταλαμβάνομεναι, αἱ μὲν ὑπὸ ΑΒΓ, ΔΕΖ τῆς ὑπὸ ΗΘΚ, αἱ δὲ ὑπὸ ΔΕΖ, ΗΘΚ τῆς ὑπὸ ΑΒΓ, καὶ ἔτι αἱ ὑπὸ ΗΘΚ, ΑΒΓ τῆς ὑπὸ ΔΕΖ, καὶ ἔστωσαν ἴσαι αἱ ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ εὐθεῖαι, καὶ ἐπεζεύχθωσαν αἱ ΑΓ, ΔΖ, ΗΚ· λέγω, ὅτι δυνατόν ἐστιν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι, τουτέστιν ὅτι τῶν ΑΓ, ΔΖ, ΗΚ δύο ὁποιοιοῦν τῆς λοιπῆς μείζονες εἰσιν.

Εἰ μὲν οὖν αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ γωνίαι ἴσαι ἀλλήλαις εἰσίν, φανερόν, ὅτι καὶ τῶν ΑΓ, ΔΖ, ΗΚ ἴσων γινομένων δυνατόν ἐστιν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι. εἰ δὲ οὐ, ἔστωσαν ἄνισοι, καὶ συνεστάτω πρὸς τῇ ΘΚ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Θ τῇ ὑπὸ ΑΒΓ γωνίᾳ ἴση ἢ ὑπὸ ΚΘΛ· καὶ χείσθω μιᾶ τῶν ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ ἴση ἢ ΘΛ, καὶ ἐπεζεύχθωσαν αἱ ΚΛ, ΗΛ. καὶ ἐπεὶ δύο αἱ ΑΒ, ΒΓ δυσὶ ταῖς ΚΘ, ΘΛ ἴσαι εἰσίν, καὶ γωνία ἡ πρὸς τῷ Β γωνία τῇ ὑπὸ ΚΘΛ ἴση, βάσεις ἄρα ἢ ΑΓ βάσει τῇ ΚΛ ἴση. καὶ ἐπεὶ αἱ ὑπὸ ΑΒΓ, ΗΘΚ τῆς

Proposition 22

If there are three plane angles, of which (the sum of any) two is greater than the remaining (one), (the angles) being taken up in any (possible way), and if equal straight-lines contain them, then it is possible to construct a triangle from (the straight-lines created by) joining the (ends of the) equal straight-lines.



Let ABC , DEF , and GHK be three plane angles, of which the sum of any two is greater than the remaining (one), (the angles) being taken up in any (possible way)—(that is), ABC and DEF (greater) than GHK , DEF and GHK (greater) than ABC , and, further, GHK and ABC (greater) than DEF . And let AB , BC , DE , EF , GH , and HK be equal straight-lines. And let AC , DF , and GK have been joined. I say that that it is possible to construct a triangle out of (straight-lines) equal to AC , DF , and GK —that is to say, that (the sum of) any two of AC , DF , and GK is greater than the remaining (one).

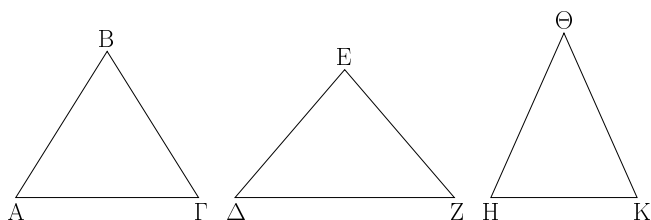
Now, if the angles ABC , DEF , and GHK are equal to one another then (it is) clear that, (with) AC , DF , and GK also becoming equal, it is possible to construct a triangle from (straight-lines) equal to AC , DF , and GK . And if not, let them be unequal, and let KHL , equal to angle ABC , have been constructed on the straight-line HK , at the point H on it. And let HL be made equal to

ὑπὸ ΔΕΖ μείζονές εἰσιν, ἴση δὲ ἡ ὑπὸ ΑΒΓ τῇ ὑπὸ ΚΘΛ, ἢ ἄρα ὑπὸ ΗΘΛ τῆς ὑπὸ ΔΕΖ μείζων ἐστίν. καὶ ἐπεὶ δύο αἱ ΗΘ, ΘΛ δύο ταῖς ΔΕ, ΕΖ ἴσαι εἰσίν, καὶ γωνία ἡ ὑπὸ ΗΘΛ γωνίας τῆς ὑπὸ ΔΕΖ μείζων, βάσις ἄρα ἡ ΗΛ βάσεως τῆς ΔΖ μείζων ἐστίν. ἀλλὰ αἱ ΗΚ, ΚΛ τῆς ΗΛ μείζονές εἰσιν. πολλῶ ἄρα αἱ ΗΚ, ΚΛ τῆς ΔΖ μείζονές εἰσιν. ἴση δὲ ἡ ΚΑ τῇ ΑΓ· αἱ ΑΓ, ΗΚ ἄρα τῆς λοιπῆς τῆς ΔΖ μείζονές εἰσιν. ὁμοίως δὴ δείξομεν, ὅτι καὶ αἱ μὲν ΑΓ, ΔΖ τῆς ΗΚ μείζονές εἰσιν, καὶ ἔτι αἱ ΔΖ, ΗΚ τῆς ΑΓ μείζονές εἰσιν. δυνατὸν ἄρα ἐστὶν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι· ὅπερ ἔδει δείξαι.

one of AB , BC , DE , EF , GH , and HK . And let KL and GL have been joined. And since the two (straight-lines) AB and BC are equal to the two (straight-lines) KH and HL (respectively), and the angle at B (is) equal to KHL , the base AC is thus equal to the base KL [Prop. 1.4]. And since (the sum of) ABC and GHK is greater than DEF , and ABC equal to KHL , GHL is thus greater than DEF . And since the two (straight-lines) GH and HL are equal to the two (straight-lines) DE and EF (respectively), and angle GHL (is) greater than DEF , the base GL is thus greater than the base DF [Prop. 1.24]. But, (the sum of) GK and KL is greater than GL [Prop. 1.20]. Thus, (the sum of) GK and KL is much greater than DF . And KL (is) equal to AC . Thus, (the sum of) AC and GK is greater than the remaining (straight-line) DF . So, similarly, we can show that (the sum of) AC and DF is greater than GK , and, further, that (the sum of) DF and GK is greater than AC . Thus, it is possible to construct a triangle from (straight-lines) equal to AC , DF , and GK . (Which is) the very thing it was required to show.

κγ'.

Ἐκ τριῶν γωνιῶν ἐπιπέδων, ὧν αἱ δύο τῆς λοιπῆς μείζονές εἰσι πάντῃ μεταλαμβανόμεναι, στερεὰν γωνίαν συστήσασθαι· δεῖ δὴ τὰς τρεῖς τεσσάρων ὀρθῶν ἐλάσσονας εἶναι.

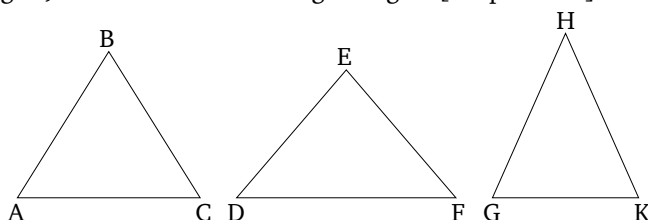


Ἐστωσαν αἱ δοθεῖσαι τρεῖς γωνίαι ἐπίπεδοι αἱ ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ, ὧν αἱ δύο τῆς λοιπῆς μείζονες ἔστωσαν πάντῃ μεταλαμβανόμεναι, ἔτι δὲ αἱ τρεῖς τεσσάρων ὀρθῶν ἐλάσσονες· δεῖ δὴ ἐκ τῶν ἴσων ταῖς ὑπὸ ΑΒΓ, ΔΕΖ, ΗΘΚ στερεὰν γωνίαν συστήσασθαι.

Ἀπειρήφθωσαν ἴσαι αἱ ΑΒ, ΒΓ, ΔΕ, ΕΖ, ΗΘ, ΘΚ, καὶ ἐπεζεύχθωσαν αἱ ΑΓ, ΔΖ, ΗΚ· δυνατὸν ἄρα ἐστὶν ἐκ τῶν ἴσων ταῖς ΑΓ, ΔΖ, ΗΚ τρίγωνον συστήσασθαι. συνεστάτω τὸ ΑΜΝ, ὥστε ἴσην εἶναι τὴν μὲν ΑΓ τῇ ΑΜ, τὴν δὲ ΔΖ τῇ ΜΝ, καὶ ἔτι τὴν ΗΚ τῇ ΝΛ, καὶ περιγεγράφθω περὶ τὸ ΑΜΝ τρίγωνον κύκλος ὁ ΑΜΝ, καὶ εἰλήφθω αὐτοῦ τὸ κέντρον καὶ ἔστω τὸ Ξ, καὶ ἐπεζεύχθωσαν αἱ ΛΞ, ΜΞ, ΝΞ·

Proposition 23

To construct a solid angle from three (given) plane angles, (the sum of) two of which is greater than the remaining (one, the angles) being taken up in any (possible way). So, it is necessary for the (sum of the) three (angles) to be less than four right-angles [Prop. 11.21].



Let ABC , DEF , and GHI be the three given plane angles, of which let (the sum of) two be greater than the remaining (one, the angles) being taken up in any (possible way), and, further, (let) the (sum of the) three (be) less than four right-angles. So, it is necessary to construct a solid angle from (plane angles) equal to ABC , DEF , and GHI .

Let AB , BC , DE , EF , GH , and HK be cut off (so as to be) equal (to one another). And let AC , DF , and GK have been joined. It is, thus, possible to construct a triangle from (straight-lines) equal to AC , DF , and GK [Prop. 11.22]. Let (such a triangle), LMN , have been constructed, such that AC is equal to LM , DF to MN , and, further, GK to NL . And let the circle LMN have been circumscribed about triangle LMN [Prop. 4.5]. And let

τῆς ΞP , καὶ ἐπεξεύχθωσαν αἱ PA , PM , PN .

Καὶ ἐπεὶ ἡ $P\Xi$ ὀρθὴ ἐστὶ πρὸς τὸ τοῦ ΛMN κύκλου ἐπίπεδον, καὶ πρὸς ἐκάστην ἄρα τῶν $\Lambda\Xi$, $M\Xi$, $N\Xi$ ὀρθὴ ἐστὶν ἡ $P\Xi$. καὶ ἐπεὶ ἴση ἐστὶν ἡ $\Lambda\Xi$ τῇ ΞM , κοινὴ δὲ καὶ πρὸς ὀρθᾶς ἡ ΞP , βάσις ἄρα ἡ PA βάσει τῇ PM ἐστὶν ἴση. διὰ τὰ αὐτὰ δὲ καὶ ἡ PN ἐκατέρω τῶν PA , PM ἐστὶν ἴση· αἱ τρεῖς ἄρα αἱ PA , PM , PN ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ὦ μείζον ἐστὶ τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς $\Lambda\Xi$, ἐκείνω ἴσον ὑπόκειται τὸ ἀπὸ τῆς ΞP , τὸ ἄρα ἀπὸ τῆς AB ἴσον ἐστὶ τοῖς ἀπὸ τῶν $\Lambda\Xi$, ΞP . τοῖς δὲ ἀπὸ τῶν $\Lambda\Xi$, ΞP ἴσον ἐστὶ τὸ ἀπὸ τῆς AP · ὀρθὴ γὰρ ἡ ὑπὸ $\Lambda\Xi P$ · τὸ ἄρα ἀπὸ τῆς AB ἴσον ἐστὶ τῷ ἀπὸ τῆς PA · ἴση ἄρα ἡ AB τῇ PA . ἀλλὰ τῇ μὲν AB ἴση ἐστὶν ἐκάστη τῶν $B\Gamma$, ΔE , EZ , $H\Theta$, ΘK , τῇ δὲ PA ἴση ἐκατέρω τῶν PM , PN · ἐκάστη ἄρα τῶν AB , $B\Gamma$, ΔE , EZ , $H\Theta$, ΘK ἐκάστη τῶν PA , PM , PN ἴση ἐστίν. καὶ ἐπεὶ δύο αἱ AP , PM δυοὶ ταῖς AB , $B\Gamma$ ἴσαι εἰσίν, καὶ βάσις ἡ ΛM βάσει τῇ $\Lambda\Gamma$ ὑπόκειται ἴση, γωνία ἄρα ἡ ὑπὸ ΛPM γωνία τῇ ὑπὸ $\Lambda B\Gamma$ ἐστὶν ἴση. διὰ τὰ αὐτὰ δὲ καὶ ἡ μὲν ὑπὸ MPN τῇ ὑπὸ ΔEZ ἐστὶν ἴση, ἡ δὲ ὑπὸ ΛPN τῇ ὑπὸ $H\Theta K$.

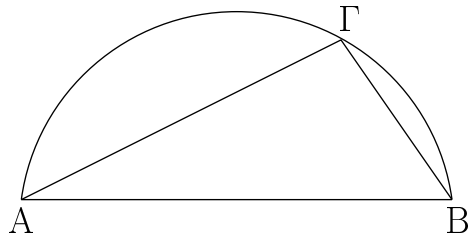
Ἐκ τριῶν ἄρα γωνιῶν ἐπιπέδων τῶν ὑπὸ ΛPM , MPN , ΛPN , αἱ εἰσὶν ἴσαι τρισὶ ταῖς δοθείσαις ταῖς ὑπὸ $AB\Gamma$, ΔEZ , $H\Theta K$, στερεὰ γωνία συνέσταται ἡ πρὸς τῷ P περιεχομένη ὑπὸ τῶν ΛPM , MPN , ΛPN γωνιῶν· ὅπερ ἔδει ποιῆσαι.

and NOL . But, (the sum of) ABC , DEF , and GHK was assumed (to be) less than four right-angles. Thus, (the sum of) LOM , MON , and NOL is much less than four right-angles. But, (it is) also equal (to four right-angles). The very thing is absurd. Thus, AB is not less than LO . And it was shown (to be) not equal either. Thus, AB (is) greater than LO .

So let OR have been set up at point O at right-angles to the plane of circle LMN [Prop. 11.12]. And let the (square) on OR be equal to that (area) by which the square on AB is greater than the (square) on LO [Prop. 11.23 lem.]. And let RL , RM , and RN have been joined.

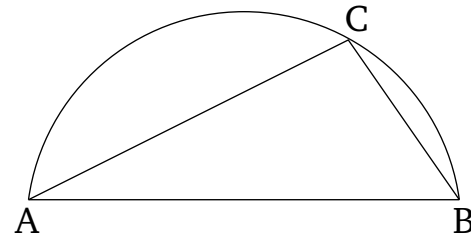
And since RO is at right-angles to the plane of circle LMN , RO is thus also at right-angles to each of LO , MO , and NO . And since LO is equal to OM , and OR is common and at right-angles, the base RL is thus equal to the base RM [Prop. 1.4]. So, for the same (reasons), RN is also equal to each of RL and RM . Thus, the three (straight-lines) RL , RM , and RN are equal to one another. And since the (square) on OR was assumed to be equal to that (area) by which the (square) on AB is greater than the (square) on LO , the (square) on AB is thus equal to the (sum of the squares) on LO and OR . And the (square) on LR is equal to the (sum of the squares) on LO and OR . For LOR (is) a right-angle [Prop. 1.47]. Thus, the (square) on AB is equal to the (square) on RL . Thus, AB (is) equal to RL . But, each of BC , DE , EF , GH , and HK is equal to AB , and each of RM and RN equal to RL . Thus, each of AB , BC , DE , EF , GH , and HK is equal to each of RL , RM , and RN . And since the two (straight-lines) LR and RM are equal to the two (straight-lines) AB and BC (respectively), and the base LM was assumed (to be) equal to the base AC , the angle LRM is thus equal to the angle ABC [Prop. 1.8]. So, for the same (reasons), MRN is also equal to DEF , and LRN to GHK .

Thus, the solid angle R , contained by the angles LRM , MRN , and LRN , has been constructed out of the three plane angles LRM , MRN , and LRN , which are equal to the three given (plane angles) ABC , DEF , and GHK (respectively). (Which is) the very thing it was required to do.



Λήμμα.

Ὅν δὲ τρόπον, ὃ μείζον ἐστὶ τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς $\Lambda\Xi$, ἐκείνῳ ἴσον λαβεῖν ἐστὶ τὸ ἀπὸ τῆς ΞP , δεῖξομεν οὕτως. ἐκκείσθωσαν αἱ AB , $\Lambda\Xi$ εὐθεῖαι, καὶ ἔστω μείζων ἢ AB , καὶ γεγράφθω ἐπ' αὐτῆς ἡμικύκλιον τὸ $AB\Gamma$, καὶ εἰς τὸ $AB\Gamma$ ἡμικύκλιον ἐνηρμόσθω τῇ $\Lambda\Xi$ εὐθείᾳ μὴ μείζονι οὕση τῆς AB διαμέτρου ἴση ἢ AG , καὶ ἐπεζεύχθω ἡ GB . ἐπεὶ οὖν ἐν ἡμικυκλίῳ τῷ AGB γωνία ἐστὶν ἡ ὑπὸ AGB , ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ AGB . τὸ ἄρα ἀπὸ τῆς AB ἴσον ἐστὶ τοῖς ἀπὸ τῶν AG , GB . ὥστε τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς AG μείζον ἐστὶ τῷ ἀπὸ τῆς GB . ἴση δὲ ἡ AG τῇ $\Lambda\Xi$. τὸ ἄρα ἀπὸ τῆς AB τοῦ ἀπὸ τῆς $\Lambda\Xi$ μείζον ἐστὶ τῷ ἀπὸ τῆς GB . ἐὰν οὖν τῇ $B\Gamma$ ἴσην τὴν ΞP ἀπολάβωμεν, ἔσται τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς $\Lambda\Xi$ μείζον τῷ ἀπὸ τῆς ΞP . ὅπερ προέκειτο ποιῆσαι.

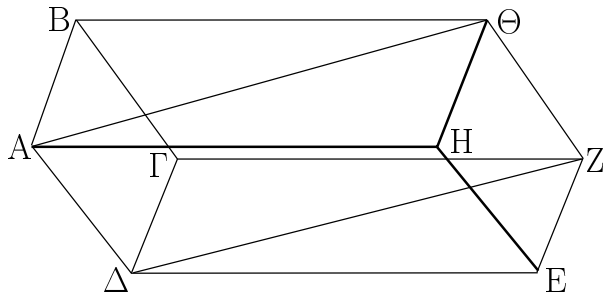


Lemma

And we can demonstrate, thusly, in which manner to take the (square) on OR equal to that (area) by which the (square) on AB is greater than the (square) on LO . Let the straight-lines AB and LO be set out, and let AB be greater, and let the semicircle ABC have been drawn around it. And let AC , equal to the straight-line LO , which is not greater than the diameter AB , have been inserted into the semicircle ABC [Prop. 4.1]. And let CB have been joined. Therefore, since the angle ACB is in the semicircle ACB , ACB is thus a right-angle [Prop. 3.31]. Thus, the (square) on AB is equal to the (sum of the) squares on AC and CB [Prop. 1.47]. Hence, the (square) on AB is greater than the (square) on AC by the (square) on CB . And AC (is) equal to LO . Thus, the (square) on AB is greater than the (square) on LO by the (square) on CB . Therefore, if we take OR equal to BC then the (square) on AB will be greater than the (square) on LO by the (square) on OR . (Which is) the very thing it was prescribed to do.

κδ'.

Ἐὰν στερεὸν ὑπὸ παραλλήλων ἐπιπέδων περιέχεται, τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν.

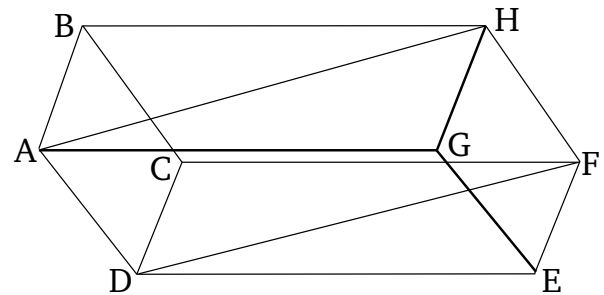


Στερεὸν γάρ τὸ $\Gamma\Delta\Theta\text{H}$ ὑπὸ παραλλήλων ἐπιπέδων περιεχέσθω τῶν AG , HZ , $A\Theta$, ΔZ , BZ , AE . λέγω, ὅτι τὰ ἀπεναντίον αὐτοῦ ἐπίπεδα ἴσα τε καὶ παραλληλόγραμμά ἐστιν.

Ἐπεὶ γὰρ δύο ἐπίπεδα παράλληλα τὰ BH , ΓE ὑπὸ ἐπιπέδου τοῦ AG τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν. παράλληλος ἄρα ἐστὶν ἡ AB τῇ $\Delta\Gamma$. πάλιν, ἐπεὶ δύο ἐπίπεδα παράλληλα τὰ BZ , AE ὑπὸ ἐπιπέδου τοῦ AG τέμνεται, αἱ κοιναὶ αὐτῶν τομαὶ παράλληλοί εἰσιν.

Proposition 24

If a solid (figure) is contained by (six) parallel planes then its opposite planes are both equal and parallelogrammic.



For let the solid (figure) $CDHG$ have been contained by the parallel planes AC , GF , and AH , DF , and BF , AE . I say that its opposite planes are both equal and parallelogrammic.

For since the two parallel planes BG and CE are cut by the plane AC , their common sections are parallel [Prop. 11.16]. Thus, AB is parallel to DC . Again, since the two parallel planes BF and AE are cut by the plane

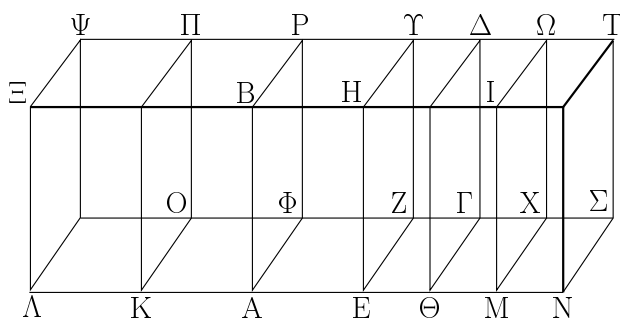
παράλληλος ἄρα ἐστὶν ἡ ΒΓ τῇ ΑΔ. ἐδείχθη δὲ καὶ ἡ ΑΒ τῇ ΔΓ παράλληλος· παραλληλόγραμμον ἄρα ἐστὶ τὸ ΑΓ. ὁμοίως δὲ δεῖξομεν, ὅτι καὶ ἕκαστον τῶν ΔΖ, ΖΗ, ΗΒ, ΒΖ, ΑΕ παραλληλόγραμμον ἐστίν.

Ἐπεξεύχθωσαν αἱ ΑΘ, ΔΖ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ μὲν ΑΒ τῇ ΔΓ, ἡ δὲ ΒΘ τῇ ΓΖ, δύο δὲ αἱ ΑΒ, ΒΘ ἀπτόμεναι ἀλλήλων παρὰ δύο εὐθείας τὰς ΔΓ, ΓΖ ἀπτόμενας ἀλλήλων εἰσὶν οὐκ ἐν τῷ αὐτῷ ἐπιπέδῳ· ἴσας ἄρα γωνίας περιέξουσιν· ἴση ἄρα ἡ ὑπὸ ΑΒΘ γωνία τῇ ὑπὸ ΔΓΖ. καὶ ἐπεὶ δύο αἱ ΑΒ, ΒΘ δυσι ταῖς ΔΓ, ΓΖ ἴσαι εἰσὶν, καὶ γωνία ἡ ὑπὸ ΑΒΘ γωνία τῇ ὑπὸ ΔΓΖ ἐστὶν ἴση, βάσις ἄρα ἡ ΑΘ βάσει τῇ ΔΖ ἐστὶν ἴση, καὶ τὸ ΑΒΘ τρίγωνον τῷ ΔΓΖ τριγώνῳ ἴσον ἐστίν. καὶ ἐστὶ τοῦ μὲν ΑΒΘ διπλάσιον τὸ ΒΗ παραλληλόγραμμον, τοῦ δὲ ΔΓΖ διπλάσιον τὸ ΓΕ παραλληλόγραμμον· ἴσον ἄρα τὸ ΒΗ παραλληλόγραμμον τῷ ΓΕ παραλληλόγραμμῳ· ὁμοίως δὲ δεῖξομεν, ὅτι καὶ τὸ μὲν ΑΓ τῷ ΗΖ ἐστὶν ἴσον, τὸ δὲ ΑΕ τῷ ΒΖ.

Ἐὰν ἄρα στερεὸν ὑπὸ παραλλήλων ἐπιπέδων περιέχεται, τὰ ἀπεναντίον αὐτοῦ ἐπιπεδὰ ἴσα τε καὶ παραλληλόγραμμά ἐστιν· ὅπερ εἶδει δεῖξαι.

κε'.

Ἐὰν στερεὸν παραλληλεπίπεδον ἐπιπέδῳ τμηθῇ παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔσται ὡς ἡ βάσις πρὸς τὴν βάσιν, οὕτως τὸ στερεὸν πρὸς τὸ στερεόν.



Στερεὸν γὰρ παραλληλεπίπεδον τὸ ΑΒΓΔ ἐπιπέδῳ τῷ ΖΗ τετμήσθω παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις τοῖς ΡΑ, ΔΘ· λέγω, ὅτι ἐστὶν ὡς ἡ ΑΕΖΦ βάσις πρὸς τὴν ΕΘΓΖ βάσιν, οὕτως τὸ ΑΒΖΥ στερεὸν πρὸς τὸ ΕΗΓΔ στερεόν.

Ἐκβεβλήσθω γὰρ ἡ ΑΘ ἐφ' ἑκάτερα τὰ μέρη, καὶ κείσθωσαν τῇ μὲν ΑΕ ἴσαι ὅσαιδηποτοῦν αἱ ΑΚ, ΚΛ, τῇ δὲ ΕΘ ἴσαι ὅσαιδηποτοῦν αἱ ΘΜ, ΜΝ, καὶ συμπληρώσθω τὰ ΛΟ, ΚΦ, ΘΧ, ΜΣ παραλληλόγραμμα καὶ τὰ ΑΠ, ΚΡ,

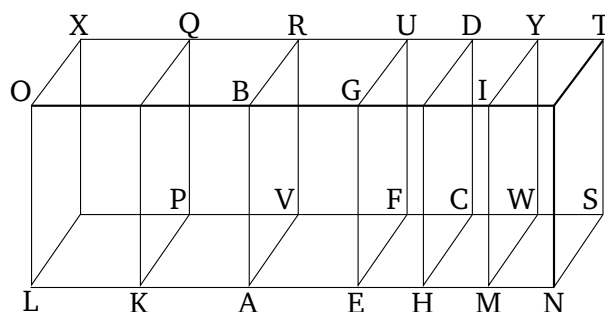
AC, their common sections are parallel [Prop. 11.16]. Thus, BC is parallel to AD. And AB was also shown (to be) parallel to DC. Thus, AC is a parallelogram. So, similarly, we can also show that DF, FG, GB, BF, and AE are each parallelograms.

Let AH and DF have been joined. And since AB is parallel to DC, and BH to CF, so the two (straight-lines) joining one another, AB and BH, are parallel to the two straight-lines joining one another, DC and CF (respectively), not (being) in the same plane. Thus, they will contain equal angles [Prop. 11.10]. Thus, angle ABH (is) equal to (angle) DCF. And since the two (straight-lines) AB and BH are equal to the two (straight-lines) DC and CF (respectively) [Prop. 1.34], and angle ABH is equal to angle DCF, the base AH is thus equal to the base DF, and triangle ABH is equal to triangle DCF [Prop. 1.4]. And parallelogram BG is double (triangle) ABH, and parallelogram CE double (triangle) DCF [Prop. 1.34]. Thus, parallelogram BG (is) equal to parallelogram CE. So, similarly, we can show that AC is also equal to GF, and AE to BF.

Thus, if a solid (figure) is contained by (six) parallel planes then its opposite planes are both equal and parallelogrammic. (Which is) the very thing it was required to show.

Proposition 25

If a paralleliped solid is cut by a plane which is parallel to the opposite planes (of the paralleliped) then as the base (is) to the base, so the solid will be to the solid.



For let the paralleliped solid ABCD have been cut by the plane FG which is parallel to the opposite planes RA and DH. I say that as the base AEFV (is) to the base EHCF, so the solid ABFU (is) to the solid EGCD.

For let AH have been produced in each direction. And let any number whatsoever (of lengths), AK and KL, be made equal to AE, and any number whatsoever (of lengths), HM and MN, equal to EH. And let the parallelograms LP, KV, HW, and MS have been completed,

ΔM , MT στερεά.

Καὶ ἐπεὶ ἴσαι εἰσὶν αἱ ΛK , KA , AE εὐθεῖαι ἀλλήλαις, ἴσα ἐστὶ καὶ τὰ μὲν ΛO , $K\Phi$, AZ παραλληλόγραμμα ἀλλήλοις, τὰ δὲ $K\Xi$, KB , AH ἀλλήλοις καὶ ἔτι τὰ $\Lambda\Psi$, $K\Pi$, AP ἀλλήλοις· ἀπεναντίον γάρ. διὰ τὰ αὐτὰ δὴ καὶ τὰ μὲν $E\Gamma$, ΘX , $M\Sigma$ παραλληλόγραμμα ἴσα εἰσὶν ἀλλήλοις, τὰ δὲ ΘH , ΘI , IN ἴσα εἰσὶν ἀλλήλοις, καὶ ἔτι τὰ $\Delta\Theta$, $M\Omega$, NT · τρία ἄρα ἐπίπεδα τῶν $\Lambda\Pi$, KP , $A\Upsilon$ στερεῶν τρισὶν ἐπιπέδοις ἐστὶν ἴσα. ἀλλὰ τὰ τρία τρισὶ τοῖς ἀπεναντίον ἐστὶν ἴσα· τὰ ἄρα τρία στερεὰ τὰ $\Lambda\Pi$, KP , $A\Upsilon$ ἴσα ἀλλήλοις ἐστὶν. διὰ τὰ αὐτὰ δὴ καὶ τὰ τρία στερεὰ τὰ $E\Delta$, ΔM , MT ἴσα ἀλλήλοις ἐστὶν· ὅσα-πλασίον ἄρα ἐστὶν ἡ AZ βάσις τῆς AZ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ $A\Upsilon$ στερεὸν τοῦ $A\Upsilon$ στερεοῦ. διὰ τὰ αὐτὰ δὴ ὅσαπλασίον ἐστὶν ἡ NZ βάσις τῆς $Z\Theta$ βάσεως, τοσαυταπλάσιόν ἐστι καὶ τὸ $N\Upsilon$ στερεὸν τοῦ $\Theta\Upsilon$ στερεοῦ. καὶ εἰ ἴση ἐστὶν ἡ AZ βάσις τῇ NZ βάσει, ἴσον ἐστὶ καὶ τὸ $A\Upsilon$ στερεὸν τῷ $N\Upsilon$ στερεῷ, καὶ εἰ ὑπερέχει ἡ AZ βάσις τῆς NZ βάσεως, ὑπερέχει καὶ τὸ $A\Upsilon$ στερεὸν τοῦ $N\Upsilon$ στερεοῦ, καὶ εἰ ἐλλείπει, ἐλλείπει. τεσσάρων δὲ ὄντων μεγεθῶν, δύο μὲν βάσεων τῶν AZ , $Z\Theta$, δύο δὲ στερεῶν τῶν $A\Upsilon$, $\Upsilon\Theta$, εἴληπται ἰσάκεις πολλαπλάσια τῆς μὲν AZ βάσεως καὶ τοῦ $A\Upsilon$ στερεοῦ ἢ τε AZ βάσις καὶ τὸ $A\Upsilon$ στερεόν, τῆς δὲ ΘZ βάσεως καὶ τοῦ $\Theta\Upsilon$ στερεοῦ ἢ τε NZ βάσις καὶ τὸ $N\Upsilon$ στερεόν, καὶ δέδεικται, ὅτι εἰ ὑπερέχει ἡ AZ βάσις τῆς ZN βάσεως, ὑπερέχει καὶ τὸ $A\Upsilon$ στερεὸν τοῦ $N\Upsilon$ [στερεοῦ], καὶ εἰ ἴση, ἴσον, καὶ εἰ ἐλλείπει, ἐλλείπει. ἔστιν ἄρα ὡς ἡ AZ βάσις πρὸς τὴν $Z\Theta$ βάσιν, οὕτως τὸ $A\Upsilon$ στερεὸν πρὸς τὸ $\Upsilon\Theta$ στερεόν· ὅπερ ἔδει δεῖξαι.

and the solids LQ , KR , DM , and MT .

And since the straight-lines LK , KA , and AE are equal to one another, the parallelograms LP , KV , and AF are also equal to one another, and KO , KB , and AG (are equal) to one another, and, further, LX , KQ , and AR (are equal) to one another. For (they are) opposite [Prop. 11.24]. So, for the same (reasons), the parallelograms EC , HW , and MS are also equal to one another, and HG , HI , and IN are equal to one another, and, further, DH , MY , and NT (are equal to one another). Thus, three planes of (one of) the solids LQ , KR , and AU are equal to the (corresponding) three planes (of the others). But, the three planes (in one of the solids) are equal to the three opposite planes [Prop. 11.24]. Thus, the three solids LQ , KR , and AU are equal to one another [Def. 11.10]. So, for the same (reasons), the three solids ED , DM , and MT are also equal to one another. Thus, as many multiples as the base LF is of the base AF , so many multiples is the solid LU also of the solid AU . So, for the same (reasons), as many multiples as the base NF is of the base FH , so many multiples is the solid NU also of the solid HU . And if the base LF is equal to the base NF then the solid LU is also equal to the solid NU .[†] And if the base LF exceeds the base NF then the solid LU also exceeds the solid NU . And if (LF) is less than (NF) then (LU) is (also) less than (NU). So, there are four magnitudes, the two bases AF and FH , and the two solids AU and UH , and equal multiples have been taken of the base AF and the solid AU — (namely), the base LF and the solid LU —and of the base FH and the solid HU —(namely), the base NF and the solid NU . And it has been shown that if the base LF exceeds the base NF then the solid LU also exceeds the [solid] NU , and if (LF is) equal (to NF) then (LU is) equal (to NU), and if (LF is) less than (NF) then (LU is) less than (NU). Thus, as the base AF is to the base FH , so the solid AU (is) to the solid UH [Def. 5.5]. (Which is) the very thing it was required to show.

[†] Here, Euclid assumes that $LF \geq NF$ implies $LU \geq NU$. This is easily demonstrated.

κτ'.

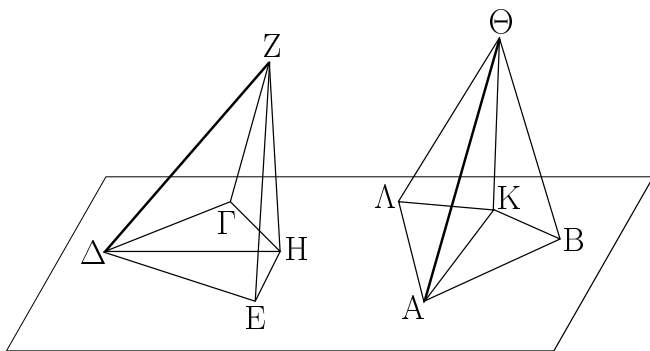
Proposition 26

Πρὸς τῇ δοθείσῃ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῇ δοθείσῃ στερεᾷ γωνίᾳ ἴσην στερεὰν γωνίαν συστήσασθαι.

Ἐστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ AB , τὸ δὲ πρὸς αὐτῇ δοθὲν σημεῖον τὸ A , ἡ δὲ δοθεῖσα στερεὰ γωνία ἡ πρὸς τῷ Δ περιεχομένη ὑπὸ τῶν ὑπὸ $E\Delta\Gamma$, $E\Delta Z$, $Z\Delta\Gamma$ γωνιῶν ἐπιπέδων· δεῖ δὴ πρὸς τῇ AB εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ A τῇ πρὸς τῷ Δ στερεᾷ γωνίᾳ ἴσην στερεὰν γωνίαν συστήσασθαι.

To construct a solid angle equal to a given solid angle on a given straight-line, and at a given point on it.

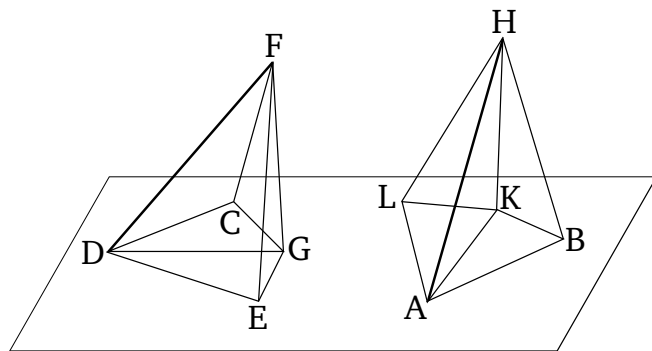
Let AB be the given straight-line, and A the given point on it, and D the given solid angle, contained by the plane angles EDC , EDF , and FDC . So, it is necessary to construct a solid angle equal to the solid angle D on the straight-line AB , and at the point A on it.



Εἰλήφθω γάρ ἐπὶ τῆς ΔZ τυχὸν σημεῖον τὸ Z , καὶ ἤχθω ἀπὸ τοῦ Z ἐπὶ τὸ διὰ τῶν $ΕΔ$, $\Delta\Gamma$ ἐπίπεδον κάθετος ἡ ZH , καὶ συμβαλλέτω τῷ ἐπίπεδῳ κατὰ τὸ H , καὶ ἐπεζεύχθω ἡ ΔH , καὶ συνεστιάτω πρὸς τῇ AB εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ A τῇ μὲν ὑπὸ $ΕΔ\Gamma$ γωνίᾳ ἴση ἢ ὑπὸ BAL , τῇ δὲ ὑπὸ $ΕΔH$ ἴση ἢ ὑπὸ BAK , καὶ κείσθω τῇ ΔH ἴση ἢ AK , καὶ ἀνεστιάτω ἀπὸ τοῦ K σημείου τῷ διὰ τῶν BAL ἐπίπεδῳ πρὸς ὀρθὰς ἡ $K\Theta$, καὶ κείσθω ἴση τῇ HZ ἢ $K\Theta$, καὶ ἐπεζεύχθω ἡ ΘA . λέγω, ὅτι ἡ πρὸς τῷ A στερεὰ γωνία περιεχομένη ὑπὸ τῶν BAL , $BA\Theta$, $\Theta A\Delta$ γωνιῶν ἴση ἐστὶ τῇ πρὸς τῷ Δ στερεᾷ γωνίᾳ τῇ περιεχομένῃ ὑπὸ τῶν $ΕΔ\Gamma$, $ΕΔZ$, $Z\Delta\Gamma$ γωνιῶν.

Ἀπειλήφθωσαν γάρ ἴσαι αἱ AB , ΔE , καὶ ἐπεζεύχθωσαν αἱ ΘB , KB , ZE , HE . καὶ ἐπεὶ ἡ ZH ὀρθὴ ἐστὶ πρὸς τὸ ὑποκείμενον ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ ὑποκειμένῳ ἐπίπεδῳ ὀρθὰς ποιήσει γωνίας· ὀρθὴ ἄρα ἐστὶν ἑκάτερα τῶν ὑπὸ $ZH\Delta$, ZHE γωνιῶν. διὰ τὰ αὐτὰ δὲ καὶ ἑκάτερα τῶν ὑπὸ ΘKA , ΘKB γωνιῶν ὀρθὴ ἐστὶν. καὶ ἐπεὶ δύο αἱ KA , AB δύο ταῖς $H\Delta$, ΔE ἴσαι εἰσὶν ἑκάτερα ἑκατέρῃ, καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ KB βάσει τῇ HE ἴση ἐστίν. ἐστὶ δὲ καὶ ἡ $K\Theta$ τῇ HZ ἴση· καὶ γωνίας ὀρθὰς περιέχουσιν· ἴση ἄρα καὶ ἡ ΘB τῇ ZE . πάλιν ἐπεὶ δύο αἱ AK , $K\Theta$ δυοὶ ταῖς ΔH , HZ ἴσαι εἰσὶν, καὶ γωνίας ὀρθὰς περιέχουσιν, βάσις ἄρα ἡ $A\Theta$ βάσει τῇ $Z\Delta$ ἴση ἐστίν. ἐστὶ δὲ καὶ ἡ AB τῇ ΔE ἴση· δύο δὲ αἱ ΘA , AB δύο ταῖς ΔZ , ΔE ἴσαι εἰσὶν. καὶ βάσις ἡ ΘB βάσει τῇ ZE ἴση· γωνία ἄρα ἡ ὑπὸ $BA\Theta$ γωνία τῇ ὑπὸ $E\Delta Z$ ἐστὶν ἴση. διὰ τὰ αὐτὰ δὲ καὶ ἡ ὑπὸ $\Theta A\Delta$ τῇ ὑπὸ $Z\Delta\Gamma$ ἐστὶν ἴση. ἐστὶ δὲ καὶ ἡ ὑπὸ BAL τῇ ὑπὸ $ΕΔ\Gamma$ ἴση.

Πρὸς ἄρα τῇ δοθείσῃ εὐθείᾳ τῇ AB καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ A τῇ δοθείσῃ στερεᾷ γωνίᾳ τῇ πρὸς τῷ Δ ἴση συνέσταται· ὅπερ ἔδει ποιῆσαι.



For let some random point F have been taken on DF , and let FG have been drawn from F perpendicular to the plane through ED and DC [Prop. 11.11], and let it meet the plane at G , and let DG have been joined. And let BAL , equal to the angle EDC , and BAK , equal to EDG , have been constructed on the straight-line AB at the point A on it [Prop. 1.23]. And let AK be made equal to DG . And let KH have been set up at the point K at right-angles to the plane through BAL [Prop. 11.12]. And let KH be made equal to GF . And let HA have been joined. I say that the solid angle at A , contained by the (plane) angles BAL , BAH , and HAL , is equal to the solid angle at D , contained by the (plane) angles EDC , EDF , and FDC .

For let AB and DE have been cut off (so as to be) equal, and let HB , KB , FE , and GE have been joined. And since FG is at right-angles to the reference plane (EDC), it will also make right-angles with all of the straight-lines joined to it which are also in the reference plane [Def. 11.3]. Thus, the angles FGD and FGE are right-angles. So, for the same (reasons), the angles HKA and HKB are also right-angles. And since the two (straight-lines) KA and AB are equal to the two (straight-lines) GD and DE , respectively, and they contain equal angles, the base KB is thus equal to the base GE [Prop. 1.4]. And KH is also equal to GF . And they contain right-angles (with the respective bases). Thus, HB (is) also equal to FE [Prop. 1.4]. Again, since the two (straight-lines) AK and KH are equal to the two (straight-lines) DG and GF (respectively), and they contain right-angles, the base HA is thus equal to the base FD [Prop. 1.4]. And AB (is) also equal to DE . So, the two (straight-lines) HA and AB are equal to the two (straight-lines) DF and DE (respectively). And the base HB (is) equal to the base FE . Thus, the angle BAH is equal to the angle EDF [Prop. 1.8]. So, for the same (reasons), HAL is also equal to FDC . And BAL is also equal to EDC .

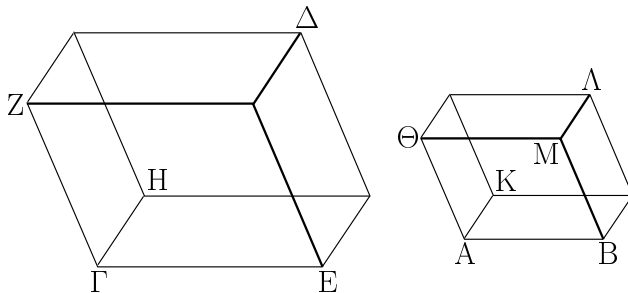
Thus, (a solid angle) has been constructed, equal to the given solid angle at D , on the given straight-line AB ,

κζ'.

Ἀπὸ τῆς δοθείσης εὐθείας τῷ δοθέντι στερεῷ παραλληλεπίπεδω ὁμοίων τε καὶ ὁμοίως κείμενον στερεὸν παραλληλεπίπεδον ἀναγράψαι.

Ἐστω ἡ μὲν δοθεῖσα εὐθεῖα ἡ AB , τὸ δὲ δοθὲν στερεὸν παραλληλεπίπεδον τὸ $\Gamma\Delta$. δεῖ δὲ ἀπὸ τῆς δοθείσης εὐθείας τῆς AB τῷ δοθέντι στερεῷ παραλληλεπίπεδω τῷ $\Gamma\Delta$ ὁμοίων τε καὶ ὁμοίως κείμενον στερεὸν παραλληλεπίπεδον ἀναγράψαι.

Συνεστάτω γὰρ πρὸς τῇ AB εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ A τῇ πρὸς τῷ Γ στερεῷ γωνία ἴση ἡ περιεχομένη ὑπὸ τῶν $BA\Theta$, ΘAK , KAB , ὥστε ἴσῃ εἶναι τὴν μὲν ὑπὸ $BA\Theta$ γωνίαν τῇ ὑπὸ EGZ , τὴν δὲ ὑπὸ BAK τῇ ὑπὸ EGH , τὴν δὲ ὑπὸ $KA\Theta$ τῇ ὑπὸ HGZ . καὶ γεγονέτω ὡς μὲν ἡ EG πρὸς τὴν GH , οὕτως ἡ BA πρὸς τὴν AK , ὡς δὲ ἡ HG πρὸς τὴν GZ , οὕτως ἡ KA πρὸς τὴν $A\Theta$. καὶ δι' ἴσου ἄρα ἐστὶν ὡς ἡ EG πρὸς τὴν GZ , οὕτως ἡ BA πρὸς τὴν $A\Theta$. καὶ συμπληρώσθω τὸ ΘB παραλληλόγραμμον καὶ τὸ AL στερεόν.



Καὶ ἐπεὶ ἐστὶν ὡς ἡ EG πρὸς τὴν GH , οὕτως ἡ BA πρὸς τὴν AK , καὶ περὶ ἴσας γωνίας τὰς ὑπὸ EGH , BAK αἱ πλευραὶ ἀνάλογόν εἰσιν, ὁμοίων ἄρα ἐστὶ τὸ HE παραλληλόγραμμον τῷ KB παραλληλόγραμμῳ. διὰ τὰ αὐτὰ δὲ καὶ τὸ μὲν $K\Theta$ παραλληλόγραμμον τῷ HZ παραλληλόγραμμῳ ὁμοίων ἐστὶ καὶ ἔτι τὸ ZE τῷ ΘB . τρία ἄρα παραλληλόγραμμα τοῦ $\Gamma\Delta$ στερεοῦ τρισὶ παραλληλόγραμμοις τοῦ AL στερεοῦ ὁμοία ἐστὶν. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστι καὶ ὁμοία, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστὶ καὶ ὁμοία· ὅλον ἄρα τὸ $\Gamma\Delta$ στερεὸν ὅλῳ τῷ AL στερεῷ ὁμοίων ἐστὶν.

Ἀπὸ τῆς δοθείσης ἄρα εὐθείας τῆς AB τῷ δοθέντι στερεῷ παραλληλεπίπεδω τῷ $\Gamma\Delta$ ὁμοίων τε καὶ ὁμοίως κείμενον ἀναγράφεται τὸ AL . ὅπερ ἔδει ποιῆσαι.

κη'.

Ἐὰν στερεὸν παραλληλεπίπεδον ἐπιπέδῳ τμηθῇ κατὰ

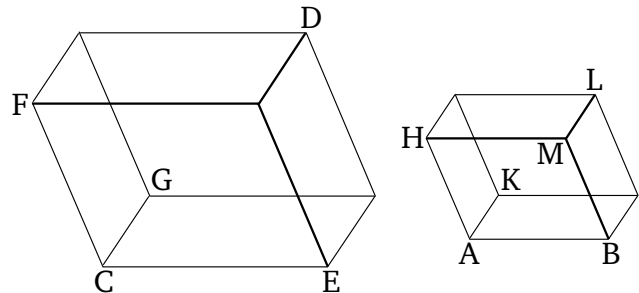
at the given point A on it. (Which is) the very thing it was required to do.

Proposition 27

To describe a parallelepiped solid similar, and similarly laid out, to a given parallelepiped solid on a given straight-line.

Let the given straight-line be AB , and the given parallelepiped solid CD . So, it is necessary to describe a parallelepiped solid similar, and similarly laid out, to the given parallelepiped solid CD on the given straight-line AB .

For, let a (solid angle) contained by the (plane angles) BAH , HAK , and KAB have been constructed, equal to solid angle at C , on the straight-line AB at the point A on it [Prop. 11.26], such that angle BAH is equal to ECF , and BAK to ECG , and KAH to GCF . And let it have been contrived that as EC (is) to CG , so BA (is) to AK , and as GC (is) to CF , so KA (is) to AH [Prop. 6.12]. And thus, via equality, as EC is to CF , so BA (is) to AH [Prop. 5.22]. And let the parallelogram HB have been completed, and the solid AL .



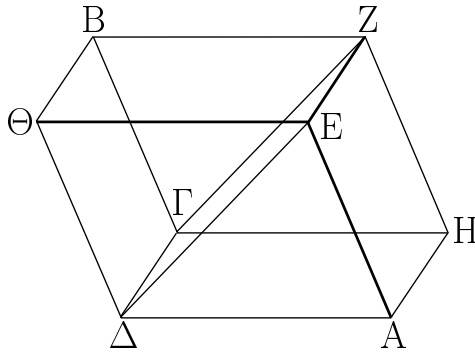
And since as EC is to CG , so BA (is) to AK , and the sides about the equal angles ECG and BAK are (thus) proportional, the parallelogram GE is thus similar to the parallelogram KB . So, for the same (reasons), the parallelogram KH is also similar to the parallelogram GF , and, further, FE (is similar) to HB . Thus, three of the parallelograms of solid CD are similar to three of the parallelograms of solid AL . But, the (former) three are equal and similar to the three opposite, and the (latter) three are equal and similar to the three opposite. Thus, the whole solid CD is similar to the whole solid AL [Def. 11.9].

Thus, AL , similar, and similarly laid out, to the given parallelepiped solid CD , has been described on the given straight-lines AB . (Which is) the very thing it was required to do.

Proposition 28

If a parallelepiped solid is cut by a plane (passing)

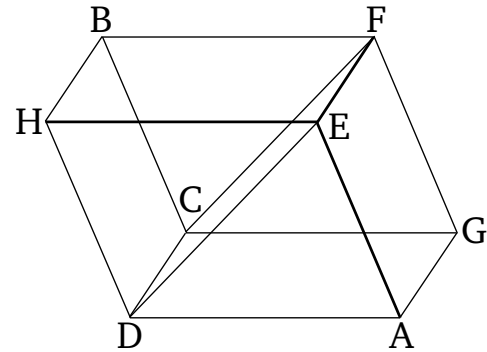
τὰς διαγωνίους τῶν ἀπεναντίον ἐπιπέδων, δίχα τμηθήσεται τὸ στερεὸν ὑπὸ τοῦ ἐπιπέδου.



Στερεὸν γὰρ παραλληλεπίπεδον τὸ AB ἐπιπέδῳ τῷ ΓΔΕΖ τετμήσθω κατὰ τὰς διαγωνίους τῶν ἀπεναντίον ἐπιπέδων τὰς ΓΖ, ΔΕ· λέγω, ὅτι δίχα τμηθήσεται τὸ AB στερεὸν ὑπὸ τοῦ ΓΔΕΖ ἐπιπέδου.

Ἐπεὶ γὰρ ἴσον ἐστὶ τὸ μὲν ΓΗΖ τρίγωνον τῷ ΓΖΒ τριγώνῳ, τὸ δὲ ΑΔΕ τῷ ΔΕΘ, ἔστι δὲ καὶ τὸ μὲν ΓΑ παραλληλόγραμμον τῷ ΕΒ ἴσον· ἀπεναντίον γάρ· τὸ δὲ ΗΕ τῷ ΓΘ, καὶ τὸ πρίσμα ἄρα τὸ περιεχόμενον ὑπὸ δύο μὲν τριγώνων τῶν ΓΗΖ, ΑΔΕ, τριῶν δὲ παραλληλογράμμων τῶν ΗΕ, ΑΓ, ΓΕ ἴσον ἐστὶ τῷ πρίσματι τῷ περιεχομένῳ ὑπὸ δύο μὲν τριγώνων τῶν ΓΖΒ, ΔΕΘ, τριῶν δὲ παραλληλογράμμων τῶν ΓΘ, ΒΕ, ΓΕ· ὑπὸ γὰρ ἴσων ἐπιπέδων περιέχονται τῷ τε πλήθει καὶ τῷ μεγέθει. ὥστε ὅλον τὸ AB στερεὸν δίχα τέτμηται ὑπὸ τοῦ ΓΔΕΖ ἐπιπέδου· ὅπερ εἶδει δεῖξαι.

through the diagonals of (a pair of) opposite planes then the solid will be cut in half by the plane.



For let the parallelepiped solid AB have been cut by the plane CDEF (passing) through the diagonals of the opposite planes CF and DE.[†] I say that the solid AB will be cut in half by the plane CDEF.

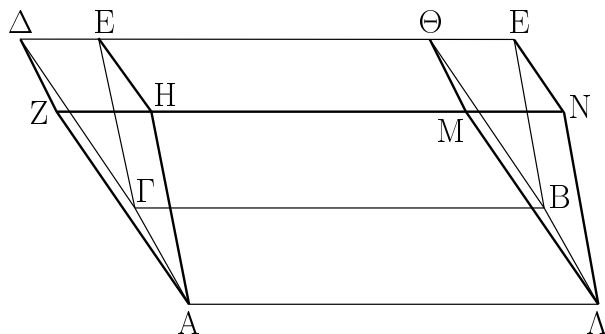
For since triangle CGF is equal to triangle CFB, and ADE (is equal) to DEH [Prop. 1.34], and parallelogram CA is also equal to EB—for (they are) opposite [Prop. 11.24]—and GE (equal) to CH, thus the prism contained by the two triangles CGF and ADE, and the three parallelograms GE, AC, and CE, is also equal to the prism contained by the two triangles CFB and DEH, and the three parallelograms CH, BE, and CE. For they are contained by planes (which are) equal in number and in magnitude [Def. 11.10].[‡] Thus, the whole of solid AB is cut in half by the plane CDEF. (Which is) the very thing it was required to show.

[†] Here, it is assumed that the two diagonals lie in the same plane. The proof is easily supplied.

[‡] However, strictly speaking, the prisms are not similarly arranged, being mirror images of one another.

κθ'.

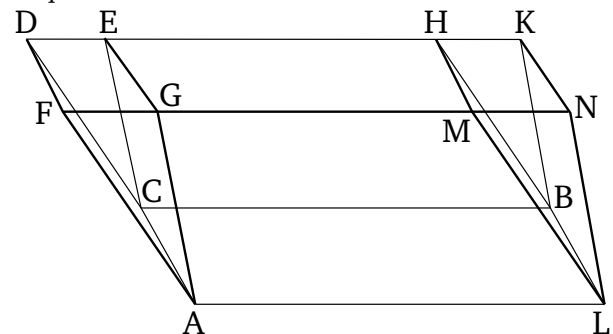
Τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφ'esτῶσαι ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν.



Ἐστω ἐπὶ τῆς αὐτῆς βάσεως τῆς AB στερεὰ παραλλη-

Proposition 29

Parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are on the same straight-lines, are equal to one another.



For let the parallelepiped solids CM and CN be on

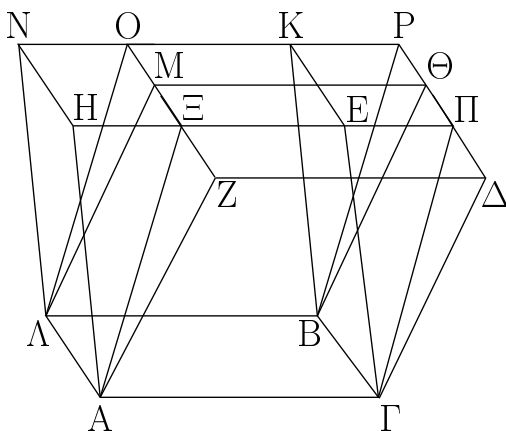
λεπίπεδα τὰ ΓΜ, ΓΝ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΗ, ΑΖ, ΑΜ, ΑΝ, ΓΔ, ΓΕ, ΒΘ, ΒΚ ἐπὶ τῶν αὐτῶν εὐθειῶν ἔστωσαν τῶν ΖΝ, ΔΚ· λέγω, ὅτι ἴσον ἐστὶ τὸ ΓΜ στερεὸν τῷ ΓΝ στερεῷ.

Ἐπεὶ γὰρ παραλληλόγραμμον ἐστὶν ἐκάτερον τῶν ΓΘ, ΓΚ, ἴση ἐστὶν ἡ ΓΒ ἐκατέρᾳ τῶν ΔΘ, ΕΚ· ὥστε καὶ ἡ ΔΘ τῇ ΕΚ ἐστὶν ἴση. κοινὴ ἀφηρήσθω ἡ ΕΘ· λοιπὴ ἄρα ἡ ΔΕ λοιπὴ τῇ ΘΚ ἐστὶν ἴση. ὥστε καὶ τὸ μὲν ΔΓΕ τρίγωνον τῷ ΘΒΚ τριγώνῳ ἴσον ἐστίν, τὸ δὲ ΔΗ παραλληλόγραμμον τῷ ΘΝ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ ΑΖΗ τρίγωνον τῷ ΜΑΝ τριγώνῳ ἴσον ἐστίν. ἔστι δὲ καὶ τὸ μὲν ΓΖ παραλληλόγραμμον τῷ ΒΜ παραλληλογράμμῳ ἴσον, τὸ δὲ ΓΗ τῷ ΒΝ· ἀπεναντίον γάρ· καὶ τὸ πρίσμα ἄρα τὸ περιχόμενον ὑπὸ δύο μὲν τριγώνων τῶν ΑΖΗ, ΔΓΕ, τριῶν δὲ παραλληλογράμμων τῶν ΑΔ, ΔΗ, ΓΗ ἴσον ἐστὶ τῷ πρίσματι τῷ περιχομένῳ ὑπὸ δύο μὲν τριγώνων τῶν ΜΑΝ, ΘΒΚ, τριῶν δὲ παραλληλογράμμων τῶν ΒΜ, ΘΝ, ΒΝ. κοινὸν προσκείσθω τὸ στερεόν, οὗ βάσις μὲν τὸ ΑΒ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΗΕΘΜ· ὅλον ἄρα τὸ ΓΜ στερεὸν παραλληλεπίπεδον ὅλῳ τῷ ΓΝ στερεῷ παραλληλεπίπεδῳ ἴσον ἐστίν.

Τὰ ἄρα ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι ἐπὶ τῶν αὐτῶν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

λ΄.

Τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν.



Ἐστω ἐπὶ τῆς αὐτῆς βάσεως τῆς ΑΒ στερεὰ παραλληλεπίπεδα τὰ ΓΜ, ΓΝ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΖ, ΑΗ, ΑΜ, ΑΝ, ΓΔ, ΓΕ, ΒΘ, ΒΚ μὴ ἔστωσαν ἐπὶ τῶν

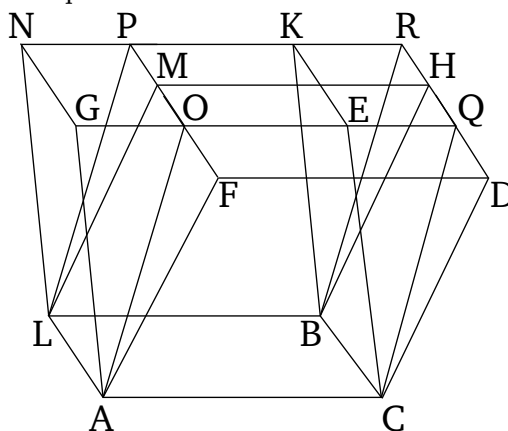
the same base AB , and (have) the same height, and let the (ends of the straight-lines) standing up in them, AG , AF , LM , LN , CD , CE , BH , and BK , be on the same straight-lines, FN and DK . I say that solid CM is equal to solid CN .

For since CH and CK are each parallelograms, CB is equal to each of DH and EK [Prop. 1.34]. Hence, DH is also equal to EK . Let EH have been subtracted from both. Thus, the remainder DE is equal to the remainder HK . Hence, triangle DCE is also equal to triangle HBK [Props. 1.4, 1.8], and parallelogram DG to parallelogram HN [Prop. 1.36]. So, for the same (reasons), triangle AFG is also equal to triangle MLN . And parallelogram CF is also equal to parallelogram BM , and CG to BN [Prop. 11.24]. For they are opposite. Thus, the prism contained by the two triangles AFG and DCE , and the three parallelograms AD , DG , and CG , is equal to the prism contained by the two triangles MLN and HBK , and the three parallelograms BM , HN , and BN . Let the solid whose base (is) parallelogram AB , and (whose) opposite (face is) $GEHM$, have been added to both (prisms). Thus, the whole parallelepiped solid CM is equal to the whole parallelepiped solid CN .

Thus, parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up (are) on the same straight-lines, are equal to one another. (Which is) the very thing it was required to show.

Proposition 30

Parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are not on the same straight-lines, are equal to one another.



Let the parallelepiped solids CM and CN be on the same base, AB , and (have) the same height, and let the (ends of the straight-lines) standing up in them, AF , AG ,

αὐτῶν εὐθειῶν· λέγω, ὅτι ἴσον ἐστὶ τὸ ΓΜ στερεὸν τῷ ΓΝ στερεῷ.

Ἐκβεβλήσθωσαν γὰρ αἱ ΝΚ, ΔΘ καὶ συμπιπτεύωσαν ἀλλήλαις κατὰ τὸ Ρ, καὶ ἔτι ἐκβεβλήσθωσαν αἱ ΖΜ, ΗΕ ἐπὶ τὰ Ο, Π, καὶ ἐπεζεύχθωσαν αἱ ΑΞ, ΛΟ, ΓΠ, ΒΡ. ἴσον δὴ ἐστὶ τὸ ΓΜ στερεόν, οὗ βάσις μὲν τὸ ΑΓΒΑ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΖΔΘΜ, τῷ ΓΟ στερεῷ, οὗ βάσις μὲν τὸ ΑΓΒΑ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΞΠΡΟ· ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως εἰσι τῆς ΑΓΒΑ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΖ, ΑΞ, ΑΜ, ΛΟ, ΓΔ, ΓΠ, ΒΘ, ΒΡ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν ΖΟ, ΔΡ. ἀλλὰ τὸ ΓΟ στερεόν, οὗ βάσις μὲν ἐστὶ τὸ ΑΓΒΑ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΞΠΡΟ, ἴσον ἐστὶ τῷ ΓΝ στερεῷ, οὗ βάσις μὲν τὸ ΑΓΒΑ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ ΗΕΚΝ· ἐπὶ τε γὰρ πάλιν τῆς αὐτῆς βάσεως εἰσι τῆς ΑΓΒΑ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ ΑΗ, ΑΞ, ΓΕ, ΓΠ, ΑΝ, ΛΟ, ΒΚ, ΒΡ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν ΗΠ, ΝΡ. ὥστε καὶ τὸ ΓΜ στερεὸν ἴσον ἐστὶ τῷ ΓΝ στερεῷ.

Τὰ ἄρα ἐπὶ τῆς αὐτῆς βάσεως στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι οὐκ εἰσιν ἐπὶ τῶν αὐτῶν εὐθειῶν, ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

λα'.

Τὰ ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν.

Ἐστω ἐπὶ ἴσων βάσεων τῶν ΑΒ, ΓΔ στερεὰ παραλληλεπίπεδα τὰ ΑΕ, ΓΖ ὑπὸ τὸ αὐτὸ ὕψος. λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΕ στερεὸν τῷ ΓΖ στερεῷ.

Ἐστωσαν δὴ πρότερον αἱ ἐφεστηκυῖαι αἱ ΘΚ, ΒΕ, ΑΗ, ΑΜ, ΟΠ, ΔΖ, ΓΞ, ΡΣ πρὸς ὀρθὰς ταῖς ΑΒ, ΓΔ βάσεσιν, καὶ ἐκβεβλήσθω ἐπ' εὐθείας τῇ ΓΡ εὐθεῖα ἡ ΡΤ, καὶ συνεστάτω πρὸς τῇ ΡΤ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Ρ τῇ ὑπὸ ΑΑΒ γωνίᾳ ἴση ἡ ὑπὸ ΤΡΥ, καὶ κείσθω τῇ μὲν ΑΑ ἴση ἡ ΡΤ, τῇ δὲ ΑΒ ἴση ἡ ΡΥ, καὶ συμπεπληρώσθω ἡ τε ΡΧ βάσις καὶ τὸ ΨΥ στερεόν.

$LM, LN, CD, CE, BH,$ and BK , not be on the same straight-lines. I say that the solid CM is equal to the solid CN .

For let NK and DH have been produced, and let them have joined one another at R . And, further, let FM and GE have been produced to P and Q (respectively). And let $AO, LP, CQ,$ and BR have been joined. So, solid CM , whose base (is) parallelogram $ACBL$, and opposite (face) $FDHM$, is equal to solid CP , whose base (is) parallelogram $ACBL$, and opposite (face) $OQRP$. For they are on the same base, $ACBL$, and (have) the same height, and the (ends of the straight-lines) standing up in them, $AF, AO, LM, LP, CD, CQ, BH,$ and BR , are on the same straight-lines, FP and DR [Prop. 11.29]. But, solid CP , whose base is parallelogram $ACBL$, and opposite (face) $OQRP$, is equal to solid CN , whose base (is) parallelogram $ACBL$, and opposite (face) $GEKN$. For, again, they are on the same base, $ACBL$, and (have) the same height, and the (ends of the straight-lines) standing up in them, $AG, AO, CE, CQ, LN, LP, BK,$ and BR , are on the same straight-lines, GQ and NR [Prop. 11.29]. Hence, solid CM is also equal to solid CN .

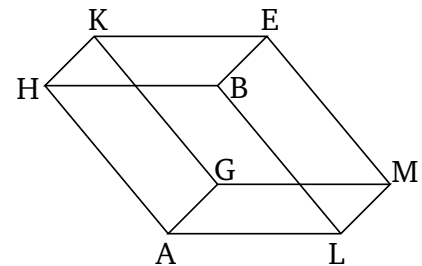
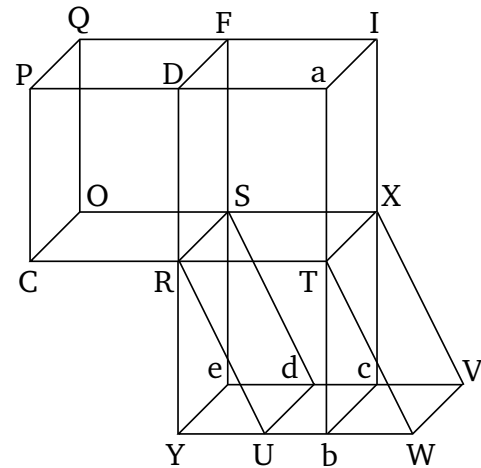
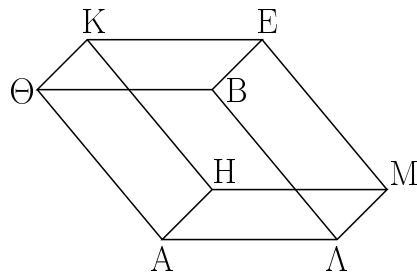
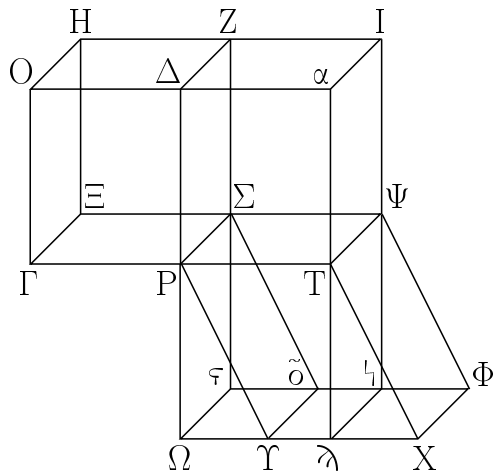
Thus, parallelepiped solids (which are) on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are not on the same straight-lines, are equal to one another. (Which is) the very thing it was required to show.

Proposition 31

Parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another.

Let the parallelepiped solids AE and CF be on the equal bases AB and CD (respectively), and (have) the same height. I say that solid AE is equal to solid CF .

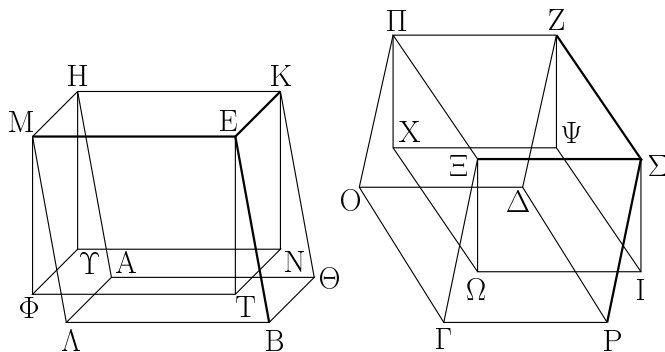
So, let the (straight-lines) standing up, $HK, BE, AG, LM, PQ, DF, CO,$ and RS , first of all, be at right-angles to the bases AB and CD . And let RT have been produced in a straight-line with CR . And let (angle) TRU , equal to angle ALB , have been constructed on the straight-line RT , at the point R on it [Prop. 1.23]. And let RT be made equal to AL , and RU to LB . And let the base RW , and the solid XU , have been completed.



Καὶ ἐπεὶ δύο αἱ TP , PY δυσὶ ταῖς AA , AB ἴσαι εἰσὶν, καὶ γωνίας ἴσας περιέχουσιν, ἴσον ἄρα καὶ ὅμοιον τὸ PX παραλληλόγραμμον τῷ ΘA παραλληλογράμμῳ. καὶ ἐπεὶ πάλιν ἴση μὲν ἡ AA τῇ PT , ἡ δὲ AM τῇ $P\Sigma$, καὶ γωνίας ὀρθὰς περιέχουσιν, ἴσον ἄρα καὶ ὅμοιον ἐστὶ τὸ $P\Psi$ παραλληλόγραμμον τῷ AM παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ AE τῷ ΣY ἴσον τέ ἐστὶ καὶ ὅμοιον· τρία ἄρα παραλληλόγραμμα τοῦ AE στερεοῦ τρισὶ παραλληλογράμμοις τοῦ ΨY στερεοῦ ἴσα τέ ἐστὶ καὶ ὅμοια. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα τέ ἐστὶ καὶ ὅμοια, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον· ὅλον ἄρα τὸ AE στερεὸν παραλληλεπίπεδον ὅλῳ τῷ ΨY στερεῷ παραλληλεπίπεδῳ ἴσον ἐστίν. διήχθωσαν αἱ ΔP , $X Y$ καὶ συμπιπτεύωσαν ἀλλήλαις κατὰ τὸ Ω , καὶ διὰ τοῦ T τῇ $\Delta\Omega$ παράλληλος ἤχθῃ ἡ $\alpha T\lambda$, καὶ ἐκβεβλήσθῃ ἡ $O\Delta$ κατὰ τὸ α , καὶ συμπληρώσθῃ τὰ $\Omega\Psi$, PI στερεά. ἴσον δὴ ἐστὶ τὸ $\Psi\Omega$ στερεόν, οὗ βάσις μὲν ἐστὶ τὸ $P\Psi$ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ $\Omega\iota$, τῷ ΨY στερεῷ, οὗ βάσις μὲν τὸ $P\Psi$ παραλληλόγραμμον, ἀπεναντίον δὲ τὸ $Y\Phi$ · ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως εἰσι τῆς $P\Psi$ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι αἱ $P\Omega$, PY , $T\lambda$, TX , $\Sigma\tau$, $\Sigma\delta$, $\Psi\iota$, $\Psi\Phi$ ἐπὶ τῶν αὐτῶν εἰσιν εὐθειῶν τῶν ΩX , $\tau\Phi$. ἀλλὰ τὸ ΨY στερεὸν τῷ AE ἐστὶν ἴσον· καὶ τὸ $\Psi\Omega$ ἄρα στερεὸν τῷ AE στερεῷ ἐστὶν ἴσον. καὶ ἐπεὶ ἴσον ἐστὶ τὸ $PYXT$ παραλληλόγραμμον τῷ ΩT παραλληλογράμμῳ· ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως εἰσι τῆς PT καὶ ἐν ταῖς αὐταῖς παραλλήλοις ταῖς PT , ΩX · ἀλλὰ τὸ $PYXT$ τῷ $\Gamma\Delta$ ἐστὶν ἴσον, ἐπεὶ καὶ τῷ AB , καὶ τὸ ΩT ἄρα παραλληλόγραμμον

And since the two (straight-lines) TR and RU are equal to the two (straight-lines) AL and LB (respectively), and they contain equal angles, parallelogram RW is thus equal and similar to parallelogram HL [Prop. 6.14]. And, again, since AL is equal to RT , and LM to RS , and they contain right-angles, parallelogram RX is thus equal and similar to parallelogram AM [Prop. 6.14]. So, for the same (reasons), LE is also equal and similar to SU . Thus, three parallelograms of solid AE are equal and similar to three parallelograms of solid XU . But, the three (faces of the former solid) are equal and similar to the three opposite (faces), and the three (faces of the latter solid) to the three opposite (faces) [Prop. 11.24]. Thus, the whole parallelepiped solid AE is equal to the whole parallelepiped solid XU [Def. 11.10]. Let DR and WU have been drawn across, and let them have met one another at Y . And let aTb have been drawn through T parallel to DY . And let PD have been produced to a . And let the solids YX and RI have been completed. So, solid XY , whose base is parallelogram RX , and opposite (face) Yc , is equal to solid XU , whose base (is) parallelogram RX , and opposite (face) UV . For they are on the same base RX , and (have) the same height, and the (ends of the straight-lines) standing up in them, RY , RU , Tb , TW , Se , Sd , Xc and XV , are on the same straight-lines, YW and eV [Prop. 11.29]. But, solid XU is equal to AE . Thus,

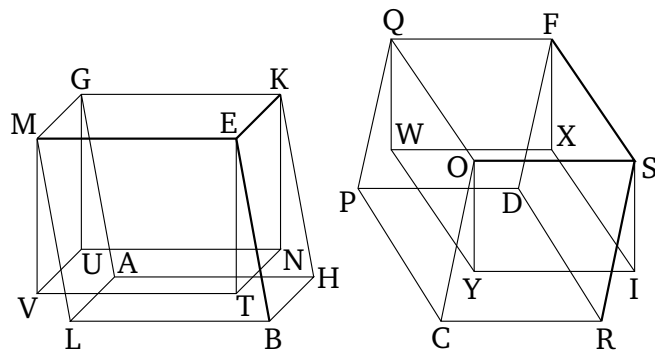
τῷ ΓΔ ἐστὶν ἴσον. ἄλλο δὲ τὸ ΔΤ· ἔστιν ἄρα ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ, οὕτως ἡ ΩΤ πρὸς τὴν ΔΤ. καὶ ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΠΙ ἐπιπέδῳ τῷ ΡΖ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ βάσιν, οὕτως τὸ ΓΖ στερεὸν πρὸς τὸ ΠΙ στερεόν. διὰ τὰ αὐτὰ δὴ, ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΟΙ ἐπιπέδῳ τῷ ΡΨ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ὡς ἡ ΩΤ βάσις πρὸς τὴν ΤΛ βάσιν, οὕτως τὸ ΩΨ στερεὸν πρὸς τὸ ΠΙ. ἀλλ' ὡς ἡ ΓΔ βάσις πρὸς τὴν ΔΤ, οὕτως ἡ ΩΤ πρὸς τὴν ΔΤ· καὶ ὡς ἄρα τὸ ΓΖ στερεὸν πρὸς τὸ ΠΙ στερεόν, οὕτως τὸ ΩΨ στερεὸν πρὸς τὸ ΠΙ. ἐκάτερον ἄρα τῶν ΓΖ, ΩΨ στερεῶν πρὸς τὸ ΠΙ τὸν αὐτὸν ἔχει λόγον· ἴσον ἄρα ἐστὶ τὸ ΓΖ στερεὸν τῷ ΩΨ στερεῷ. ἀλλὰ τὸ ΩΨ τῷ ΑΕ ἐδείχθη ἴσον· καὶ τὸ ΑΕ ἄρα τῷ ΓΖ ἐστὶν ἴσον.



Μὴ ἔστωσαν δὴ αἱ ἐφεστηκυῖαι αἱ ΑΗ, ΘΚ, ΒΕ, ΑΜ, ΓΞ, ΟΠ, ΔΖ, ΡΣ πρὸς ὀρθὰς ταῖς ΑΒ, ΓΔ βάσεσιν· λέγω πάλιν, ὅτι ἴσον τὸ ΑΕ στερεὸν τῷ ΓΖ στερεῷ. ἤχθωσαν γὰρ ἀπὸ τῶν Κ, Ε, Η, Μ, Π, Ζ, Ξ, Σ σημείων ἐπὶ τὸ ὑποκείμενον ἐπίπεδον κάθετοι αἱ ΚΝ, ΕΤ, ΗΥ, ΜΦ, ΠΧ, ΖΨ, ΞΩ, ΣΙ, καὶ συμβαλλέτωσαν τῷ ἐπιπέδῳ κατὰ τὰ Ν, Τ, Υ, Φ, Χ, Ψ, Ω, Ι σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΝΤ, ΝΥ, ΥΦ, ΤΦ, ΧΨ, ΧΩ, ΩΙ, ΙΨ. ἴσον δὴ ἐστὶ τὸ ΚΦ στερεὸν τῷ ΠΙ στερεῷ· ἐπὶ τε γὰρ ἴσων βάσεων εἰσι τῶν ΚΜ, ΠΣ καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι πρὸς ὀρθὰς εἰσι ταῖς βάσεσιν. ἀλλὰ τὸ μὲν ΚΦ στερεὸν τῷ ΑΕ στερεῷ ἐστὶν ἴσον, τὸ δὲ ΠΙ τῷ ΓΖ· ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως εἰσι καὶ ὑπὸ τὸ αὐτὸ ὕψος, ὧν αἱ ἐφεστῶσαι οὐκ εἰσιν ἐπὶ τῶν αὐτῶν εὐθειῶν. καὶ τὸ ΑΕ ἄρα στερεὸν τῷ ΓΖ στερεῷ ἐστὶν ἴσον.

Τὰ ἄρα ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

solid XY is also equal to solid AE . And since parallelogram $RUWT$ is equal to parallelogram YT . For they are on the same base RT , and between the same parallels RT and YW [Prop. 1.35]. But, $RUWT$ is equal to CD , since (it is) also (equal) to AB . Parallelogram YT is thus also equal to CD . And DT is another (parallelogram). Thus, as base CD is to DT , so YT (is) to DT [Prop. 5.7]. And since the parallelepiped solid CI has been cut by the plane RF , which is parallel to the opposite planes (of CI), as base CD is to base DT , so solid CF (is) to solid RI [Prop. 11.25]. So, for the same (reasons), since the parallelepiped solid YI has been cut by the plane RX , which is parallel to the opposite planes (of YI), as base YT is to base TD , so solid YX (is) to solid RI [Prop. 11.25]. But, as base CD (is) to DT , so YT (is) to DT . And, thus, as solid CF (is) to solid RI , so solid YX (is) to solid RI . Thus, solids CF and YX each have the same ratio to RI [Prop. 5.11]. Thus, solid CF is equal to solid YX [Prop. 5.9]. But, YX was show (to be) equal to AE . Thus, AE is also equal to CF .



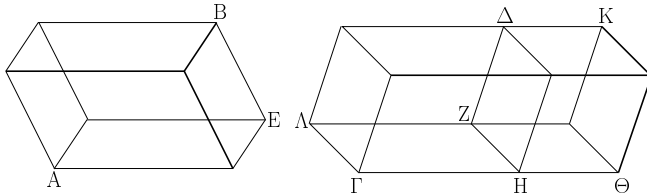
And so let the (straight-lines) standing up, AG , HK , BE , LM , CO , PQ , DF , and RS , not be at right-angles to the bases AB and CD . Again, I say that solid AE (is) equal to solid CF . For let KN , ET , GU , MV , QW , FX , OY , and SI have been drawn from points K , E , G , M , Q , F , O , and S (respectively) perpendicular to the reference plane (i.e., the plane of the bases AB and CD), and let them have met the plane at points N , T , U , V , W , X , Y , and I (respectively). And let NT , NU , UV , TV , WX , WY , YI , and IX have been joined. So solid KV is equal to solid QI . For they are on the equal bases KM and QS , and (have) the same height, and the (straight-lines) standing up in them are at right-angles to their bases (see first part of proposition). But, solid KV is equal to solid AE , and QI to CF . For they are on the same base, and (have) the same height, and the (straight-lines) standing up in them are not on the same straight-lines [Prop. 11.30]. Thus, solid AE is also equal to solid CF .

Thus, parallelepiped solids which are on equal bases,

and (have) the same height, are equal to one another. (Which is) the very thing it was required to show.

λβ'.

Τὰ ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις.



Ἐστω ὑπὸ τὸ αὐτὸ ὕψος στερεὰ παραλληλεπίπεδα τὰ AB , $\Gamma\Delta$. λέγω, ὅτι τὰ AB , $\Gamma\Delta$ στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις, τουτέστιν ὅτι ἐστὶν ὡς ἡ AE βάσις πρὸς τὴν ΓZ βάσιν, οὕτως τὸ AB στερεὸν πρὸς τὸ $\Gamma\Delta$ στερεόν.

Παραβεβλήσθω γὰρ παρὰ τὴν ZH τῷ AE ἴσον τὸ $Z\Theta$, καὶ ἀπὸ βάσεως μὲν τῆς $Z\Theta$, ὕψους δὲ τοῦ αὐτοῦ τῷ $\Gamma\Delta$ στερεὸν παραλληλεπίπεδον συμπληρώσθω τὸ HK . ἴσον δὴ ἐστὶ τὸ AB στερεὸν τῷ HK στερεῷ· ἐπὶ τε γὰρ ἴσων βάσεων εἰσι τῶν AE , $Z\Theta$ καὶ ὑπὸ τὸ αὐτὸ ὕψος. καὶ ἐπεὶ στερεὸν παραλληλεπίπεδον τὸ ΓK ἐπιπέδῳ τῷ ΔH τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἐστὶν ἄρα ὡς ἡ ΓZ βάσις πρὸς τὴν $Z\Theta$ βάσιν, οὕτως τὸ $\Gamma\Delta$ στερεὸν πρὸς τὸ $\Delta\Theta$ στερεόν. ἴση δὲ ἡ μὲν $Z\Theta$ βάσις τῇ AE βάσει, τὸ δὲ HK στερεὸν τῷ AB στερεῷ· ἐστὶν ἄρα καὶ ὡς ἡ AE βάσις πρὸς τὴν ΓZ βάσιν, οὕτως τὸ AB στερεὸν πρὸς τὸ $\Gamma\Delta$ στερεόν.

Τὰ ἄρα ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

λγ'.

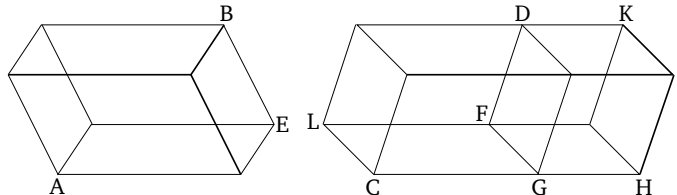
Τὰ ὅμοια στερεὰ παραλληλεπίπεδα πρὸς ἄλληλα ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν.

Ἐστω ὅμοια στερεὰ παραλληλεπίπεδα τὰ AB , $\Gamma\Delta$, ὁμόλογος δὲ ἔστω ἡ AE τῇ ΓZ . λέγω, ὅτι τὸ AB στερεὸν πρὸς τὸ $\Gamma\Delta$ στερεὸν τριπλασίονα λόγον ἔχει, ἥπερ ἡ AE πρὸς τὴν ΓZ .

Ἐκβεβλήσθωσαν γὰρ ἐπ' εὐθείας ταῖς AE , HE , ΘE αἱ EK , EL , EM , καὶ κείσθω τῇ μὲν ΓZ ἴση ἡ EK , τῇ δὲ ZN ἴση ἡ EL , καὶ ἔτι τῇ ZP ἴση ἡ EM , καὶ συμπληρώσθω τὸ KL παραλληλόγραμμον καὶ τὸ KO στερεόν.

Proposition 32

Parallelepiped solids which (have) the same height are to one another as their bases.



Let AB and CD be parallelepiped solids (having) the same height. I say that the parallelepiped solids AB and CD are to one another as their bases. That is to say, as base AE is to base CF , so solid AB (is) to solid CD .

For let FH , equal to AE , have been applied to FG (in the angle FGH equal to angle LCG) [Prop. 1.45]. And let the parallelepiped solid GK , (having) the same height as CD , have been completed on the base FH . So solid AB is equal to solid GK . For they are on the equal bases AE and FH , and (have) the same height [Prop. 11.31]. And since the parallelepiped solid CK has been cut by the plane DG , which is parallel to the opposite planes (of CK), thus as the base CF is to the base FH , so the solid CD (is) to the solid DH [Prop. 11.25]. And base FH (is) equal to base AE , and solid GK to solid AB . And thus as base AE is to base CF , so solid AB (is) to solid CD .

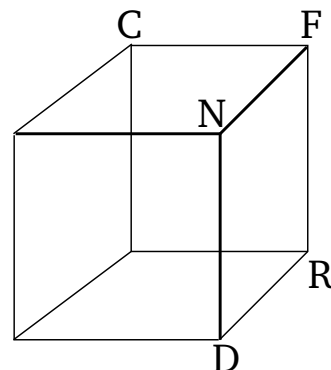
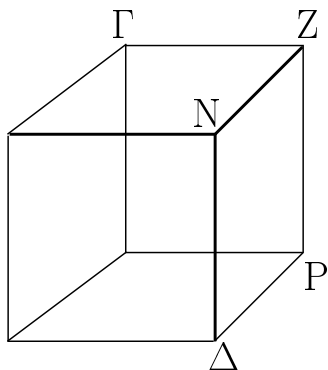
Thus, parallelepiped solids which (have) the same height are to one another as their bases. (Which is) the very thing it was required to show.

Proposition 33

Similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides.

Let AB and CD be similar parallelepiped solids, and let AE correspond to CF . I say that solid AB has to solid CD the cubed ratio that AE (has) to CF .

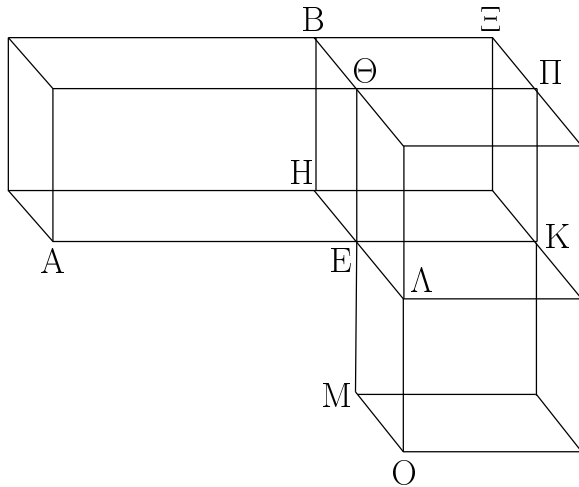
For let EK , EL , and EM have been produced in a straight-line with AE , GE , and HE (respectively). And let EK be made equal to CF , and EL equal to FN , and, further, EM equal to FR . And let the parallelogram KL have been completed, and the solid KP .



Καὶ ἐπεὶ δύο αἱ KE , EA δυσὶ ταῖς ΓZ , ZN ἴσαι εἰσίν, ἀλλὰ καὶ γωνία ἡ ὑπὸ KEA γωνία τῇ ὑπὸ ΓZN ἐστὶν ἴση, ἐπειδὴ περ καὶ ἡ ὑπὸ AEH τῇ ὑπὸ ΓZN ἐστὶν ἴση διὰ τὴν ὁμοιότητα τῶν AB , $\Gamma\Delta$ στερεῶν, ἴσον ἄρα ἐστὶ [καὶ ὅμοιον] τὸ $ΚΑ$ παραλληλόγραμμον τῷ ΓN παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν KM παραλληλόγραμμον ἴσον ἐστὶ καὶ ὅμοιον τῷ ΓP [παραλληλογράμμῳ] καὶ ἔτι τὸ EO τῷ ΔZ · τρία ἄρα παραλληλόγραμμα τοῦ KO στερεοῦ τρισὶ παραλληλογράμμοις τοῦ $\Gamma\Delta$ στερεοῦ ἴσα ἐστὶ καὶ ὅμοια. ἀλλὰ τὰ μὲν τρία τρισὶ τοῖς ἀπεναντίον ἴσα ἐστὶ καὶ ὅμοια, τὰ δὲ τρία τρισὶ τοῖς ἀπεναντίον ἴσα ἐστὶ καὶ ὅμοια· ὅλον ἄρα τὸ KO στερεὸν ὅλῳ τῷ $\Gamma\Delta$ στερεῷ ἴσον ἐστὶ καὶ ὅμοιον. συμπεπληρώσθω τὸ HK παραλληλόγραμμον, καὶ ἀπὸ βάσεων μὲν τῶν HK , $ΚΑ$ παραλληλόγραμμων, ὕψους δὲ τοῦ αὐτοῦ τῷ AB στερεᾷ συμπεπληρώσθω τὰ $E\Xi$, $\Lambda\Pi$. καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν AB , $\Gamma\Delta$ στερεῶν ἐστὶν ὡς ἡ AE πρὸς τὴν ΓZ , οὕτως ἡ EH πρὸς τὴν ZN , καὶ ἡ $E\Theta$ πρὸς τὴν ZP , ἴση δὲ ἡ μὲν ΓZ τῇ EK , ἡ δὲ ZN τῇ EA , ἡ δὲ ZP τῇ EM , ἔστιν ἄρα ὡς ἡ AE πρὸς τὴν EK , οὕτως ἡ HE πρὸς τὴν EA καὶ ἡ ΘE πρὸς τὴν EM . ἀλλ' ὡς μὲν ἡ AE πρὸς τὴν EK , οὕτως τὸ AH [παραλληλόγραμμον] πρὸς τὸ HK παραλληλόγραμμον, ὡς δὲ ἡ HE πρὸς τὴν EA , οὕτως τὸ HK πρὸς τὸ $ΚΑ$, ὡς δὲ ἡ ΘE πρὸς EM , οὕτως τὸ ΠE πρὸς τὸ KM · καὶ ὡς ἄρα τὸ AH παραλληλόγραμμον πρὸς τὸ HK , οὕτως τὸ HK πρὸς τὸ $ΚΑ$ καὶ τὸ ΠE πρὸς τὸ KM . ἀλλ' ὡς μὲν τὸ AH πρὸς τὸ HK , οὕτως τὸ AB στερεὸν πρὸς τὸ $E\Xi$ στερεόν, ὡς δὲ τὸ HK πρὸς τὸ $ΚΑ$, οὕτως τὸ ΞE στερεὸν πρὸς τὸ $\Pi\Lambda$ στερεόν, ὡς δὲ τὸ ΠE πρὸς τὸ KM , οὕτως τὸ $\Pi\Lambda$ στερεὸν πρὸς τὸ KO στερεόν· καὶ ὡς ἄρα τὸ AB στερεὸν πρὸς τὸ $E\Xi$, οὕτως τὸ $E\Xi$ πρὸς τὸ $\Pi\Lambda$ καὶ τὸ $\Pi\Lambda$ πρὸς τὸ KO . εἰ δὲ τέσσαρα μεγέθη κατὰ τὸ συνεχὲς ἀνάλογον ἦ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχει ἥπερ πρὸς τὸ δεύτερον· τὸ AB ἄρα στερεὸν πρὸς τὸ KO τριπλασίονα λόγον ἔχει ἥπερ τὸ AB πρὸς τὸ $E\Xi$. ἀλλ' ὡς τὸ AB πρὸς τὸ $E\Xi$, οὕτως τὸ AH παραλληλόγραμμον πρὸς τὸ HK καὶ ἡ AE εὐθεῖα πρὸς τὴν EK · ὥστε καὶ τὸ AB στερεὸν πρὸς τὸ KO τριπλασίονα λόγον ἔχει ἥπερ ἡ AE πρὸς τὴν EK . ἴσον δὲ τὸ [μὲν] KO στερεὸν τῷ $\Gamma\Delta$ στερεῷ, ἡ δὲ EK εὐθεῖα τῇ ΓZ · καὶ τὸ AB ἄρα στερεὸν πρὸς τὸ $\Gamma\Delta$ στερεόν τρι-

And since the two (straight-lines) KE and EL are equal to the two (straight-lines) CF and FN , but angle KEL is also equal to angle CFN , inasmuch as AEG is also equal to CFN , on account of the similarity of the solids AB and CD , parallelogram KL is thus equal [and similar] to parallelogram CN . So, for the same (reasons), parallelogram KM is also equal and similar to [parallelogram] CR , and, further, EP to DF . Thus, three parallelograms of solid KP are equal and similar to three parallelograms of solid CD . But the three (former parallelograms) are equal and similar to the three opposite (parallelograms), and the three (latter parallelograms) are equal and similar to the three opposite (parallelograms) [Prop. 11.24]. Thus, the whole of solid KP is equal and similar to the whole of solid CD [Def. 11.10]. Let parallelogram GK have been completed. And let the solids EO and LQ , with bases the parallelograms GK and KL (respectively), and with the same height as AB , have been completed. And since, on account of the similarity of solids AB and CD , as AE is to CF , so EG (is) to FN , and EH to FR [Defs. 6.1, 11.9], and CF (is) equal to EK , and FN to EL , and FR to EM , thus as AE is to EK , so GE (is) to EL , and HE to EM . But, as AE (is) to EK , so [parallelogram] AG (is) to parallelogram GK , and as GE (is) to EL , so GK (is) to KL , and as HE (is) to EM , so QE (is) to KM [Prop. 6.1]. And thus as parallelogram AG (is) to GK , so GK (is) to KL , and QE (is) to KM . But, as AG (is) to GK , so solid AB (is) to solid EO , and as GK (is) to KL , so solid OE (is) to solid QL , and as QE (is) to KM , so solid QL (is) to solid KP [Prop. 11.32]. And, thus, as solid AB is to EO , so EO (is) to QL , and QL to KP . And if four magnitudes are continuously proportional then the first has to the fourth the cubed ratio that (it has) to the second [Def. 5.10]. Thus, solid AB has to KP the cubed ratio which AB (has) to EO . But, as AB (is) to EO , so parallelogram AG (is) to GK , and the straight-line AE to EK [Prop. 6.1]. Hence, solid AB also has to KP the cubed ratio that AE (has) to EK . And solid KP (is)

πλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος αὐτοῦ πλευρὰ ἡ AE πρὸς τὴν ὁμόλογον πλευρὰν τὴν ΓZ .



Τὰ ἄρα ὅμοια στερεὰ παραλληλεπίπεδα ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν· ὅπερ ἔδει δεῖξαι.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ὦσιν, ἔσται ὡς ἡ πρώτη πρὸς τὴν τετάρτην, οὕτω τὸ ἀπὸ τῆς πρώτης στερεὸν παραλληλεπίπεδον πρὸς τὸ ἀπὸ τῆς δευτέρας τὸ ὅμοιον καὶ ὁμοίως ἀναγραφόμενον, ἐπεὶπερ καὶ ἡ πρώτη πρὸς τὴν τετάρτην τριπλασίονα λόγον ἔχει ἥπερ πρὸς τὴν δευτέραν.

λδ'.

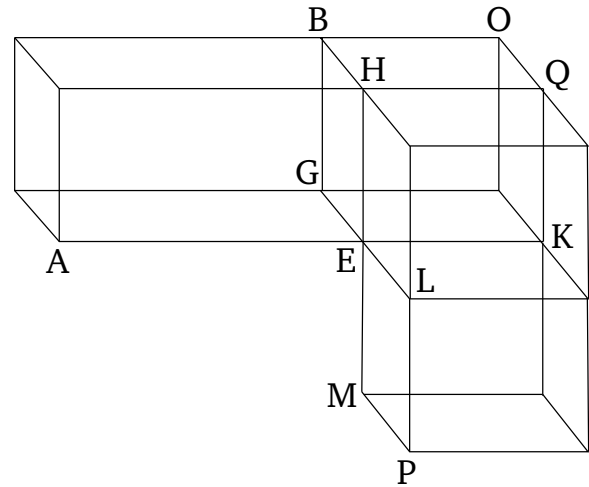
Τῶν ἴσων στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσα ἐστὶν ἐκείνα.

Ἐστω ἴσα στερεὰ παραλληλεπίπεδα τὰ AB , $\Gamma\Delta$ · λέγω, ὅτι τῶν AB , $\Gamma\Delta$ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἐστὶν ὡς ἡ EO βάσις πρὸς τὴν NI βάσιν, οὕτως τὸ τοῦ $\Gamma\Delta$ στερεοῦ ὕψος πρὸς τὸ τοῦ AB στερεοῦ ὕψος.

Ἐστώσαν γὰρ πρότερον αἱ ἐφεστηκυῖαι αἱ AH , EZ , AB , ΘK , GM , $N\Xi$, OD , IP πρὸς ὀρθὰς ταῖς βάσεσιν αὐτῶν· λέγω, ὅτι ἐστὶν ὡς ἡ EO βάσις πρὸς τὴν NI βάσιν, οὕτως ἡ GM πρὸς τὴν AH .

Εἰ μὲν οὖν ἴση ἐστὶν ἡ EO βάσις τῇ NI βάσει, ἔστι δὲ καὶ τὸ AB στερεὸν τῷ $\Gamma\Delta$ στερεῷ ἴσον, ἔσται καὶ ἡ GM τῇ AH ἴση. τὰ γὰρ ὑπὸ τὸ αὐτὸ ὕψος στερεὰ παραλληλεπίπεδα πρὸς ἄλληλά ἐστιν ὡς αἱ βάσεις. καὶ ἔσται ὡς ἡ EO βάσις πρὸς τὴν NI , οὕτως ἡ GM πρὸς τὴν AH , καὶ φανερόν, ὅτι

equal to solid CD , and straight-line EK to CF . Thus, solid AB also has to solid CD the cubed ratio which its corresponding side AE (has) to the corresponding side CF .



Thus, similar parallelepiped solids are to one another as the cubed ratio of their corresponding sides. (Which is) the very thing it was required to show.

Corollary

So, (it is) clear, from this, that if four straight-lines are (continuously) proportional then as the first is to the fourth, so the parallelepiped solid on the first will be to the similar, and similarly described, parallelepiped solid on the second, since the first also has to the fourth the cubed ratio that (it has) to the second.

Proposition 34[†]

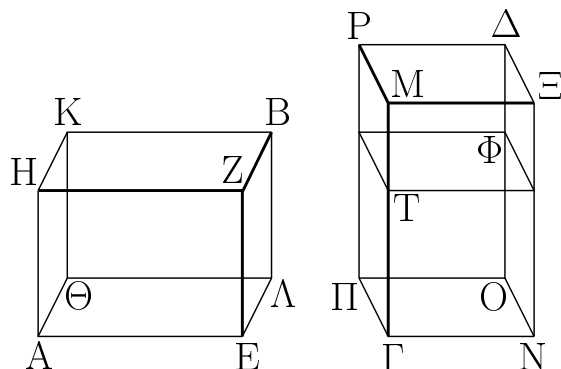
The bases of equal parallelepiped solids are reciprocally proportional to their heights. And those parallelepiped solids whose bases are reciprocally proportional to their heights are equal.

Let AB and CD be equal parallelepiped solids. I say that the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights, and (so) as base EH is to base NQ , so the height of solid CD (is) to the height of solid AB .

For, first of all, let the (straight-lines) standing up, AG , EF , LB , HK , CM , NO , PD , and QR , be at right-angles to their bases. I say that as base EH is to base NQ , so CM (is) to AG .

Therefore, if base EH is equal to base NQ , and solid AB is also equal to solid CD , CM will also be equal to AG . For parallelepiped solids of the same height are to one another as their bases [Prop. 11.32]. And as base

τῶν AB , $\Gamma\Delta$ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.



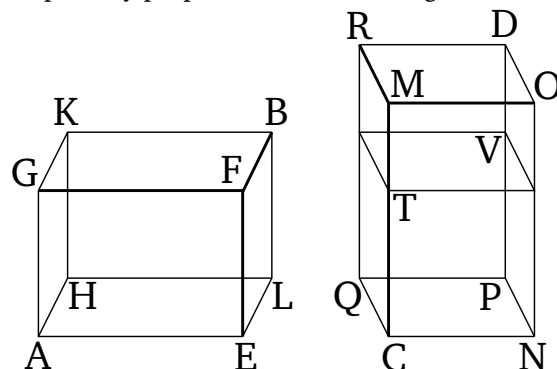
Μὴ ἔστω δὴ ἴση ἡ $E\Theta$ βάσις τῇ NI βάσει, ἀλλ' ἔστω μείζων ἡ $E\Theta$. ἔστι δὲ καὶ τὸ AB στερεὸν τῷ $\Gamma\Delta$ στερεῷ ἴσον· μείζων ἄρα ἐστὶ καὶ ἡ GM τῆς AH . κείσθω οὖν τῇ AH ἴση ἡ GT , καὶ συμπληρώσθω ἀπὸ βάσεως μὲν τῆς NI , ὕψους δὲ τοῦ GT , στερεὸν παραλληλεπίπεδον τὸ $\Phi\Gamma$. καὶ ἐπεὶ ἴσον ἐστὶ τὸ AB στερεὸν τῷ $\Gamma\Delta$ στερεῷ, ἔξωθεν δὲ τὸ $\Gamma\Phi$, τὰ δὲ ἴσα πρὸς τὸ αὐτὸ τὸν αὐτὸν ἔχει λόγον, ἔστιν ἄρα ὡς τὸ AB στερεὸν πρὸς τὸ $\Gamma\Phi$ στερεόν, οὕτως τὸ $\Gamma\Delta$ στερεὸν πρὸς τὸ $\Gamma\Phi$ στερεόν. ἀλλ' ὡς μὲν τὸ AB στερεὸν πρὸς τὸ $\Gamma\Phi$ στερεόν, οὕτως ἡ $E\Theta$ βάσις πρὸς τὴν NI βάσιν· ἰσοῦψή γὰρ τὰ AB , $\Gamma\Phi$ στερεά· ὡς δὲ τὸ $\Gamma\Delta$ στερεὸν πρὸς τὸ $\Gamma\Phi$ στερεόν, οὕτως ἡ MI βάσις πρὸς τὴν TI βάσιν καὶ ἡ GM πρὸς τὴν GT · καὶ ὡς ἄρα ἡ $E\Theta$ βάσις πρὸς τὴν NI βάσιν, οὕτως ἡ MI πρὸς τὴν GT . ἴση δὲ ἡ GT τῇ AH · καὶ ὡς ἄρα ἡ $E\Theta$ βάσις πρὸς τὴν NI βάσιν, οὕτως ἡ MI πρὸς τὴν AH . τῶν AB , $\Gamma\Delta$ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Πάλιν δὴ τῶν AB , $\Gamma\Delta$ στερεῶν παραλληλεπιπέδων ἀντιπεπονθέντων αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ $E\Theta$ βάσις πρὸς τὴν NI βάσιν, οὕτως τὸ τοῦ $\Gamma\Delta$ στερεοῦ ὕψος πρὸς τὸ τοῦ AB στερεοῦ ὕψος· λέγω, ὅτι ἴσον ἐστὶ τὸ AB στερεὸν τῷ $\Gamma\Delta$ στερεῷ.

Ἐστῶσαν [γὰρ] πάλιν αἱ ἐφεστηκυῖαι πρὸς ὀρθὰς ταῖς βάσεσιν. καὶ εἰ μὲν ἴση ἐστὶν ἡ $E\Theta$ βάσις τῇ NI βάσει, καὶ ἐστὶν ὡς ἡ $E\Theta$ βάσις πρὸς τὴν NI βάσιν, οὕτως τὸ τοῦ $\Gamma\Delta$ στερεοῦ ὕψος πρὸς τὸ τοῦ AB στερεοῦ ὕψος, ἴσον ἄρα ἐστὶ καὶ τὸ τοῦ $\Gamma\Delta$ στερεοῦ ὕψος τῷ τοῦ AB στερεοῦ ὕψει. τὰ δὲ ἐπὶ ἴσων βάσεων στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ AB στερεὸν τῷ $\Gamma\Delta$ στερεῷ.

Μὴ ἔστω δὴ ἡ $E\Theta$ βάσις τῇ NI [βάσει] ἴση, ἀλλ' ἔστω μείζων ἡ $E\Theta$ · μείζον ἄρα ἐστὶ καὶ τὸ τοῦ $\Gamma\Delta$ στερεοῦ ὕψος τοῦ τοῦ AB στερεοῦ ὕψους, τουτέστιν ἡ GM τῆς AH . κείσθω τῇ AH ἴση πάλιν ἡ GT , καὶ συμπληρώσθω ὁμοίως τὸ $\Gamma\Phi$ στερεόν. ἐπεὶ ἐστὶν ὡς ἡ $E\Theta$ βάσις πρὸς τὴν NI βάσιν, οὕτως ἡ MI πρὸς τὴν AH , ἴση δὲ ἡ AH τῇ GT ,

EH (is) to NQ , so CM will be to AG . And (so it is) clear that the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.

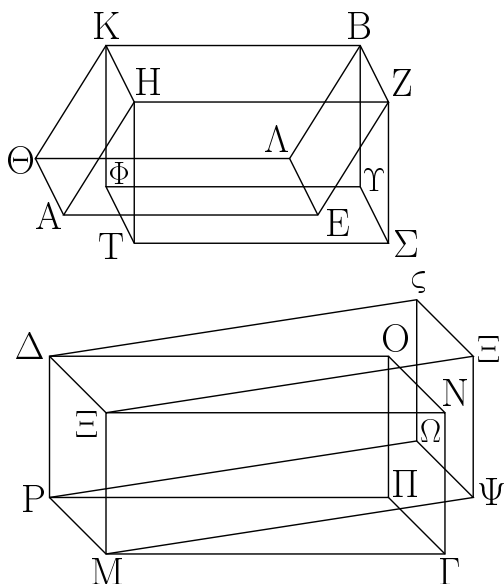


So let base EH not be equal to base NQ , but let EH be greater. And solid AB is also equal to solid CD . Thus, CM is also greater than AG . Therefore, let CT be made equal to AG . And let the parallelepiped solid VC have been completed on the base NQ , with height CT . And since solid AB is equal to solid CD , and CV (is) extrinsic (to them), and equal (magnitudes) have the same ratio to the same (magnitude) [Prop. 5.7], thus as solid AB is to solid CV , so solid CD (is) to solid CV . But, as solid AB (is) to solid CV , so base EH (is) to base NQ . For the solids AB and CV (are) of equal height [Prop. 11.32]. And as solid CD (is) to solid CV , so base MQ (is) to base TQ [Prop. 11.25], and CM to CT [Prop. 6.1]. And, thus, as base EH is to base NQ , so MC (is) to AG . And CT (is) equal to AG . And thus as base EH (is) to base NQ , so MC (is) to AG . Thus, the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.

So, again, let the bases of the parallelepiped solids AB and CD be reciprocally proportional to their heights, and let base EH be to base NQ , as the height of solid CD (is) to the height of solid AB . I say that solid AB is equal to solid CD . [For] let the (straight-lines) standing up again be at right-angles to the bases. And if base EH is equal to base NQ , and as base EH is to base NQ , so the height of solid CD (is) to the height of solid AB , the height of solid CD is thus also equal to the height of solid AB . And parallelepiped solids on equal bases, and also with the same height, are equal to one another [Prop. 11.31]. Thus, solid AB is equal to solid CD .

So, let base EH not be equal to [base] NQ , but let EH be greater. Thus, the height of solid CD is also greater than the height of solid AB , that is to say CM (greater) than AG . Let CT again be made equal to AG , and let the solid CV have been similarly completed. Since as base EH is to base NQ , so MC (is) to AG ,

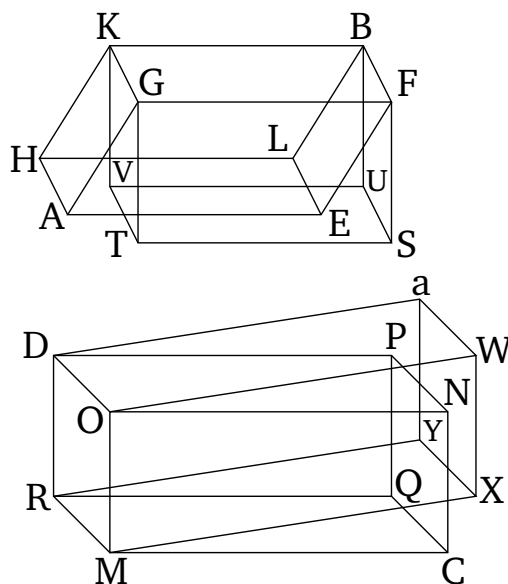
ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως ἡ ΓΜ πρὸς τὴν ΓΤ. ἀλλ' ὡς μὲν ἡ ΕΘ [βάσις] πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν· ἰσοῦψή γάρ ἐστι τὰ ΑΒ, ΓΦ στερεά· ὡς δὲ ἡ ΓΜ πρὸς τὴν ΓΤ, οὕτως ἡ τε ΜΠ βάσις πρὸς τὴν ΠΤ βάσιν καὶ τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν. καὶ ὡς ἄρα τὸ ΑΒ στερεὸν πρὸς τὸ ΓΦ στερεόν, οὕτως τὸ ΓΔ στερεὸν πρὸς τὸ ΓΦ στερεόν· ἐκάτερον ἄρα τῶν ΑΒ, ΓΔ πρὸς τὸ ΓΦ τὸν αὐτὸν ἔχει λόγον. ἴσον ἄρα ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.



Μὴ ἔστωσαν δὴ αἱ ἐφεστηκυῖαι αἱ ΖΕ, ΒΛ, ΗΑ, ΚΘ, ΕΝ, ΔΟ, ΜΓ, ΡΠ πρὸς ὀρθὰς ταῖς βάσεσιν αὐτῶν, καὶ ἤχθωσαν ἀπὸ τῶν Ζ, Η, Β, Κ, Ξ, Μ, Ρ, Δ σημείων ἐπὶ τὰ διὰ τῶν ΕΘ, ΝΠ ἐπίπεδα κάθετοι καὶ συμβαλλέτωσαν τοῖς ἐπιπέδοις κατὰ τὰ Σ, Τ, Υ, Φ, Χ, Ψ, Ω, ς, καὶ συμπληρώσθω τὰ ΖΦ, ΞΩ στερεά· λέγω, ὅτι καὶ οὕτως ἴσων ὄντων τῶν ΑΒ, ΓΔ στερεῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἐστὶν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος.

Ἐπεὶ ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ, ἀλλὰ τὸ μὲν ΑΒ τῷ ΒΤ ἐστὶν ἴσον· ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως εἰσι τῆς ΖΚ καὶ ὑπὸ τὸ αὐτὸ ὕψος· τὸ δὲ ΓΔ στερεὸν τῷ ΔΨ ἐστὶν ἴσον· ἐπὶ τε γὰρ πάλιν τῆς αὐτῆς βάσεως εἰσι τῆς ΡΞ καὶ ὑπὸ τὸ αὐτὸ ὕψος· καὶ τὸ ΒΤ ἄρα στερεὸν τῷ ΔΨ στερεῷ ἴσον ἐστίν. ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. ἴση δὲ ἡ μὲν ΖΚ βάσις τῇ ΕΘ βάσει, ἡ δὲ ΞΡ βάσις τῇ ΝΠ βάσει· ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. τὰ δ' αὐτὰ ὕψη ἐστὶ τῶν ΔΨ, ΒΤ στερεῶν καὶ τῶν ΔΓ, ΒΑ· ἔστιν ἄρα ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ

and AG (is) equal to CT , thus as base EH (is) to base NQ , so CM (is) to CT . But, as [base] EH (is) to base NQ , so solid AB (is) to solid CV . For solids AB and CV are of equal heights [Prop. 11.32]. And as CM (is) to CT , so (is) base MQ to base QT [Prop. 6.1], and solid CD to solid CV [Prop. 11.25]. And thus as solid AB (is) to solid CV , so solid CD (is) to solid CV . Thus, AB and CD each have the same ratio to CV . Thus, solid AB is equal to solid CD [Prop. 5.9].



So, let the (straight-lines) standing up, FE , BL , GA , KH , ON , DP , MC , and RQ , not be at right-angles to their bases. And let perpendiculars have been drawn to the planes through EH and NQ from points F , G , B , K , O , M , R , and D , and let them have joined the planes at (points) S , T , U , V , W , X , Y , and a (respectively). And let the solids FV and OY have been completed. In this case, also, I say that the solids AB and CD being equal, their bases are reciprocally proportional to their heights, and (so) as base EH is to base NQ , so the height of solid CD (is) to the height of solid AB .

Since solid AB is equal to solid CD , but AB is equal to BT . For they are on the same base FK , and (have) the same height [Props. 11.29, 11.30]. And solid CD is equal to DX . For, again, they are on the same base RO , and (have) the same height [Props. 11.29, 11.30]. Solid BT is thus also equal to solid DX . Thus, as base FK (is) to base OR , so the height of solid DX (is) to the height of solid BT (see first part of proposition). And base FK (is) equal to base EH , and base OR to NQ . Thus, as base EH is to base NQ , so the height of solid DX (is) to

βάσιν, οὕτως τὸ τοῦ ΔΓ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος. τῶν ΑΒ, ΓΔ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Πάλιν δὴ τῶν ΑΒ, ΓΔ στερεῶν παραλληλεπιπέδων ἀντιπεπονθέντων αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒ στερεὸν τῷ ΓΔ στερεῷ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἐστὶν ὡς ἡ ΕΘ βάσις πρὸς τὴν ΝΠ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος, ἴση δὲ ἡ μὲν ΕΘ βάσις τῇ ΖΚ βάσει, ἡ δὲ ΝΠ τῇ ΞΡ, ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΓΔ στερεοῦ ὕψος πρὸς τὸ τοῦ ΑΒ στερεοῦ ὕψος. τὰ δ' αὐτὰ ὕψη ἐστὶ τῶν ΑΒ, ΓΔ στερεῶν καὶ τῶν ΒΤ, ΔΨ· ἔστιν ἄρα ὡς ἡ ΖΚ βάσις πρὸς τὴν ΞΡ βάσιν, οὕτως τὸ τοῦ ΔΨ στερεοῦ ὕψος πρὸς τὸ τοῦ ΒΤ στερεοῦ ὕψος. τῶν ΒΤ, ΔΨ ἄρα στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· ἴσον ἄρα ἐστὶ τὸ ΒΤ στερεὸν τῷ ΔΨ στερεῷ. ἀλλὰ τὸ μὲν ΒΤ τῷ ΒΑ ἴσον ἐστίν· ἐπὶ τε γὰρ τῆς αὐτῆς βάσεως [εἰσι] τῆς ΖΚ καὶ ὑπὸ τὸ αὐτὸ ὕψος. τὸ δὲ ΔΨ στερεὸν τῷ ΔΓ στερεῷ ἴσον ἐστίν. καὶ τὸ ΑΒ ἄρα στερεὸν τῷ ΓΔ στερεῷ ἐστὶν ἴσον· ὅπερ ἔδει δεῖξαι.

the height of solid BT . And solids DX , BT are the same height as (solids) DC , BA (respectively). Thus, as base EH is to base NQ , so the height of solid DC (is) to the height of solid AB . Thus, the bases of the parallelepiped solids AB and CD are reciprocally proportional to their heights.

So, again, let the bases of the parallelepiped solids AB and CD be reciprocally proportional to their heights, and (so) let base EH be to base NQ , as the height of solid CD (is) to the height of solid AB . I say that solid AB is equal to solid CD .

For, with the same construction (as before), since as base EH is to base NQ , so the height of solid CD (is) to the height of solid AB , and base EH (is) equal to base FK , and NQ to OR , thus as base FK is to base OR , so the height of solid CD (is) to the height of solid AB . And solids AB , CD are the same height as (solids) BT , DX (respectively). Thus, as base FK is to base OR , so the height of solid DX (is) to the height of solid BT . Thus, the bases of the parallelepiped solids BT and DX are reciprocally proportional to their heights. Thus, solid BT is equal to solid DX (see first part of proposition). But, BT is equal to BA . For [they are] on the same base FK , and (have) the same height [Props. 11.29, 11.30]. And solid DX is equal to solid DC [Props. 11.29, 11.30]. Thus, solid AB is also equal to solid CD . (Which is) the very thing it was required to show.

† This proposition assumes that (a) if two parallelepipeds are equal, and have equal bases, then their heights are equal, and (b) if the bases of two equal parallelepipeds are unequal, then that solid which has the lesser base has the greater height.

λε'.

Proposition 35

Ἐάν ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι, ἐπὶ δὲ τῶν κορυφῶν αὐτῶν μετέωροι εὐθεῖαι ἐπισταθῶσιν ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, ἐπὶ δὲ τῶν μετέωρων ληφθῇ τυχόντα σημεία, καὶ ἀπ' αὐτῶν ἐπὶ τὰ ἐπίπεδα, ἐν οἷς εἰσιν αἱ ἐξ ἀρχῆς γωνίαι, κάθετοι ἄχθῶσιν, ἀπὸ δὲ τῶν γενομένων σημείων ἐν τοῖς ἐπίπεδοις ἐπὶ τὰς ἐξ ἀρχῆς γωνίας ἐπιζευχθῶσιν εὐθεῖαι, ἴσας γωνίας περιέξουσιν μετὰ τῶν μετέωρων.

Ἐστωσαν δύο γωνίαι εὐθύγραμμοι ἴσαι αἱ ὑπὸ ΒΑΓ, ΕΔΖ, ἀπὸ δὲ τῶν Α, Δ σημείων μετέωροι εὐθεῖαι ἐφεστώσαν αἱ ΑΗ, ΔΜ ἴσας γωνίας περιέχουσιν μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, τὴν μὲν ὑπὸ ΜΔΕ τῇ ὑπὸ ΗΑΒ, τὴν δὲ ὑπὸ ΜΔΖ τῇ ὑπὸ ΗΑΓ, καὶ εἰλήφθω ἐπὶ τῶν ΑΗ, ΔΜ τυχόντα σημεία τὰ Η, Μ, καὶ ἤχθωσαν ἀπὸ τῶν Η, Μ σημείων ἐπὶ τὰ διὰ τῶν ΒΑΓ, ΕΔΖ ἐπίπεδα κάθετοι αἱ ΗΛ, ΜΝ, καὶ συμβαλλέτωσαν τοῖς ἐπίπεδοις κατὰ τὰ Α, Ν, καὶ ἐπεζεύχθωσαν αἱ ΛΑ, ΝΔ· λέγω, ὅτι ἴση ἐστὶν ἡ ὑπὸ ΗΑΛ γωνία τῇ ὑπὸ ΜΔΝ γωνία.

If there are two equal plane angles, and raised straight-lines are stood on the apexes of them, containing equal angles respectively with the original straight-lines (forming the angles), and random points are taken on the raised (straight-lines), and perpendiculars are drawn from them to the planes in which the original angles are, and straight-lines are joined from the points created in the planes to the (vertices of the) original angles, then they will enclose equal angles with the raised (straight-lines).

Let BAC and EDF be two equal rectilinear angles. And let the raised straight-lines AG and DM have been stood on points A and D , containing equal angles respectively with the original straight-lines. (That is) MDE (equal) to GAB , and MDF (to) GAC . And let the random points G and M have been taken on AG and DM (respectively). And let the GL and MN have been drawn from points G and M perpendicular to the planes through

καὶ ἡ $ΑΓ$ τῇ $ΔΖ$ ἴση· δύο δὲ αἱ $ΑΓ$, $ΓΚ$ δυοὶ ταῖς $ΔΖ$, $ΖΝ$ ἴσαι εἰσὶν· καὶ ὀρθὰς γωνίας περιέχουσιν. βάσις ἄρα ἡ $ΑΚ$ βάσει τῇ $ΔΝ$ ἴση ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ $ΑΘ$ τῇ $ΔΜ$, ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς $ΑΘ$ τῷ ἀπὸ τῆς $ΔΜ$. ἀλλὰ τῷ μὲν ἀπὸ τῆς $ΑΘ$ ἴσα ἐστὶ τὰ ἀπὸ τῶν $ΑΚ$, $ΚΘ$ · ὀρθὴ γὰρ ἡ ὑπὸ $ΑΚΘ$ · τῷ δὲ ἀπὸ τῆς $ΔΜ$ ἴσα τὰ ἀπὸ τῶν $ΔΝ$, $ΝΜ$ · ὀρθὴ γὰρ ἡ ὑπὸ $ΔΝΜ$ · τὰ ἄρα ἀπὸ τῶν $ΑΚ$, $ΚΘ$ ἴσα ἐστὶ τοῖς ἀπὸ τῶν $ΔΝ$, $ΝΜ$, ὧν τὸ ἀπὸ τῆς $ΑΚ$ ἴσον ἐστὶ τῷ ἀπὸ τῆς $ΔΝ$ · λοιπὸν ἄρα τὸ ἀπὸ τῆς $ΚΘ$ ἴσον ἐστὶ τῷ ἀπὸ τῆς $ΝΜ$ · ἴση ἄρα ἡ $ΘΚ$ τῇ $ΜΝ$. καὶ ἐπεὶ δύο αἱ $ΘΑ$, $ΑΚ$ δυοὶ ταῖς $ΜΔ$, $ΔΝ$ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ βάσις ἡ $ΘΚ$ βάσει τῇ $ΜΝ$ ἐδείχθη ἴση, γωνία ἄρα ἡ ὑπὸ $ΘΑΚ$ γωνία τῇ ὑπὸ $ΜΔΝ$ ἐστὶν ἴση.

Ἐὰν ἄρα ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι καὶ τὰ ἐξῆς τῆς προτάσεως [ὅπερ ἔδει δεῖξαι].

Thus, angle ACB (is) equal to DFE . And the right-angle ACK is also equal to the right-angle DFN . Thus, the remainder BCK is equal to the remainder EFN . So, for the same (reasons), CBK is also equal to FEN . So, BCK and EFN are two triangles having two angles equal to two angles, respectively, and one side equal to one side—(namely), that by the equal angles—(that is), BC (equal) to EF . Thus, they will also have the remaining sides equal to the remaining sides (respectively) [Prop. 1.26]. Thus, CK is equal to FN . And AC (is) also equal to DF . So, the two (straight-lines) AC and CK are equal to the two (straight-lines) DF and FN (respectively). And they enclose right-angles. Thus, base AK is equal to base DN [Prop. 1.4]. And since AH is equal to DM , the (square) on AH is also equal to the (square) on DM . But, the the (sum of the squares) on AK and KH is equal to the (square) on AH . For angle AKH (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on DN and NM (is) equal to the square on DM . For angle DNM (is) a right-angle [Prop. 1.47]. Thus, the (sum of the squares) on AK and KH is equal to the (sum of the squares) on DN and NM , of which the (square) on AK is equal to the (square) on DN . Thus, the remaining (square) on KH is equal to the (square) on NM . Thus, HK (is) equal to MN . And since the two (straight-lines) HA and AK are equal to the two (straight-lines) MD and DN , respectively, and base HK was shown (to be) equal to base MN , angle HAK is thus equal to angle MDN [Prop. 1.8].

Thus, if there are two equal plane angles, and so on of the proposition. [(Which is) the very thing it was required to show].

Πόρισμα.

Ἐκ δὲ τούτου φανερόν, ὅτι, ἐὰν ὦσι δύο γωνίαι ἐπίπεδοι ἴσαι, ἐπισταθῶσι δὲ ἐπ' αὐτῶν μετέωροι εὐθεῖαι ἴσαι ἴσας γωνίας περιέχουσαι μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἑκατέραν ἑκατέρᾳ, αἱ ἀπ' αὐτῶν κάθεται ἀγόμεναι ἐπὶ τὰ ἐπίπεδα, ἐν οἷς εἰσιν αἱ ἐξ ἀρχῆς γωνίαι, ἴσαι ἀλλήλαις εἰσὶν. ὅπερ ἔδει δεῖξαι.

λζ'.

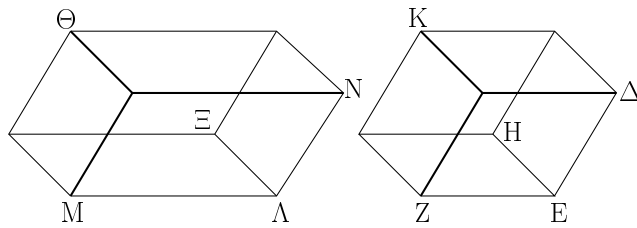
Ἐὰν τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, τὸ ἐκ τῶν τριῶν στερεὸν παραλληλεπίπεδον ἴσον ἐστὶ τῷ ἀπὸ τῆς μέσης στερεῷ παραλληλεπίπεδῳ ἰσοπλευρῷ μὲν, ἰσογωνίῳ δὲ τῷ προειρημένῳ.

Corollary

So, it is clear, from this, that if there are two equal plane angles, and equal raised straight-lines are stood on them (at their apexes), containing equal angles respectively with the original straight-lines (forming the angles), then the perpendiculars drawn from (the raised ends of) them to the planes in which the original angles lie are equal to one another. (Which is) the very thing it was required to show.

Proposition 36

If three straight-lines are (continuously) proportional then the parallelepiped solid (formed) from the three (straight-lines) is equal to the equilateral parallelepiped solid on the middle (straight-line which is) equiangular to the aforementioned (parallelepiped solid).



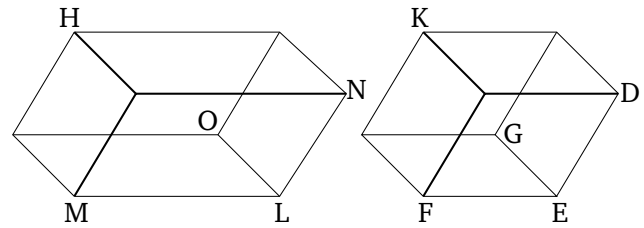
A _____
 B _____
 Γ _____

Ἐστωσαν τρεῖς εὐθεῖαι ἀνάλογον αἱ A, B, Γ , ὥς ἡ A πρὸς τὴν B , οὕτως ἡ B πρὸς τὴν Γ . λέγω, ὅτι τὸ ἐκ τῶν A, B, Γ στερεὸν ἴσον ἐστὶ τῷ ἀπὸ τῆς B στερεῷ ἰσοπλευρῷ μὲν, ἰσογωνίῳ δὲ τῷ προειρημένῳ.

Ἐκκείσθω στερεὰ γωνία ἡ πρὸς τῷ E περιεχομένη ὑπὸ τῶν ὑπὸ $\Delta E H, H E Z, Z E \Delta$, καὶ κείσθω τῇ μὲν B ἴση ἐκάστη τῶν $\Delta E, H E, E Z$, καὶ συμπληρώσθω τὸ $E K$ στερεὸν παραλληλεπίπεδον, τῇ δὲ A ἴση ἡ ΛM , καὶ συνεστάτω πρὸς τῇ ΛM εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Λ τῇ πρὸς τῷ E στερεᾷ γωνίᾳ ἴση στερεὰ γωνία ἡ περιεχομένη ὑπὸ τῶν $\Lambda \Lambda \Xi, \Xi \Lambda M, M \Lambda N$, καὶ κείσθω τῇ μὲν B ἴση ἡ $\Lambda \Xi$, τῇ δὲ Γ ἴση ἡ ΛN . καὶ ἐπεὶ ἐστὶν ὥς ἡ A πρὸς τὴν B , οὕτως ἡ B πρὸς τὴν Γ , ἴση δὲ ἡ μὲν A τῇ ΛM , ἡ δὲ B ἐκατέρᾳ τῶν $\Lambda \Xi, E \Delta$, ἡ δὲ Γ τῇ ΛN , ἐστὶν ἄρα ὥς ἡ ΛM πρὸς τὴν $E Z$, οὕτως ἡ ΔE πρὸς τὴν ΛN . καὶ περὶ ἴσας γωνίας τὰς ὑπὸ $\Lambda M, \Delta E Z$ αἱ πλευραὶ ἀντιπεπόνθασιν· ἴσον ἄρα ἐστὶ τὸ $M N$ παραλληλόγραμμον τῷ ΔZ παραλληλογράμμῳ. καὶ ἐπεὶ δύο γωνίαι ἐπίπεδοι εὐθύγραμμοι ἴσαι εἰσὶν αἱ ὑπὸ $\Delta E Z, \Lambda M$, καὶ ἐπ' αὐτῶν μετέωροι εὐθεῖαι ἐφεστᾶσιν αἱ $\Lambda \Xi, E H$ ἴσαι τε ἀλλήλαις καὶ ἴσας γωνίας περιέχουσιν μετὰ τῶν ἐξ ἀρχῆς εὐθειῶν ἐκατέραν ἐκατέρᾳ, αἱ ἄρα ἀπὸ τῶν H, Ξ σημείων κάθετοι ἀγόμεναι ἐπὶ τὰ διὰ τῶν $\Lambda M, \Delta E Z$ ἐπίπεδα ἴσαι ἀλλήλαις εἰσὶν· ὥστε τὰ $\Lambda \Theta, E K$ στερεὰ ὑπὸ τὸ αὐτὸ ὕψος ἐστίν. τὰ δὲ ἐπὶ ἴσων βάσεων στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ $\Theta \Lambda$ στερεὸν τῷ $E K$ στερεῷ. καὶ ἐστὶ τὸ μὲν $\Lambda \Theta$ τὸ ἐκ τῶν A, B, Γ στερεόν, τὸ δὲ $E K$ τὸ ἀπὸ τῆς B στερεόν· τὸ ἄρα ἐκ τῶν A, B, Γ στερεὸν παραλληλεπίπεδον ἴσον ἐστὶ τῷ ἀπὸ τῆς B στερεῷ ἰσοπλευρῷ μὲν, ἰσογωνίῳ δὲ τῷ προειρημένῳ· ὃπερ εἶδει δεῖξαι.

λζ'.

Ἐὰν τέσσαρες εὐθεῖαι ἀνάλογον ᾤσιν, καὶ τὰ ἀπ' αὐτῶν



A _____
 B _____
 C _____

Let A, B , and C be three (continuously) proportional straight-lines, (such that) as A (is) to B , so B (is) to C . I say that the (parallelepiped) solid (formed) from A, B , and C is equal to the equilateral solid on B (which is) equiangular with the aforementioned (solid).

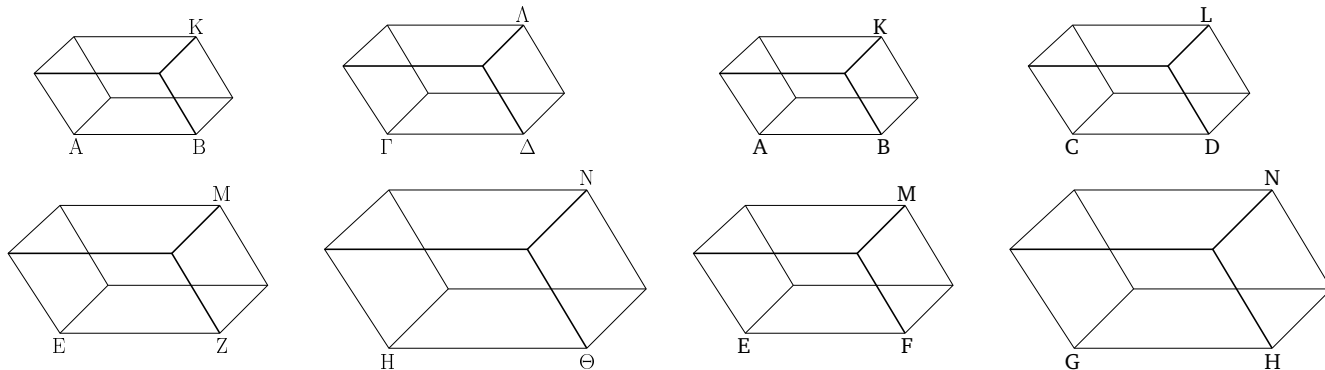
Let the solid angle at E , contained by DEG, GEF , and FED , be set out. And let DE, GE , and EF each be made equal to B . And let the parallelepiped solid $E K$ have been completed. And (let) LM (be made) equal to A . And let the solid angle contained by NLO, OLM , and MLN have been constructed on the straight-line LM , and at the point L on it, (so as to be) equal to the solid angle E [Prop. 11.23]. And let LO be made equal to B , and LN equal to C . And since as A (is) to B , so B (is) to C , and A (is) equal to LM , and B to each of LO and ED , and C to LN , thus as LM (is) to EF , so DE (is) to LN . And (so) the sides around the equal angles NLM and DEF are reciprocally proportional. Thus, parallelogram MN is equal to parallelogram DF [Prop. 6.14]. And since the two plane rectilinear angles DEF and NLM are equal, and the raised straight-lines stood on them (at their apexes), LO and EG , are equal to one another, and contain equal angles respectively with the original straight-lines (forming the angles), the perpendiculars drawn from points G and O to the planes through NLM and DEF (respectively) are thus equal to one another [Prop. 11.35 corr.]. Thus, the solids LH and $E K$ (have) the same height. And parallelepiped solids on equal bases, and with the same height, are equal to one another [Prop. 11.31]. Thus, solid HL is equal to solid $E K$. And LH is the solid (formed) from A, B , and C , and $E K$ the solid on B . Thus, the parallelepiped solid (formed) from A, B , and C is equal to the equilateral solid on B (which is) equiangular with the aforementioned (solid). (Which is) the very thing it was required to show.

Proposition 37[†]

If four straight-lines are proportional then the similar,

στερεὰ παραλληλεπίπεδα ὁμοία τε καὶ ὁμοίως ἀναγραφόμενα ἀνάλογον ἔσται· καὶ ἐὰν τὰ ἀπ' αὐτῶν στερεὰ παραλληλεπίπεδα ὁμοία τε καὶ ὁμοίως ἀναγραφόμενα ἀνάλογον ᾖ, καὶ αὐταὶ αἱ εὐθεῖαι ἀνάλογον ἔσονται.

and similarly described, parallelepiped solids on them will also be proportional. And if the similar, and similarly described, parallelepiped solids on them are proportional then the straight-lines themselves will be proportional.



Ἐστωσαν τέσσαρες εὐθεῖαι ἀνάλογον αἱ AB , $\Gamma\Delta$, EZ , $H\Theta$, ὡς ἡ AB πρὸς τὴν $\Gamma\Delta$, οὕτως ἡ EZ πρὸς τὴν $H\Theta$, καὶ ἀναγεγράφθωσαν ἀπὸ τῶν AB , $\Gamma\Delta$, EZ , $H\Theta$ ὁμοία τε καὶ ὁμοίως κείμενα στερεὰ παραλληλεπίπεδα τὰ KA , $\Lambda\Gamma$, ME , NH · λέγω, ὅτι ἔστιν ὡς τὸ KA πρὸς τὸ $\Lambda\Gamma$, οὕτως τὸ ME πρὸς τὸ NH .

Ἐπεὶ γὰρ ὁμοίον ἐστὶ τὸ KA στερεὸν παραλληλεπίπεδον τῷ $\Lambda\Gamma$, τὸ KA ἄρα πρὸς τὸ $\Lambda\Gamma$ τριπλασίονα λόγον ἔχει ἢ περ ἡ AB πρὸς τὴν $\Gamma\Delta$. διὰ τὰ αὐτὰ δὴ καὶ τὸ ME πρὸς τὸ NH τριπλασίονα λόγον ἔχει ἢ περ ἡ EZ πρὸς τὴν $H\Theta$. καὶ ἐστὶν ὡς ἡ AB πρὸς τὴν $\Gamma\Delta$, οὕτως ἡ EZ πρὸς τὴν $H\Theta$. καὶ ὡς ἄρα τὸ AK πρὸς τὸ $\Lambda\Gamma$, οὕτως τὸ ME πρὸς τὸ NH .

Ἀλλὰ δὴ ἔστω ὡς τὸ AK στερεὸν πρὸς τὸ $\Lambda\Gamma$ στερεόν, οὕτως τὸ ME στερεὸν πρὸς τὸ NH · λέγω, ὅτι ἔστιν ὡς ἡ AB εὐθεῖα πρὸς τὴν $\Gamma\Delta$, οὕτως ἡ EZ πρὸς τὴν $H\Theta$.

Ἐπεὶ γὰρ πάλιν τὸ KA πρὸς τὸ $\Lambda\Gamma$ τριπλασίονα λόγον ἔχει ἢ περ ἡ AB πρὸς τὴν $\Gamma\Delta$, ἔχει δὲ καὶ τὸ ME πρὸς τὸ NH τριπλασίονα λόγον ἢ περ ἡ EZ πρὸς τὴν $H\Theta$, καὶ ἐστὶν ὡς τὸ KA πρὸς τὸ $\Lambda\Gamma$, οὕτως τὸ ME πρὸς τὸ NH , καὶ ὡς ἄρα ἡ AB πρὸς τὴν $\Gamma\Delta$, οὕτως ἡ EZ πρὸς τὴν $H\Theta$.

Ἐὰν ἄρα τέσσαρες εὐθεῖαι ἀνάλογον ᾖσι καὶ τὰ ἐξῆς τῆς προτάσεως· ὅπερ ἔδει δεῖξαι.

Let AB , CD , EF , and GH , be four proportional straight-lines, (such that) as AB (is) to CD , so EF (is) to GH . And let the similar, and similarly laid out, parallelepiped solids KA , LC , ME and NG have been described on AB , CD , EF , and GH (respectively). I say that as KA is to LC , so ME (is) to NG .

For since the parallelepiped solid KA is similar to LC , KA thus has to LC the cubed ratio that AB (has) to CD [Prop. 11.33]. So, for the same (reasons), ME also has to NG the cubed ratio that EF (has) to GH [Prop. 11.33]. And since as AB is to CD , so EF (is) to GH , thus, also, as AK (is) to LC , so ME (is) to NG .

And so let solid AK be to solid LC , as solid ME (is) to NG . I say that as straight-line AB is to CD , so EF (is) to GH .

For, again, since KA has to LC the cubed ratio that AB (has) to CD [Prop. 11.33], and ME also has to NG the cubed ratio that EF (has) to GH [Prop. 11.33], and as KA is to LC , so ME (is) to NG , thus, also, as AB (is) to CD , so EF (is) to GH .

Thus, if four straight-lines are proportional, and so on of the proposition. (Which is) the very thing it was required to show.

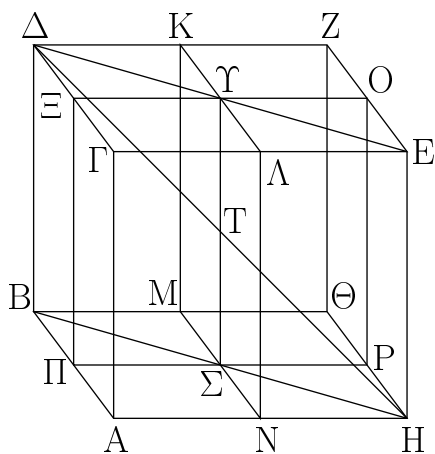
† This proposition assumes that if two ratios are equal then the cube of the former is also equal to the cube of the latter, and *vice versa*.

λη'.

Proposition 38

Ἐὰν κύβου τῶν ἀπεναντίον ἐπιπέδων αἱ πλευραὶ δίχα τμηθῶσιν, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβληθῇ, ἡ κοινὴ τομὴ τῶν ἐπιπέδων καὶ ἡ τοῦ κύβου διάμετρος δίχα τέμνουσιν ἀλλήλας.

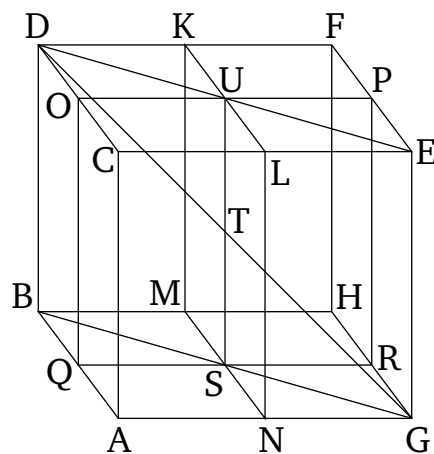
If the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half.



Κύβου γάρ τοῦ ΑΖ τῶν ἀπεναντίον ἐπιπέδων τῶν ΓΖ, ΑΘ αἱ πλευραὶ δίχα τετμήσθωσαν κατὰ τὰ Κ, Λ, Μ, Ν, Ξ, Π, Ο, Ρ σημεῖα, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβεβλήσθω τὰ ΚΝ, ΞΡ, κοινὴ δὲ τομὴ τῶν ἐπιπέδων ἔστω ἡ ΥΣ, τοῦ δὲ ΑΖ κύβου διαγώνιος ἡ ΔΗ. λέγω, ὅτι ἴση ἐστὶν ἡ μὲν ΥΤ τῇ ΤΣ, ἡ δὲ ΔΤ τῇ ΤΗ.

Ἐπεζεύχθωσαν γάρ αἱ ΔΥ, ΥΕ, ΒΣ, ΣΗ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΔΞ τῇ ΟΕ, αἱ ἐναλλάξ γωνίαι αἱ ὑπὸ ΔΞΥ, ΥΟΕ ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν ΔΞ τῇ ΟΕ, ἡ δὲ ΞΥ τῇ ΥΟ, καὶ γωνίας ἴσας περιέχουσιν, βάσεις ἄρα ἡ ΔΥ τῇ ΥΕ ἐστὶν ἴση, καὶ τὸ ΔΞΥ τρίγωνον τῷ ΟΥΕ τριγώνῳ ἐστὶν ἴσον καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι. ἴση ἄρα ἡ ὑπὸ ΞΥΔ γωνία τῇ ὑπὸ ΟΥΕ γωνίᾳ. διὰ δὲ τοῦτο εὐθεῖα ἐστὶν ἡ ΔΥΕ. διὰ τὰ αὐτὰ δὲ καὶ ΒΣΗ εὐθεῖα ἐστὶν, καὶ ἴση ἡ ΒΣ τῇ ΣΗ. καὶ ἐπεὶ ἡ ΓΑ τῇ ΔΒ ἴση ἐστὶ καὶ παράλληλος, ἀλλὰ ἡ ΓΑ καὶ τῇ ΕΗ ἴση τέ ἐστὶ καὶ παράλληλος, καὶ ἡ ΔΒ ἄρα τῇ ΕΗ ἴση τέ ἐστὶ καὶ παράλληλος. καὶ ἐπιζευγνύουσιν αὐτὰς εὐθεῖαι αἱ ΔΕ, ΒΗ· παράλληλος ἄρα ἐστὶν ἡ ΔΕ τῇ ΒΗ. ἴση ἄρα ἡ μὲν ὑπὸ ΕΔΤ γωνία τῇ ὑπὸ ΒΗΤ· ἐναλλάξ γάρ· ἡ δὲ ὑπὸ ΔΤΥ τῇ ὑπὸ ΗΤΣ. δύο δὲ τρίγωνα ἐστὶ τὰ ΔΤΥ, ΗΤΣ τὰς δύο γωνίας ταῖς δυσὶ γωνίαις ἴσας ἔχοντα καὶ μίαν πλευρὰν μιᾷ πλευρᾷ ἴσην τὴν ὑποτείνουσαν ὑπὸ μίαν τῶν ἴσων γωνιῶν τὴν ΔΥ τῇ ΗΣ· ἡμίσειαι γάρ εἰσι τῶν ΔΕ, ΒΗ· καὶ τὰς λοιπὰς πλευρὰς ταῖς λοιπαῖς πλευραῖς ἴσας ἔξει. ἴση ἄρα ἡ μὲν ΔΤ τῇ ΤΗ, ἡ δὲ ΥΤ τῇ ΤΣ.

Ἐὰν ἄρα κύβου τῶν ἀπεναντίον ἐπιπέδων αἱ πλευραὶ δίχα τμηθῶσιν, διὰ δὲ τῶν τομῶν ἐπίπεδα ἐκβληθῇ, ἡ κοινὴ τομὴ τῶν ἐπιπέδων καὶ ἡ τοῦ κύβου διάμετρος δίχα τέμνουσιν ἀλλήλας· ὅπερ ἔδει δεῖξαι.



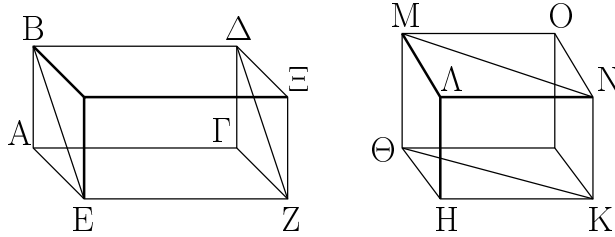
For let the opposite planes CF and AH of the cube AF have been cut in half at the points K, L, M, N, O, Q, P , and R . And let the planes KN and OR have been produced through the pieces. And let US be the common section of the planes, and DG the diameter of cube AF . I say that UT is equal to TS , and DT to TG .

For let DU, UE, BS , and SG have been joined. And since DO is parallel to PE , the alternate angles DOU and UPE are equal to one another [Prop. 1.29]. And since DO is equal to PE , and OU to UP , and they contain equal angles, base DU is thus equal to base UE , and triangle DOU is equal to triangle PUE , and the remaining angles (are) equal to the remaining angles [Prop. 1.4]. Thus, angle ODU (is) equal to angle PUE . So, for this (reason), DUE is a straight-line [Prop. 1.14]. So, for the same (reason), BSG is also a straight-line, and BS equal to SG . And since CA is equal and parallel to DB , but CA is also equal and parallel to EG , DB is thus also equal and parallel to EG [Prop. 11.9]. And the straight-lines DE and BG join them. DE is thus parallel to BG [Prop. 1.33]. Thus, angle EDT (is) equal to BGT . For (they are) alternate [Prop. 1.29]. And (angle) DTU (is equal) to GTS [Prop. 1.15]. So, DTU and GTS are two triangles having two angles equal to two angles, and one side equal to one side—(namely), that subtended by one of the equal angles—(that is), DU (equal) to GS . For they are halves of DE and BG (respectively). (Thus), they will also have the remaining sides equal to the remaining sides [Prop. 1.26]. Thus, DT (is) equal to TG , and UT to TS .

Thus, if the sides of the opposite planes of a cube are cut in half, and planes are produced through the pieces, then the common section of the (latter) planes and the diameter of the cube cut one another in half. (Which is) the very thing it was required to show.

λθ'.

Ἐάν ᾗ δύο πρίσματα ἰσοῦψῃ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ᾗ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἔσται τὰ πρίσματα.



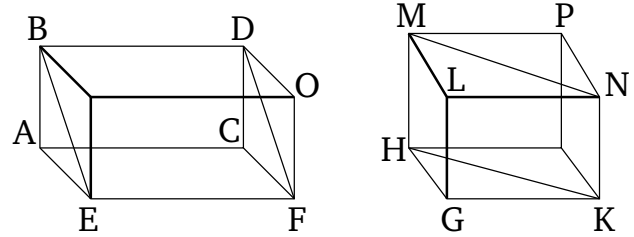
Ἐστω δύο πρίσματα ἰσοῦψῃ τὰ ΑΒΓΔΕΖ, ΗΘΚΛΜΝ, καὶ τὸ μὲν ἔχέτω βάσιν τὸ ΑΖ παραλληλόγραμμον, τὸ δὲ τὸ ΗΘΚ τρίγωνον, διπλάσιον δὲ ἔστω τὸ ΑΖ παραλληλόγραμμον τοῦ ΗΘΚ τριγώνου· λέγω, ὅτι ἴσον ἐστὶ τὸ ΑΒΓΔΕΖ πρίσμα τῷ ΗΘΚΛΜΝ πρίσματι.

Συμπεπληρώσθω γὰρ τὰ ΑΞ, ΗΟ στερεά. ἐπεὶ διπλάσιόν ἐστι τὸ ΑΖ παραλληλόγραμμον τοῦ ΗΘΚ τριγώνου, ἔστι δὲ καὶ τὸ ΘΚ παραλληλόγραμμον διπλάσιον τοῦ ΗΘΚ τριγώνου, ἴσον ἄρα ἐστὶ τὸ ΑΖ παραλληλόγραμμον τῷ ΘΚ παραλληλογράμῳ. τὰ δὲ ἐπὶ ἴσων βάσεων ὄντα στερεὰ παραλληλεπίπεδα καὶ ὑπὸ τὸ αὐτὸ ὕψος ἴσα ἀλλήλοις ἐστίν· ἴσον ἄρα ἐστὶ τὸ ΑΞ στερεὸν τῷ ΗΟ στερεῷ. καὶ ἐστὶ τοῦ μὲν ΑΞ στερεοῦ ἡμισυ τὸ ΑΒΓΔΕΖ πρίσμα, τοῦ δὲ ΗΟ στερεοῦ ἡμισυ τὸ ΗΘΚΛΜΝ πρίσμα· ἴσον ἄρα ἐστὶ τὸ ΑΒΓΔΕΖ πρίσμα τῷ ΗΘΚΛΜΝ πρίσματι.

Ἐάν ἄρα ᾗ δύο πρίσματα ἰσοῦψῃ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ᾗ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἔσται τὰ πρίσματα· ὅπερ ἔδει δεῖξαι.

Proposition 39

If there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms will be equal.



Let $ABCDEF$ and $GHKLMN$ be two equal height prisms, and let the former have the parallelogram AF , and the latter the triangle GHK , as a base. And let parallelogram AF be twice triangle GHK . I say that prism $ABCDEF$ is equal to prism $GHKLMN$.

For let the solids AO and GP have been completed. Since parallelogram AF is double triangle GHK , and parallelogram HK is also double triangle GHK [Prop. 1.34], parallelogram AF is thus equal to parallelogram HK . And parallelepiped solids which are on equal bases, and (have) the same height, are equal to one another [Prop. 11.31]. Thus, solid AO is equal to solid GP . And prism $ABCDEF$ is half of solid AO , and prism $GHKLMN$ half of solid GP [Prop. 11.28]. Prism $ABCDEF$ is thus equal to prism $GHKLMN$.

Thus, if there are two equal height prisms, and one has a parallelogram, and the other a triangle, (as a) base, and the parallelogram is double the triangle, then the prisms are equal. (Which is) the very thing it was required to show.

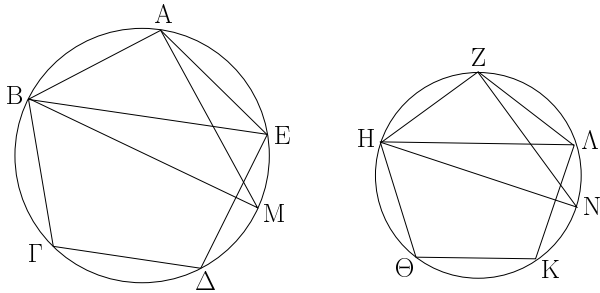
ELEMENTS BOOK 12

Proportional Stereometry[†]

[†]The novel feature of this book is the use of the so-called *method of exhaustion* (see Prop. 10.1), a precursor to integration which is generally attributed to Eudoxus of Cnidus.

α'.

Τὰ ἐν τοῖς κύκλοις ὅμοια πολύγωνα πρὸς ἀλλήλα ἐστὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα.



Ἐστωσαν κύκλοι οἱ ΑΒΓ, ΖΗΘ, καὶ ἐν αὐτοῖς ὅμοια πολύγωνα ἔστω τὰ ΑΒΓΔΕ, ΖΗΘΚΛ, διαμέτροι δὲ τῶν κύκλων ἔστωσαν ΒΜ, ΗΝ· λέγω, ὅτι ἐστὶν ὡς τὸ ἀπὸ τῆς ΒΜ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΝ τετράγωνον, οὕτως τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΕ, ΑΜ, ΗΛ, ΖΝ. καὶ ἐπεὶ ὁμοῖον τὸ ΑΒΓΔΕ πολύγωνον τῷ ΖΗΘΚΛ πολυγώνῳ, ἴση ἐστὶ καὶ ἡ ὑπὸ ΒΑΕ γωνία τῇ ὑπὸ ΗΖΛ, καὶ ἐστὶν ὡς ἡ ΒΑ πρὸς τὴν ΑΕ, οὕτως ἡ ΗΖ πρὸς τὴν ΖΛ. δύο δὲ τρίγωνά ἐστι τὰ ΒΑΕ, ΗΖΛ μίαν γωνίαν μιᾷ γωνίᾳ ἴσην ἔχοντα τὴν ὑπὸ ΒΑΕ τῇ ὑπὸ ΗΖΛ, περὶ δὲ τὰς ἴσας γωνίας τὰς πλευρὰς ἀνάλογον· ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΕ τρίγωνον τῷ ΖΗΛ τριγώνῳ. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΑΕΒ γωνία τῇ ὑπὸ ΖΑΗ. ἀλλ' ἡ μὲν ὑπὸ ΑΕΒ τῇ ὑπὸ ΑΜΒ ἐστὶν ἴση· ἐπὶ γὰρ τῆς αὐτῆς περιφερείας βεβήκασιν· ἡ δὲ ὑπὸ ΖΑΗ τῇ ὑπὸ ΖΝΗ· καὶ ἡ ὑπὸ ΑΜΒ ἄρα τῇ ὑπὸ ΖΝΗ ἐστὶν ἴση. ἐστὶ δὲ καὶ ὀρθὴ ἡ ὑπὸ ΒΑΜ ὀρθὴ τῇ ὑπὸ ΗΖΝ ἴση· καὶ ἡ λοιπὴ ἄρα τῇ λοιπῇ ἐστὶν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ΑΒΜ τρίγωνον τῷ ΖΗΝ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΜ πρὸς τὴν ΗΝ, οὕτως ἡ ΒΑ πρὸς τὴν ΗΖ. ἀλλὰ τοῦ μὲν τῆς ΒΜ πρὸς τὴν ΗΝ λόγον διπλασίων ἐστὶν ὁ τοῦ ἀπὸ τῆς ΒΜ τετραγώνου πρὸς τὸ ἀπὸ τῆς ΗΝ τετράγωνον, τοῦ δὲ τῆς ΒΑ πρὸς τὴν ΗΖ διπλασίων ἐστὶν ὁ τοῦ ΑΒΓΔΕ πολυγώνου πρὸς τὸ ΖΗΘΚΛ πολύγωνον· καὶ ὡς ἄρα τὸ ἀπὸ τῆς ΒΜ τετράγωνον πρὸς τὸ ἀπὸ τῆς ΗΝ τετράγωνον, οὕτως τὸ ΑΒΓΔΕ πολύγωνον πρὸς τὸ ΖΗΘΚΛ πολύγωνον.

Τὰ ἄρα ἐν τοῖς κύκλοις ὅμοια πολύγωνα πρὸς ἀλλήλα ἐστὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα· ὅπερ εἶδει δεῖξαι.

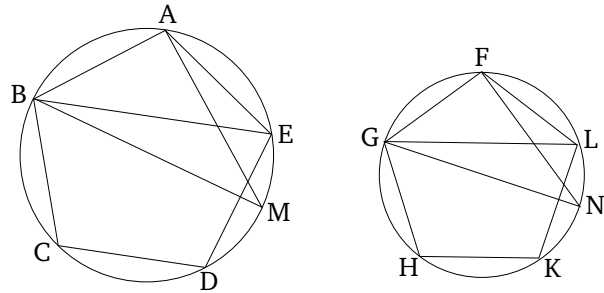
β'.

Οἱ κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα.

Ἐστωσαν κύκλοι οἱ ΑΒΓΔ, ΕΖΗΘ, διαμέτροι δὲ αὐτῶν

Proposition 1

Similar polygons (inscribed) in circles are to one another as the squares on the diameters (of the circles).



Let ABC and FGH be circles, and let $ABCDE$ and $FGHKL$ be similar polygons (inscribed) in them (respectively), and let BM and GN be the diameters of the circles (respectively). I say that as the square on BM is to the square on GN , so polygon $ABCDE$ (is) to polygon $FGHKL$.

For let BE , AM , GL , and FN have been joined. And since polygon $ABCDE$ (is) similar to polygon $FGHKL$, angle BAE is also equal to (angle) GFL , and as BA is to AE , so GF (is) to FL [Def. 6.1]. So, BAE and GFL are two triangles having one angle equal to one angle, (namely), BAE (equal) to GFL , and the sides around the equal angles proportional. Triangle ABE is thus equiangular with triangle FGL [Prop. 6.6]. Thus, angle AEB is equal to (angle) FLG . But, AEB is equal to AMB , and FLG to FNG , for they stand on the same circumference [Prop. 3.27]. Thus, AMB is also equal to FNG . And the right-angle BAM is also equal to the right-angle GFN [Prop. 3.31]. Thus, the remaining (angle) is also equal to the remaining (angle) [Prop. 1.32]. Thus, triangle ABM is equiangular with triangle FGN . Thus, proportionally, as BM is to GN , so BA (is) to GF [Prop. 6.4]. But, the (ratio) of the square on BM to the square on GN is the square of the ratio of BM to GN , and the (ratio) of polygon $ABCDE$ to polygon $FGHKL$ is the square of the (ratio) of BA to GF [Prop. 6.20]. And, thus, as the square on BM (is) to the square on GN , so polygon $ABCDE$ (is) to polygon $FGHKL$.

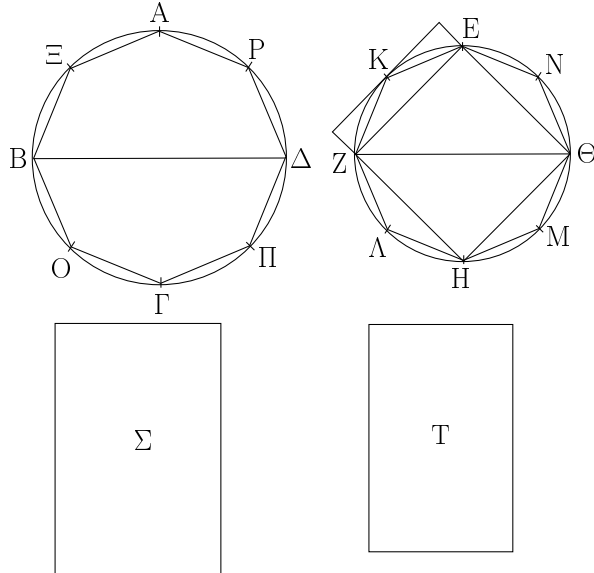
Thus, similar polygons (inscribed) in circles are to one another as the squares on the diameters (of the circles). (Which is) the very thing it was required to show.

Proposition 2

Circles are to one another as the squares on (their) diameters.

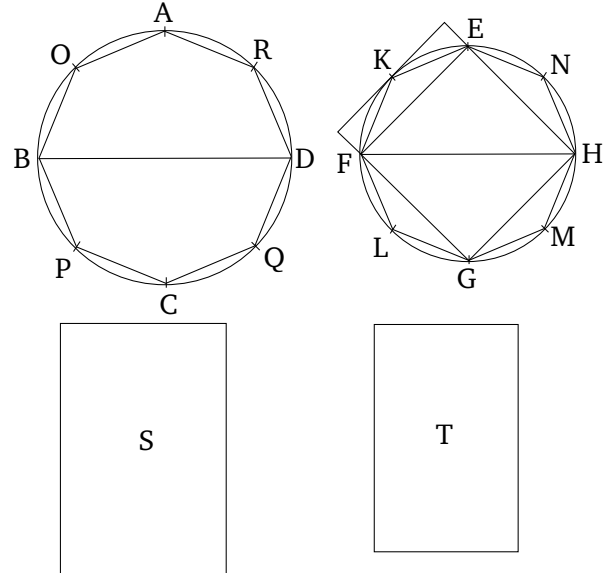
Let $ABCD$ and $EFGH$ be circles, and [let] BD and

[ἔστωσαν] αἱ $B\Delta$, $Z\Theta$ · λέγω, ὅτι ἐστὶν ὡς ὁ $AB\Gamma\Delta$ κύκλος πρὸς τὸν $EZH\Theta$ κύκλον, οὕτως τὸ ἀπὸ τῆς $B\Delta$ τετράγωνον πρὸς τὸ ἀπὸ τῆς $Z\Theta$ τετράγωνον.



Εἰ γὰρ μὴ ἐστὶν ὡς ὁ $AB\Gamma\Delta$ κύκλος πρὸς τὸν $EZH\Theta$, οὕτως τὸ ἀπὸ τῆς $B\Delta$ τετράγωνον πρὸς τὸ ἀπὸ τῆς $Z\Theta$, ἔσται ὡς τὸ ἀπὸ τῆς $B\Delta$ πρὸς τὸ ἀπὸ τῆς $Z\Theta$, οὕτως ὁ $AB\Gamma\Delta$ κύκλος ἢτοι πρὸς ἑλάσσον τι τοῦ $EZH\Theta$ κύκλου χωρίον ἢ πρὸς μείζον. ἔστω πρότερον πρὸς ἑλάσσον τὸ Σ . καὶ ἐγγεγράφω εἰς τὸν $EZH\Theta$ κύκλον τετράγωνον τὸ $EZH\Theta$. τὸ δὴ ἐγγεγραμμένον τετράγωνον μείζον ἐστὶν ἢ τὸ ἥμισυ τοῦ $EZH\Theta$ κύκλου, ἐπειδὴ περ ἐὰν διὰ τῶν E, Z, H, Θ σημείων ἐφαπτομένης [εὐθείας] τοῦ κύκλου ἀγάγωμεν, τοῦ περιγραφομένου περὶ τὸν κύκλον τετραγώνου ἥμισυ ἐστὶ τὸ $EZH\Theta$ τετράγωνον, τοῦ δὲ περιγραφέντος τετραγώνου ἐλάττων ἐστὶν ὁ κύκλος· ὥστε τὸ $EZH\Theta$ ἐγγεγραμμένον τετράγωνον μείζον ἐστὶ τοῦ ἡμίσεως τοῦ $EZH\Theta$ κύκλου. τετμήσθωσαν δίχα αἱ $EZ, ZH, H\Theta, \Theta E$ περιφέρειαι κατὰ τὰ K, Λ, M, N σημεία, καὶ ἐπεζεύχθωσαν αἱ $EK, KZ, Z\Lambda, \Lambda H, H M, M\Theta, \Theta N, N E$ · καὶ ἕκαστον ἄρα τῶν $EKZ, Z\Lambda H, H M\Theta, \Theta N E$ τριγώνων μείζον ἐστὶν ἢ τὸ ἥμισυ τοῦ καθ' ἑαυτὸ τμήματος τοῦ κύκλου, ἐπειδὴ περ ἐὰν διὰ τῶν K, Λ, M, N σημείων ἐφαπτομένης τοῦ κύκλου ἀγάγωμεν καὶ ἀναπληρώσωμεν τὰ ἐπὶ τῶν $EZ, ZH, H\Theta, \Theta E$ εὐθειῶν παραλληλόγραμμα, ἕκαστον τῶν $EKZ, Z\Lambda H, H M\Theta, \Theta N E$ τριγώνων ἥμισυ ἔσται τοῦ καθ' ἑαυτὸ παραλληλογράμμου, ἀλλὰ τὸ καθ' ἑαυτὸ τμήμα ἐλαττόν ἐστὶ τοῦ παραλληλογράμμου· ὥστε ἕκαστον τῶν $EKZ, Z\Lambda H, H M\Theta, \Theta N E$ τριγώνων μείζον ἐστὶ τοῦ ἡμίσεως τοῦ καθ' ἑαυτὸ τμήματος τοῦ κύκλου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγύνοντες εὐθείας καὶ τοῦτο αἰ ποιοῦντες καταλείβομεν τινα ἀποτμήματα τοῦ κύκλου, ἃ ἔσται ἐλάσσονα τῆς ὑπεροχῆς, ἢ ὑπερέχει ὁ $EZH\Theta$ κύκλος τοῦ Σ χωρίου.

FH [be] their diameters. I say that as circle $ABCD$ is to circle $EFGH$, so the square on BD (is) to the square on FH .



For if the circle $ABCD$ is not to the (circle) $EFGH$, as the square on BD (is) to the (square) on FH , then as the (square) on BD (is) to the (square) on FH , so circle $ABCD$ will be to some area either less than, or greater than, circle $EFGH$. Let it, first of all, be (in that ratio) to (some) lesser (area), S . And let the square $EFGH$ have been inscribed in circle $EFGH$ [Prop. 4.6]. So the inscribed square is greater than half of circle $EFGH$, inasmuch as if we draw tangents to the circle through the points E, F, G , and H , then square $EFGH$ is half of the square circumscribed about the circle [Prop. 1.47], and the circle is less than the circumscribed square. Hence, the inscribed square $EFGH$ is greater than half of circle $EFGH$. Let the circumferences EF, FG, GH , and HE have been cut in half at points K, L, M , and N (respectively), and let $EK, KF, FL, LG, GM, MH, HN$, and NE have been joined. And, thus, each of the triangles EKF, FLG, GMH , and HNE is greater than half of the segment of the circle about it, inasmuch as if we draw tangents to the circle through points K, L, M , and N , and complete the parallelograms on the straight-lines EF, FG, GH , and HE , then each of the triangles EKF, FLG, GMH , and HNE will be half of the parallelogram about it, but the segment about it is less than the parallelogram. Hence, each of the triangles EKF, FLG, GMH , and HNE is greater than half of the segment of the circle about it. So, by cutting the circumferences remaining behind in half, and joining straight-lines, and doing this continually, we will (even-

ἐδείχθη γὰρ ἐν τῷ πρώτῳ θεωρήματι τοῦ δεκάτου βιβλίου, ὅτι δύο μεγεθῶν ἀνίσων ἐκκειμένων, ἐὰν ἀπὸ τοῦ μείζονος ἀφαιρεθῇ μείζον ἢ τὸ ἥμισυ καὶ τοῦ καταλειπομένου μείζον ἢ τὸ ἥμισυ, καὶ τοῦτο αἰεὶ γίγνηται, λειφθήσεται τι μέγεθος, ὃ ἔσται ἔλασσον τοῦ ἐκκειμένου ἐλάσσονος μεγέθους. λελείφθω οὖν, καὶ ἔστω τὰ ἐπὶ τῶν EK , KZ , ZA , AH , HM , MO , ON , NE τμήματα τοῦ $EZH\Theta$ κύκλου ἐλάττονα τῆς ὑπεροχῆς, ἢ ὑπερέχει ὁ $EZH\Theta$ κύκλος τοῦ Σ χωρίου. λοιπὸν ἄρα τὸ $EKZAHM\Theta N$ πολύγωνον μείζον ἔστι τοῦ Σ χωρίου. ἐγγεγράφθω καὶ εἰς τὸν $AB\Gamma\Delta$ κύκλον τῷ $EKZAHM\Theta N$ πολυγώνῳ ὁμοιον πολύγωνον τὸ $A\Xi B O\Gamma\Pi\Delta P$. ἔστιν ἄρα ὡς τὸ ἀπὸ τῆς $B\Delta$ τετράγωνον πρὸς τὸ ἀπὸ τῆς $Z\Theta$ τετράγωνον, οὕτως τὸ $A\Xi B O\Gamma\Pi\Delta P$ πολύγωνον πρὸς τὸ $EKZAHM\Theta N$ πολύγωνον. ἀλλὰ καὶ ὡς τὸ ἀπὸ τῆς $B\Delta$ τετράγωνον πρὸς τὸ ἀπὸ τῆς $Z\Theta$, οὕτως ὁ $AB\Gamma\Delta$ κύκλος πρὸς τὸ Σ χωρίον· καὶ ὡς ἄρα ὁ $AB\Gamma\Delta$ κύκλος πρὸς τὸ Σ χωρίον, οὕτως τὸ $A\Xi B O\Gamma\Pi\Delta P$ πολύγωνον πρὸς τὸ $EKZAHM\Theta N$ πολύγωνον· ἐναλλάξ ἄρα ὡς ὁ $AB\Gamma\Delta$ κύκλος πρὸς τὸ ἐν αὐτῷ πολύγωνον, οὕτως τὸ Σ χωρίον πρὸς τὸ $EKZAHM\Theta N$ πολύγωνον. μείζων δὲ ὁ $AB\Gamma\Delta$ κύκλος τοῦ ἐν αὐτῷ πολυγώνου· μείζον ἄρα καὶ τὸ Σ χωρίον τοῦ $EKZAHM\Theta N$ πολυγώνου. ἀλλὰ καὶ ἔλαττον· ὅπερ ἔστιν ἀδύνατον. οὐκ ἄρα ἔστιν ὡς τὸ ἀπὸ τῆς $B\Delta$ τετράγωνον πρὸς τὸ ἀπὸ τῆς $Z\Theta$, οὕτως ὁ $AB\Gamma\Delta$ κύκλος πρὸς ἔλασσόν τι τοῦ $EZH\Theta$ κύκλου χωρίου. ὁμοίως δὲ δεῖξομεν, ὅτι οὐδὲ ὡς τὸ ἀπὸ $Z\Theta$ πρὸς τὸ ἀπὸ $B\Delta$, οὕτως ὁ $EZH\Theta$ κύκλος πρὸς ἔλασσόν τι τοῦ $AB\Gamma\Delta$ κύκλου χωρίου.

Λέγω δὴ, ὅτι οὐδὲ ὡς τὸ ἀπὸ τῆς $B\Delta$ πρὸς τὸ ἀπὸ τῆς $Z\Theta$, οὕτως ὁ $AB\Gamma\Delta$ κύκλος πρὸς μείζον τι τοῦ $EZH\Theta$ κύκλου χωρίου.

Εἰ γὰρ δυνατόν, ἔστω πρὸς μείζον τὸ Σ . ἀνάπαλιν ἄρα [ἔστιν] ὡς τὸ ἀπὸ τῆς $Z\Theta$ τετράγωνον πρὸς τὸ ἀπὸ τῆς $B\Delta$, οὕτως τὸ Σ χωρίον πρὸς τὸν $AB\Gamma\Delta$ κύκλον. ἀλλ' ὡς τὸ Σ χωρίον πρὸς τὸν $AB\Gamma\Delta$ κύκλον, οὕτως ὁ $EZH\Theta$ κύκλος πρὸς ἔλαττον τι τοῦ $AB\Gamma\Delta$ κύκλου χωρίου· καὶ ὡς ἄρα τὸ ἀπὸ τῆς $Z\Theta$ πρὸς τὸ ἀπὸ τῆς $B\Delta$, οὕτως ὁ $EZH\Theta$ κύκλος πρὸς ἔλασσόν τι τοῦ $AB\Gamma\Delta$ κύκλου χωρίου· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ἔστιν ὡς τὸ ἀπὸ τῆς $B\Delta$ τετράγωνον πρὸς τὸ ἀπὸ τῆς $Z\Theta$, οὕτως ὁ $AB\Gamma\Delta$ κύκλος πρὸς μείζον τι τοῦ $EZH\Theta$ κύκλου χωρίου. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλασσον· ἔστιν ἄρα ὡς τὸ ἀπὸ τῆς $B\Delta$ τετράγωνον πρὸς τὸ ἀπὸ τῆς $Z\Theta$, οὕτως ὁ $AB\Gamma\Delta$ κύκλος πρὸς τὸν $EZH\Theta$ κύκλον.

Οἱ ἄρα κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ἀπὸ τῶν διαμέτρων τετράγωνα· ὅπερ ἔδει δεῖξαι.

tually) leave behind some segments of the circle whose (sum) will be less than the excess by which circle $EFGH$ exceeds the area S . For we showed in the first theorem of the tenth book that if two unequal magnitudes are laid out, and if (a part) greater than a half is subtracted from the greater, and (if from) the remainder (a part) greater than a half (is subtracted), and this happens continually, then some magnitude will (eventually) be left which will be less than the lesser laid out magnitude [Prop. 10.1]. Therefore, let the (segments) have been left, and let the (sum of the) segments of the circle $EFGH$ on EK , KF , FL , LG , GM , MH , HN , and NE be less than the excess by which circle $EFGH$ exceeds area S . Thus, the remaining polygon $EKFLGMHN$ is greater than area S . And let the polygon $AOBPCQDR$, similar to the polygon $EKFLGMHN$, have been inscribed in circle $ABCD$. Thus, as the square on BD is to the square on FH , so polygon $AOBPCQDR$ (is) to polygon $EKFLGMHN$ [Prop. 12.1]. But, also, as the square on BD (is) to the square on FH , so circle $ABCD$ (is) to area S . And, thus, as circle $ABCD$ (is) to area S , so polygon $AOBPCQDR$ (is) to polygon $EKFLGMHN$ [Prop. 5.11]. Thus, alternately, as circle $ABCD$ (is) to the polygon (inscribed) within it, so area S (is) to polygon $EKFLGMHN$ [Prop. 5.16]. And circle $ABCD$ (is) greater than the polygon (inscribed) within it. Thus, area S is also greater than polygon $EKFLGMHN$. But, (it is) also less. The very thing is impossible. Thus, the square on BD is not to the (square) on FH , as circle $ABCD$ (is) to some area less than circle $EFGH$. So, similarly, we can show that the (square) on FH (is) not to the (square) on BD as circle $EFGH$ (is) to some area less than circle $ABCD$ either.

So, I say that neither (is) the (square) on BD to the (square) on FH , as circle $ABCD$ (is) to some area greater than circle $EFGH$.

For, if possible, let it be (in that ratio) to (some) greater (area), S . Thus, inversely, as the square on FH [is] to the (square) on DB , so area S (is) to circle $ABCD$ [Prop. 5.7 corr.]. But, as area S (is) to circle $ABCD$, so circle $EFGH$ (is) to some area less than circle $ABCD$ (see lemma). And, thus, as the (square) on FH (is) to the (square) on BD , so circle $EFGH$ (is) to some area less than circle $ABCD$ [Prop. 5.11]. The very thing was shown (to be) impossible. Thus, as the square on BD is to the (square) on FH , so circle $ABCD$ (is) not to some area greater than circle $EFGH$. And it was shown that neither (is it in that ratio) to (some) lesser (area). Thus, as the square on BD is to the (square) on FH , so circle $ABCD$ (is) to circle $EFGH$.

Thus, circles are to one another as the squares on

(their) diameters. (Which is) the very thing it was required to show.

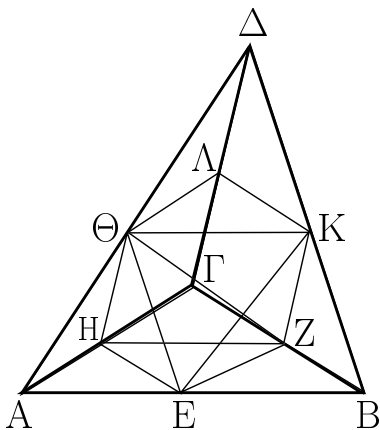
Λήμμα.

Λέγω δὴ, ὅτι τοῦ Σ χωρίου μείζονος ὄντος τοῦ ΕΖΗΘ κύκλου ἐστὶν ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἑλαττόν τι τοῦ ΑΒΓΔ κύκλου χωρίον.

Γεγονέντω γὰρ ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς τὸ Τ χωρίον. λέγω, ὅτι ἑλαττόν ἐστι τὸ Τ χωρίον τοῦ ΑΒΓΔ κύκλου. ἐπεὶ γὰρ ἐστὶν ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς τὸ Τ χωρίον, ἐναλλάξ ἐστὶν ὡς τὸ Σ χωρίον πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΒΓΔ κύκλος πρὸς τὸ Τ χωρίον. μείζον δὲ τὸ Σ χωρίον τοῦ ΕΖΗΘ κύκλου· μείζων ἄρα καὶ ὁ ΑΒΓΔ κύκλος τοῦ Τ χωρίου. ὥστε ἐστὶν ὡς τὸ Σ χωρίον πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΖΗΘ κύκλος πρὸς ἑλαττόν τι τοῦ ΑΒΓΔ κύκλου χωρίον· ὅπερ ἔδει δείξαι.

Υ'.

Πᾶσα πυραμὶς τρίγωνον ἔχουσα βάσιν διαιρεῖται εἰς δύο πυραμίδας ἴσας τε καὶ ὁμοίας ἀλλήλαις καὶ [ὁμοίας] τῇ ὅλῃ τριγώνου ἔχουσας βάσεις καὶ εἰς δύο πρίσματα ἴσα· καὶ τὰ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος.



Ἐστω πυραμὶς, ἥς βάσις μὲν ἐστὶ τὸ ΑΒΓ τρίγωνον, κορυφή δὲ τὸ Δ σημεῖον· λέγω, ὅτι ἡ ΑΒΓΔ πυραμὶς διαιρεῖται εἰς δύο πυραμίδας ἴσας ἀλλήλαις τριγώνους βάσεις ἔχουσας καὶ ὁμοίας τῇ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα· καὶ τὰ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος.

Τετμήσθωσαν γὰρ αἱ ΑΒ, ΒΓ, ΓΑ, ΑΔ, ΔΒ, ΔΓ δίχα κατὰ τὰ Ε, Ζ, Η, Θ, Κ, Λ σημεῖα, καὶ ἐπεξεύχθωσαν αἱ ΘΕ, ΕΗ, ΗΘ, ΘΚ, ΚΛ, ΛΘ, ΚΖ, ΖΗ. ἐπεὶ ἴση ἐστὶν ἡ μὲν ΑΕ τῇ ΕΒ, ἡ δὲ ΑΘ τῇ ΔΘ, παράλληλος ἄρα ἐστὶν ἡ ΕΘ τῇ ΔΒ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΘΚ τῇ ΑΒ παράλληλός ἐστιν.

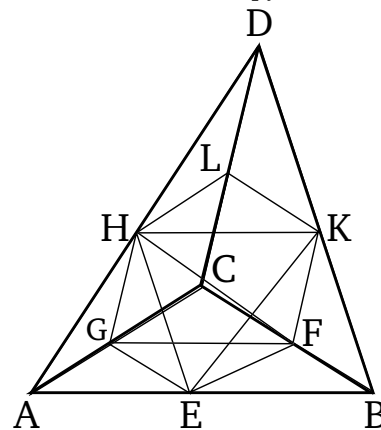
Lemma

So, I say that, area S being greater than circle $EFGH$, as area S is to circle $ABCD$, so circle $EFGH$ (is) to some area less than circle $ABCD$.

For let it have been contrived that as area S (is) to circle $ABCD$, so circle $EFGH$ (is) to area T . I say that area T is less than circle $ABCD$. For since as area S is to circle $ABCD$, so circle $EFGH$ (is) to area T , alternately, as area S is to circle $EFGH$, so circle $ABCD$ (is) to area T [Prop. 5.16]. And area S (is) greater than circle $EFGH$. Thus, circle $ABCD$ (is) also greater than area T [Prop. 5.14]. Hence, as area S is to circle $ABCD$, so circle $EFGH$ (is) to some area less than circle $ABCD$. (Which is) the very thing it was required to show.

Proposition 3

Any pyramid having a triangular base is divided into two pyramids having triangular bases (which are) equal, similar to one another, and [similar] to the whole, and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid.



Let there be a pyramid whose base is triangle ABC , and (whose) apex (is) point D . I say that pyramid $ABCD$ is divided into two pyramids having triangular bases (which are) equal to one another, and similar to the whole, and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid.

For let AB , BC , CA , AD , DB , and DC have been cut in half at points E , F , G , H , K , and L (respectively). And let HE , EG , GH , HK , KL , LH , KF , and FG have been joined. Since AE is equal to EB , and AH to DH ,

παραλληλόγραμμον ἄρα ἐστὶ τὸ ΘΕΒΚ· ἴση ἄρα ἐστὶν ἡ ΘΚ τῇ ΕΒ. ἀλλὰ ἡ ΕΒ τῇ ΕΑ ἐστὶν ἴση· καὶ ἡ ΑΕ ἄρα τῇ ΘΚ ἐστὶν ἴση. ἔστι δὲ καὶ ἡ ΑΘ τῇ ΘΔ ἴση· δύο δὲ αἱ ΕΑ, ΑΘ δυσὶ ταῖς ΚΘ, ΘΔ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΕΑΘ γωνία τῇ ὑπὸ ΚΘΔ ἴση· βάσις ἄρα ἡ ΕΘ βάσει τῇ ΚΔ ἐστὶν ἴση. ἴσον ἄρα καὶ ὁμοίον ἐστὶ τὸ ΑΕΘ τρίγωνον τῷ ΘΚΔ τριγώνῳ. διὰ τὰ αὐτὰ δὲ καὶ τὸ ΑΘΗ τρίγωνον τῷ ΘΛΔ τριγώνῳ ἴσον τέ ἐστὶ καὶ ὁμοίον. καὶ ἐπεὶ δύο εὐθεῖαι ἀπτόμεναι ἀλλήλων αἱ ΕΘ, ΘΗ παρὰ δύο εὐθείας ἀπτομένας ἀλλήλων τὰς ΚΔ, ΔΛ εἰσιν οὐκ ἐν τῷ αὐτῷ ἐπιπέδῳ οὔσαι, ἴσας γωνίας περιέξουσιν. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΕΘΗ γωνία τῇ ὑπὸ ΚΔΛ γωνίᾳ. καὶ ἐπεὶ δύο εὐθεῖαι αἱ ΕΘ, ΘΗ δυσὶ ταῖς ΚΔ, ΔΛ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ, καὶ γωνία ἡ ὑπὸ ΕΘΗ γωνία τῇ ὑπὸ ΚΔΛ ἐστὶν ἴση, βάσις ἄρα ἡ ΕΗ βάσει τῇ ΚΛ [ἐστὶν] ἴση· ἴσον ἄρα καὶ ὁμοίον ἐστὶ τὸ ΕΘΗ τρίγωνον τῷ ΚΔΛ τριγώνῳ. διὰ τὰ αὐτὰ δὲ καὶ τὸ ΑΕΗ τρίγωνον τῷ ΘΚΛ τριγώνῳ ἴσον τε καὶ ὁμοίον ἐστὶν. ἡ ἄρα πυραμὶς, ἥς βάσις μὲν ἐστὶ τὸ ΑΕΗ τρίγωνον, κορυφὴ δὲ τὸ Θ σημεῖον, ἴση καὶ ὁμοία ἐστὶ πυραμίδι, ἥς βάσις μὲν ἐστὶ τὸ ΘΚΛ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον. καὶ ἐπεὶ τριγώνου τοῦ ΑΔΒ παρὰ μίαν τῶν πλευρῶν τὴν ΑΒ ἥκται ἡ ΘΚ, ἰσογώνιον ἐστὶ τὸ ΑΔΒ τρίγωνον τῷ ΔΘΚ τριγώνῳ, καὶ τὰς πλευρὰς ἀνάλογον ἔχουσιν· ὁμοίον ἄρα ἐστὶ τὸ ΑΔΒ τρίγωνον τῷ ΔΘΚ τριγώνῳ. διὰ τὰ αὐτὰ δὲ καὶ τὸ μὲν ΔΒΓ τρίγωνον τῷ ΔΚΛ τριγώνῳ ὁμοίον ἐστὶν, τὸ δὲ ΑΔΓ τῷ ΔΛΘ. καὶ ἐπεὶ δύο εὐθεῖαι ἀπτόμεναι ἀλλήλων αἱ ΒΑ, ΑΓ παρὰ δύο εὐθείας ἀπτομένας ἀλλήλων τὰς ΚΘ, ΘΛ εἰσιν οὐκ ἐν τῷ αὐτῷ ἐπιπέδῳ, ἴσας γωνίας περιέξουσιν. ἴση ἄρα ἐστὶν ἡ ὑπὸ ΒΑΓ γωνία τῇ ὑπὸ ΚΘΛ. καὶ ἐστὶν ὡς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΚΘ πρὸς τὴν ΘΛ· ὁμοίον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον τῷ ΘΚΛ τριγώνῳ. καὶ πυραμὶς ἄρα, ἥς βάσις μὲν ἐστὶ τὸ ΑΒΓ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ὁμοία ἐστὶ πυραμίδι, ἥς βάσις μὲν ἐστὶ τὸ ΘΚΛ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον. ἀλλὰ πυραμὶς, ἥς βάσις μὲν [ἐστὶ] τὸ ΘΚΛ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ὁμοία ἐδείχθη πυραμίδι, ἥς βάσις μὲν ἐστὶ τὸ ΑΕΗ τρίγωνον, κορυφὴ δὲ τὸ Θ σημεῖον. ἑκατέρα ἄρα τῶν ΑΕΗΘ, ΘΚΛΔ πυραμίδων ὁμοία ἐστὶ τῇ ὅλῃ τῇ ΑΒΓΔ πυραμίδι.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΒΖ τῇ ΖΓ, διπλάσιον ἐστὶ τὸ ΕΒΖΗ παραλληλόγραμμον τοῦ ΗΖΓ τριγώνου. καὶ ἐπεὶ, ἐὰν ᾗ δύο πρίσματα ἰσοϋψῆ, καὶ τὸ μὲν ἔχῃ βάσιν παραλληλόγραμμον, τὸ δὲ τρίγωνον, διπλάσιον δὲ ᾗ τὸ παραλληλόγραμμον τοῦ τριγώνου, ἴσα ἐστὶ τὰ πρίσματα, ἴσον ἄρα ἐστὶ τὸ πρίσμα τὸ περιεχόμενον ὑπὸ δύο μὲν τριγώνων τῶν ΒΚΖ, ΕΘΗ, τριῶν δὲ παραλληλογράμμων τῶν ΕΒΖΗ, ΕΒΚΘ, ΘΚΖΗ τῷ πρισματι τῷ περιεχομένῳ ὑπὸ δύο μὲν τριγώνων τῶν ΗΖΓ, ΘΚΛ, τριῶν δὲ παραλληλογράμμων τῶν ΚΖΓΛ, ΛΓΗΘ, ΘΚΖΗ. καὶ φανερόν, ὅτι ἑκάτρων τῶν πρισμάτων, οὗ τε βάσις τὸ ΕΒΖΗ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘΚ εὐθεῖα, καὶ οὗ βάσις τὸ ΗΖΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΘΚΛ τρίγωνον, μεῖζόν ἐστὶν ἑκατέρας

EH is thus parallel to DB [Prop. 6.2]. So, for the same (reasons), HK is also parallel to AB . Thus, $HEBK$ is a parallelogram. Thus, HK is equal to EB [Prop. 1.34]. But, EB is equal to EA . Thus, AE is also equal to HK . And AH is also equal to HD . So the two (straight-lines) EA and AH are equal to the two (straight-lines) KH and HD , respectively. And angle EAH (is) equal to angle KHD [Prop. 1.29]. Thus, base EH is equal to base KD [Prop. 1.4]. Thus, triangle AEH is equal and similar to triangle HKD [Prop. 1.4]. So, for the same (reasons), triangle AHG is also equal and similar to triangle HLD . And since EH and HG are two straight-lines joining one another (which are respectively) parallel to two straight-lines joining one another, KD and DL , not being in the same plane, they will contain equal angles [Prop. 11.10]. Thus, angle EHG is equal to angle KDL . And since the two straight-lines EH and HG are equal to the two straight-lines KD and DL , respectively, and angle EHG is equal to angle KDL , base EG [is] thus equal to base KL [Prop. 1.4]. Thus, triangle EHG is equal and similar to triangle KDL . So, for the same (reasons), triangle AEG is also equal and similar to triangle HKL . Thus, the pyramid whose base is triangle AEG , and apex the point H , is equal and similar to the pyramid whose base is triangle HKL , and apex the point D [Def. 11.10]. And since HK has been drawn parallel to one of the sides, AB , of triangle ADB , triangle ADB is equiangular to triangle DHK [Prop. 1.29], and they have proportional sides. Thus, triangle ADB is similar to triangle DHK [Def. 6.1]. So, for the same (reasons), triangle DBC is also similar to triangle DKL , and ADC to DLH . And since two straight-lines joining one another, BA and AC , are parallel to two straight-lines joining one another, KH and HL , not in the same plane, they will contain equal angles [Prop. 11.10]. Thus, angle BAC is equal to (angle) KHL . And as BA is to AC , so KH (is) to HL . Thus, triangle ABC is similar to triangle HKL [Prop. 6.6]. And, thus, the pyramid whose base is triangle ABC , and apex the point D , is similar to the pyramid whose base is triangle HKL , and apex the point D [Def. 11.9]. But, the pyramid whose base [is] triangle HKL , and apex the point D , was shown (to be) similar to the pyramid whose base is triangle AEG , and apex the point H . Thus, each of the pyramids $AEGH$ and $HKLD$ is similar to the whole pyramid $ABCD$.

And since BF is equal to FC , parallelogram $EBFG$ is double triangle GFC [Prop. 1.41]. And since, if two prisms (have) equal heights, and the former has a parallelogram as a base, and the latter a triangle, and the parallelogram (is) double the triangle, then the prisms are equal [Prop. 11.39], the prism contained by the two

τῶν πυραμίδων, ὧν βάσεις μὲν τὰ AEH , ΘKA τρίγωνα, κορυφαί, δὲ τὰ Θ , Δ σημεία, ἐπειδήπερ [καί] ἐὰν ἐπιζεύξωμεν τὰς EZ , EK εὐθείας, τὸ μὲν πρίσμα, οὗ βάσις τὸ $EBZH$ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘK εὐθεΐα, μείζον ἐστὶ τῆς πυραμίδος, ἥς βάσις τὸ EBZ τρίγωνον, κορυφή δὲ τὸ K σημεῖον. ἀλλ' ἡ πυραμίς, ἥς βάσις τὸ EBZ τρίγωνον, κορυφή δὲ τὸ K σημεῖον, ἴση ἐστὶ πυραμίδι, ἥς βάσις τὸ AEH τρίγωνον, κορυφή δὲ τὸ Θ σημεῖον· ὅσον δὲ τὸ μὲν πρίσμα, οὗ βάσις μὲν τὸ $EBZH$ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘK εὐθεΐα, μείζον ἐστὶ πυραμίδος, ἥς βάσις μὲν τὸ AEH τρίγωνον, κορυφή δὲ τὸ Θ σημεῖον. ἴσον δὲ τὸ μὲν πρίσμα, οὗ βάσις τὸ $EBZH$ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ ΘK εὐθεΐα, τῷ πρίσματι, οὗ βάσις μὲν τὸ HZG τρίγωνον, ἀπεναντίον δὲ τὸ ΘKA τρίγωνον· ἡ δὲ πυραμίς, ἥς βάσις τὸ AEH τρίγωνον, κορυφή δὲ τὸ Θ σημεῖον, ἴση ἐστὶ πυραμίδι, ἥς βάσις τὸ ΘKA τρίγωνον, κορυφή δὲ τὸ Δ σημεῖον. τὰ ἄρα εἰρημένα δύο πρίσματα μείζονά ἐστι τῶν εἰρημένων δύο πυραμίδων, ὧν βάσεις μὲν τὰ AEH , ΘKA τρίγωνα, κορυφαί δὲ τὰ Θ , Δ σημεία.

Ἡ ἄρα ὅλη πυραμίς, ἥς βάσις τὸ ABG τρίγωνον, κορυφή δὲ τὸ Δ σημεῖον, διήρηται εἰς τε δύο πυραμίδας ἴσας ἀλλήλαις [καὶ ὁμοίας τῇ ὅλῃ] καὶ εἰς δύο πρίσματα ἴσα, καὶ τὰ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος· ὅπερ ἔδει δεῖξαι.

δ'.

Ἐὰν ὦσι δύο πυραμίδες ὑπὸ τὸ αὐτὸ ὕψος τριγώνους ἔχουσαι βάσεις, διαιρεθῇ δὲ ἑκάτερα αὐτῶν εἰς τε δύο πυραμίδας ἴσας ἀλλήλαις καὶ ὁμοίας τῇ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα, ἔσται ὡς ἡ τῆς μιᾶς πυραμίδος βάσις πρὸς τὴν τῆς ἐτέρας πυραμίδος βάσιν, οὕτως τὰ ἐν τῇ μιᾷ πυραμίδι πρίσματα πάντα πρὸς τὰ ἐν τῇ ἐτέρᾳ πυραμίδι πρίσματα πάντα ἰσοπληθῆ.

Ἐστῶσαν δύο πυραμίδες ὑπὸ τὸ αὐτὸ ὕψος τριγώνους ἔχουσαι βάσεις τὰς ABG , ΔEZ , κορυφὰς δὲ τὰ H , Θ σημεία, καὶ διηρήσθω ἑκάτερα αὐτῶν εἰς τε δύο πυραμίδας ἴσας ἀλλήλαις καὶ ὁμοίας τῇ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα· λέγω,

triangles BKF and EHG , and the three parallelograms $EBFG$, $EBKH$, and $HKFG$, is thus equal to the prism contained by the two triangles GFC and HKL , and the three parallelograms $KFCL$, $LCGH$, and $HKFG$. And (it is) clear that each of the prisms whose base (is) parallelogram $EBFG$, and opposite (side) straight-line HK , and whose base (is) triangle GFC , and opposite (plane) triangle HKL , is greater than each of the pyramids whose bases are triangles AEG and HKL , and apexes the points H and D (respectively), inasmuch as, if we [also] join the straight-lines EF and EK then the prism whose base (is) parallelogram $EBFG$, and opposite (side) straight-line HK , is greater than the pyramid whose base (is) triangle EBF , and apex the point K . But the pyramid whose base (is) triangle EBF , and apex the point K , is equal to the pyramid whose base is triangle AEG , and apex point H . For they are contained by equal and similar planes. And, hence, the prism whose base (is) parallelogram $EBFG$, and opposite (side) straight-line HK , is greater than the pyramid whose base (is) triangle AEG , and apex the point H . And the prism whose base is parallelogram $EBFG$, and opposite (side) straight-line HK , (is) equal to the prism whose base (is) triangle GFC , and opposite (plane) triangle HKL . And the pyramid whose base (is) triangle AEG , and apex the point H , is equal to the pyramid whose base (is) triangle HKL , and apex the point D . Thus, the (sum of the) aforementioned two prisms is greater than the (sum of the) aforementioned two pyramids, whose bases (are) triangles AEG and HKL , and apexes the points H and D (respectively).

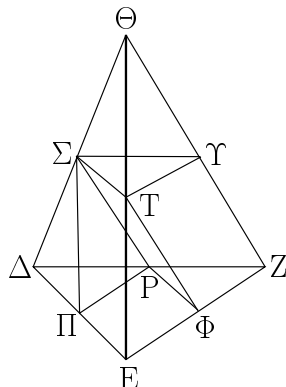
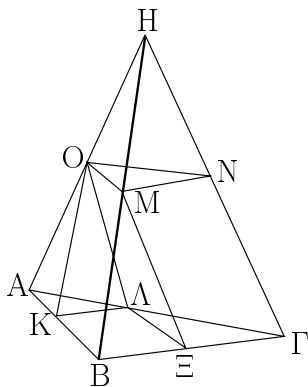
Thus, the whole pyramid, whose base (is) triangle ABC , and apex the point D , has been divided into two pyramids (which are) equal to one another [and similar to the whole], and into two equal prisms. And the (sum of the) two prisms is greater than half of the whole pyramid. (Which is) the very thing it was required to show.

Proposition 4

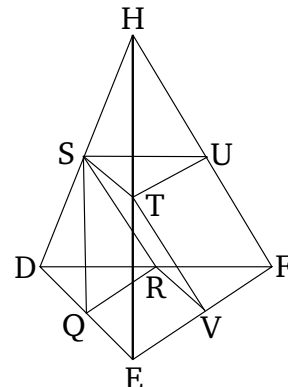
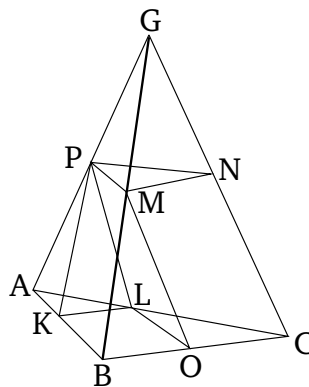
If there are two pyramids with the same height, having triangular bases, and each of them is divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms then as the base of one pyramid (is) to the base of the other pyramid, so (the sum of) all the prisms in one pyramid will be to (the sum of) all the equal number of prisms in the other pyramid.

Let there be two pyramids with the same height, having the triangular bases ABC and DEF , (with) apexes the points G and H (respectively). And let each of them have been divided into two pyramids equal to one an-

ὅτι ἐστὶν ὡς ἡ $ABΓ$ βάσις πρὸς τὴν $ΔΕΖ$ βάσιν, οὕτως τὰ ἐν τῇ $ABΓΗ$ πυραμίδι πρίσματα πάντα πρὸς τὰ ἐν τῇ $ΔΕΖΘ$ πυραμίδι πρίσματα ἰσοπληθῆ.



other, and similar to the whole, and into two equal prisms [Prop. 12.3]. I say that as base ABC is to base DEF , so (the sum of) all the prisms in pyramid $ABCG$ (is) to (the sum of) all the equal number of prisms in pyramid $DEFH$.



Ἐπεὶ γὰρ ἴση ἐστὶν ἡ μὲν $ΒΞ$ τῇ $ΞΓ$, ἡ δὲ $ΑΛ$ τῇ $ΛΓ$, παράλληλος ἄρα ἐστὶν ἡ $ΛΞ$ τῇ $ΑΒ$ καὶ ὅμοιον τὸ $ΑΒΓ$ τρίγωνον τῷ $ΛΞΓ$ τριγώνῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ $ΔΕΖ$ τρίγωνον τῷ $ΡΦΖ$ τριγώνῳ ὅμοιον ἐστὶν. καὶ ἐπεὶ διπλασίον ἐστὶν ἡ μὲν $ΒΓ$ τῆς $ΓΞ$, ἡ δὲ $ΕΖ$ τῆς $ΖΦ$, ἔστιν ἄρα ὡς ἡ $ΒΓ$ πρὸς τὴν $ΓΞ$, οὕτως ἡ $ΕΖ$ πρὸς τὴν $ΖΦ$. καὶ ἀναγέγραπται ἀπὸ μὲν τῶν $ΒΓ$, $ΓΞ$ ὁμοιά τε καὶ ὁμοίως κείμενα εὐθύγραμμα τὰ $ΑΒΓ$, $ΛΞΓ$, ἀπὸ δὲ τῶν $ΕΖ$, $ΖΦ$ ὁμοιά τε καὶ ὁμοίως κείμενα [εὐθύγραμμα] τὰ $ΔΕΖ$, $ΡΦΖ$. ἔστιν ἄρα ὡς τὸ $ΑΒΓ$ τρίγωνον πρὸς τὸ $ΛΞΓ$ τρίγωνον, οὕτως τὸ $ΔΕΖ$ τρίγωνον πρὸς τὸ $ΡΦΖ$ τρίγωνον· ἐναλλάξ ἄρα ἐστὶν ὡς τὸ $ΑΒΓ$ τρίγωνον πρὸς τὸ $ΔΕΖ$ [τρίγωνον], οὕτως τὸ $ΛΞΓ$ [τρίγωνον] πρὸς τὸ $ΡΦΖ$ τρίγωνον. ἀλλ' ὡς τὸ $ΛΞΓ$ τρίγωνον πρὸς τὸ $ΡΦΖ$ τρίγωνον, οὕτως τὸ πρίσμα, οὗ βάσις μὲν [ἐστὶ] τὸ $ΛΞΓ$ τρίγωνον, ἀπεναντίον δὲ τὸ $ΟΜΝ$, πρὸς τὸ πρίσμα, οὗ βάσις μὲν τὸ $ΡΦΖ$ τρίγωνον, ἀπεναντίον δὲ τὸ $ΣΤΥ$. καὶ ὡς ἄρα τὸ $ΑΒΓ$ τρίγωνον πρὸς τὸ $ΔΕΖ$ τρίγωνον, οὕτως τὸ πρίσμα, οὗ βάσις μὲν τὸ $ΛΞΓ$ τρίγωνον, ἀπεναντίον δὲ τὸ $ΟΜΝ$, πρὸς τὸ πρίσμα, οὗ βάσις μὲν τὸ $ΡΦΖ$ τρίγωνον, ἀπεναντίον δὲ τὸ $ΣΤΥ$. ὡς δὲ τὰ εἰρημένα πρίσματα πρὸς ἄλληλα, οὕτως τὸ πρίσμα, οὗ βάσις μὲν τὸ $ΚΒΞΛ$ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ $ΟΜ$ εὐθεῖα, πρὸς τὸ πρίσμα, οὗ βάσις μὲν τὸ $ΠΕΦΡ$ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ $ΣΤ$ εὐθεῖα. καὶ τὰ δύο ἄρα πρίσματα, οὗ τε βάσις μὲν τὸ $ΚΒΞΛ$ παραλληλόγραμμον, ἀπεναντίον δὲ ἡ $ΟΜ$, καὶ οὗ βάσις μὲν τὸ $ΛΞΓ$, ἀπεναντίον δὲ τὸ $ΟΜΝ$, πρὸς τὰ πρίσματα, οὗ τε βάσις μὲν τὸ $ΠΕΦΡ$, ἀπεναντίον δὲ ἡ $ΣΤ$ εὐθεῖα, καὶ οὗ βάσις μὲν τὸ $ΡΦΖ$ τρίγωνον, ἀπεναντίον δὲ τὸ $ΣΤΥ$. καὶ ὡς ἄρα ἡ $ΑΒΓ$ βάσις πρὸς τὴν $ΔΕΖ$ βάσιν, οὕτως τὰ εἰρημένα δύο πρίσματα πρὸς τὰ εἰρημένα δύο πρίσματα.

Καὶ ὁμοίως, ἐὰν διαιρεθῶσιν αἱ $ΟΜΝΗ$, $ΣΤΥΘ$ πυραμίδες εἰς τε δύο πρίσματα καὶ δύο πυραμίδας, ἔσται ὡς ἡ

For since BO is equal to OC , and AL to LC , LO is thus parallel to AB , and triangle ABC similar to triangle LOC [Prop. 12.3]. So, for the same (reasons), triangle DEF is also similar to triangle RVF . And since BC is double CO , and EF (double) FV , thus as BC (is) to CO , so EF (is) to FV . And the similar, and similarly laid out, rectilinear (figures) ABC and LOC have been described on BC and CO (respectively), and the similar, and similarly laid out, [rectilinear] (figures) DEF and RVF on EF and FV (respectively). Thus, as triangle ABC is to triangle LOC , so triangle DEF (is) to triangle RVF [Prop. 6.22]. Thus, alternately, as triangle ABC is to [triangle] DEF , so [triangle] LOC (is) to triangle RVF [Prop. 5.16]. But, as triangle LOC (is) to triangle RVF , so the prism whose base [is] triangle LOC , and opposite (plane) PMN , (is) to the prism whose base (is) triangle RVF , and opposite (plane) STU (see lemma). And, thus, as triangle ABC (is) to triangle DEF , so the prism whose base (is) triangle LOC , and opposite (plane) PMN , (is) to the prism whose base (is) triangle RVF , and opposite (plane) STU . And as the aforementioned prisms (are) to one another, so the prism whose base (is) parallelogram $KBOL$, and opposite (side) straight-line PM , (is) to the prism whose base (is) parallelogram $QEV R$, and opposite (side) straight-line ST [Props. 11.39, 12.3]. Thus, also, (is) the (sum of the) two prisms—that whose base (is) parallelogram $KBOL$, and opposite (side) PM , and that whose base (is) LOC , and opposite (plane) PMN —to (the sum of) the (two) prisms—that whose base (is) $QEV R$, and opposite (side) straight-line ST , and that whose base (is) triangle RVF , and opposite (plane) STU [Prop. 5.12]. And, thus, as base ABC (is) to base DEF , so the (sum

OMN βάσις πρὸς τὴν ΣΤΥ βάσιν, οὕτως τὰ ἐν τῇ OMNH πυραμίδι δύο πρίσματα πρὸς τὰ ἐν τῇ ΣΤΥΘ πυραμίδι δύο πρίσματα. ἀλλ' ὥς ἡ OMN βάσις πρὸς τὴν ΣΤΥ βάσιν, οὕτως ἡ ABΓ βάσις πρὸς τὴν ΔΕΖ βάσιν· ἴσον γὰρ ἑκάτερον τῶν OMN, ΣΤΥ τριγώνων ἑκατέρω τῶν ΛΕΓ, ΡΦΖ. καὶ ὥς ἄρα ἡ ABΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ τέσσαρα πρίσματα πρὸς τὰ τέσσαρα πρίσματα. ὁμοίως δὲ καὶ τὰς ὑπολειπομένας πυραμίδας διέλωμεν εἰς τε δύο πυραμίδας καὶ εἰς δύο πρίσματα, ἔσται ὥς ἡ ABΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ ἐν τῇ ABΓH πυραμίδι πρίσματα πάντα πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα πάντα ἰσοπληθῆ· ὅπερ ἔδει δεῖξαι.

Λήμμα.

Ὅτι δέ ἐστιν ὥς τὸ ΛΕΓ τρίγωνον πρὸς τὸ ΡΦΖ τρίγωνον, οὕτως τὸ πρίσμα, οὗ βάσις τὸ ΛΕΓ τρίγωνον, ἀπεναντίον δὲ τὸ OMN, πρὸς τὸ πρίσμα, οὗ βάσις μὲν τὸ ΡΦΖ [τρίγωνον], ἀπεναντίον δὲ τὸ ΣΤΥ, οὕτω δεικτέον.

Ἐπὶ γὰρ τῆς αὐτῆς καταγραφῆς νενοήσθωσαν ἀπὸ τῶν H, Θ κάθετοι ἐπὶ τὰ ABΓ, ΔΕΖ ἐπίπεδα, ἴσαι δηλαδὴ τυγχάνουσαι διὰ τὸ ἰσοῦσθαι ὑποκεῖσθαι τὰς πυραμίδας. καὶ ἐπεὶ δύο εὐθεῖαι ἢ τε ΗΓ καὶ ἡ ἀπὸ τοῦ H κάθετος ὑπὸ παραλλήλων ἐπιπέδων τῶν ABΓ, OMN τέμνονται, εἰς τοὺς αὐτοὺς λόγους τμηθίσονται. καὶ τέμνηται ἡ ΗΓ δίχα ὑπὸ τοῦ OMN ἐπιπέδου κατὰ τὸ N· καὶ ἡ ἀπὸ τοῦ H ἄρα κάθετος ἐπὶ τὸ ABΓ ἐπίπεδον δίχα τμηθήσεται ὑπὸ τοῦ OMN ἐπιπέδου. διὰ τὰ αὐτὰ δὴ καὶ ἡ ἀπὸ τοῦ Θ κάθετος ἐπὶ τὸ ΔΕΖ ἐπίπεδον δίχα τμηθήσεται ὑπὸ τοῦ ΣΤΥ ἐπιπέδου. καὶ εἰσιν ἴσαι αἱ ἀπὸ τῶν H, Θ κάθετοι ἐπὶ τὰ ABΓ, ΔΕΖ ἐπίπεδα· ἴσαι ἄρα καὶ αἱ ἀπὸ τῶν OMN, ΣΤΥ τριγώνων ἐπὶ τὰ ABΓ, ΔΕΖ κάθετοι. ἰσοῦσθ' ἄρα [ἐστὶ] τὰ πρίσματα, ὧν βάσεις μὲν εἰσι τὰ ΛΕΓ, ΡΦΖ τρίγωνα, ἀπεναντίον δὲ τὰ OMN, ΣΤΥ. ὥστε καὶ τὰ στερεὰ παραλληλεπίπεδα τὰ ἀπὸ τῶν εἰρημένων πρισμαμάτων ἀναγραφόμενα ἰσοῦσθ' καὶ πρὸς ἄλληλά [εἰσιν] ὥς αἱ βάσεις· καὶ τὰ ἡμίση ἄρα ἐστὶν ὥς ἡ ΛΕΓ βάσις πρὸς τὴν ΡΦΖ βάσιν, οὕτως τὰ εἰρημένα πρίσματα πρὸς ἄλληλα· ὅπερ ἔδει δεῖξαι.

of the first) aforementioned two prisms (is) to the (sum of the second) aforementioned two prisms.

And, similarly, if pyramids *PMNG* and *STUH* are divided into two prisms, and two pyramids, as base *PMN* (is) to base *STU*, so (the sum of) the two prisms in pyramid *PMNG* will be to (the sum of) the two prisms in pyramid *STUH*. But, as base *PMN* (is) to base *STU*, so base *ABC* (is) to base *DEF*. For the triangles *PMN* and *STU* (are) equal to *LOC* and *RVF*, respectively. And, thus, as base *ABC* (is) to base *DEF*, so (the sum of) the four prisms (is) to (the sum of) the four prisms [Prop. 5.12]. So, similarly, even if we divide the pyramids left behind into two pyramids and into two prisms, as base *ABC* (is) to base *DEF*, so (the sum of) all the prisms in pyramid *ABCG* will be to (the sum of) all the equal number of prisms in pyramid *DEFH*. (Which is) the very thing it was required to show.

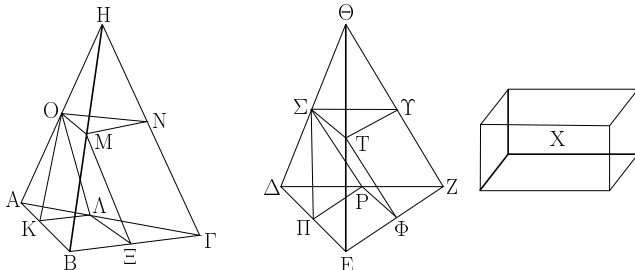
Lemma

And one may show, as follows, that as triangle *LOC* is to triangle *RVF*, so the prism whose base (is) triangle *LOC*, and opposite (plane) *PMN*, (is) to the prism whose base (is) [triangle] *RVF*, and opposite (plane) *STU*.

For, in the same figure, let perpendiculars have been conceived (drawn) from (points) *G* and *H* to the planes *ABC* and *DEF* (respectively). These clearly turn out to be equal, on account of the pyramids being assumed (to be) of equal height. And since two straight-lines, *GC* and the perpendicular from *G*, are cut by the parallel planes *ABC* and *PMN* they will be cut in the same ratios [Prop. 11.17]. And *GC* was cut in half by the plane *PMN* at *N*. Thus, the perpendicular from *G* to the plane *ABC* will also be cut in half by the plane *PMN*. So, for the same (reasons), the perpendicular from *H* to the plane *DEF* will also be cut in half by the plane *STU*. And the perpendiculars from *G* and *H* to the planes *ABC* and *DEF* (respectively) are equal. Thus, the perpendiculars from the triangles *PMN* and *STU* to *ABC* and *DEF* (respectively, are) also equal. Thus, the prisms whose bases are triangles *LOC* and *RVF*, and opposite (sides) *PMN* and *STU* (respectively), [are] of equal height. And, hence, the parallelepiped solids described on the aforementioned prisms [are] of equal height and (are) to one another as their bases [Prop. 11.32]. Likewise, the halves (of the solids) [Prop. 11.28]. Thus, as base *LOC* is to base *RVF*, so the aforementioned prisms (are) to one another. (Which is) the very thing it was required to show.

ε'.

Αἱ ὑπὸ τὸ αὐτὸ ὕψος οὔσαι πυραμίδες καὶ τριγώνους ἔχουσαι βάσεις πρὸς ἀλλήλας εἰσὶν ὡς αἱ βάσεις.



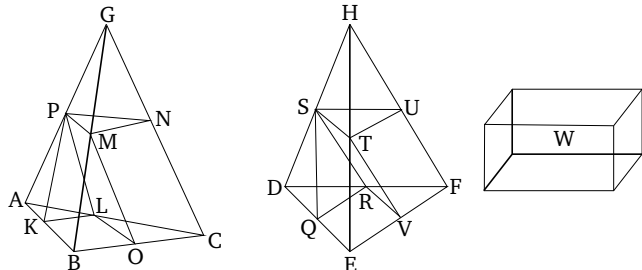
Ἐστωσαν ὑπὸ τὸ αὐτὸ ὕψος πυραμίδες, ὧν βάσεις μὲν τὰ ABΓ, ΔΕΖ τρίγωνα, κορυφαὶ δὲ τὰ H, Θ σημεῖα· λέγω, ὅτι ἐστὶν ὡς ἡ ABΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ABΓH πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα.

Εἰ γὰρ μὴ ἐστὶν ὡς ἡ ABΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ABΓH πυραμὶς πρὸς τὴν ΔΕΖΘ πυραμίδα, ἔσται ὡς ἡ ABΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ABΓH πυραμὶς ἢτοι πρὸς ἔλασσόν τι τῆς ΔΕΖΘ πυραμίδος στερεὸν ἢ πρὸς μείζον. ἔστω πρότερον πρὸς ἔλασσον τὸ X, καὶ διηρήσθω ἡ ΔΕΖΘ πυραμὶς εἰς τε δύο πυραμίδας ἴσας ἀλλήλαις καὶ ὁμοίας τῇ ὅλῃ καὶ εἰς δύο πρίσματα ἴσα· τὰ δὲ δύο πρίσματα μείζονά ἐστιν ἢ τὸ ἥμισυ τῆς ὅλης πυραμίδος. καὶ πάλιν αἱ ἐκ τῆς διαίρεσεως γινόμεναι πυραμίδες ὁμοίως διηρήσθωσαν, καὶ τοῦτο αἰετὶ γινέσθω, ἕως οὗ λειψθῶσί τινες πυραμίδες ἀπὸ τῆς ΔΕΖΘ πυραμίδος, αἱ εἰσὶν ἐλάττωες τῆς ὑπεροχῆς, ἣ ὑπερέχει ἡ ΔΕΖΘ πυραμὶς τοῦ X στερεοῦ. λειψθῶσαν καὶ ἔστωσαν λόγου ἔνεκεν αἱ ΔΠΡΣ, ΣΤΥΘ· λοιπὰ ἄρα τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα μείζονά ἐστι τοῦ X στερεοῦ. διηρήσθω καὶ ἡ ABΓH πυραμὶς ὁμοίως καὶ ἰσοπληθῶς τῇ ΔΕΖΘ πυραμίδι· ἔστιν ἄρα ὡς ἡ ABΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὰ ἐν τῇ ABΓH πυραμίδι πρίσματα πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα, ἀλλὰ καὶ ὡς ἡ ABΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ABΓH πυραμὶς πρὸς τὸ X στερεόν· καὶ ὡς ἄρα ἡ ABΓH πυραμὶς πρὸς τὸ X στερεόν, οὕτως τὰ ἐν τῇ ABΓH πυραμίδι πρίσματα πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα· ἐναλλάξ ἄρα ὡς ἡ ABΓH πυραμὶς πρὸς τὰ ἐν αὐτῇ πρίσματα, οὕτως τὸ X στερεόν πρὸς τὰ ἐν τῇ ΔΕΖΘ πυραμίδι πρίσματα. μείζων δὲ ἡ ABΓH πυραμὶς τῶν ἐν αὐτῇ πρισμάτων· μείζων ἄρα καὶ τὸ X στερεόν τῶν ἐν τῇ ΔΕΖΘ πυραμίδι πρισμάτων. ἀλλὰ καὶ ἔλαττον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐστὶν ὡς ἡ ABΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ABΓH πυραμὶς πρὸς ἔλασσόν τι τῆς ΔΕΖΘ πυραμίδος στερεόν. ὁμοίως δὲ δειχθήσεται, ὅτι οὐδὲ ὡς ἡ ΔΕΖ βάσις πρὸς τὴν ABΓ βάσιν, οὕτως ἡ ΔΕΖΘ πυραμὶς πρὸς ἔλαττον τι τῆς ABΓH πυραμίδος στερεόν.

Λέγω δὴ, ὅτι οὐκ ἔστιν οὐδὲ ὡς ἡ ABΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως ἡ ABΓH πυραμὶς πρὸς μείζον τι τῆς ΔΕΖΘ πυραμίδος στερεόν.

Proposition 5

Pyramids which are of the same height, and have triangular bases, are to one another as their bases.



Let there be pyramids of the same height whose bases (are) the triangles ABC and DEF , and apexes the points G and H (respectively). I say that as base ABC is to base DEF , so pyramid $ABCG$ (is) to pyramid $DEFH$.

For if base ABC is not to base DEF , as pyramid $ABCG$ (is) to pyramid $DEFH$, then base ABC will be to base DEF , as pyramid $ABCG$ (is) to some solid either less than, or greater than, pyramid $DEFH$. Let it, first of all, be (in this ratio) to (some) lesser (solid), W . And let pyramid $DEFH$ have been divided into two pyramids equal to one another, and similar to the whole, and into two equal prisms. So, the (sum of the) two prisms is greater than half of the whole pyramid [Prop. 12.3]. And, again, let the pyramids generated by the division have been similarly divided, and let this be done continually until some pyramids are left from pyramid $DEFH$ which (when added together) are less than the excess by which pyramid $DEFH$ exceeds the solid W [Prop. 10.1]. Let them have been left, and, for the sake of argument, let them be $DQRS$ and $STUH$. Thus, the (sum of the) remaining prisms within pyramid $DEFH$ is greater than solid W . Let pyramid $ABCG$ also have been divided similarly, and a similar number of times, as pyramid $DEFH$. Thus, as base ABC is to base DEF , so the (sum of the) prisms within pyramid $ABCG$ (is) to the (sum of the) prisms within pyramid $DEFH$ [Prop. 12.4]. But, also, as base ABC (is) to base DEF , so pyramid $ABCG$ (is) to solid W . And, thus, as pyramid $ABCG$ (is) to solid W , so the (sum of the) prisms within pyramid $ABCG$ (is) to the (sum of the) prisms within pyramid $DEFH$ [Prop. 5.11]. Thus, alternately, as pyramid $ABCG$ (is) to the (sum of the) prisms within it, so solid W (is) to the (sum of the) prisms within pyramid $DEFH$ [Prop. 5.16]. And pyramid $ABCG$ (is) greater than the (sum of the) prisms within it. Thus, solid W (is) also greater than the (sum of the) prisms within pyramid $DEFH$ [Prop. 5.14]. But, (it is) also less. This very thing is impossible. Thus, as base ABC is to base DEF , so pyramid $ABCG$ (is)

Εἰ γὰρ δυνατόν, ἔστω πρὸς μείζον τὸ X · ἀνάπαλιν ἄρα ἔστιν ὡς ἡ ΔEZ βάσις πρὸς τὴν $AB\Gamma$ βάσιν, οὕτως τὸ X στερεὸν πρὸς τὴν $AB\Gamma H$ πυραμίδα. ὡς δὲ τὸ X στερεὸν πρὸς τὴν $AB\Gamma H$ πυραμίδα, οὕτως ἡ $\Delta EZ\Theta$ πυραμὶς πρὸς ἑλασσόν τι τῆς $AB\Gamma H$ πυραμίδος, ὡς ἔμπροσθεν ἐδείχθη· καὶ ὡς ἄρα ἡ ΔEZ βάσις πρὸς τὴν $AB\Gamma$ βάσιν, οὕτως ἡ $\Delta EZ\Theta$ πυραμὶς πρὸς ἑλασσόν τι τῆς $AB\Gamma H$ πυραμίδος· ὅπερ ἄτοπον ἐδείχθη. οὐκ ἄρα ἔστιν ὡς ἡ $AB\Gamma$ βάσις πρὸς τὴν ΔEZ βάσιν, οὕτως ἡ $AB\Gamma H$ πυραμὶς πρὸς μείζον τι τῆς $\Delta EZ\Theta$ πυραμίδος στερεόν. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἑλασσόν. ἔστιν ἄρα ὡς ἡ $AB\Gamma$ βάσις πρὸς τὴν ΔEZ βάσιν, οὕτως ἡ $AB\Gamma H$ πυραμὶς πρὸς τὴν $\Delta EZ\Theta$ πυραμίδα· ὅπερ ἔδει δεῖξαι.

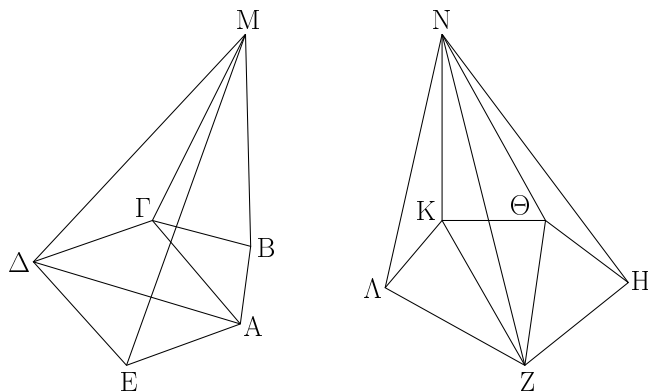
not to some solid less than pyramid $DEFH$. So, similarly, we can show that base DEF is not to base ABC , as pyramid $DEFH$ (is) to some solid less than pyramid $ABCG$ either.

So, I say that neither is base ABC to base DEF , as pyramid $ABCG$ (is) to some solid greater than pyramid $DEFH$.

For, if possible, let it be (in this ratio) to some greater (solid), W . Thus, inversely, as base DEF (is) to base ABC , so solid W (is) to pyramid $ABCG$ [Prop. 5.7. corr.]. And as solid W (is) to pyramid $ABCG$, so pyramid $DEFH$ (is) to some (solid) less than pyramid $ABCG$, as shown before [Prop. 12.2 lem.]. And, thus, as base DEF (is) to base ABC , so pyramid $DEFH$ (is) to some (solid) less than pyramid $ABCG$ [Prop. 5.11]. The very thing was shown (to be) absurd. Thus, base ABC is not to base DEF , as pyramid $ABCG$ (is) to some solid greater than pyramid $DEFH$. And, it was shown that neither (is it in this ratio) to a lesser (solid). Thus, as base ABC is to base DEF , so pyramid $ABCG$ (is) to pyramid $DEFH$. (Which is) the very thing it was required to show.

τ'.

Αἱ ὑπὸ τὸ αὐτὸ ὕψος οὔσαι πυραμίδες καὶ πολυγώνους ἔχουσαι βάσεις πρὸς ἀλλήλας εἰσὶν ὡς αἱ βάσεις.

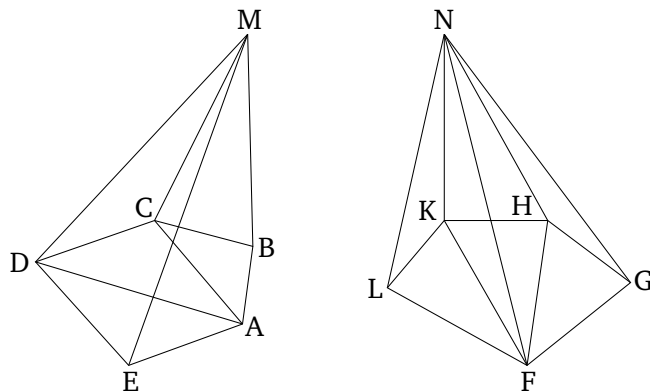


Ἐστωσαν ὑπὸ τὸ αὐτὸ ὕψος πυραμίδες, ὧν [αἱ] βάσεις μὲν τὰ $AB\Gamma\Delta E$, $ZH\Theta K\Lambda$ πολύγωνα, κορυφαὶ δὲ τὰ M , N σημεία· λέγω, ὅτι ἔστιν ὡς ἡ $AB\Gamma\Delta E$ βάσις πρὸς τὴν $ZH\Theta K\Lambda$ βάσιν, οὕτως ἡ $AB\Gamma\Delta E M$ πυραμὶς πρὸς τὴν $ZH\Theta K\Lambda N$ πυραμίδα.

Ἐπεξεύχθωσαν γὰρ αἱ AG , AD , $Z\Theta$, ZK . ἐπεὶ οὖν δύο πυραμίδες εἰσὶν αἱ $AB\Gamma M$, $AG\Delta M$ τριγώνους ἔχουσαι βάσεις καὶ ὕψος ἴσον, πρὸς ἀλλήλας εἰσὶν ὡς αἱ βάσεις· ἔστιν ἄρα ὡς ἡ $AB\Gamma$ βάσις πρὸς τὴν $AG\Delta$ βάσιν, οὕτως ἡ $AB\Gamma M$ πυραμὶς πρὸς τὴν $AG\Delta M$ πυραμίδα. καὶ συνθέντι ὡς ἡ $AB\Gamma\Delta$ βάσις πρὸς τὴν $AG\Delta$ βάσιν, οὕτως ἡ $AB\Gamma\Delta M$

Proposition 6

Pyramids which are of the same height, and have polygonal bases, are to one another as their bases.



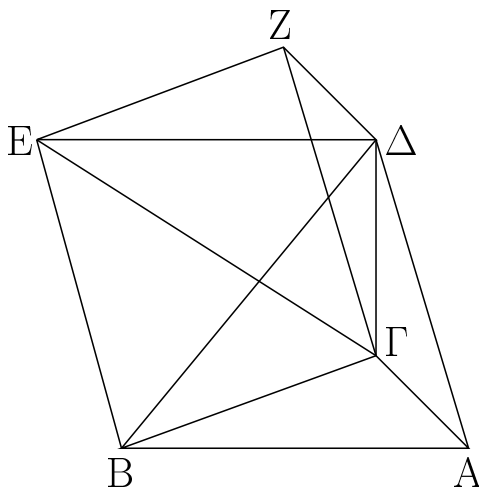
Let there be pyramids of the same height whose bases (are) the polygons $ABCDE$ and $FGHLK$, and apexes the points M and N (respectively). I say that as base $ABCDE$ is to base $FGHLK$, so pyramid $ABCDEM$ (is) to pyramid $FGHLKN$.

For let AC , AD , FH , and FK have been joined. Therefore, since $ABCM$ and $ACDM$ are two pyramids having triangular bases and equal height, they are to one another as their bases [Prop. 12.5]. Thus, as base ABC is to base ACD , so pyramid $ABCM$ (is) to pyramid $ACDM$. And, via composition, as base $ABCD$

πυραμὶς πρὸς τὴν ΑΓΔΜ πυραμίδα. ἀλλὰ καὶ ὡς ἡ ΑΓΔ βάσις πρὸς τὴν ΑΔΕ βάσιν, οὕτως ἡ ΑΓΔΜ πυραμὶς πρὸς τὴν ΑΔΕΜ πυραμίδα. δι' ἴσου ἄρα ὡς ἡ ΑΒΓΔ βάσις πρὸς τὴν ΑΔΕ βάσιν, οὕτως ἡ ΑΒΓΔΜ πυραμὶς πρὸς τὴν ΑΔΕΜ πυραμίδα. καὶ συνθέντι πάλιν, ὡς ἡ ΑΒΓΔΕ βάσις πρὸς τὴν ΑΔΕ βάσιν, οὕτως ἡ ΑΒΓΔΕΜ πυραμὶς πρὸς τὴν ΑΔΕΜ πυραμίδα. ὁμοίως δὲ δειχθήσεται, ὅτι καὶ ὡς ἡ ΖΗΘΚΑ βάσις πρὸς τὴν ΖΗΘ βάσιν, οὕτως καὶ ἡ ΖΗΘΚΑΝ πυραμὶς πρὸς τὴν ΖΗΘΝ πυραμίδα. καὶ ἐπεὶ δύο πυραμίδες εἰσὶν αἱ ΑΔΕΜ, ΖΗΘΝ τριγώνους ἔχουσαι βάσεις καὶ ὕψος ἴσον, ἔστιν ἄρα ὡς ἡ ΑΔΕ βάσις πρὸς τὴν ΖΗΘ βάσιν, οὕτως ἡ ΑΔΕΜ πυραμὶς πρὸς τὴν ΖΗΘΝ πυραμίδα. ἀλλ' ὡς ἡ ΑΔΕ βάσις πρὸς τὴν ΑΒΓΔΕ βάσιν, οὕτως ἡ ΑΔΕΜ πυραμὶς πρὸς τὴν ΑΒΓΔΕΜ πυραμίδα. καὶ δι' ἴσου ἄρα ὡς ἡ ΑΒΓΔΕ βάσις πρὸς τὴν ΖΗΘ βάσιν, οὕτως ἡ ΑΒΓΔΕΜ πυραμὶς πρὸς τὴν ΖΗΘΝ πυραμίδα. ἀλλὰ μὴν καὶ ὡς ἡ ΖΗΘ βάσις πρὸς τὴν ΖΗΘΚΑ βάσιν, οὕτως ἡ ΖΗΘΝ πυραμὶς πρὸς τὴν ΖΗΘΚΑΝ πυραμίδα, καὶ δι' ἴσου ἄρα ὡς ἡ ΑΒΓΔΕ βάσις πρὸς τὴν ΖΗΘΚΑ βάσιν, οὕτως ἡ ΑΒΓΔΕΜ πυραμὶς πρὸς τὴν ΖΗΘΚΑΝ πυραμίδα· ὅπερ ἔδει δεῖξαι.

ζ'.

Πᾶν πρίσμα τριγώνον ἔχον βάσιν διαιρεῖται εἰς τρεῖς πυραμίδας ἴσας ἀλλήλαις τριγώνους βάσεις ἔχούσας.



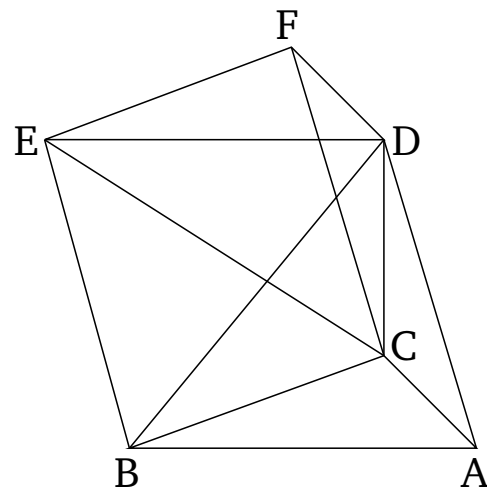
Ἐστω πρίσμα, οὗ βάσις μὲν τὸ ΑΒΓ τρίγωνον, ἀπεναντίον δὲ τὸ ΔΕΖ· λέγω, ὅτι τὸ ΑΒΓΔΕΖ πρίσμα διαιρεῖται εἰς τρεῖς πυραμίδας ἴσας ἀλλήλαις τριγώνους ἔχούσας βάσεις.

Ἐπεζεύχθωσαν γὰρ αἱ ΒΔ, ΕΓ, ΓΔ. ἐπεὶ παραλληλόγραμμον ἔστι τὸ ΑΒΕΔ, διάμετρος δὲ αὐτοῦ ἔστιν ἡ ΒΔ, ἴσον ἄρα ἔστι τὸ ΑΒΔ τρίγωνον τῷ ΕΒΔ τριγώνῳ·

(is) to base ACD , so pyramid $ABCDM$ (is) to pyramid $ACDM$ [Prop. 5.18]. But, as base ACD (is) to base ADE , so pyramid $ACDM$ (is) also to pyramid $ADEM$ [Prop. 12.5]. Thus, via equality, as base $ABCD$ (is) to base ADE , so pyramid $ABCDM$ (is) to pyramid $ADEM$ [Prop. 5.22]. And, again, via composition, as base $ABCDE$ (is) to base ADE , so pyramid $ABCDEM$ (is) to pyramid $ADEM$ [Prop. 5.18]. So, similarly, it can also be shown that as base $FGHKL$ (is) to base FGH , so pyramid $FGHKLN$ (is) also to pyramid $FGHN$. And since $ADEM$ and $FGHN$ are two pyramids having triangular bases and equal height, thus as base ADE (is) to base FGH , so pyramid $ADEM$ (is) to pyramid $FGHN$ [Prop. 12.5]. But, as base ADE (is) to base $ABCDE$, so pyramid $ADEM$ (was) to pyramid $ABCDEM$. Thus, via equality, as base $ABCDE$ (is) to base FGH , so pyramid $ABCDEM$ (is) also to pyramid $FGHN$ [Prop. 5.22]. But, furthermore, as base FGH (is) to base $FGHKL$, so pyramid $FGHN$ was also to pyramid $FGHKLN$. Thus, via equality, as base $ABCDE$ (is) to base $FGHKL$, so pyramid $ABCDEM$ (is) also to pyramid $FGHKLN$ [Prop. 5.22]. (Which is) the very thing it was required to show.

Proposition 7

Any prism having a triangular base is divided into three pyramids having triangular bases (which are) equal to one another.



Let there be a prism whose base (is) triangle ABC , and opposite (plane) DEF . I say that prism $ABCDEF$ is divided into three pyramids having triangular bases (which are) equal to one another.

For let BD , EC , and CD have been joined. Since $ABED$ is a parallelogram, and BD is its diagonal, triangle ABD is thus equal to triangle EBD [Prop. 1.34].

καὶ ἡ πυραμὶς ἄρα, ἥς βάσις μὲν τὸ $AB\Delta$ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, ἴση ἐστὶ πυραμίδι, ἥς βάσις μὲν ἐστὶ τὸ ΔEB τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον. ἀλλὰ ἡ πυραμὶς, ἥς βάσις μὲν ἐστὶ τὸ ΔEB τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, ἡ αὐτὴ ἐστὶ πυραμίδι, ἥς βάσις μὲν ἐστὶ τὸ $EB\Gamma$ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον· ὑπὸ γὰρ τῶν αὐτῶν ἐπιπέδων περιέχεται. καὶ πυραμὶς ἄρα, ἥς βάσις μὲν ἐστὶ τὸ $AB\Delta$ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, ἴση ἐστὶ πυραμίδι, ἥς βάσις μὲν ἐστὶ τὸ $EB\Gamma$ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον. πάλιν, ἐπεὶ παραλληλόγραμμον ἐστὶ τὸ $Z\Gamma BE$, διάμετρος δὲ ἐστὶν αὐτοῦ ἡ ΓE , ἴσον ἐστὶ τὸ ΓEZ τρίγωνον τῷ ΓBE τριγώνῳ. καὶ πυραμὶς ἄρα, ἥς βάσις μὲν ἐστὶ τὸ ΓBE τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ἴση ἐστὶ πυραμίδι, ἥς βάσις μὲν ἐστὶ τὸ ΓEZ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον. ἡ δὲ πυραμὶς, ἥς βάσις μὲν ἐστὶ τὸ ΓBE τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ἴση ἐδείχθη πυραμίδι, ἥς βάσις μὲν ἐστὶ τὸ $AB\Delta$ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον· καὶ πυραμὶς ἄρα, ἥς βάσις μὲν ἐστὶ τὸ ΓEZ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, ἴση ἐστὶ πυραμίδι, ἥς βάσις μὲν ἐστὶ τὸ $AB\Delta$ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον· διήρηται ἄρα τὸ $AB\Gamma\Delta EZ$ πρίσμα εἰς τρεῖς πυραμίδας ἴσας ἀλλήλαις τριγώνους ἔχουσας βάσεις.

Καὶ ἐπεὶ πυραμὶς, ἥς βάσις μὲν ἐστὶ τὸ $AB\Delta$ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, ἡ αὐτὴ ἐστὶ πυραμίδι, ἥς βάσις τὸ ΓAB τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον· ὑπὸ γὰρ τῶν αὐτῶν ἐπιπέδων περιέχονται· ἡ δὲ πυραμὶς, ἥς βάσις τὸ $AB\Delta$ τρίγωνον, κορυφὴ δὲ τὸ Γ σημεῖον, τρίτον ἐδείχθη τοῦ πρίσματος, οὗ βάσις τὸ $AB\Gamma$ τρίγωνον, ἀπεναντίον δὲ τὸ ΔEZ , καὶ ἡ πυραμὶς ἄρα, ἥς βάσις τὸ $AB\Gamma$ τρίγωνον, κορυφὴ δὲ τὸ Δ σημεῖον, τρίτον ἐστὶ τοῦ πρίσματος τοῦ ἔχοντος βάσις τὴν αὐτὴν τὸ $AB\Gamma$ τρίγωνον, ἀπεναντίον δὲ τὸ ΔEZ .

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι πᾶσα πυραμὶς τρίτον μέρος ἐστὶ τοῦ πρίσματος τοῦ τὴν αὐτὴν βάσιν ἔχοντος αὐτῇ καὶ ὕψος ἴσον· ὅπερ ἔδει δεῖξαι.

η'.

Αἱ ὅμοιαι πυραμίδες καὶ τριγώνους ἔχουσαι βάσεις ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν.

Ἐστωσαν ὅμοιαι καὶ ὁμοίως κείμεναι πυραμίδες, ὧν βάσεις μὲν εἰσὶ τὰ $AB\Gamma$, ΔEZ τρίγωνα, κορυφαὶ δὲ τὰ H , Θ σημεία· λέγω, ὅτι ἡ $AB\Gamma H$ πυραμὶς πρὸς τὴν $\Delta EZ\Theta$ πυραμίδα τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Gamma$ πρὸς τὴν EZ .

And, thus, the pyramid whose base (is) triangle ABD , and apex the point C , is equal to the pyramid whose base is triangle DEB , and apex the point C [Prop. 12.5]. But, the pyramid whose base is triangle DEB , and apex the point C , is the same as the pyramid whose base is triangle EBC , and apex the point D . For they are contained by the same planes. And, thus, the pyramid whose base is ABD , and apex the point C , is equal to the pyramid whose base is EBC and apex the point D . Again, since $FCBE$ is a parallelogram, and CE is its diagonal, triangle CEF is equal to triangle CBE [Prop. 1.34]. And, thus, the pyramid whose base is triangle BCE , and apex the point D , is equal to the pyramid whose base is triangle ECF , and apex the point D [Prop. 12.5]. And the pyramid whose base is triangle BCE , and apex the point D , was shown (to be) equal to the pyramid whose base is triangle ABD , and apex the point C . Thus, the pyramid whose base is triangle CEF , and apex the point D , is also equal to the pyramid whose base [is] triangle ABD , and apex the point C . Thus, the prism $ABCDEF$ has been divided into three pyramids having triangular bases (which are) equal to one another.

And since the pyramid whose base is triangle ABD , and apex the point C , is the same as the pyramid whose base is triangle CAB , and apex the point D . For they are contained by the same planes. And the pyramid whose base (is) triangle ABD , and apex the point C , was shown (to be) a third of the prism whose base is triangle ABC , and opposite (plane) DEF , thus the pyramid whose base is triangle ABC , and apex the point D , is also a third of the pyramid having the same base, triangle ABC , and opposite (plane) DEF .

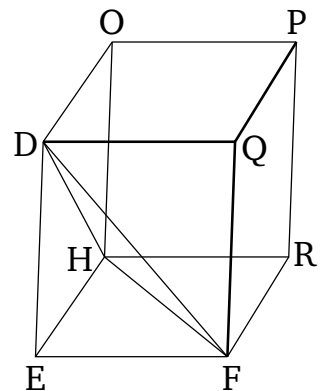
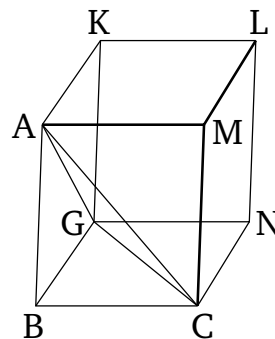
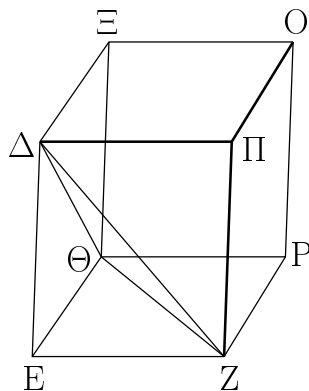
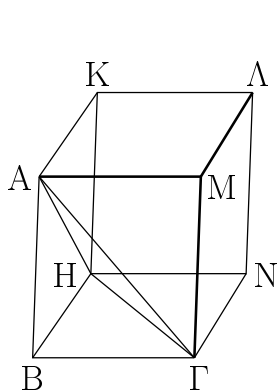
Corollary

And, from this, (it is) clear that any pyramid is the third part of the prism having the same base as it, and an equal height. (Which is) the very thing it was required to show.

Proposition 8

Similar pyramids which also have triangular bases are in the cubed ratio of their corresponding sides.

Let there be similar, and similarly laid out, pyramids whose bases are triangles ABC and DEF , and apexes the points G and H (respectively). I say that pyramid $ABCG$ has to pyramid $DEFH$ the cubed ratio of that BC (has) to EF .



Συμπεπληρώσθω γὰρ τὰ $BHML$, $EΘΠΟ$ στερεὰ παραλληλεπίπεδα. καὶ ἐπεὶ ὁμοία ἐστὶν ἡ $ABΓH$ πυραμὶς τῇ $ΔΕΖΘ$ πυραμίδι, ἴση ἄρα ἐστὶν ἡ μὲν ὑπὸ $ABΓ$ γωνία τῇ ὑπὸ $ΔΕΖ$ γωνίᾳ, ἡ δὲ ὑπὸ $HΒΓ$ τῇ ὑπὸ $ΘΕΖ$, ἡ δὲ ὑπὸ ABH τῇ ὑπὸ $ΔΕΘ$, καὶ ἐστὶν ὡς ἡ AB πρὸς τὴν $ΔΕ$, οὕτως ἡ $BΓ$ πρὸς τὴν $ΕΖ$, καὶ ἡ BH πρὸς τὴν $ΕΘ$. καὶ ἐπεὶ ἐστὶν ὡς ἡ AB πρὸς τὴν $ΔΕ$, οὕτως ἡ $BΓ$ πρὸς τὴν $ΕΖ$, καὶ περὶ ἴσας γωνίας αἱ πλευραὶ ἀνάλογόν εἰσιν, ὅμοιον ἄρα ἐστὶ τὸ BM παραλληλόγραμμον τῷ $ΕΠ$ παραλληλογράμμῳ. διὰ τὰ αὐτὰ δὴ καὶ τὸ μὲν BN τῷ $ΕΡ$ ὁμοιὸν ἐστὶ, τὸ δὲ BK τῷ $ΕΞ$. τὰ τρία ἄρα τὰ MB , BK , BN τρισὶ τοῖς $ΕΠ$, $ΕΞ$, $ΕΡ$ ὁμοία ἐστὶν. ἀλλὰ τὰ μὲν τρία τὰ MB , BK , BN τρισὶ τοῖς ἀπεναντίον ἴσα τε καὶ ὁμοία ἐστὶν, τὰ δὲ τρία τὰ $ΕΠ$, $ΕΞ$, $ΕΡ$ τρισὶ τοῖς ἀπεναντίον ἴσα τε καὶ ὁμοία ἐστὶν. τὰ $BHML$, $EΘΠΟ$ ἄρα στερεὰ ὑπὸ ὁμοίων ἐπιπέδων ἴσων τὸ πλῆθος περιέχεται. ὅμοιον ἄρα ἐστὶ τὸ $BHML$ στερεὸν τῷ $EΘΠΟ$ στερεῷ. τὰ δὲ ὁμοία στερεὰ παραλληλεπίπεδα ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν. τὸ $BHML$ ἄρα στερεὸν πρὸς τὸ $EΘΠΟ$ στερεὸν τριπλασίονα λόγον ἔχει ἥπερ ἡ ὁμόλογος πλευρὰ ἡ $BΓ$ πρὸς τὴν ὁμόλογον πλευρὰν τὴν $ΕΖ$. ὡς δὲ τὸ $BHML$ στερεὸν πρὸς τὸ $EΘΠΟ$ στερεόν, οὕτως ἡ $ABΓH$ πυραμὶς πρὸς τὴν $ΔΕΖΘ$ πυραμίδα, ἐπειδὴ περ ἡ πυραμὶς ἕκτον μέρος ἐστὶ τοῦ στερεοῦ διὰ τὸ καὶ τὸ πρίσμα ἡμισυὸν τοῦ στερεοῦ παραλληλεπιπέδου τριπλασίον εἶναι τῆς πυραμίδος. καὶ ἡ $ABΓH$ ἄρα πυραμὶς πρὸς τὴν $ΔΕΖΘ$ πυραμίδα τριπλασίονα λόγον ἔχει ἥπερ ἡ $BΓ$ πρὸς τὴν $ΕΖ$. ὅπερ εἶδει δεῖξαι.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι καὶ αἱ πολυγώνους ἔχουσαι βάσεις ὁμοίαι πυραμίδες πρὸς ἀλλήλας ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν. διαιρεθεισῶν γὰρ αὐτῶν εἰς τὰς ἐν αὐταῖς πυραμίδας τριγώνους βάσεις ἐχούσας τῷ καὶ τὰ ὁμοία πολύγωνα τῶν βάσεων εἰς ὁμοία τρίγωνα διαιρεῖσθαι καὶ ἴσα τῷ πλήθει καὶ ὁμόλογα τοῖς ὅλοις ἔσται

For let the parallelepiped solids $BGML$ and $EHQP$ have been completed. And since pyramid $ABCG$ is similar to pyramid $DEFH$, angle ABC is thus equal to angle DEF , and GBC to HEF , and ABG to DEH . And as AB is to DE , so BC (is) to EF , and BG to EH [Def. 11.9]. And since as AB is to DE , so BC (is) to EF , and (so) the sides around equal angles are proportional, parallelogram BM is thus similar to parallelogram EQ . So, for the same (reasons), BN is also similar to ER , and BK to EO . Thus, the three (parallelograms) MB , BK , and BN are similar to the three (parallelograms) EQ , EO , ER (respectively). But, the three (parallelograms) MB , BK , and BN are (both) equal and similar to the three opposite (parallelograms), and the three (parallelograms) EQ , EO , and ER are (both) equal and similar to the three opposite (parallelograms) [Prop. 11.24]. Thus, the solids $BGML$ and $EHQP$ are contained by equal numbers of similar (and similarly laid out) planes. Thus, solid $BGML$ is similar to solid $EHQP$ [Def. 11.9]. And similar parallelepiped solids are in the cubed ratio of corresponding sides [Prop. 11.33]. Thus, solid $BGML$ has to solid $EHQP$ the cubed ratio that the corresponding side BC (has) to the corresponding side EF . And as solid $BGML$ (is) to solid $EHQP$, so pyramid $ABCG$ (is) to pyramid $DEFH$, inasmuch as the pyramid is the sixth part of the solid, on account of the prism, being half of the parallelepiped solid [Prop. 11.28], also being three times the pyramid [Prop. 12.7]. Thus, pyramid $ABCG$ also has to pyramid $DEFH$ the cubed ratio that BC (has) to EF . (Which is) the very thing it was required to show.

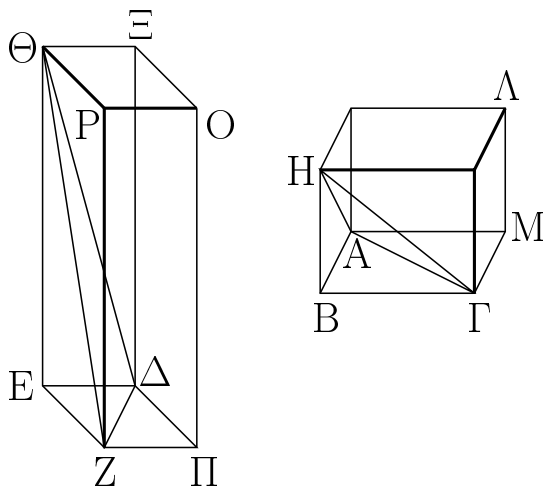
Corollary

So, from this, (it is) also clear that similar pyramids having polygonal bases (are) to one another as the cubed ratio of their corresponding sides. For, dividing them into the pyramids (contained) within them which have triangular bases, with the similar polygons of the bases also being divided into similar triangles (which are)

ὥς [ἡ] ἐν τῇ ἐτέρᾳ μία πυραμὶς τρίγωνον ἔχουσα βάσιν πρὸς τὴν ἐν τῇ ἐτέρᾳ μίαν πυραμίδα τρίγωνον ἔχουσαν βάσιν, οὕτως καὶ ἅπασαι αἱ ἐν τῇ ἐτέρᾳ πυραμίδι πυραμίδες τριγώνους ἔχουσαι βάσεις πρὸς τὰς ἐν τῇ ἐτέρᾳ πυραμίδι πυραμίδας τριγώνους βάσεις ἔχούσας, τουτέστιν αὐτὴ ἡ πολύγωνον βάσιν ἔχουσα πυραμὶς πρὸς τὴν πολύγωνον βάσιν ἔχουσαν πυραμίδα. ἡ δὲ τρίγωνον βάσιν ἔχουσα πυραμὶς πρὸς τὴν τρίγωνον βάσιν ἔχουσαν ἐν τριπλασίονι λόγῳ ἐστὶ τῶν ὁμολόγων πλευρῶν· καὶ ἡ πολύγωνον ἄρα βάσιν ἔχουσα πρὸς τὴν ὁμοίαν βάσιν ἔχουσαν τριπλασίονα λόγον ἔχει ἢ περ ἢ πλευρὰ πρὸς τὴν πλευράν.

θ'.

Τῶν ἴσων πυραμίδων καὶ τριγώνους βάσεις ἔχουσῶν ἀντιτεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν πυραμίδων τριγώνους βάσεις ἔχουσῶν ἀντιτεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσαι εἰσὶν ἐκείναι.



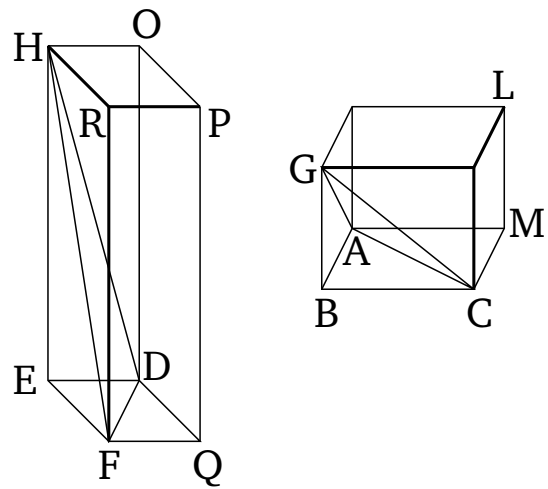
Ἐστωσαν γὰρ ἴσαι πυραμίδες τριγώνους βάσεις ἔχουσαι τὰς ABΓ, ΔΕΖ, κορυφὰς δὲ τὰ H, Θ σημεία· λέγω, ὅτι τῶν ABΓH, ΔΕΖΘ πυραμίδων ἀντιτεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἐστὶν ὥς ἡ ABΓ βάσις πρὸς τὴν ΔΕΖ βάσιν, οὕτως τὸ τῆς ΔΕΖΘ πυραμίδος ὕψος πρὸς τὸ τῆς ABΓH πυραμίδος ὕψος.

Συμπεπληρώσθω γὰρ τὰ BHMA, EΘΠO στερεὰ παραλληλεπίπεδα. καὶ ἐπεὶ ἴση ἐστὶν ἡ ABΓH πυραμὶς τῇ ΔΕΖΘ πυραμίδι, καὶ ἐστὶ τῆς μὲν ABΓH πυραμίδος ἕξαπλάσιον τὸ BHMA στερεόν, τῆς δὲ ΔΕΖΘ πυραμίδος ἕξαπλάσιον τὸ EΘΠO στερεόν, ἴσον ἄρα ἐστὶ τὸ BHMA στερεόν τῷ EΘΠO στερεῷ. τῶν δὲ ἴσων στερεῶν παραλληλεπιπύδων

both equal in number, and corresponding, to the wholes [Prop. 6.20]. As one pyramid having a triangular base in the former (pyramid having a polygonal base is) to one pyramid having a triangular base in the latter (pyramid having a polygonal base), so (the sum of) all the pyramids having triangular bases in the former pyramid will also be to (the sum of) all the pyramids having triangular bases in the latter pyramid [Prop. 5.12]—that is to say, the (former) pyramid itself having a polygonal base to the (latter) pyramid having a polygonal base. And a pyramid having a triangular base is to a (pyramid) having a triangular base in the cubed ratio of corresponding sides [Prop. 12.8]. Thus, a (pyramid) having a polygonal base also has to to a (pyramid) having a similar base the cubed ratio of a (corresponding) side to a (corresponding) side.

Proposition 9

The bases of equal pyramids which also have triangular bases are reciprocally proportional to their heights. And those pyramids which have triangular bases whose bases are reciprocally proportional to their heights are equal.



For let there be (two) equal pyramids having the triangular bases ABC and DEF , and apexes the points G and H (respectively). I say that the bases of the pyramids $ABCG$ and $DEFH$ are reciprocally proportional to their heights, and (so) that as base ABC is to base DEF , so the height of pyramid $DEFH$ (is) to the height of pyramid $ABCG$.

For let the parallelepiped solids $BGML$ and $EHQP$ have been completed. And since pyramid $ABCG$ is equal to pyramid $DEFH$, and solid $BGML$ is six times pyramid $ABCG$ (see previous proposition), and solid $EHQP$ (is) six times pyramid $DEFH$, solid $BGML$ is

ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· ἔστιν ἄρα ὡς ἡ BM βάσις πρὸς τὴν $ΕΠ$ βάσιν, οὕτως τὸ τοῦ $ΕΘΠΟ$ στερεοῦ ὕψος πρὸς τὸ τοῦ $BHMA$ στερεοῦ ὕψος. ἀλλ' ὡς ἡ BM βάσις πρὸς τὴν $ΕΠ$, οὕτως τὸ $ABΓ$ τρίγωνον πρὸς τὸ $ΔΕΖ$ τρίγωνον. καὶ ὡς ἄρα τὸ $ABΓ$ τρίγωνον πρὸς τὸ $ΔΕΖ$ τρίγωνον, οὕτως τὸ τοῦ $ΕΘΠΟ$ στερεοῦ ὕψος πρὸς τὸ τοῦ $BHMA$ στερεοῦ ὕψος. ἀλλὰ τὸ μὲν τοῦ $ΕΘΠΟ$ στερεοῦ ὕψος τὸ αὐτὸ ἐστὶ τῷ τῆς $ΔΕΖΘ$ πυραμίδος ὕψει, τὸ δὲ τοῦ $BHMA$ στερεοῦ ὕψος τὸ αὐτὸ ἐστὶ τῷ τῆς $ABΓΗ$ πυραμίδος ὕψει· ἔστιν ἄρα ὡς ἡ $ABΓ$ βάσις πρὸς τὴν $ΔΕΖ$ βάσιν, οὕτως τὸ τῆς $ΔΕΖΘ$ πυραμίδος ὕψος πρὸς τὸ τῆς $ABΓΗ$ πυραμίδος ὕψος. τῶν $ABΓΗ$, $ΔΕΖΘ$ ἄρα πυραμίδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Ἀλλὰ δὴ τῶν $ABΓΗ$, $ΔΕΖΘ$ πυραμίδων ἀντιπεπονθέντων αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ $ABΓ$ βάσις πρὸς τὴν $ΔΕΖ$ βάσιν, οὕτως τὸ τῆς $ΔΕΖΘ$ πυραμίδος ὕψος πρὸς τὸ τῆς $ABΓΗ$ πυραμίδος ὕψος· λέγω, ὅτι ἴση ἐστὶν ἡ $ABΓΗ$ πυραμὶς τῇ $ΔΕΖΘ$ πυραμίδι.

Τῶν γὰρ αὐτῶν κατασκευασθέντων, ἐπεὶ ἐστὶν ὡς ἡ $ABΓ$ βάσις πρὸς τὴν $ΔΕΖ$ βάσιν, οὕτως τὸ τῆς $ΔΕΖΘ$ πυραμίδος ὕψος πρὸς τὸ τῆς $ABΓΗ$ πυραμίδος ὕψος, ἀλλ' ὡς ἡ $ABΓ$ βάσις πρὸς τὴν $ΔΕΖ$ βάσιν, οὕτως τὸ BM παραλληλόγραμμον πρὸς τὸ $ΕΠ$ παραλληλόγραμμον, καὶ ὡς ἄρα τὸ BM παραλληλόγραμμον πρὸς τὸ $ΕΠ$ παραλληλόγραμμον, οὕτως τὸ τῆς $ΔΕΖΘ$ πυραμίδος ὕψος πρὸς τὸ τῆς $ABΓΗ$ πυραμίδος ὕψος. ἀλλὰ τὸ [μὲν] τῆς $ΔΕΖΘ$ πυραμίδος ὕψος τὸ αὐτὸ ἐστὶ τῷ τοῦ $ΕΘΠΟ$ παραλληλεπιπέδου ὕψει, τὸ δὲ τῆς $ABΓΗ$ πυραμίδος ὕψος τὸ αὐτὸ ἐστὶ τῷ τοῦ $BHMA$ παραλληλεπιπέδου ὕψει· ἔστιν ἄρα ὡς ἡ BM βάσις πρὸς τὴν $ΕΠ$ βάσιν, οὕτως τὸ τοῦ $ΕΘΠΟ$ παραλληλεπιπέδου ὕψος πρὸς τὸ τοῦ $BHMA$ παραλληλεπιπέδου ὕψος. ὦν δὲ στερεῶν παραλληλεπιπέδων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσα ἐστὶν ἐκεῖνα· ἴσον ἄρα ἐστὶ τὸ $BHMA$ στερεὸν παραλληλεπίπεδον τῷ $ΕΘΠΟ$ στερεῷ παραλληλεπίπεδῳ. καὶ ἐστὶ τοῦ μὲν $BHMA$ ἕκτον μέρος ἡ $ABΓΗ$ πυραμὶς, τοῦ δὲ $ΕΘΠΟ$ παραλληλεπιπέδου ἕκτον μέρος ἡ $ΔΕΖΘ$ πυραμὶς· ἴση ἄρα ἡ $ABΓΗ$ πυραμὶς τῇ $ΔΕΖΘ$ πυραμίδι.

Τῶν ἄρα ἴσων πυραμίδων καὶ τριγώνους βάσεις ἔχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὦν πυραμίδων τριγώνους βάσεις ἔχουσῶν ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσαι εἰσὶν ἐκεῖναι· ὅπερ ἔδει δεῖξαι.

ι'.

Πᾶς κῶνος κυλίνδρου τρίτον μέρος ἐστὶ τοῦ τὴν αὐτὴν βάσιν ἔχοντος αὐτῷ καὶ ὕψος ἴσον.

Ἐχέτω γὰρ κῶνος κυλίνδρῳ βάσιν τε τὴν αὐτὴν τὸν

thus equal to solid $EHQP$. And the bases of equal parallelepiped solids are reciprocally proportional to their heights [Prop. 11.34]. Thus, as base BM is to base EQ , so the height of solid $EHQP$ (is) to the height of solid $BGML$. But, as base BM (is) to base EQ , so triangle ABC (is) to triangle DEF [Prop. 1.34]. And, thus, as triangle ABC (is) to triangle DEF , so the height of solid $EHQP$ (is) to the height of solid $BGML$ [Prop. 5.11]. But, the height of solid $EHQP$ is the same as the height of pyramid $DEFH$, and the height of solid $BGML$ is the same as the height of pyramid $ABCG$. Thus, as base ABC is to base DEF , so the height of pyramid $DEFH$ (is) to the height of pyramid $ABCG$. Thus, the bases of pyramids $ABCG$ and $DEFH$ are reciprocally proportional to their heights.

And so, let the bases of pyramids $ABCG$ and $DEFH$ be reciprocally proportional to their heights, and (thus) let base ABC be to base DEF , as the height of pyramid $DEFH$ (is) to the height of pyramid $ABCG$. I say that pyramid $ABCG$ is equal to pyramid $DEFH$.

For, with the same construction, since as base ABC is to base DEF , so the height of pyramid $DEFH$ (is) to the height of pyramid $ABCG$, but as base ABC (is) to base DEF , so parallelogram BM (is) to parallelogram EQ [Prop. 1.34], thus as parallelogram BM (is) to parallelogram EQ , so the height of pyramid $DEFH$ (is) also to the height of pyramid $ABCG$ [Prop. 5.11]. But, the height of pyramid $DEFH$ is the same as the height of parallelepiped $EHQP$, and the height of pyramid $ABCG$ is the same as the height of parallelepiped $BGML$. Thus, as base BM is to base EQ , so the height of parallelepiped $EHQP$ (is) to the height of parallelepiped $BGML$. And those parallelepiped solids whose bases are reciprocally proportional to their heights are equal [Prop. 11.34]. Thus, the parallelepiped solid $BGML$ is equal to the parallelepiped solid $EHQP$. And pyramid $ABCG$ is a sixth part of $BGML$, and pyramid $DEFH$ a sixth part of parallelepiped $EHQP$. Thus, pyramid $ABCG$ is equal to pyramid $DEFH$.

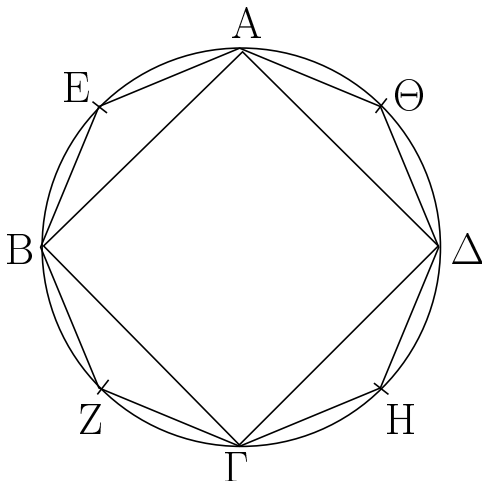
Thus, the bases of equal pyramids which also have triangular bases are reciprocally proportional to their heights. And those pyramids having triangular bases whose bases are reciprocally proportional to their heights are equal. (Which is) the very thing it was required to show.

Proposition 10

Every cone is the third part of the cylinder which has the same base as it, and an equal height.

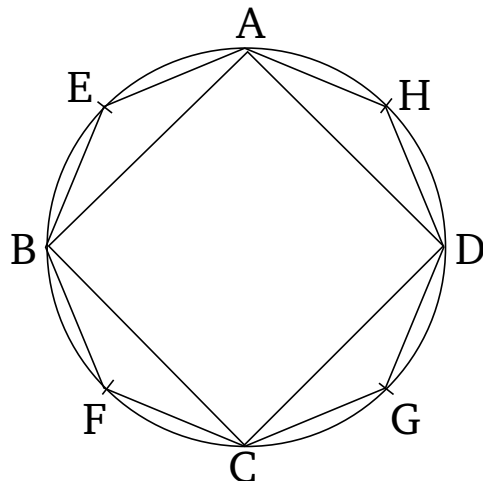
For let there be a cone (with) the same base as a cylin-

ΑΒΓΔ κύκλον καὶ ὕψος ἴσον· λέγω, ὅτι ὁ κώνος τοῦ κυλίνδρου τρίτον ἐστὶ μέρος, τουτέστιν ὅτι ὁ κύλινδρος τοῦ κώνου τριπλασίων ἐστίν.



Εἰ γὰρ μὴ ἐστὶν ὁ κύλινδρος τοῦ κώνου τριπλασίων, ἔσται ὁ κύλινδρος τοῦ κώνου ἥτοι μείζων ἢ τριπλασίων ἢ ἐλάσσων ἢ τριπλασίων. ἔστω πρότερον μείζων ἢ τριπλασίων, καὶ ἐγγεγράφθω εἰς τὸν ΑΒΓΔ κύκλον τετράγωνον τὸ ΑΒΓΔ· τὸ δὲ ΑΒΓΔ τετράγωνον μείζον ἐστὶν ἢ τὸ ἥμισυ τοῦ ΑΒΓΔ κύκλου. καὶ ἀνεστιάτω ἀπὸ τοῦ ΑΒΓΔ τετραγώνου πρίσμα ἰσοῦψές τῷ κυλίνδρῳ. τὸ δὲ ἀνιστάμενον πρίσμα μείζον ἐστὶν ἢ τὸ ἥμισυ τοῦ κυλίνδρου, ἐπειδὴ περὶ τὸν ΑΒΓΔ κύκλον τετράγωνον περιγράψωμεν, τὸ ἐγγεγραμμένον εἰς τὸν ΑΒΓΔ κύκλον τετράγωνον ἥμισυ ἐστὶ τοῦ περιγεγραμμένου· καὶ ἐστὶ τὰ ἀπ' αὐτῶν ἀνιστάμενα στερεὰ παραλληλεπίπεδα πρίσματα ἰσοῦψή· τὰ δὲ ὑπὸ τὸ αὐτὸ ὕψος ὄντα στερεὰ παραλληλεπίπεδα πρὸς ἀλλήλα ἐστὶν ὡς αἱ βάσεις· καὶ τὸ ἐπὶ τοῦ ΑΒΓΔ ἄρα τετραγώνου ἀνασταθέν πρίσμα ἥμισυ ἐστὶ τοῦ ἀνασταθέντος πρίσματος ἀπὸ τοῦ περὶ τὸν ΑΒΓΔ κύκλον περιγραφέντος τετραγώνου· καὶ ἐστὶν ὁ κύλινδρος ἐλάττω τοῦ πρίσματος τοῦ ἀνατραθέντος ἀπὸ τοῦ περὶ τὸν ΑΒΓΔ κύκλον περιγραφέντος τετραγώνου· τὸ ἄρα πρίσμα τὸ ἀνασταθέν ἀπὸ τοῦ ΑΒΓΔ τετραγώνου ἰσοῦψές τῷ κυλίνδρῳ μείζον ἐστὶ τοῦ ἡμίσεως τοῦ κυλίνδρου. τετμήσθωσαν αἱ ΑΒ, ΒΓ, ΓΔ, ΔΑ περιφέρειαι δίχα κατὰ τὰ Ε, Ζ, Η, Θ σημεῖα, καὶ ἐπεζεύχθωσαν αἱ ΑΕ, ΕΒ, ΒΖ, ΖΓ, ΓΗ, ΗΔ, ΔΘ, ΘΑ· καὶ ἕκαστον ἄρα τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων μείζον ἐστὶν ἢ τὸ ἥμισυ τοῦ καθ' ἑαυτὸ τμήματος τοῦ ΑΒΓΔ κύκλου, ὡς ἔμπροσθεν ἐδείκνυμεν. ἀνεστιάτω ἐφ' ἑκάστου τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων πρίσματα ἰσοῦψή τῷ κυλίνδρῳ· καὶ ἕκαστον ἄρα τῶν ἀνασταθέντων πρισμάτων μείζον ἐστὶν ἢ τὸ ἥμισυ μέρος τοῦ καθ' ἑαυτὸ τμήματος τοῦ κυλίνδρου, ἐπειδὴ περὶ ἐὰν διὰ τῶν Ε, Ζ, Η, Θ σημείων παραλλήλους ταῖς ΑΒ, ΒΓ, ΓΔ, ΔΑ ἀγάγωμεν, καὶ συμπληρώσωμεν τὰ ἐπὶ τῶν ΑΒ, ΒΓ, ΓΔ, ΔΑ παραλ-

der, (namely) the circle $ABCD$, and an equal height. I say that the cone is the third part of the cylinder—that is to say, that the cylinder is three times the cone.



For if the cylinder is not three times the cone then the cylinder will be either more than three times, or less than three times, (the cone). Let it, first of all, be more than three times (the cone). And let the square $ABCD$ have been inscribed in circle $ABCD$ [Prop. 4.6]. So, square $ABCD$ is more than half of circle $ABCD$ [Prop. 12.2]. And let a prism of equal height to the cylinder have been set up on square $ABCD$. So, the prism set up is more than half of the cylinder, inasmuch as if we also circumscribe a square around circle $ABCD$ [Prop. 4.7] then the square inscribed in circle $ABCD$ is half of the circumscribed (square). And the solids set up on them are parallelepiped prisms of equal height. And parallelepiped solids having the same height are to one another as their bases [Prop. 11.32]. And, thus, the prism set up on square $ABCD$ is half of the prism set up on the square circumscribed about circle $ABCD$. And the cylinder is less than the prism set up on the square circumscribed about circle $ABCD$. Thus, the prism set up on square $ABCD$ of the same height as the cylinder is more than half of the cylinder. Let the circumferences AB , BC , CD , and DA have been cut in half at points E , F , G , and H . And let AE , EB , BF , FC , CG , GD , DH , and HA have been joined. And thus each of the triangles AEB , BFC , CGD , and DHA is more than half of the segment of circle $ABCD$ about it, as was shown previously [Prop. 12.2]. Let prisms of equal height to the cylinder have been set up on each of the triangles AEB , BFC , CGD , and DHA . And each of the prisms set up is greater than the half part of the segment of the cylinder about it—inasmuch as if we draw (straight-lines) parallel to AB , BC , CD , and DA through points E , F , G , and H

ληλόγραμμα, καὶ ἀπ' αὐτῶν ἀναστήσωμεν στερεὰ παραλλη-
λεπίπεδα ἰσοῦψῃ τῷ κυλίνδρῳ, ἐκάστου τῶν ἀνασταθέντων
ἡμίση ἐστὶ τὰ πρίσματα τὰ ἐπὶ τῶν AEB , $BZΓ$, $ΓΗΔ$, $ΔΘΑ$
τριγώνων· καὶ ἐστὶ τὰ τοῦ κυλίνδρου τμήματα ἐλάττωνα
τῶν ἀνασταθέντων στερεῶν παραλληλεπίπεδων· ὥστε καὶ
τὰ ἐπὶ τῶν AEB , $BZΓ$, $ΓΗΔ$, $ΔΘΑ$ τριγώνων πρίσματα
μεῖζονά ἐστιν ἢ τὸ ἥμισυ τῶν καθ' ἑαυτὰ τοῦ κυλίνδρου
τμημάτων. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας
δίχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐφ' ἐκάστου
τῶν τριγώνων πρίσματα ἰσοῦψῃ τῷ κυλίνδρῳ καὶ τοῦτο αἶ
ποιοῦντες καταλείβομεν τινὰ ἀποτμήματα τοῦ κυλίνδρου,
ἃ ἔσται ἐλάττωνα τῆς ὑπεροχῆς, ἢ ὑπερέχει ὁ κυλίνδρος
τοῦ τριπλασίου τοῦ κώνου. λελείφθω, καὶ ἔστω τὰ AE ,
 EB , BZ , $ZΓ$, $ΓΗ$, $ΗΔ$, $ΔΘ$, $ΘΑ$ · λοιπὸν ἄρα τὸ πρίσμα, οὗ
βάσις μὲν τὸ $AEBZΓΗΔΘ$ πολύγωνον, ὕψος δὲ τὸ αὐτὸ
τῷ κυλίνδρῳ, μεῖζόν ἐστὶν ἢ τριπλάσιον τοῦ κώνου. ἀλλὰ
τὸ πρίσμα, οὗ βάσις μὲν ἐστὶ τὸ $AEBZΓΗΔΘ$ πολύγωνον,
ὕψος δὲ τὸ αὐτὸ τῷ κυλίνδρῳ, τριπλάσιόν ἐστι τῆς πυ-
ραμίδος, ἥς βάσις μὲν ἐστὶ τὸ $AEBZΓΗΔΘ$ πολύγωνον,
κορυφὴ δὲ ἡ αὐτὴ τῷ κώνῳ· καὶ ἡ πυραμὶς ἄρα, ἥς βάσις
μὲν [ἐστὶ] τὸ $AEBZΓΗΔΘ$ πολύγωνον, κορυφὴ δὲ ἡ αὐτὴ
τῷ κώνῳ, μεῖζων ἐστὶ τοῦ κώνου τοῦ βάσιν ἔχοντες τὸν
 $ABΓΔ$ κύκλον. ἀλλὰ καὶ ἐλάττων· ἐμπεριέχεται γὰρ ὑπ'
αὐτοῦ· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ἐστὶν ὁ κύλινδρος
τοῦ κώνου μεῖζων ἢ τριπλάσιος.

Λέγω δὴ, ὅτι οὐδὲ ἐλάττων ἐστὶν ἢ τριπλάσιος ὁ
κύλινδρος τοῦ κώνου.

Εἰ γὰρ δυνατόν, ἔστω ἐλάττων ἢ τριπλάσιος ὁ κύλινδρος
τοῦ κώνου· ἀνάπαλιν ἄρα ὁ κώνος τοῦ κυλίνδρου μεῖζων
ἐστὶν ἢ τρίτον μέρος. ἐγγεγράφθω δὴ εἰς τὸν $ABΓΔ$ κύκλον
τετράγωνον τὸ $ABΓΔ$ · τὸ $ABΓΔ$ ἄρα τετράγωνον μεῖζόν
ἐστὶν ἢ τὸ ἥμισυ τοῦ $ABΓΔ$ κύκλου. καὶ ἀνεστάτω ἀπὸ τοῦ
 $ABΓΔ$ τετραγώνου πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ
κώνῳ· ἢ ἄρα ἀνασταθεῖσα πυραμὶς μεῖζων ἐστὶν ἢ τὸ ἥμισυ
μέρος τοῦ κώνου, ἐπειδὴ περ, ὡς ἔμπροσθεν ἐδείκνυμεν,
ὅτι ἐὰν περὶ τὸν κύκλον τετράγωνον περιγράψωμεν, ἔσται
τὸ $ABΓΔ$ τετράγωνον ἥμισυ τοῦ περὶ τὸν κύκλον περι-
γεγραμμένου τετραγώνου· καὶ ἐὰν ἀπὸ τῶν τετραγώνων
στερεὰ παραλληλεπίπεδα ἀναστήσωμεν ἰσοῦψῃ τῷ κώνῳ, ἃ
καλεῖται πρίσματα, ἔσται τὸ ἀνασταθέν ἀπὸ τοῦ $ABΓΔ$
τετραγώνου ἥμισυ τοῦ ἀνασταθέντος ἀπὸ τοῦ περὶ τὸν
κύκλον περιγραφέντος τετραγώνου· πρὸς ἄλληλα γὰρ εἰσιν
ὡς αἱ βάσεις. ὥστε καὶ τὰ τρίτα· καὶ πυραμὶς ἄρα, ἥς
βάσις τὸ $ABΓΔ$ τετράγωνον, ἥμισυ ἐστὶ τῆς πυραμίδος τῆς
ἀνασταθείσης ἀπὸ τοῦ περὶ τὸν κύκλον περιγραφέντος τε-
τραγώνου. καὶ ἐστὶ μεῖζων ἢ πυραμὶς ἢ ἀνασταθεῖσα ἀπὸ
τοῦ περὶ τὸν κύκλον τετραγώνου τοῦ κώνου· ἐμπεριέχει
γὰρ αὐτόν. ἢ ἄρα πυραμὶς, ἥς βάσις τὸ $ABΓΔ$ τετράγωνον,
κορυφὴ δὲ ἡ αὐτὴ τῷ κώνῳ, μεῖζων ἐστὶν ἢ τὸ ἥμισυ τοῦ
κώνου. τετμήσθωσαν αἱ AB , $BΓ$, $ΓΔ$, $ΔΑ$ περιφέρειαι
δίχα κατὰ τὰ E , Z , H , $Θ$ σημεία, καὶ ἐπεξεύχθωσαν αἱ

(respectively), and complete the parallelograms on AB ,
 BC , CD , and DA , and set up parallelepiped solids of
equal height to the cylinder on them, then the prisms on
triangles AEB , BFC , CGD , and DHA are each half of
the set up (parallelepipeds). And the segments of the
cylinder are less than the set up parallelepiped solids.
Hence, the prisms on triangles AEB , BFC , CGD , and
 DHA are also greater than half of the segments of the
cylinder about them. So (if) the remaining circumfer-
ences are cut in half, and straight-lines are joined, and
prisms of equal height to the cylinder are set up on each
of the triangles, and this is done continually, then we will
(eventually) leave some segments of the cylinder whose
(sum) is less than the excess by which the cylinder ex-
ceeds three times the cone [Prop. 10.1]. Let them have
been left, and let them be AE , EB , BF , FC , CG , GD ,
 DH , and HA . Thus, the remaining prism whose base
(is) polygon $AEBFCGDH$, and height the same as the
cylinder, is greater than three times the cone. But, the
prism whose base is polygon $AEBFCGDH$, and height
the same as the cylinder, is three times the pyramid whose
base is polygon $AEBFCGDH$, and apex the same as the
cone [Prop. 12.7 corr.]. And thus the pyramid whose
base [is] polygon $AEBFCGDH$, and apex the same as
the cone, is greater than the cone having (as) base circle
 $ABCD$. But (it is) also less. For it is encompassed by it.
The very thing (is) impossible. Thus, the cylinder is not
more than three times the cone.

So, I say that neither (is) the cylinder less than three
times the cone.

For, if possible, let the cylinder be less than three times
the cone. Thus, inversely, the cone is greater than the
third part of the cylinder. So, let the square $ABCD$ have
been inscribed in circle $ABCD$ [Prop. 4.6]. Thus, square
 $ABCD$ is greater than half of circle $ABCD$. And let a
pyramid having the same apex as the cone have been set
up on square $ABCD$. Thus, the pyramid set up is greater
than the half part of the cone, inasmuch as we showed
previously that if we circumscribe a square about the cir-
cle [Prop. 4.7] then the square $ABCD$ will be half of the
square circumscribed about the circle [Prop. 12.2]. And
if we set up on the squares parallelepiped solids—which
are also called prisms—of the same height as the cone,
then the (prism) set up on square $ABCD$ will be half
of the (prism) set up on the square circumscribed about
the circle. For they are to one another as their bases
[Prop. 11.32]. Hence, (the same) also (goes for) the
thirds. Thus, the pyramid whose base is square $ABCD$
is half of the pyramid set up on the square circumscribed
about the circle [Prop. 12.7 corr.]. And the pyramid set
up on the square circumscribed about the circle is greater

ΑΕ, ΕΒ, ΒΖ, ΖΓ, ΓΗ, ΗΔ, ΔΘ, ΘΑ· καὶ ἕκαστον ἄρα τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων μεῖζόν ἐστιν ἢ τὸ ἥμισυ μέρος τοῦ καθ' ἑαυτὸ τμήματος τοῦ ΑΒΓΔ κύκλου. καὶ ἀνεστάτωσαν ἐφ' ἑκάστου τῶν ΑΕΒ, ΒΖΓ, ΓΗΔ, ΔΘΑ τριγώνων πυραμίδες τὴν αὐτὴν κορυφὴν ἔχουσαι τῷ κώνῳ· καὶ ἑκάστη ἄρα τῶν ἀνασταθμισῶν πυραμίδων κατὰ τὸν αὐτὸν τρόπον μεῖζων ἐστὶν ἢ τὸ ἥμισυ μέρος τοῦ καθ' ἑαυτὴν τμήματος τοῦ κώνου. τέμνοντες δὲ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγύνοντες εὐθείας καὶ ἀνιστάντες ἐφ' ἑκάστου τῶν τριγώνων πυραμίδα τὴν αὐτὴν κορυφὴν ἔχουσιν τῷ κώνῳ καὶ τοῦτο αἰεὶ ποιοῦντες καταλείψομεν τινα ἀποτμήματα τοῦ κώνου, ἃ ἔσται ἐλάττωνα τῆς ὑπεροχῆς, ἢ ὑπερέχει ὁ κώνος τοῦ τρίτου μέρους τοῦ κυλίνδρου. λελείφθω, καὶ ἔστω τὰ ἐπὶ τῶν ΑΕ, ΕΒ, ΒΖ, ΖΓ, ΓΗ, ΗΔ, ΔΘ, ΘΑ· λοιπὴ ἄρα ἡ πυραμὶς, ἥς βάσις μὲν ἐστὶ τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, κορυφὴ δὲ ἡ αὐτὴ τῷ κώνῳ, μεῖζων ἐστὶν ἢ τρίτον μέρος τοῦ κυλίνδρου. ἀλλ' ἡ πυραμὶς, ἥς βάσις μὲν ἐστὶ τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, κορυφὴ δὲ ἡ αὐτὴ τῷ κώνῳ, τρίτον ἐστὶ μέρος τοῦ πρίσματος, οὗ βάσις μὲν ἐστὶ τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ κυλίνδρῳ· τὸ ἄρα πρίσμα, οὗ βάσις μὲν ἐστὶ τὸ ΑΕΒΖΓΗΔΘ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ κυλίνδρῳ, μεῖζόν ἐστὶ τοῦ κυλίνδρου, οὗ βάσις ἐστὶν ὁ ΑΒΓΔ κύκλος. ἀλλὰ καὶ ἔλαττον· ἐμπεριέχεται γὰρ ὑπ' αὐτοῦ· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ κύλινδρος τοῦ κώνου ἐλάττω ἐστὶν ἢ τριπλάσιος. ἐδείχθη δέ, ὅτι οὐδὲ μεῖζων ἢ τριπλάσιος· τριπλάσιος ἄρα ὁ κύλινδρος τοῦ κώνου· ὥστε ὁ κώνος τρίτον ἐστὶ μέρος τοῦ κυλίνδρου.

Πᾶς ἄρα κώνος κυλίνδρου τρίτον μέρος ἐστὶ τοῦ τὴν αὐτὴν βάσιν ἔχοντος αὐτῷ καὶ ὕψος ἴσον· ὅπερ ἔδει δεῖξαι.

ια'.

Οἱ ὑπο τὸ αὐτὸ ὕψος ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις.

Ἔστωσαν ὑπὸ τὸ αὐτὸ ὕψος κῶνοι καὶ κύλινδροι, ὧν βάσεις μὲν [εἰσιν] οἱ ΑΒΓΔ, ΕΖΗΘ κύκλοι, ἄξονες δὲ οἱ ΚΛ, ΜΝ, διαμέτροι δὲ τῶν βάσεων αἱ ΑΓ, ΕΗ· λέγω, ὅτι ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς τὸν ΕΝ κῶνον.

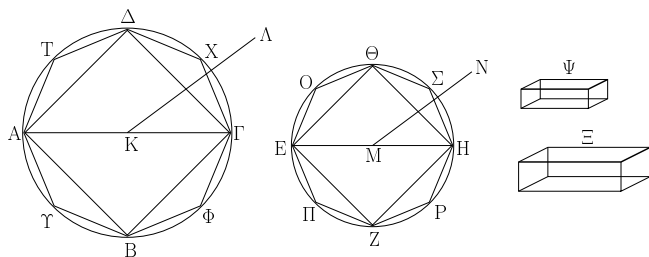
than the cone. For it encompasses it. Thus, the pyramid whose base is square $ABCD$, and apex the same as the cone, is greater than half of the cone. Let the circumferences AB , BC , CD , and DA have been cut in half at points E , F , G , and H (respectively). And let AE , EB , BF , FC , CG , GD , DH , and HA have been joined. And, thus, each of the triangles AEB , BFC , CGD , and DHA is greater than the half part of the segment of circle $ABCD$ about it [Prop. 12.2]. And let pyramids having the same apex as the cone have been set up on each of the triangles AEB , BFC , CGD , and DHA . And, thus, in the same way, each of the pyramids set up is more than the half part of the segment of the cone about it. So, (if) the remaining circumferences are cut in half, and straight-lines are joined, and pyramids having the same apex as the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone whose (sum) is less than the excess by which the cone exceeds the third part of the cylinder [Prop. 10.1]. Let them have been left, and let them be the (segments) on AE , EB , BF , FC , CG , GD , DH , and HA . Thus, the remaining pyramid whose base is polygon $AEBFCGDH$, and apex the same as the cone, is greater than the third part of the cylinder. But, the pyramid whose base is polygon $AEBFCGDH$, and apex the same as the cone, is the third part of the prism whose base is polygon $AEBFCGDH$, and height the same as the cylinder [Prop. 12.7 corr.]. Thus, the prism whose base is polygon $AEBFCGDH$, and height the same as the cylinder, is greater than the cylinder whose base is circle $ABCD$. But, (it is) also less. For it is encompassed by it. The very thing is impossible. Thus, the cylinder is not less than three times the cone. And it was shown that neither (is it) greater than three times (the cone). Thus, the cylinder (is) three times the cone. Hence, the cone is the third part of the cylinder.

Thus, every cone is the third part of the cylinder which has the same base as it, and an equal height. (Which is) the very thing it was required to show.

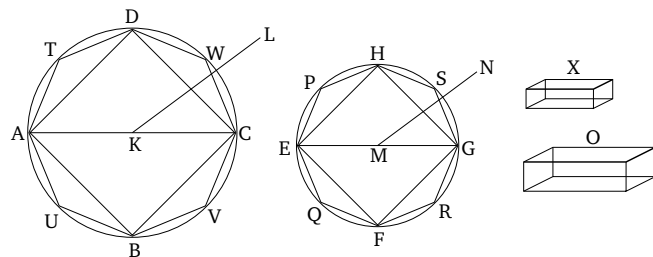
Proposition 11

Cones and cylinders having the same height are to one another as their bases.

Let there be cones and cylinders of the same height whose bases [are] the circles $ABCD$ and $EFGH$, axes KL and MN , and diameters of the bases AC and EG (respectively). I say that as circle $ABCD$ is to circle $EFGH$, so cone AL (is) to cone EN .



Εἰ γὰρ μή, ἔσται ὡς ὁ $AB\Gamma\Delta$ κύκλος πρὸς τὸν $EZH\Theta$ κύκλον, οὕτως ὁ AA κῶνος ἦτοι πρὸς ἑλασσόν τι τοῦ EN κώνου στερεὸν ἢ πρὸς μείζον. ἔστω πρότερον πρὸς ἑλασσόν τὸ Ξ , καὶ ὅ ἑλασσόν ἐστι τὸ Ξ στερεὸν τοῦ EN κώνου, ἐκεῖνῳ ἴσον ἔστω τὸ Ψ στερεόν· ὁ EN κῶνος ἄρα ἴσος ἐστὶ τοῖς Ξ , Ψ στερεοῖς. ἐγγεγράφθω εἰς τὸν $EZH\Theta$ κύκλον τετράγωνον τὸ $EZH\Theta$ · τὸ ἄρα τετράγωνον μείζον ἐστὶν ἢ τὸ ἥμισυ τοῦ κύκλου. ἀνεστάτω ἀπὸ τοῦ $EZH\Theta$ τετραγώνου πυραμὶς ἰσοϋψῆς τῷ κώνῳ· ἡ ἄρα ἀνασταθεῖσα πυραμὶς μείζων ἐστὶν ἢ τὸ ἥμισυ τοῦ κώνου, ἐπειδὴ περ ἐὰν περιγράψωμεν περὶ τὸν κύκλον τετράγωνον, καὶ ἀπ' αὐτοῦ ἀναστήσωμεν πυραμίδα ἰσοϋψῆ τῷ κώνῳ, ἡ ἐγγραφεῖσα πυραμὶς ἥμισυ ἐστὶ τῆς περιγραφείσης· πρὸς ἀλλήλας γὰρ εἰσιν ὡς αἱ βάσεις· ἐλάττων δὲ ὁ κῶνος τῆς περιγραφείσης πυραμίδος. τετμήσθωσαν αἱ EZ , ZH , $H\Theta$, ΘE περιφέρειαι διχα κατὰ τὰ O , Π , P , Σ σημεῖα, καὶ ἐπεξεύχθωσαν αἱ ΘO , $O E$, $E\Pi$, ΠZ , ZP , $P H$, $H\Sigma$, $\Sigma\Theta$. ἕκαστον ἄρα τῶν $\Theta O E$, $E\Pi Z$, $ZP H$, $H\Sigma\Theta$ τριγώνων μείζον ἐστὶν ἢ τὸ ἥμισυ τοῦ καθ' ἑαυτὸ τμήματος τοῦ κύκλου. ἀνεστάτω ἐφ' ἑκάστου τῶν $\Theta O E$, $E\Pi Z$, $ZP H$, $H\Sigma\Theta$ τριγώνων πυραμὶς ἰσοϋψῆς τῷ κώνῳ· καὶ ἑκάστη ἄρα τῶν ἀνασταθεισῶν πυραμίδων μείζων ἐστὶν ἢ τὸ ἥμισυ τοῦ καθ' ἑαυτὴν τμήματος τοῦ κώνου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας διχα καὶ ἐπιζευγνύντες εὐθείας καὶ ἀνιστάντες ἐπὶ ἑκάστου τῶν τριγώνων πυραμίδας ἰσοϋψεῖς τῷ κώνῳ καὶ αἰε τοῦτο ποιοῦντες καταλείψομεν τινα ἀποτμήματα τοῦ κώνου, ἃ ἔσται ἐλάσσονα τοῦ Ψ στερεοῦ. λελείθω, καὶ ἔστω τὰ ἐπὶ τῶν $\Theta O E$, $E\Pi Z$, $ZP H$, $H\Sigma\Theta$ λοιπὴ ἄρα ἡ πυραμὶς, ἥς βάσις τὸ $\Theta O E\Pi ZP H\Sigma$ πολύγωνον, ὕψος δὲ τὸ αὐτὸ τῷ κώνῳ, μείζων ἐστὶ τοῦ Ξ στερεοῦ. ἐγγεγράφθω καὶ εἰς τὸν $AB\Gamma\Delta$ κύκλον τῷ $\Theta O E\Pi ZP H\Sigma$ πολύγωνῳ ὁμοίον τε καὶ ὁμοίως κείμενον πολύγωνον τὸ $\Delta T A Y B \Phi \Gamma X$, καὶ ἀνεστάτω ἐπ' αὐτοῦ πυραμὶς ἰσοϋψῆς τῷ AA κώνῳ. ἐπεὶ οὖν ἐστὶν ὡς τὸ ἀπὸ τῆς AG πρὸς τὸ ἀπὸ τῆς EH , οὕτως τὸ $\Delta T A Y B \Phi \Gamma X$ πολύγωνον πρὸς τὸ $\Theta O E\Pi ZP H\Sigma$ πολύγωνον, ὥς δὲ τὸ ἀπὸ τῆς AG πρὸς τὸ ἀπὸ τῆς EH , οὕτως ὁ $AB\Gamma\Delta$ κύκλος πρὸς τὸν $EZH\Theta$ κύκλον, καὶ ὡς ἄρα ὁ $AB\Gamma\Delta$ κύκλος πρὸς τὸν $EZH\Theta$ κύκλον, οὕτως τὸ $\Delta T A Y B \Phi \Gamma X$ πολύγωνον πρὸς τὸ $\Theta O E\Pi ZP H\Sigma$ πολύγωνον. ὥς δὲ ὁ $AB\Gamma\Delta$ κύκλος πρὸς τὸν $EZH\Theta$ κύκλον, οὕτως ὁ AA κῶνος πρὸς τὸ Ξ στερεόν, ὥς δὲ τὸ $\Delta T A Y B \Phi \Gamma X$ πολύγωνον πρὸς τὸ $\Theta O E\Pi ZP H\Sigma$ πολύγωνον, οὕτως ἡ πυραμὶς, ἥς βάσις μὲν τὸ $\Delta T A Y B \Phi \Gamma X$ πολύγωνον, κορυφὴ δὲ τὸ A σημεῖον, πρὸς



For if not, then as circle $ABCD$ (is) to circle $EFGH$, so cone AL will be to some solid either less than, or greater than, cone EN . Let it, first of all, be (in this ratio) to (some) lesser (solid), O . And let solid X be equal to that (magnitude) by which solid O is less than cone EN . Thus, cone EN is equal to (the sum of) solids O and X . Let the square $EFGH$ have been inscribed in circle $EFGH$ [Prop. 4.6]. Thus, the square is greater than half of the circle [Prop. 12.2]. Let a pyramid of the same height as the cone have been set up on square $EFGH$. Thus, the pyramid set up is greater than half of the cone, inasmuch as, if we circumscribe a square about the circle [Prop. 4.7], and set up on it a pyramid of the same height as the cone, then the inscribed pyramid is half of the circumscribed pyramid. For they are to one another as their bases [Prop. 12.6]. And the cone (is) less than the circumscribed pyramid. Let the circumferences EF , FG , GH , and HE have been cut in half at points P , Q , R , and S . And let HP , PE , EQ , QF , FR , RG , GS , and SH have been joined. Thus, each of the triangles HPE , EQF , FRG , and GSH is greater than half of the segment of the circle about it [Prop. 12.2]. Let pyramids of the same height as the cone have been set up on each of the triangles HPE , EQF , FRG , and GSH . And, thus, each of the pyramids set up is greater than half of the segment of the cone about it [Prop. 12.10]. So, (if) the remaining circumferences are cut in half, and straight-lines are joined, and pyramids of equal height to the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone (the sum of) which is less than solid X [Prop. 10.1]. Let them have been left, and let them be the (segments) on HPE , EQF , FRG , and GSH . Thus, the remaining pyramid whose base is polygon $HPEQFRGS$, and height the same as the cone, is greater than solid O [Prop. 6.18]. And let the polygon $DTAUBVCW$, similar, and similarly laid out, to polygon $HPEQFRGS$, have been inscribed in circle $ABCD$. And on it let a pyramid of the same height as cone AL have been set up. Therefore, since as the (square) on AC is to the (square) on EG , so polygon $DTAUBVCW$ (is) to polygon $HPEQFRGS$ [Prop. 12.1], and as the (square) on AC (is) to the (square) on EG , so circle $ABCD$ (is)

τὴν πυραμίδα, ἥς βάσις μὲν τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, κορυφή δὲ τὸ Ν σημείον. καὶ ὡς ἄρα ὁ ΑΛ κῶνος πρὸς τὸ Ξ στερεόν, οὕτως ἡ πυραμὶς, ἥς βάσις μὲν τὸ ΔΤΑΥΒΦΓΧ πολύγωνον, κορυφή δὲ τὸ Λ σημείον, πρὸς τὴν πυραμίδα, ἥς βάσις μὲν τὸ ΘΟΕΠΖΡΗΣ πολύγωνον, κορυφή δὲ τὸ Ν σημείον· ἐναλλάξ ἄρα ἐστὶν ὡς ὁ ΑΛ κῶνος πρὸς τὴν ἐν αὐτῷ πυραμίδα, οὕτως τὸ Ξ στερεόν πρὸς τὴν ἐν τῷ ΕΝ κώνῳ πυραμίδα. μείζων δὲ ὁ ΑΛ κῶνος τῆς ἐν αὐτῷ πυραμίδος· μείζον ἄρα καὶ τὸ Ξ στερεόν τῆς ἐν τῷ ΕΝ κώνῳ πυραμίδος. ἀλλὰ καὶ ἔλασσον· ὅπερ ἄτοπον. οὐκ ἄρα ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς ἔλασσόν τι τοῦ ΕΝ κώνου στερεόν. ὁμοίως δὲ δείξομεν, ὅτι οὐδὲ ἐστὶν ὡς ὁ ΕΖΗΘ κύκλος πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΝ κῶνος πρὸς ἔλασσόν τι τοῦ ΑΛ κώνου στερεόν.

Λέγω δὴ, ὅτι οὐδὲ ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς μείζον τι τοῦ ΕΝ κώνου στερεόν.

Εἰ γὰρ δυνατόν, ἔστω πρὸς μείζον τὸ Ξ· ἀνάπαλιν ἄρα ἐστὶν ὡς ὁ ΕΖΗΘ κύκλος πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως τὸ Ξ στερεόν πρὸς τὸν ΑΛ κῶνον. ἀλλ' ὡς τὸ Ξ στερεόν πρὸς τὸν ΑΛ κῶνον, οὕτως ὁ ΕΝ κῶνος πρὸς ἔλασσόν τι τοῦ ΑΛ κώνου στερεόν· καὶ ὡς ἄρα ὁ ΕΖΗΘ κύκλος πρὸς τὸν ΑΒΓΔ κύκλον, οὕτως ὁ ΕΝ κῶνος πρὸς ἔλασσόν τι τοῦ ΑΛ κώνου στερεόν· ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ἐστὶν ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς μείζον τι τοῦ ΕΝ κώνου στερεόν. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλασσον· ἔστιν ἄρα ὡς ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως ὁ ΑΛ κῶνος πρὸς τὸν ΕΝ κῶνον.

Ἄλλ' ὡς ὁ κῶνος πρὸς τὸν κῶνον, ὁ κύλινδρος πρὸς τὸν κύλινδρον· τριπλασίων γὰρ ἑκάτερος ἑκατέρου. καὶ ὡς ἄρα ὁ ΑΒΓΔ κύκλος πρὸς τὸν ΕΖΗΘ κύκλον, οὕτως οἱ ἐπ' αὐτῶν ἰσοῦψεῖς.

Οἱ ἄρα ὑπὸ τὸ αὐτὸ ὕψος ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις· ὅπερ ἔδει δεῖξαι.

ιβ'.

Οἱ ὅμοιοι κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ἐν ταῖς βάσεσι διαμέτρων.

Ἐστῶσαν ὅμοιοι κῶνοι καὶ κύλινδροι, ὧν βάσεις μὲν οἱ ΑΒΓΔ, ΕΖΗΘ κύκλοι, διάμετροι δὲ τῶν βάσεων αἱ ΒΔ, ΖΘ, ἄξονες δὲ τῶν κώνων καὶ κυλίνδρων οἱ ΚΛ, ΜΝ· λέγω,

to circle $EFGH$ [Prop. 12.2], thus as circle $ABCD$ (is) to circle $EFGH$, so polygon $DTAUBVCW$ also (is) to polygon $HPEQFRGS$. And as circle $ABCD$ (is) to circle $EFGH$, so cone AL (is) to solid O . And as polygon $DTAUBVCW$ (is) to polygon $HPEQFRGS$, so the pyramid whose base is polygon $DTAUBVCW$, and apex the point L , (is) to the pyramid whose base is polygon $HPEQFRGS$, and apex the point N [Prop. 12.6]. And, thus, as cone AL (is) to solid O , so the pyramid whose base is $DTAUBVCW$, and apex the point L , (is) to the pyramid whose base is polygon $HPEQFRGS$, and apex the point N [Prop. 5.11]. Thus, alternately, as cone AL is to the pyramid within it, so solid O (is) to the pyramid within cone EN [Prop. 5.16]. But, cone AL (is) greater than the pyramid within it. Thus, solid O (is) also greater than the pyramid within cone EN [Prop. 5.14]. But, (it is) also less. The very thing (is) absurd. Thus, circle $ABCD$ is not to circle $EFGH$, as cone AL (is) to some solid less than cone EN . So, similarly, we can show that neither is circle $EFGH$ to circle $ABCD$, as cone EN (is) to some solid less than cone AL .

So, I say that neither is circle $ABCD$ to circle $EFGH$, as cone AL (is) to some solid greater than cone EN .

For, if possible, let it be (in this ratio) to (some) greater (solid), O . Thus, inversely, as circle $EFGH$ is to circle $ABCD$, so solid O (is) to cone AL [Prop. 5.7 corr.]. But, as solid O (is) to cone AL , so cone EN (is) to some solid less than cone AL [Prop. 12.2 lem.]. And, thus, as circle $EFGH$ (is) to circle $ABCD$, so cone EN (is) to some solid less than cone AL . The very thing was shown (to be) impossible. Thus, circle $ABCD$ is not to circle $EFGH$, as cone AL (is) to some solid greater than cone EN . And, it was shown that neither (is it in this ratio) to (some) lesser (solid). Thus, as circle $ABCD$ is to circle $EFGH$, so cone AL (is) to cone EN .

But, as the cone (is) to the cone, (so) the cylinder (is) to the cylinder. For each (is) three times each [Prop. 12.10]. Thus, circle $ABCD$ (is) also to circle $EFGH$, as (the ratio of the cylinders) on them (having) the same height.

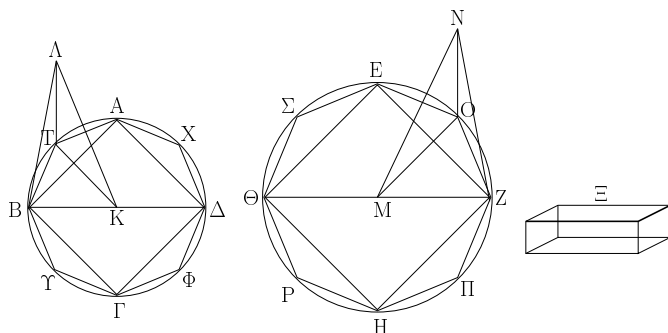
Thus, cones and cylinders having the same height are to one another as their bases. (Which is) the very thing it was required to show.

Proposition 12

Similar cones and cylinders are to one another in the cubed ratio of the diameters of their bases.

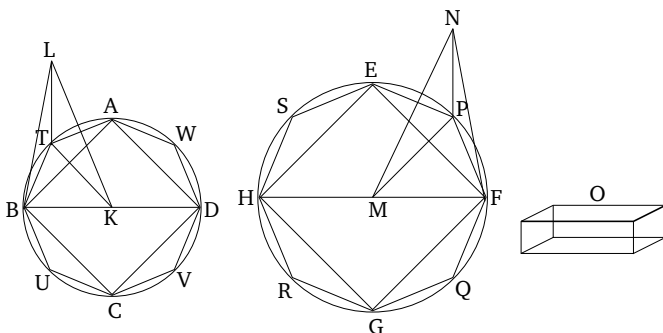
Let there be similar cones and cylinders of which the bases (are) the circles $ABCD$ and $EFGH$, the diameters of the bases (are) BD and FH , and the axes of the cones

ὅτι ὁ κῶνος, οὗ βάσις μὲν [ἐστίν] ὁ $AB\Gamma\Delta$ κύκλος, κορυφή δὲ τὸ Λ σημεῖον, πρὸς τὸν κῶνον, οὗ βάσις μὲν [ἐστίν] ὁ $EZH\Theta$ κύκλος, κορυφή δὲ τὸ N σημεῖον, τριπλασίονα λόγον ἔχει ἢ πρὸς τὴν $Z\Theta$.



Εἰ γὰρ μὴ ἔχει ὁ $AB\Gamma\Delta\Lambda$ κῶνος πρὸς τὸν $EZH\Theta N$ κῶνον πριπλασίονα λόγον ἢ πρὸς τὴν $Z\Theta$, ἔξει ὁ $AB\Gamma\Delta\Lambda$ κῶνος ἢ πρὸς ἔλασσόν τι τοῦ $EZH\Theta N$ κῶνου στερεὸν τριπλασίονα λόγον ἢ πρὸς μείζον. ἐχέτω πρότερον πρὸς ἔλασσον τὸ Ξ , καὶ ἐγγεγράφθω εἰς τὸν $EZH\Theta$ κύκλον τετράγωνον τὸ $EZH\Theta$. τὸ ἄρα $EZH\Theta$ τετράγωνον μείζον ἐστίν ἢ τὸ ἥμισυ τοῦ $EZH\Theta$ κύκλου. καὶ ἀνεστάτω ἐπὶ τοῦ $EZH\Theta$ τετραγώνου πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κῶνῳ· ἢ ἄρα ἀνασταθεῖσα πυραμὶς μείζων ἐστίν ἢ τὸ ἥμισυ μέρος τοῦ κῶνου. τεμήσθωσαν δὴ αἱ EZ , ZH , $H\Theta$, ΘE περιφέρειαι δίχα κατὰ τὰ O , Π , P , Σ σημεία, καὶ ἐπεζεύχθωσαν αἱ EO , OZ , $Z\Pi$, ΠH , $H P$, $P\Theta$, $\Theta\Sigma$, ΣE . καὶ ἕκαστον ἄρα τῶν EOZ , $Z\Pi H$, $H P\Theta$, $\Theta\Sigma E$ τριγώνων μείζον ἐστίν ἢ τὸ ἥμισυ μέρος τοῦ καθ' ἑαυτὸ τμήματος τοῦ $EZH\Theta$ κύκλου. καὶ ἀνεστάτω ἐφ' ἑκάστου τῶν EOZ , $Z\Pi H$, $H P\Theta$, $\Theta\Sigma E$ τριγώνων πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κῶνῳ· καὶ ἑκάστη ἄρα τῶν ἀνασταθεῖσων πυραμίδων μείζων ἐστίν ἢ τὸ ἥμισυ μέρος τοῦ καθ' ἑαυτὴν τμήματος τοῦ κῶνου. τέμνοντες δὴ τὰς ὑπολειπομένας περιφερείας δίχα καὶ ἐπιζευγύνντες εὐθείας καὶ ἀνιστάντες ἐφ' ἑκάστου τῶν τριγώνων πυραμίδας τὴν αὐτὴν κορυφὴν ἔχουσας τῷ κῶνῳ καὶ τοῦτο αἰ ποιοῦντες καταλείβομεν τινα ἀποτμήματα τοῦ κῶνου, ἃ ἔσται ἐλάσσονα τῆς ὑπεροχῆς, ἣ ὑπερέχει ὁ $EZH\Theta N$ κῶνος τοῦ Ξ στερεοῦ. λελείφθω, καὶ ἔστω τὰ ἐπὶ τῶν EO , OZ , $Z\Pi$, ΠH , $H P$, $P\Theta$, $\Theta\Sigma$, ΣE λοιπὴ ἄρα ἢ πυραμὶς, ἥς βάσις μὲν ἐστὶ τὸ $EOZ\Pi H P\Theta\Sigma$ πολύγωνον, κορυφὴ δὲ τὸ N σημεῖον, μείζων ἐστὶ τοῦ Ξ στερεοῦ. ἐγγεγράφθω καὶ εἰς τὸν $AB\Gamma\Delta$ κύκλον τῷ $EOZ\Pi H P\Theta\Sigma$ πολυγώνῳ ὁμοίον τε καὶ ὁμοίως κείμενον πολύγωνον τὸ $ATB\Upsilon\Gamma\Phi\Delta X$, καὶ ἀνεστάτω ἐπὶ τοῦ $ATB\Upsilon\Gamma\Phi\Delta X$ πολυγώνου πυραμὶς τὴν αὐτὴν κορυφὴν ἔχουσα τῷ κῶνῳ, καὶ τῶν μὲν περιεχόντων τὴν πυραμίδα, ἥς βάσις μὲν ἐστὶ τὸ $ATB\Upsilon\Gamma\Phi\Delta X$ πολύγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, ἐν τρίγωνον ἔστω τὸ ΛBT , τῶν δὲ περιεχόντων τὴν πυραμίδα, ἥς βάσις μὲν ἐστὶ τὸ $EOZ\Pi H P\Theta\Sigma$ πολύγωνον,

and cylinders (are) KL and MN (respectively). I say that the cone whose base [is] circle $ABCD$, and apex the point L , has to the cone whose base [is] circle $EFGH$, and apex the point N , the cubed ratio that BD (has) to FH .



For if cone $ABCDL$ does not have to cone $EFGHN$ the cubed ratio that BD (has) to FH then cone $ABCDL$ will have the cubed ratio to some solid either less than, or greater than, cone $EFGHN$. Let it, first of all, have (such a ratio) to (some) lesser (solid), O . And let the square $EFGH$ have been inscribed in circle $EFGH$ [Prop. 4.6]. Thus, square $EFGH$ is greater than half of circle $EFGH$ [Prop. 12.2]. And let a pyramid having the same apex as the cone have been set up on square $EFGH$. Thus, the pyramid set up is greater than the half part of the cone [Prop. 12.10]. So, let the circumferences EF , FG , GH , and HE have been cut in half at points P , Q , R , and S (respectively). And let EP , PF , FQ , QG , GR , RH , HS , and SE have been joined. And, thus, each of the triangles EPF , FQG , GRH , and HSE is greater than the half part of the segment of circle $EFGH$ about it [Prop. 12.2]. And let a pyramid having the same apex as the cone have been set up on each of the triangles EPF , FQG , GRH , and HSE . And thus each of the pyramids set up is greater than the half part of the segment of the cone about it [Prop. 12.10]. So, (if) the the remaining circumferences are cut in half, and straight-lines are joined, and pyramids having the same apex as the cone are set up on each of the triangles, and this is done continually, then we will (eventually) leave some segments of the cone whose (sum) is less than the excess by which cone $EFGHN$ exceeds solid O [Prop. 10.1]. Let them have been left, and let them be the (segments) on EP , PF , FQ , QG , GR , RH , HS , and SE . Thus, the remaining pyramid whose base is polygon $EPFQGRHS$, and apex the point N , is greater than solid O . And let the polygon $ATBUCVDW$, similar, and similarly laid out, to polygon $EPFQGRHS$, have been inscribed in circle $ABCD$ [Prop. 6.18]. And let a pyramid having the same apex as the cone have been set up on polygon $ATBUCVDW$.

κορυφή δὲ τὸ Ν σημεῖον, ἐν τρίγωνον ἔστω τὸ ΝΖΟ, καὶ ἐπεξεύχθωσαν αἱ ΚΤ, ΜΟ. καὶ ἐπεὶ ὁμοίως ἐστὶν ὁ ΑΒΓΔΛ κῶνος τῷ ΕΖΗΘΝ κῶνῳ, ἔστιν ἄρα ὡς ἡ ΒΔ πρὸς τὴν ΖΘ, οὕτως ὁ ΚΑ ἄξων πρὸς τὸν ΜΝ ἄξονα. ὡς δὲ ἡ ΒΔ πρὸς τὴν ΖΘ, οὕτως ἡ ΒΚ πρὸς τὴν ΖΜ· καὶ ὡς ἄρα ἡ ΒΚ πρὸς τὴν ΖΜ, οὕτως ἡ ΚΑ πρὸς τὴν ΜΝ. καὶ ἐναλλάξ ὡς ἡ ΒΚ πρὸς τὴν ΚΑ, οὕτως ἡ ΖΜ πρὸς τὴν ΜΝ. καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΒΚΑ, ΖΜΝ αἱ πλευραὶ ἀνάλογόν εἰσιν· ὁμοιον ἄρα ἐστὶ τὸ ΒΚΑ τρίγωνον τῷ ΖΜΝ τριγώνῳ. πάλιν, ἐπεὶ ἐστὶν ὡς ἡ ΒΚ πρὸς τὴν ΚΤ, οὕτως ἡ ΖΜ πρὸς τὴν ΜΟ, καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΒΚΤ, ΖΜΟ, ἐπειδήπερ, ὁ μέρος ἐστὶν ἡ ὑπὸ ΒΚΤ γωνία τῶν πρὸς τῷ Κ κέντρῳ τεσσάρων ὀρθῶν, τὸ αὐτὸ μέρος ἐστὶ καὶ ἡ ὑπὸ ΖΜΟ γωνία τῶν πρὸς τῷ Μ κέντρῳ τεσσάρων ὀρθῶν· ἐπεὶ οὖν περὶ ἴσας γωνίας αἱ πλευραὶ ἀνάλογόν εἰσιν, ὁμοιον ἄρα ἐστὶ τὸ ΒΚΤ τρίγωνον τῷ ΖΜΟ τριγώνῳ. πάλιν, ἐπεὶ ἐδείχθη ὡς ἡ ΒΚ πρὸς τὴν ΚΑ, οὕτως ἡ ΖΜ πρὸς τὴν ΜΝ, ἴση δὲ ἡ μὲν ΒΚ τῇ ΚΤ, ἡ δὲ ΖΜ τῇ ΟΜ, ἔστιν ἄρα ὡς ἡ ΤΚ πρὸς τὴν ΚΑ, οὕτως ἡ ΟΜ πρὸς τὴν ΜΝ. καὶ περὶ ἴσας γωνίας τὰς ὑπὸ ΤΚΑ, ΟΜΝ· ὀρθαὶ γάρ· αἱ πλευραὶ ἀνάλογόν εἰσιν· ὁμοιον ἄρα ἐστὶ τὸ ΑΚΤ τρίγωνον τῷ ΝΜΟ τριγώνῳ. καὶ ἐπεὶ διὰ τὴν ὁμοιότητα τῶν ΑΚΒ, ΝΜΖ τριγώνων ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΒΚ, οὕτως ἡ ΝΖ πρὸς τὴν ΖΜ, διὰ δὲ τὴν ὁμοιότητα τῶν ΒΚΤ, ΖΜΟ τριγώνων ἐστὶν ὡς ἡ ΚΒ πρὸς τὴν ΒΤ, οὕτως ἡ ΜΖ πρὸς τὴν ΖΟ, δι' ἴσου ἄρα ὡς ἡ ΑΒ πρὸς τὴν ΒΤ, οὕτως ἡ ΝΖ πρὸς τὴν ΖΟ. πάλιν, ἐπεὶ διὰ τὴν ὁμοιότητα τῶν ΑΤΚ, ΝΟΜ τριγώνων ἐστὶν ὡς ἡ ΑΤ πρὸς τὴν ΤΚ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΜ, διὰ δὲ τὴν ὁμοιότητα τῶν ΤΚΒ, ΟΜΖ τριγώνων ἐστὶν ὡς ἡ ΚΤ πρὸς τὴν ΤΒ, οὕτως ἡ ΜΟ πρὸς τὴν ΟΖ, δι' ἴσου ἄρα ὡς ἡ ΑΤ πρὸς τὴν ΤΒ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΖ. ἐδείχθη δὲ καὶ ὡς ἡ ΤΒ πρὸς τὴν ΒΑ, οὕτως ἡ ΟΖ πρὸς τὴν ΖΝ. δι' ἴσου ἄρα ὡς ἡ ΤΑ πρὸς τὴν ΑΒ, οὕτως ἡ ΟΝ πρὸς τὴν ΝΖ. τῶν ΑΤΒ, ΝΟΖ ἄρα τριγώνων ἀνάλογόν εἰσιν αἱ πλευραὶ· ἰσογώνια ἄρα ἐστὶ τὰ ΑΤΒ, ΝΟΖ τρίγωνα· ὥστε καὶ ὅμοια. καὶ πυραμῖς ἄρα, ἥς βάσεις μὲν τὸ ΒΚΤ τρίγωνον, κορυφὴ δὲ τὸ Α σημεῖον, ὅμοια ἐστὶ πυραμίδι, ἥς βάσεις μὲν τὸ ΖΜΟ τρίγωνον, κορυφὴ δὲ τὸ Ν σημεῖον· ὑπὸ γὰρ ὁμοίων ἐπιπέδων περιέχονται ἴσων τὸ πλῆθος. αἱ δὲ ὅμοια πυραμίδες καὶ τριγώνους ἔχουσιν βάσεις ἐν τριπλασίονι λόγῳ εἰς τῶν ὁμολόγων πλευρῶν. ἡ ἄρα ΒΚΤΑ πυραμῖς πρὸς τὴν ΖΜΟΝ πυραμίδα τριπλασίονα λόγον ἔχει ἢ περὶ ἡ ΒΚ πρὸς τὴν ΖΜ. ὁμοίως δὲ ἐπιζευγνύντες ἀπὸ τῶν Α, Χ, Δ, Φ, Γ, Υ ἐπὶ τὸ Κ εὐθείας καὶ ἀπὸ τῶν Ε, Σ, Θ, Ρ, Η, Π ἐπὶ τὸ Μ καὶ ἀνιστάντες ἐφ' ἐκάστου τῶν τριγώνων πυραμίδας τὴν αὐτὴν κορυφὴν ἔχούσας τοῖς κῶνις δειξομεν, ὅτι καὶ ἐκάστη τῶν ὁμοταγῶν πυραμίδων πρὸς ἐκάστην ὁμοταγῇ πυραμίδα τριπλασίονα λόγον ἔξει ἢ περὶ ἡ ΒΚ ὁμόλογος πλευρὰ πρὸς τὴν ΖΜ ὁμόλογον πλευράν, τουτέστιν ἢ περὶ ἡ ΒΔ πρὸς τὴν ΖΘ. καὶ ὡς ἐν τῶν ἡγουμένων πρὸς ἐν τῶν ἐπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἐπόμενα· ἔστιν ἄρα

And let LBT be one of the triangles containing the pyramid whose base is polygon $ATBUCVDW$, and apex the point L . And let NFP be one of the triangles containing the pyramid whose base is triangle $EPFQGRHS$, and apex the point N . And let KT and MP have been joined. And since cone $ABCDL$ is similar to cone $EFGHN$, thus as BD is to FH , so axis KL (is) to axis MN [Def. 11.24]. And as BD (is) to FH , so BK (is) to FM . And, thus, as BK (is) to FM , so KL (is) to MN . And, alternately, as BK (is) to KL , so FM (is) to MN [Prop. 5.16]. And the sides around the equal angles BKL and FMN are proportional. Thus, triangle BKL is similar to triangle FMN [Prop. 6.6]. Again, since as BK (is) to KT , so FM (is) to MP , and (they are) about the equal angles BKT and FMP , inasmuch as whatever part angle BKT is of the four right-angles at the center K , angle FMP is also the same part of the four right-angles at the center M . Therefore, since the sides about equal angles are proportional, triangle BKT is thus similar to triangle FMP [Prop. 6.6]. Again, since it was shown that as BK (is) to KL , so FM (is) to MN , and BK (is) equal to KT , and FM to PM , thus as TK (is) to KL , so PM (is) to MN . And the sides about the equal angles TKL and PMN —for (they are both) right-angles—are proportional. Thus, triangle LKT (is) similar to triangle NMP [Prop. 6.6]. And since, on account of the similarity of triangles LKB and NMF , as LB (is) to BK , so NF (is) to FM , and, on account of the similarity of triangles BKT and FMP , as KB (is) to BT , so MF (is) to FP [Def. 6.1], thus, via equality, as LB (is) to BT , so NF (is) to FP [Prop. 5.22]. Again, since, on account of the similarity of triangles LTK and NPM , as LT (is) to TK , so NP (is) to PM , and, on account of the similarity of triangles TKB and PMF , as KT (is) to TB , so MP (is) to PF , thus, via equality, as LT (is) to TB , so NP (is) to PF [Prop. 5.22]. And it was shown that as TB (is) to BL , so PF (is) to FN . Thus, via equality, as TL (is) to LB , so PN (is) to NF [Prop. 5.22]. Thus, the sides of triangles LTB and NPF are proportional. Thus, triangles LTB and NPF are equiangular [Prop. 6.5]. And, hence, (they are) similar [Def. 6.1]. And, thus, the pyramid whose base is triangle BKT , and apex the point L , is similar to the pyramid whose base is triangle FMP , and apex the point N . For they are contained by equal numbers of similar planes [Def. 11.9]. And similar pyramids which also have triangular bases are in the cubed ratio of corresponding sides [Prop. 12.8]. Thus, pyramid $BKTL$ has to pyramid $FMPN$ the cubed ratio that BK (has) to FM . So, similarly, joining straight-lines from (points) A, W, D, V, C , and U to (center) K , and from (points) E, S, H, R, G , and Q to (center) M , and set-

καὶ ὡς ἡ $BKTL$ πυραμὶς πρὸς τὴν $ZMON$ πυραμίδα, οὕτως ἡ ὅλη πυραμὶς, ἥς βάσις τὸ $ATBYΓΦΔX$ πολύγωνον, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὴν ὅλην πυραμίδα, ἥς βάσις μὲν τὸ $EOZΠHPΘΣ$ πολύγωνον, κορυφὴ δὲ τὸ N σημεῖον· ὥστε καὶ πυραμὶς, ἥς βάσις μὲν τὸ $ATBYΓΦΔX$, κορυφὴ δὲ τὸ Λ , πρὸς τὴν πυραμίδα, ἥς βάσις [μὲν] τὸ $EOZΠHPΘΣ$ πολύγωνον, κορυφὴ δὲ τὸ N σημεῖον, τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Delta$ πρὸς τὴν $Z\Theta$. ὑπόκειται δὲ καὶ ὁ κῶνος, οὗ βάσις [μὲν] ὁ $ABΓΔ$ κύκλος, κορυφὴ δὲ τὸ Λ σημεῖον, πρὸς τὸ Ξ στερεὸν τριπλασίονα λόγον ἔχων ἥπερ ἡ $B\Delta$ πρὸς τὴν $Z\Theta$. ἔστιν ἄρα ὡς ὁ κῶνος, οὗ βάσις μὲν ἐστὶν ὁ $ABΓΔ$ κύκλος, κορυφὴ δὲ τὸ Λ , πρὸς τὸ Ξ στερεόν, οὕτως ἡ πυραμὶς, ἥς βάσις μὲν τὸ $ATBYΓΦΔX$ [πολύγωνον], κορυφὴ δὲ τὸ Λ , πρὸς τὴν πυραμίδα, ἥς βάσις μὲν ἐστὶ τὸ $EOZΠHPΘΣ$ πολύγωνον, κορυφὴ δὲ τὸ N . ἐναλλάξ ἄρα, ὡς ὁ κῶνος, οὗ βάσις μὲν ὁ $ABΓΔ$ κύκλος, κορυφὴ δὲ τὸ Λ , πρὸς τὴν ἐν αὐτῷ πυραμίδα, ἥς βάσις μὲν τὸ $ATBYΓΦΔX$ πολύγωνον, κορυφὴ δὲ τὸ Λ , οὕτως τὸ Ξ [στερεόν] πρὸς τὴν πυραμίδα, ἥς βάσις μὲν ἐστὶ τὸ $EOZΠHPΘΣ$ πολύγωνον, κορυφὴ δὲ τὸ N . μεῖζον δὲ ὁ εἰρημένος κῶνος τῆς ἐν αὐτῷ πυραμίδος· ἐμπεριέχει γὰρ αὐτήν. μεῖζον ἄρα καὶ τὸ Ξ στερεὸν τῆς πυραμίδος, ἥς βάσις μὲν ἐστὶ τὸ $EOZΠHPΘΣ$ πολύγωνον, κορυφὴ δὲ τὸ N . ἀλλὰ καὶ ἔλαττον· ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα ὁ κῶνος, οὗ βάσις ὁ $ABΓΔ$ κύκλος, κορυφὴ δὲ τὸ Λ [σημεῖον], πρὸς ἔλαττον τι τοῦ κῶνου στερεόν, οὗ βάσις μὲν ὁ $EZH\Theta$ κύκλος, κορυφὴ δὲ τὸ N σημεῖον, τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Delta$ πρὸς τὴν $Z\Theta$. ὁμοίως δὲ δείξομεν, ὅτι οὐδὲ ὁ $EZH\Theta$ κῶνος πρὸς ἔλαττον τι τοῦ $ABΓΔ\Lambda$ κῶνου στερεόν τριπλασίονα λόγον ἔχει ἥπερ ἡ $Z\Theta$ πρὸς τὴν $B\Delta$.

Λέγω δὴ, ὅτι οὐδὲ ὁ $ABΓΔ\Lambda$ κῶνος πρὸς μεῖζόν τι τοῦ $EZH\Theta$ κῶνου στερεόν τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Delta$ πρὸς τὴν $Z\Theta$.

Εἰ γὰρ δυνατόν, ἐχέτω πρὸς μεῖζον τὸ Ξ . ἀνάπαλιν ἄρα τὸ Ξ στερεὸν πρὸς τὸν $ABΓΔ\Lambda$ κῶνον τριπλασίονα λόγον ἔχει ἥπερ ἡ $Z\Theta$ πρὸς τὴν $B\Delta$. ὡς δὲ τὸ Ξ στερεὸν πρὸς τὸν $ABΓΔ\Lambda$ κῶνον, οὕτως ὁ $EZH\Theta$ κῶνος πρὸς ἔλαττον τι τοῦ $ABΓΔ\Lambda$ κῶνου στερεόν. καὶ ὁ $EZH\Theta$ ἄρα κῶνος πρὸς ἔλαττον τι τοῦ $ABΓΔ\Lambda$ κῶνου στερεόν τριπλασίονα λόγον ἔχει ἥπερ ἡ $Z\Theta$ πρὸς τὴν $B\Delta$. ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ὁ $ABΓΔ\Lambda$ κῶνος πρὸς μεῖζόν τι τοῦ $EZH\Theta$ κῶνου στερεόν τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Delta$ πρὸς τὴν $Z\Theta$. ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἔλαττον. ὁ $ABΓΔ\Lambda$ ἄρα κῶνος πρὸς τὸν $EZH\Theta$ κῶνον τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Delta$ πρὸς τὴν $Z\Theta$.

Ὡς δὲ ὁ κῶνος πρὸς τὸν κῶνον, ὁ κύλινδρος πρὸς τὸν κύλινδρον· τριπλάσιος γὰρ ὁ κύλινδρος τοῦ κῶνου ὁ ἐπὶ τῆς αὐτῆς βάσεως τῷ κῶνῳ καὶ ἰσοϋψῆς αὐτῷ. καὶ ὁ κύλινδρος ἄρα πρὸς τὸν κύλινδρον τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Delta$ πρὸς τὴν $Z\Theta$.

Οἱ ἄρα ὅμοιοι κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους ἐν

ting up pyramids having the same apexes as the cones on each of the triangles (so formed), we can also show that each of the pyramids (on base $ABCD$ taken) in order will have to each of the pyramids (on base $EFGH$ taken) in order the cubed ratio that the corresponding side BK (has) to the corresponding side FM —that is to say, that BD (has) to FH . And (for two sets of proportional magnitudes) as one of the leading (magnitudes is) to one of the following, so (the sum of) all of the leading (magnitudes is) to (the sum of) all of the following (magnitudes) [Prop. 5.12]. And, thus, as pyramid $BKTL$ (is) to pyramid $FMPN$, so the whole pyramid whose base is polygon $ATBUCVDW$, and apex the point L , (is) to the whole pyramid whose base is polygon $EPFQGRHS$, and apex the point N . And, hence, the pyramid whose base is polygon $ATBUCVDW$, and apex the point L , has to the pyramid whose base is polygon $EPFQGRHS$, and apex the point N , the cubed ratio that BD (has) to FH . And it was also assumed that the cone whose base is circle $ABCD$, and apex the point L , has to solid O the cubed ratio that BD (has) to FH . Thus, as the cone whose base is circle $ABCD$, and apex the point L , is to solid O , so the pyramid whose base (is) [polygon] $ATBUCVDW$, and apex the point L , (is) to the pyramid whose base is polygon $EPFQGRHS$, and apex the point N . Thus, alternately, as the cone whose base (is) circle $ABCD$, and apex the point L , (is) to the pyramid within it whose base (is) the polygon $ATBUCVDW$, and apex the point L , so the [solid] O (is) to the pyramid whose base is polygon $EPFQGRHS$, and apex the point N [Prop. 5.16]. And the aforementioned cone (is) greater than the pyramid within it. For it encompasses it. Thus, solid O (is) also greater than the pyramid whose base is polygon $EPFQGRHS$, and apex the point N . But, (it is) also less. The very thing is impossible. Thus, the cone whose base (is) circle $ABCD$, and apex the [point] L , does not have to some solid less than the cone whose base (is) circle $EFGH$, and apex the point N , the cubed ratio that BD (has) to EH . So, similarly, we can show that neither does cone $EFGHN$ have to some solid less than cone $ABCDL$ the cubed ratio that FH (has) to BD .

So, I say that neither does cone $ABCDL$ have to some solid greater than cone $EFGHN$ the cubed ratio that BD (has) to FH .

For, if possible, let it have (such a ratio) to a greater (solid), O . Thus, inversely, solid O has to cone $ABCDL$ the cubed ratio that FH (has) to BD [Prop. 5.7 corr.]. And as solid O (is) to cone $ABCDL$, so cone $EFGHN$ (is) to some solid less than cone $ABCDL$ [12.2 lem.]. Thus, cone $EFGHN$ also has to some solid less than cone $ABCDL$ the cubed ratio that FH (has) to BD . The very

τριπλασίονι λόγῳ εἰσὶ τῶν ἐν ταῖς βάσεσι διαμέτρων· ὅπερ ἔδει δεῖξαι.

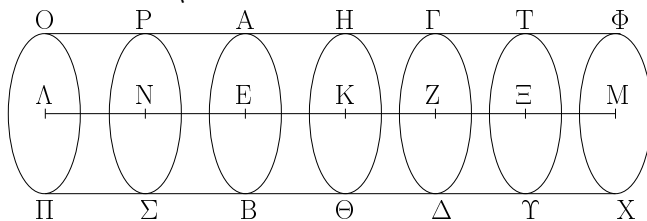
thing was shown (to be) impossible. Thus, cone $ABCDL$ does not have to some solid greater than cone $EFGHN$ the cubed ratio than BD (has) to FH . And it was shown that neither (does it have such a ratio) to a lesser (solid). Thus, cone $ABCDL$ has to cone $EFGHN$ the cubed ratio that BD (has) to FG .

And as the cone (is) to the cone, so the cylinder (is) to the cylinder. For a cylinder is three times a cone on the same base as the cone, and of the same height as it [Prop. 12.10]. Thus, the cylinder also has to the cylinder the cubed ratio that BD (has) to FH .

Thus, similar cones and cylinders are in the cubed ratio of the diameters of their bases. (Which is) the very thing it was required to show.

ιγ'.

Ἐὰν κύλινδρος ἐπιπέδῳ τμηθῇ παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔσται ὡς ὁ κύλινδρος πρὸς τὸν κύλινδρον, οὕτως ὁ ἄξων πρὸς τὸν ἄξονα.

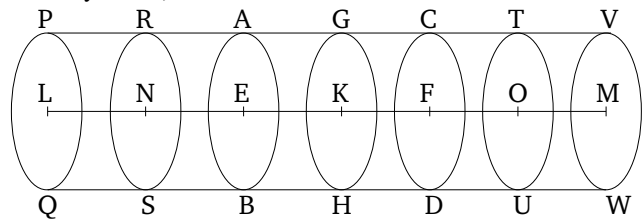


Κύλινδρος γάρ ὁ $A\Delta$ ἐπιπέδῳ τῷ $H\Theta$ τετμήσθω παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις τοῖς AB , $\Gamma\Delta$, καὶ συμβαλλέτω τῷ ἄξονι τὸ $H\Theta$ ἐπίπεδον κατὰ τὸ K σημεῖον· λέγω, ὅτι ἐστὶν ὡς ὁ BH κύλινδρος πρὸς τὸν $H\Delta$ κύλινδρον, οὕτως ὁ EK ἄξων πρὸς τὸν KZ ἄξονα.

Ἐκβεβλήσθω γάρ ὁ EZ ἄξων ἐφ' ἑκάτερα τὰ μέρη ἐπὶ τὰ Λ , M σημεία, καὶ ἐκχείσθωσαν τῷ EK ἄξονι ἴσοι ὅσοιδηποτοῦν οἱ EN , $N\Lambda$, τῷ δὲ ZK ἴσοι ὅσοιδηποτοῦν οἱ $Z\Xi$, ΞM , καὶ νοείσθω ὁ ἐπὶ τοῦ ΛM ἄξονος κύλινδρος ὁ OX , οὗ βάσεις οἱ $O\Pi$, ΦX κύκλοι. καὶ ἐκβεβλήσθω διὰ τῶν N , Ξ σημείων ἐπίπεδα παράλληλα τοῖς AB , $\Gamma\Delta$ καὶ ταῖς βάσεσι τοῦ OX κυλίνδρου καὶ ποιείτωσαν τοὺς $P\Sigma$, $T\Upsilon$ κύκλους περὶ τὰ N , Ξ κέντρα. καὶ ἐπεὶ οἱ ΛN , $N\Xi$, EK ἄξονες ἴσοι εἰσὶν ἀλλήλοις, οἱ ἄρα ΠP , PB , BH κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις. ἴσαι δὲ εἰσὶν αἱ βάσεις· ἴσοι ἄρα καὶ οἱ ΠP , PB , BH κύλινδροι ἀλλήλοις. ἐπεὶ οὖν οἱ ΛN , $N\Xi$, EK ἄξονες ἴσοι εἰσὶν ἀλλήλοις, εἰσὶ δὲ καὶ οἱ ΠP , PB , BH κύλινδροι ἴσοι ἀλλήλοις, καὶ ἐστὶν ἴσον τὸ πλῆθος τῷ πλῆθει, ὅσαπλασίον ἄρα ὁ $K\Lambda$ ἄξων τοῦ EK ἄξονος, τοσαυταπλασίον ἔσται καὶ ὁ ΠH κύλινδρος τοῦ $H\Delta$ κυλίνδρου. διὰ τὰ αὐτὰ δὲ καὶ ὅσαπλασίον ἐστὶν ὁ MK ἄξων τοῦ KZ ἄξονος, τοσαυταπλασίον ἔσται καὶ ὁ XH κύλινδρος τοῦ $H\Delta$ κυλίνδρου. καὶ εἰ μὲν ἴσος ἐστὶν ὁ $K\Lambda$ ἄξων τῷ KM ἄξονι, ἴσος ἔσται καὶ ὁ ΠH κύλινδρος τῷ HX κυλίνδρῳ,

Proposition 13

If a cylinder is cut by a plane which is parallel to the opposite planes (of the cylinder) then as the cylinder (is) to the cylinder, so the axis will be to the axis.



For let the cylinder AD have been cut by the plane GH which is parallel to the opposite planes (of the cylinder), AB and CD . And let the plane GH have met the axis at point K . I say that as cylinder BG is to cylinder GD , so axis EK (is) to axis KF .

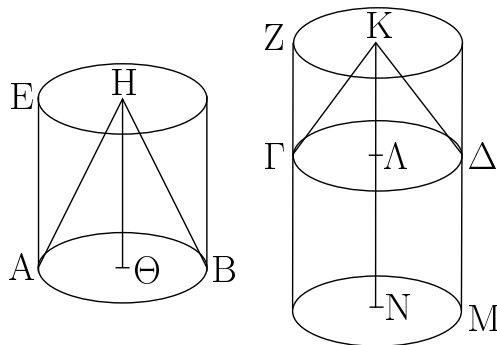
For let axis EF have been produced in each direction to points L and M . And let any number whatsoever (of lengths), EN and NL , equal to axis EK , be set out (on the axis EL), and any number whatsoever (of lengths), FO and OM , equal to (axis) FK , (on the axis KM). And let the cylinder PW , whose bases (are) the circles PQ and VW , have been conceived on axis LM . And let planes parallel to AB , CD , and the bases of cylinder PW , have been produced through points N and O , and let them have made the circles RS and TU around the centers N and O (respectively). And since axes LN , NE , and EK are equal to one another, the cylinders QR , RB , and BG are to one another as their bases [Prop. 12.11]. But the bases are equal. Thus, the cylinders QR , RB , and BG (are) also equal to one another. Therefore, since the axes LN , NE , and EK are equal to one another, and the cylinders QR , RB , and BG are also equal to one another, and the number (of the former) is equal to the number (of the latter), thus as many multiples as axis KL

εἰ δὲ μείζων ὁ ἄξων τοῦ ἄξονος, μείζων καὶ ὁ κύλινδρος τοῦ κυλίνδρου, καὶ εἰ ἐλάσσων, ἐλάσσων. τεσσάρων δὲ μεγεθῶν ὄντων, ἁξόνων μὲν τῶν EK , KZ , κυλίνδρων δὲ τῶν BH , $H\Delta$, εἴληπται ἰσάκεις πολλαπλάσια, τοῦ μὲν EK ἄξονος καὶ τοῦ BH κυλίνδρου ὅ τε ΛK ἄξων καὶ ὁ ΠH κύλινδρος, τοῦ δὲ KZ ἄξονος καὶ τοῦ $H\Delta$ κυλίνδρου ὅ τε KM ἄξων καὶ ὁ HX κύλινδρος, καὶ δέδεικται, ὅτι εἰ ὑπερέχει ὁ $K\Lambda$ ἄξων τοῦ KM ἄξονος, ὑπερέχει καὶ ὁ ΠH κύλινδρος τοῦ HX κυλίνδρου, καὶ εἰ ἴσος, ἴσος, καὶ εἰ ἐλάσσων, ἐλάσσων. ἔστιν ἄρα ὡς ὁ EK ἄξων πρὸς τὸν KZ ἄξονα, οὕτως ὁ BH κύλινδρος πρὸς τὸν $H\Delta$ κύλινδρον· ὅπερ ἔδει δεῖξαι.

is of axis EK , so many multiples is cylinder QG also of cylinder GB . And so, for the same (reasons), as many multiples as axis MK is of axis KF , so many multiples is cylinder WG also of cylinder GD . And if axis KL is equal to axis KM then cylinder QG will also be equal to cylinder GW , and if the axis (is) greater than the axis then the cylinder (will also be) greater than the cylinder, and if (the axis is) less then (the cylinder will also be) less. So, there are four magnitudes—the axes EK and KF , and the cylinders BG and GD —and equal multiples have been taken of axis EK and cylinder BG —(namely), axis LK and cylinder QG —and of axis KF and cylinder GD —(namely), axis KM and cylinder GW . And it has been shown that if axis KL exceeds axis KM then cylinder QG also exceeds cylinder GW , and if (the axes are) equal then (the cylinders are) equal, and if (KL is) less then (QG is) less. Thus, as axis EK is to axis KF , so cylinder BG (is) to cylinder GD [Def. 5.5]. (Which is) the very thing it was required to show.

ιδ'.

Οἱ ἐπὶ ἴσων βάσεων ὄντες κῶνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ὕψη.

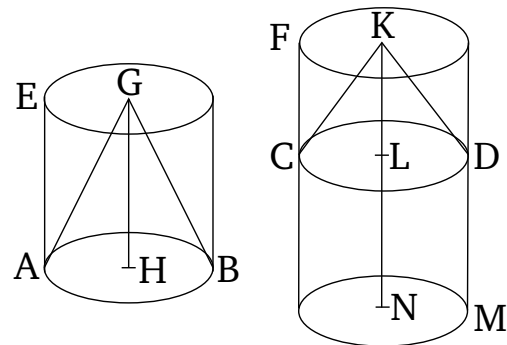


Ἐστωσαν γὰρ ἐπὶ ἴσων βάσεων τῶν AB , $\Gamma\Delta$ κύκλων κύλινδροι οἱ EB , $Z\Delta$ · λέγω, ὅτι ἐστὶν ὡς ὁ EB κύλινδρος πρὸς τὸν $Z\Delta$ κύλινδρον, οὕτως ὁ $H\Theta$ ἄξων πρὸς τὸν $K\Lambda$ ἄξονα.

Ἐκβεβλήσθω γὰρ ὁ $K\Lambda$ ἄξων ἐπὶ τὸ N σημεῖον, καὶ κείσθω τῷ $H\Theta$ ἄξονι ἴσος ὁ ΛN , καὶ περὶ ἄξονα τὸν ΛN κύλινδρος νενοήσθω ὁ ΓM . ἐπεὶ οὖν οἱ EB , ΓM κύλινδροι ὑπὸ τὸ αὐτὸ ὕψος εἰσὶν, πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις. ἴσαι δὲ εἰσὶν αἱ βάσεις ἀλλήλαις· ἴσοι ἄρα εἰσὶ καὶ οἱ EB , ΓM κύλινδροι. καὶ ἐπεὶ κύλινδρος ὁ ZM ἐπιπέδῳ τέτμηται τῷ $\Gamma\Delta$ παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις, ἔστιν ἄρα ὡς ὁ ΓM κύλινδρος πρὸς τὸν $Z\Delta$ κύλινδρον, οὕτως ὁ ΛN ἄξων πρὸς τὸν $K\Lambda$ ἄξονα. ἴσος δὲ ἐστὶν ὁ μὲν ΓM κύλινδρος τῷ EB κυλίνδρῳ, ὁ δὲ ΛN ἄξων τῷ $H\Theta$ ἄξονι· ἔστιν ἄρα ὡς ὁ EB κύλινδρος πρὸς τὸν $Z\Delta$ κύλινδρον, οὕτως ὁ $H\Theta$ ἄξων πρὸς τὸν $K\Lambda$ ἄξονα. ὡς δὲ ὁ EB κύλινδρος πρὸς τὸν $Z\Delta$

Proposition 14

Cones and cylinders which are on equal bases are to one another as their heights.



For let EB and FD be cylinders on equal bases, (namely) the circles AB and CD (respectively). I say that as cylinder EB is to cylinder FD , so axis GH (is) to axis KL .

For let the axis KL have been produced to point N . And let LN be made equal to axis GH . And let the cylinder CM have been conceived about axis LN . Therefore, since cylinders EB and CM have the same height they are to one another as their bases [Prop. 12.11]. And the bases are equal to one another. Thus, cylinders EB and CM are also equal to one another. And since cylinder FM has been cut by the plane CD , which is parallel to its opposite planes, thus as cylinder CM is to cylinder FD , so axis LN (is) to axis KL [Prop. 12.13]. And cylinder CM is equal to cylinder EB , and axis LN to axis GH . Thus, as cylinder EB is to cylinder FD , so axis GH (is)

κύλινδρον, οὕτως ὁ ABH κώνος πρὸς τὸν $\Gamma\Delta K$ κώνον. καὶ ὡς ἄρα ὁ $H\Theta$ ἄξων πρὸς τὸν $ΚΛ$ ἄξωνα, οὕτως ὁ ABH κώνος πρὸς τὸν $\Gamma\Delta K$ κώνον καὶ ὁ EB κύλινδρος πρὸς τὸν $Z\Delta$ κύλινδρον· ὅπερ ἔδει δεῖξαι.

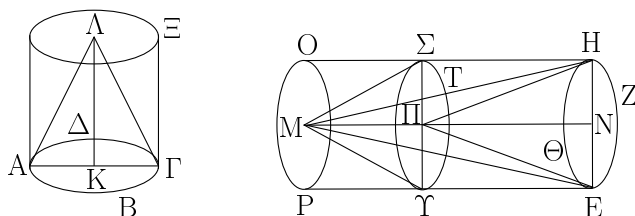
to axis KL . And as cylinder EB (is) to cylinder FD , so cone ABG (is) to cone CDK [Prop. 12.10]. Thus, also, as axis GH (is) to axis KL , so cone ABG (is) to cone CDK , and cylinder EB to cylinder FD . (Which is) the very thing it was required to show.

ιε'.

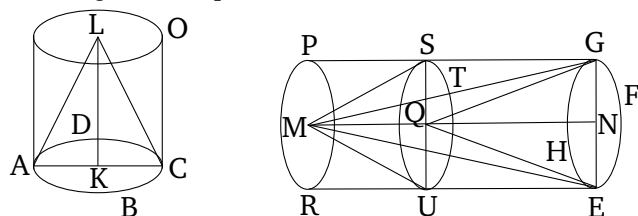
Proposition 15

Τῶν ἴσων κώνων καὶ κύλινδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν· καὶ ὧν κώνων καὶ κύλινδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, ἴσοι εἰσὶν ἐκεῖνοι.

The bases of equal cones and cylinders are reciprocally proportional to their heights. And, those cones and cylinders whose bases (are) reciprocally proportional to their heights are equal.



Ἐστωσαν ἴσοι κώνοι καὶ κύλινδροι, ὧν βάσεις μὲν οἱ $AB\Gamma\Delta$, $EZH\Theta$ κύκλοι, διαμέτροι δὲ αὐτῶν αἱ $ΑΓ$, $ΕΗ$, ἄξονες δὲ οἱ $ΚΑ$, $ΜΝ$, οἵτινες καὶ ὕψη εἰσὶ τῶν κώνων ἢ κύλινδρων, καὶ συμπεπληρώσθωσαν οἱ $A\Xi$, EO κύλινδροι. λέγω, ὅτι τῶν $A\Xi$, EO κύλινδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἐστὶν ὡς ἡ $AB\Gamma\Delta$ βάσις πρὸς τὴν $EZH\Theta$ βάσιν, οὕτως τὸ MN ὕψος πρὸς τὸ $ΚΑ$ ὕψος.



Τὸ γὰρ AK ὕψος τῷ MN ὕψει ἴσους ἐστὶν ἢ οὐ. ἔστω πρότερον ἴσους. ἔστι δὲ καὶ ὁ $A\Xi$ κύλινδρος τῷ EO κύλινδρῳ ἴσος. οἱ δὲ ὑπὸ τὸ αὐτὸ ὕψος ὄντες κώνοι καὶ κύλινδροι πρὸς ἀλλήλους εἰσὶν ὡς αἱ βάσεις· ἴση ἄρα καὶ ἡ $AB\Gamma\Delta$ βάσις τῇ $EZH\Theta$ βάσει. ὥστε καὶ ἀντιπέπονθεν, ὡς ἡ $AB\Gamma\Delta$ βάσις πρὸς τὴν $EZH\Theta$ βάσιν, οὕτως τὸ MN ὕψος πρὸς τὸ $ΚΑ$ ὕψος. ἀλλὰ δὴ μὴ ἔστω τὸ AK ὕψος τῷ MN ἴσον, ἀλλ' ἔστω μείζον τὸ MN , καὶ ἀφῃρησῶ ἀπὸ τοῦ MN ὕψους τῷ $ΚΑ$ ἴσον τὸ $ΠΝ$, καὶ διὰ τοῦ Π σημείου τετμήσθω ὁ EO κύλινδρος ἐπιπέδῳ τῷ $T\Upsilon\Sigma$ παραλλήλῳ τοῖς τῶν $EZH\Theta$, PO κύκλων ἐπιπέδοις, καὶ ἀπὸ βάσεως μὲν τοῦ $EZH\Theta$ κύκλου, ὕψους δὲ τοῦ $N\Pi$ κύλινδρος νενοήσθω ὁ $ΕΣ$. καὶ ἐπεὶ ἴσος ἐστὶν ὁ $A\Xi$ κύλινδρος τῷ EO κύλινδρῳ, ἔστιν ἄρα ὡς ὁ $A\Xi$ κύλινδρος πρὸς τὸν $ΕΣ$ κύλινδρον, οὕτως ὁ EO κύλινδρος πρὸς τὸν $ΕΣ$ κύλινδρον. ἀλλ' ὡς μὲν ὁ $A\Xi$ κύλινδρος πρὸς τὸν $ΕΣ$ κύλινδρον, οὕτως ἡ $AB\Gamma\Delta$ βάσις πρὸς τὴν $EZH\Theta$ · ὑπὸ γὰρ τὸ αὐτὸ ὕψος εἰσὶν οἱ $A\Xi$, $ΕΣ$ κύλινδροι· ὡς δὲ ὁ EO κύλινδρος πρὸς τὸν $ΕΣ$, οὕτως τὸ MN ὕψος πρὸς τὸ $ΠΝ$ ὕψος· ὁ γὰρ EO κύλινδρος ἐπιπέδῳ τέτμηται παραλλήλῳ ὄντι τοῖς ἀπεναντίον ἐπιπέδοις. ἔστιν ἄρα καὶ ὡς ἡ $AB\Gamma\Delta$ βάσις πρὸς τὴν $EZH\Theta$ βάσιν, οὕτως τὸ MN ὕψος πρὸς τὸ $ΠΝ$ ὕψος. ἴσον δὲ τὸ $ΠΝ$ ὕψος τῷ $ΚΑ$ ὕψει· ἔστιν ἄρα ὡς ἡ $AB\Gamma\Delta$ βάσις πρὸς τὴν $EZH\Theta$ βάσιν, οὕτως τὸ MN ὕψος πρὸς τὸ $ΚΑ$ ὕψος. τῶν ἄρα $A\Xi$, EO κύλινδρων ἀντιπεπόνθασιν αἱ βάσεις τοῖς ὕψεσιν.

Let there be equal cones and cylinders whose bases are the circles $ABCD$ and $EFGH$, and the diameters of (the bases) AC and EG , and (whose) axes (are) KL and MN , which are also the heights of the cones and cylinders (respectively). And let the cylinders AO and EP have been completed. I say that the bases of cylinders AO and EP are reciprocally proportional to their heights, and (so) as base $ABCD$ is to base $EFGH$, so height MN (is) to height KL .

For height LK is either equal to height MN , or not. Let it, first of all, be equal. And cylinder AO is also equal to cylinder EP . And cones and cylinders having the same height are to one another as their bases [Prop. 12.11]. Thus, base $ABCD$ (is) also equal to base $EFGH$. And, hence, reciprocally, as base $ABCD$ (is) to base $EFGH$, so height MN (is) to height KL . And so, let height LK not be equal to MN , but let MN be greater. And let QN , equal to KL , have been cut off from height MN . And let the cylinder EP have been cut, through point Q , by the plane TUS (which is) parallel to the planes of the circles $EFGH$ and RP . And let cylinder ES have been conceived, with base the circle $EFGH$, and height NQ . And since cylinder AO is equal to cylinder EP , thus, as cylinder AO (is) to cylinder ES , so cylinder EP (is) to cylinder ES [Prop. 5.7]. But, as cylinder AO (is) to cylinder ES , so base $ABCD$ (is) to base $EFGH$. For cylinders AO and ES (have) the same height [Prop. 12.11]. And as cylinder EP (is) to (cylinder) ES , so height MN (is) to height QN . For cylinder EP has been cut by a plane which is parallel to its opposite planes [Prop. 12.13]. And, thus, as base $ABCD$ is to base $EFGH$, so height MN (is) to height QN [Prop. 5.11]. And height QN

Ἀλλὰ δὴ τῶν $ΑΞ$, $ΕΟ$ κυλίνδρων ἀντιπεπονητέωσαν αἱ βάσεις τοῖς ὕψεσιν, καὶ ἔστω ὡς ἡ $ΑΒΓΔ$ βάσις πρὸς τὴν $ΕΖΗΘ$ βάσιν, οὕτως τὸ $ΜΝ$ ὕψος πρὸς τὸ $ΚΛ$ ὕψος· λέγω, ὅτι ἴσος ἐστὶν ὁ $ΑΞ$ κύλινδρος τῷ $ΕΟ$ κυλίνδρῳ.

Τῶν γὰρ αὐτῶν κατασκευασθέντων ἐπεὶ ἐστὶν ὡς ἡ $ΑΒΓΔ$ βάσις πρὸς τὴν $ΕΖΗΘ$ βάσιν, οὕτως τὸ $ΜΝ$ ὕψος πρὸς τὸ $ΚΛ$ ὕψος, ἴσον δὲ τὸ $ΚΛ$ ὕψος τῷ $ΠΝ$ ὕψει, ἔσται ἄρα ὡς ἡ $ΑΒΓΔ$ βάσις πρὸς τὴν $ΕΖΗΘ$ βάσιν, οὕτως τὸ $ΜΝ$ ὕψος πρὸς τὸ $ΠΝ$ ὕψος. ἀλλ' ὡς μὲν ἡ $ΑΒΓΔ$ βάσις πρὸς τὴν $ΕΖΗΘ$ βάσιν, οὕτως ὁ $ΑΞ$ κύλινδρος πρὸς τὸν $ΕΣ$ κύλινδρον· ὑπὸ γὰρ τὸ αὐτὸ ὕψος εἰσὶν· ὡς δὲ τὸ $ΜΝ$ ὕψος πρὸς τὸ $ΠΝ$ [ὕψος], οὕτως ὁ $ΕΟ$ κύλινδρος πρὸς τὸν $ΕΣ$ κύλινδρον· ἔστιν ἄρα ὡς ὁ $ΑΞ$ κύλινδρος πρὸς τὸν $ΕΣ$ κύλινδρον, οὕτως ὁ $ΕΟ$ κύλινδρος πρὸς τὸν $ΕΣ$. ἴσος ἄρα ὁ $ΑΞ$ κύλινδρος τῷ $ΕΟ$ κυλίνδρῳ. ὡσαύτως δὲ καὶ ἐπὶ τῶν κώνων· ὅπερ ἔδει δεῖξαι.

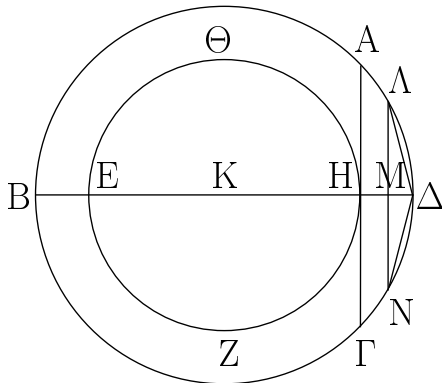
(is) equal to height KL . Thus, as base $ABCD$ is to base $EFGH$, so height MN (is) to height KL . Thus, the bases of cylinders AO and EP are reciprocally proportional to their heights.

And, so, let the bases of cylinders AO and EP be reciprocally proportional to their heights, and (thus) let base $ABCD$ be to base $EFGH$, as height MN (is) to height KL . I say that cylinder AO is equal to cylinder EP .

For, with the same construction, since as base $ABCD$ is to base $EFGH$, so height MN (is) to height KL , and height KL (is) equal to height QN , thus, as base $ABCD$ (is) to base $EFGH$, so height MN will be to height QN . But, as base $ABCD$ (is) to base $EFGH$, so cylinder AO (is) to cylinder ES . For they are the same height [Prop. 12.11]. And as height MN (is) to [height] QN , so cylinder EP (is) to cylinder ES [Prop. 12.13]. Thus, as cylinder AO is to cylinder ES , so cylinder EP (is) to (cylinder) ES [Prop. 5.11]. Thus, cylinder AO (is) equal to cylinder EP [Prop. 5.9]. In the same manner, (the proposition can) also (be demonstrated) for the cones. (Which is) the very thing it was required to show.

ιβ'.

Δύο κύκλων περὶ τὸ αὐτὸ κέντρον ὄντων εἰς τὸν μείζονα κύκλον πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον ἐγγράψαι μὴ ψαῦον τοῦ ἐλάσσονος κύκλου.

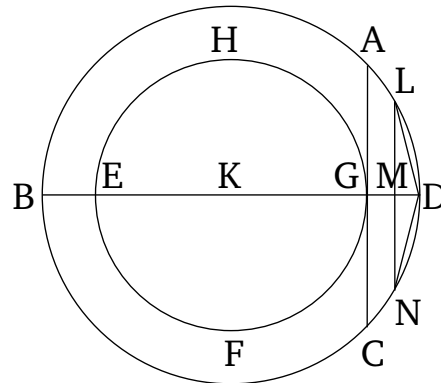


Ἐστωσαν οἱ δοθέντες δύο κύκλοι οἱ $ΑΒΓΔ$, $ΕΖΗΘ$ περὶ τὸ αὐτὸ κέντρον τὸ $Κ$ · δεῖ δὴ εἰς τὸν μείζονα κύκλον τὸν $ΑΒΓΔ$ πολύγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον ἐγγράψαι μὴ ψαῦον τοῦ $ΕΖΗΘ$ κύκλου.

Ἦχθω γὰρ διὰ τοῦ $Κ$ κέντρου εὐθεῖα ἡ $ΒΚΔ$, καὶ ἀπὸ τοῦ $Η$ σημείου τῇ $ΒΔ$ εὐθείᾳ πρὸς ὀρθὰς ἤχθω ἡ $ΗΑ$ καὶ διήχθω ἐπὶ τὸ $Γ$ · ἡ $ΑΓ$ ἄρα ἐφάπτεται τοῦ $ΕΖΗΘ$ κύκλου. τέμνοντες δὴ τὴν $ΒΑΔ$ περιφέρειαν δίχα καὶ τὴν ἡμίσειαν αὐτῆς δίχα καὶ τοῦτο αἶ ποιοῦντες καταλείψομεν περιφέρειαν ἐλάσσονα τῆς $ΑΔ$. λελείφθω, καὶ ἔστω ἡ $ΛΔ$, καὶ ἀπὸ τοῦ $Λ$ ἐπὶ τὴν $ΒΔ$ κάθετος ἤχθω ἡ $ΛΜ$ καὶ διήχθω

Proposition 16

There being two circles about the same center, to inscribe an equilateral and even-sided polygon in the greater circle, not touching the lesser circle.



Let $ABCD$ and $EFGH$ be the given two circles, about the same center, K . So, it is necessary to inscribe an equilateral and even-sided polygon in the greater circle $ABCD$, not touching circle $EFGH$.

Let the straight-line BKD have been drawn through the center K . And let GA have been drawn, at right-angles to the straight-line BD , through point G , and let it have been drawn through to C . Thus, AC touches circle $EFGH$ [Prop. 3.16 corr.]. So, (by) cutting circumference BAD in half, and the half of it in half, and doing this continually, we will (eventually) leave a circumference less

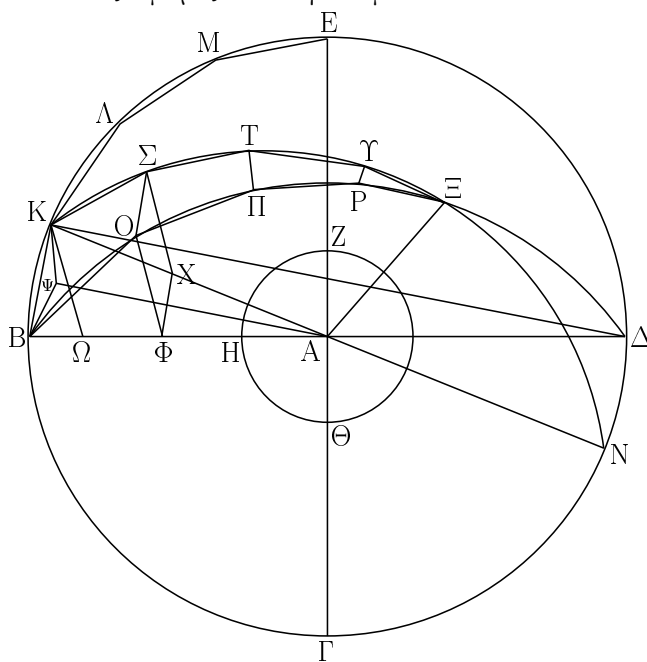
ἐπὶ τὸ Ν, καὶ ἐπεξεύχθωσαν αἱ ΛΔ, ΔΝ· ἴση ἄρα ἐστὶν ἡ ΛΔ τῇ ΔΝ. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΑΝ τῇ ΑΓ, ἡ δὲ ΑΓ ἐφάπτεται τοῦ ΕΖΗΘ κύκλου, ἡ ΑΝ ἄρα οὐκ ἐφάπτεται τοῦ ΕΖΗΘ κύκλου· πολλῶν ἄρα αἱ ΛΔ, ΔΝ οὐκ ἐφάπτονται τοῦ ΕΖΗΘ κύκλου. ἐὰν δὲ τῇ ΛΔ εὐθείᾳ ἴσας κατὰ τὸ συνεχὲς ἐναρμόσωμεν εἰς τὸν ΑΒΓΔ κύκλον, ἐγγραφήσεται εἰς τὸν ΑΒΓΔ κύκλον πολὺγωνον ἰσόπλευρόν τε καὶ ἀρτιόπλευρον μὴ ψαῦον τοῦ ἐλάσσονος κύκλου τοῦ ΕΖΗΘ· ὅπερ ἔδει ποιῆσαι.

than AD [Prop. 10.1]. Let it have been left, and let it be LD . And let LM have been drawn, from L , perpendicular to BD , and let it have been drawn through to N . And let LD and DN have been joined. Thus, LD is equal to DN [Props. 3.3, 1.4]. And since LN is parallel to AC [Prop. 1.28], and AC touches circle $EFGH$, LN thus does not touch circle $EFGH$. Thus, even more so, LD and DN do not touch circle $EFGH$. And if we continuously insert (straight-lines) equal to straight-line LD into circle $ABCD$ [Prop. 4.1] then an equilateral and even-sided polygon, not touching the lesser circle $EFGH$, will have been inscribed in circle $ABCD$.[†] (Which is) the very thing it was required to do.

[†] Note that the chord of the polygon, LN , does not touch the inner circle either.

ιζ'.

Δύο σφαιρῶν περὶ τὸ αὐτὸ κέντρον οὐσῶν εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγράφαι μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

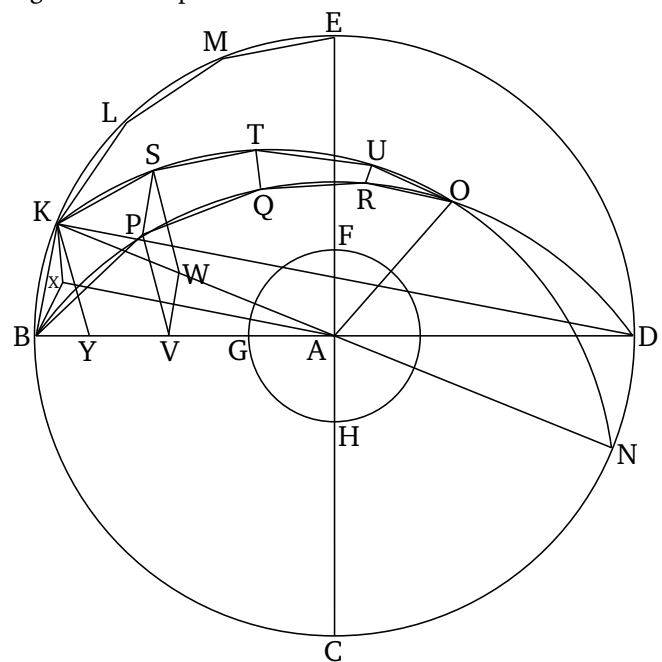


Νενοήσθωσαν δύο σφαῖραι περὶ τὸ αὐτὸ κέντρον τὸ Α· δεῖ δὲ εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγράφαι μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

Τετμήσθωσαν αἱ σφαῖραι ἐπιπέδῳ τινὶ διὰ τοῦ κέντρου· ἔσονται δὲ αἱ τομαὶ κύκλοι, ἐπειδὴ περ μενούσης τῆς διαμέτρου καὶ περιφερομένου τοῦ ἡμικυκλίου ἐγίγνετο ἡ σφαῖρα· ὥστε καὶ καθ' οἷας ἂν θέσεως ἐπινοήσωμεν τὸ ἡμικύκλιον, τὸ δι' αὐτοῦ ἐκβαλλόμενον ἐπίπεδον ποιήσει ἐπὶ τῆς ἐπιφανείας τῆς σφαίρας κύκλον. καὶ φανερόν, ὅτι καὶ μέγιστον, ἐπειδὴ περ ἡ διάμετρος τῆς σφαίρας, ἥτις

Proposition 17

There being two spheres about the same center, to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.



Let two spheres have been conceived about the same center, A . So, it is necessary to inscribe a polyhedral solid in the greater sphere, not touching the lesser sphere on its surface.

Let the spheres have been cut by some plane through the center. So, the sections will be circles, inasmuch as a sphere is generated by the diameter remaining behind, and a semi-circle being carried around [Def. 11.14]. And, hence, whatever position we conceive (of for) the semi-circle, the plane produced through it will make a

ἐστὶ καὶ τοῦ ἡμικυκλίου διάμετρος δηλαδὴ καὶ τοῦ κύκλου, μείζων ἐστὶ πασῶν τῶν εἰς τὸν κύκλον ἢ τὴν σφαῖραν διαγομένων [εὐθειῶν]. ἔστω οὖν ἐν μὲν τῇ μείζονι σφαίρᾳ κύκλος ὁ ΒΓΔΕ, ἐν δὲ τῇ ἐλάσσονι σφαίρᾳ κύκλος ὁ ΖΗΘ, καὶ ἤχθωσαν αὐτῶν δύο διαμέτροι πρὸς ὀρθὰς ἀλλήλαις αἱ ΒΔ, ΓΕ, καὶ δύο κύκλων περὶ τὸ αὐτὸ κέντρον ὄντων τῶν ΒΓΔΕ, ΖΗΘ εἰς τὸν μείζονα κύκλον τὸν ΒΓΔΕ πολὺγωνα ἰσόπλευρον καὶ ἀρτιόπλευρον ἐγγεγράφθω μὴ ψαῦον τοῦ ἐλάσσονος κύκλου τοῦ ΖΗΘ, οὗ πλευραὶ ἔστωσαν ἐν τῷ ΒΕ τεταρτημορίῳ αἱ ΒΚ, ΚΛ, ΛΜ, ΜΕ, καὶ ἐπιζευχθεῖσα ἡ ΚΑ διήχθω ἐπὶ τὸ Ν, καὶ ἀνεστάτω ἀπὸ τοῦ Α σημείου τῷ τοῦ ΒΓΔΕ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΑΞ καὶ συμβαλλέτω τῇ ἐπιφανείᾳ τῆς σφαίρας κατὰ τὸ Ξ, καὶ διὰ τῆς ΑΞ καὶ ἐκατέρας τῶν ΒΔ, ΚΝ ἐπίπεδα ἐκβεβλήσθω· ποιήσουσι δὴ διὰ τὰ εἰρημένα ἐπὶ τῆς ἐπιφανείας τῆς σφαίρας μεγίστους κύκλους. ποιείτωσαν, ὧν ἡμικύκλια ἔστω ἐπὶ τῶν ΒΔ, ΚΝ διαμέτρων τὰ ΒΞΔ, ΚΞΝ. καὶ ἐπεὶ ἡ ΞΑ ὀρθή ἐστι πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον, καὶ πάντα ἄρα τὰ διὰ τῆς ΞΑ ἐπίπεδά ἐστὶν ὀρθὰ πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον· ὥστε καὶ τὰ ΒΞΔ, ΚΞΝ ἡμικύκλια ὀρθὰ ἐστὶ πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον. καὶ ἐπεὶ ἴσα ἐστὶ τὰ ΒΕΔ, ΒΞΔ, ΚΞΝ ἡμικύκλια· ἐπὶ γὰρ ἴσων εἰσὶ διαμέτρων τῶν ΒΔ, ΚΝ· ἴσα ἐστὶ καὶ τὰ ΒΕ, ΒΞ, ΚΞ τεταρτημόρια ἀλλήλοις. ὅσαι ἄρα εἰσὶν ἐν τῷ ΒΕ τεταρτημορίῳ πλευραὶ τοῦ πολυγώνου, τοσαῦταί εἰσι καὶ ἐν τοῖς ΒΞ, ΚΞ τεταρτημορίοις ἴσαι ταῖς ΒΚ, ΚΛ, ΛΜ, ΜΕ εὐθείαις. ἐγγεγράφθωσαν καὶ ἔστωσαν αἱ ΒΟ, ΟΠ, ΠΡ, ΡΞ, ΚΣ, ΣΤ, ΤΥ, ΥΞ, καὶ ἐπεξεύχθωσαν αἱ ΣΟ, ΤΠ, ΥΡ, καὶ ἀπὸ τῶν Ο, Σ ἐπὶ τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον κάθετοι ἤχθωσαν· πεσοῦνται δὴ ἐπὶ τὰς κοινὰς τομὰς τῶν ἐπιπέδων τὰς ΒΔ, ΚΝ, ἐπειδὴ περ καὶ τὰ τῶν ΒΞΔ, ΚΞΝ ἐπίπεδα ὀρθὰ ἐστὶ πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον. πιπτέτωσαν, καὶ ἔστωσαν αἱ ΟΦ, ΣΧ, καὶ ἐπεξεύχθω ἡ ΧΦ. καὶ ἐπεὶ ἐν ἴσοις ἡμικυκλίοις τοῖς ΒΞΔ, ΚΞΝ ἴσαι ἀπειλημμεναι εἰσὶν αἱ ΒΟ, ΚΣ, καὶ κάθετοι ἡγμέναι εἰσὶν αἱ ΟΦ, ΣΧ, ἴση [ἄρα] ἐστὶν ἡ μὲν ΟΦ τῇ ΣΧ, ἡ δὲ ΒΦ τῇ ΚΧ. ἔστι δὲ καὶ ὅλη ἡ ΒΑ ὅλη τῇ ΚΑ ἴση· καὶ λοιπὴ ἄρα ἡ ΦΑ λοιπὴ τῇ ΧΑ ἐστὶν ἴση· ἔστιν ἄρα ὡς ἡ ΒΦ πρὸς τὴν ΦΑ, οὕτως ἡ ΚΧ πρὸς τὴν ΧΑ· παράλληλος ἄρα ἐστὶν ἡ ΧΦ τῇ ΚΒ. καὶ ἐπεὶ ἐκάτερα τῶν ΟΦ, ΣΧ ὀρθὴ ἐστὶ πρὸς τὸ τοῦ ΒΓΔΕ κύκλου ἐπίπεδον, παράλληλος ἄρα ἐστὶν ἡ ΟΦ τῇ ΣΧ. ἐδείχθη δὲ αὐτῇ καὶ ἴση· καὶ αἱ ΧΦ, ΣΟ ἄρα ἴσαι εἰσὶ καὶ παράλληλοι. καὶ ἐπεὶ παράλληλός ἐστιν ἡ ΧΦ τῇ ΣΟ, ἀλλὰ ἡ ΧΦ τῇ ΚΒ ἐστὶ παράλληλος, καὶ ἡ ΣΟ ἄρα τῇ ΚΒ ἐστὶ παράλληλος. καὶ ἐπιζευγνύουσιν αὐτάς αἱ ΒΟ, ΚΣ· τὸ ΚΒΟΣ ἄρα τετράπλευρον ἐν ἐνὶ ἐστὶν ἐπιπέδῳ, ἐπειδὴ περ, ἐὰν ὦσι δύο εὐθεῖαι παράλληλοι, καὶ ἐφ' ἐκατέρας αὐτῶν ληφθῇ τυχόντα σημεία, ἡ ἐπὶ τὰ σημεία ἐπιζευγνυμένη εὐθεῖα ἐν τῷ αὐτῷ ἐπιπέδῳ ἐστὶ ταῖς παραλλήλοις. διὰ τὰ αὐτὰ δὴ καὶ ἐκάτερον τῶν ΣΟΠΤ, ΤΠΡΥ τετραπλεύρων ἐν ἐνὶ ἐστὶν ἐπιπέδῳ. ἔστι δὲ καὶ τὸ ΥΡΞ τρίγωνον ἐν ἐνὶ ἐπιπέδῳ. ἐὰν δὴ νοήσωμεν ἀπὸ

circle on the surface of the sphere. And (it is) clear that (it is) also a great (circle), inasmuch as the diameter of the sphere, which is also manifestly the diameter of the semi-circle and the circle, is greater than all of the (other) [straight-lines] drawn across in the circle or the sphere [Prop. 3.15]. Therefore, let $BCDE$ be the circle in the greater sphere, and FGH the circle in the lesser sphere. And let two diameters of them have been drawn at right-angles to one another, (namely), BD and CE . And there being two circles about the same center—(namely), $BCDE$ and FGH —let an equilateral and even-sided polygon have been inscribed in the greater circle, $BCDE$, not touching the lesser circle, FGH [Prop. 12.16], of which let the sides in the quadrant BE be BK, KL, LM , and ME . And, KA being joined, let it have been drawn across to N . And let AO have been set up at point A , at right-angles to the plane of circle $BCDE$. And let it meet the surface of the (greater) sphere at O . And let planes have been produced through AO and each of BD and KN . So, according to the aforementioned (discussion), they will make great circles on the surface of the (greater) sphere. Let them make (great circles), of which let BOD and KON be semi-circles on the diameters BD and KN (respectively). And since OA is at right-angles to the plane of circle $BCDE$, all of the planes through OA are thus also at right-angles to the plane of circle $BCDE$ [Prop. 11.18]. And, hence, the semi-circles BOD and KON are also at right-angles to the plane of circle $BCDE$. And since semi-circles BED, BOD , and KON are equal—for (they are) on the equal diameters BD and KN [Def. 3.1]—the quadrants BE, BO , and KO are also equal to one another. Thus, as many sides of the polygon as are in quadrant BE , so many are also in quadrants BO and KO equal to the straight-lines BK, KL, LM , and ME . Let them have been inscribed, and let them be $BP, PQ, QR, RO, KS, ST, TU$, and UO . And let SP, TQ , and UR have been joined. And let perpendiculars have been drawn from P and S to the plane of circle $BCDE$ [Prop. 11.11]. So, they will fall on the common sections of the planes BD and KN (with $BCDE$), inasmuch as the planes of BOD and KON are also at right-angles to the plane of circle $BCDE$ [Def. 11.4]. Let them have fallen, and let them be PV and SW . And let WV have been joined. And since BP and KS are equal (circumferences) having been cut off in the equal semi-circles BOD and KON [Def. 3.28], and PV and SW are perpendiculars having been drawn (from them), PV is [thus] equal to SW , and BV to KW [Props. 3.27, 1.26]. And the whole of BA is also equal to the whole of KA . And, thus, as BV is to VA , so KW (is) to WA . WV is thus parallel to KB [Prop. 6.2]. And

τῶν $O, \Sigma, \Pi, T, P, \Upsilon$ σημείων ἐπὶ τὸ A ἐπιζευγνυμένας εὐθείας, συσταθήσεται τι σχῆμα στερεὸν πολύεδρον ματαξὺ τῶν $B\Xi, K\Xi$ περιφερειῶν ἐκ πυραμίδων συγκείμενον, ὧν βάσεις μὲν τὰ $KBO\Sigma, \Sigma O\Pi T, T\Pi P\Upsilon$ τετράπλευρα καὶ τὸ $\Upsilon P\Xi$ τρίγωνον, κορυφὴ δὲ τὸ A σημεῖον. ἐὰν δὲ καὶ ἐπὶ ἐκάστης τῶν $K\Lambda, \Lambda M, ME$ πλευρῶν καθάπερ ἐπὶ τῆς BK τὰ αὐτὰ κατασκευάσωμεν καὶ ἔτι τῶν λοιπῶν τριῶν τεταρτημορίων, συσταθήσεται τι σχῆμα πολύεδρον ἐγγεγραμμένον εἰς τὴν σφαῖραν πυραμίσι περιεχόμενον, ὧν βάσεις [μὲν] τὰ εἰρημένα τετράπλευρα καὶ τὸ $\Upsilon P\Xi$ τρίγωνον καὶ τὰ ὁμοταγῆ αὐτοῖς, κορυφὴ δὲ τὸ A σημεῖον.

Λέγω ὅτι τὸ εἰρημένον πολύεδρον οὐκ ἐφάπτεται τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν, ἐφ' ἧς ἐστὶν ὁ $ZH\Theta$ κύκλος.

Ἦχθω ἀπὸ τοῦ A σημείου ἐπὶ τὸ τοῦ $KBO\Sigma$ τετραπλεύρου ἐπίπεδον κάθετος ἡ $A\Psi$ καὶ συμβαλλέτω τῷ ἐπιπέδῳ κατὰ τὸ Ψ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ $\Psi B, \Psi K$. καὶ ἐπεὶ ἡ $A\Psi$ ὀρθὴ ἐστὶ πρὸς τὸ τοῦ $KBO\Sigma$ τετραπλεύρου ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ τοῦ τετραπλεύρου ἐπιπέδῳ ὀρθὴ ἐστὶν. ἡ $A\Psi$ ἄρα ὀρθὴ ἐστὶ πρὸς ἐκατέραν τῶν $B\Psi, \Psi K$. καὶ ἐπεὶ ἴση ἐστὶν ἡ AB τῇ AK , ἴσον ἐστὶ καὶ τὸ ἀπὸ τῆς AB τῷ ἀπὸ τῆς AK . καὶ ἐστὶ τῷ μὲν ἀπὸ τῆς AB ἴσα τὰ ἀπὸ τῶν $A\Psi, \Psi B$. ὀρθὴ γὰρ ἡ πρὸς τῷ Ψ . τῷ δὲ ἀπὸ τῆς AK ἴσα τὰ ἀπὸ τῶν $A\Psi, \Psi K$. τὰ ἄρα ἀπὸ τῶν $A\Psi, \Psi B$ ἴσα ἐστὶ τοῖς ἀπὸ τῶν $A\Psi, \Psi K$. κοινὸν ἀφηρήσθω τὸ ἀπὸ τῆς $A\Psi$. λοιπὸν ἄρα τὸ ἀπὸ τῆς $B\Psi$ λοιπὸν τῷ ἀπὸ τῆς ΨK ἴσον ἐστὶν. ἴση ἄρα ἡ $B\Psi$ τῇ ΨK . ὁμοίως δὲ δεῖξομεν, ὅτι καὶ αἱ ἀπὸ τοῦ Ψ ἐπὶ τὰ O, Σ ἐπιζευγνύμεναι εὐθεῖαι ἴσαι εἰσὶν ἐκατέρᾳ τῶν $B\Psi, \Psi K$. ὁ ἄρα κέντρῳ τῷ Ψ καὶ διαστήματι ἐνὶ τῶν $\Psi B, \Psi K$ γραφόμενος κύκλος ἥξει καὶ διὰ τῶν O, Σ , καὶ ἔσται ἐν κύκλῳ τὸ $KBO\Sigma$ τετράπλευρον.

Καὶ ἐπεὶ μείζων ἐστὶν ἡ KB τῆς $X\Phi$, ἴση δὲ ἡ $X\Phi$ τῇ ΣO , μείζων ἄρα ἡ KB τῆς ΣO . ἴση δὲ ἡ KB ἐκατέρᾳ τῶν $K\Sigma, BO$. καὶ ἐκατέρᾳ ἄρα τῶν $K\Sigma, BO$ τῆς ΣO μείζων ἐστὶν. καὶ ἐπεὶ ἐν κύκλῳ τετράπλευρόν ἐστὶ τὸ $KBO\Sigma$, καὶ ἴσαι αἱ $KB, BO, K\Sigma$, καὶ ἐλάττων ἡ $O\Sigma$, καὶ ἐκ τοῦ κέντρου τοῦ κύκλου ἐστὶν ἡ $B\Psi$, τὸ ἄρα ἀπὸ τῆς KB τοῦ ἀπὸ τῆς $B\Psi$ μείζον ἐστὶν ἢ διπλάσιον. ἦχθω ἀπὸ τοῦ K ἐπὶ τὴν $B\Phi$ κάθετος ἡ $K\Omega$. καὶ ἐπεὶ ἡ $B\Delta$ τῆς $\Delta\Omega$ ἐλάττων ἐστὶν ἢ διπλῇ, καὶ ἐστὶν ὡς ἡ $B\Delta$ πρὸς τὴν $\Delta\Omega$, οὕτως τὸ ὑπὸ τῶν $\Delta B, B\Omega$ πρὸς τὸ ὑπὸ [τῶν] $\Delta\Omega, \Omega B$, ἀναγραφόμενου ἀπὸ τῆς $B\Omega$ τετραγώνου καὶ συμπληρουμένου τοῦ ἐπὶ τῆς $\Omega\Delta$ παραλληλογράμμου καὶ τὸ ὑπὸ $\Delta B, B\Omega$ ἄρα τοῦ ὑπὸ $\Delta\Omega, \Omega B$ ἐλαττόν ἐστὶν ἢ διπλάσιον. καὶ ἐστὶ τῆς $K\Delta$ ἐπιζευγνυμένης τὸ μὲν ὑπὸ $\Delta B, B\Omega$ ἴσον τῷ ἀπὸ τῆς BK , τὸ δὲ ὑπὸ τῶν $\Delta\Omega, \Omega B$ ἴσον τῷ ἀπὸ τῆς $K\Omega$. τὸ ἄρα ἀπὸ τῆς KB τοῦ ἀπὸ τῆς $K\Omega$ ἐλασσόν ἐστὶν ἢ διπλάσιον. ἀλλὰ τὸ ἀπὸ τῆς KB τοῦ ἀπὸ τῆς $B\Psi$ μείζον ἐστὶν ἢ διπλάσιον. μείζον ἄρα τὸ ἀπὸ τῆς $K\Omega$ τοῦ ἀπὸ τῆς $B\Psi$. καὶ ἐπεὶ ἴση ἐστὶν ἡ BA τῇ KA , ἴσον ἐστὶ τὸ ἀπὸ τῆς BA τῷ ἀπὸ τῆς AK . καὶ

since PV and SW are each at right-angles to the plane of circle $BCDE$, PV is thus parallel to SW [Prop. 11.6]. And it was also shown (to be) equal to it. And, thus, WV and SP are equal and parallel [Prop. 1.33]. And since WV is parallel to SP , but WV is parallel to KB , SP is thus also parallel to KB [Prop. 11.1]. And BP and KS join them. Thus, the quadrilateral $KBPS$ is in one plane, inasmuch as if there are two parallel straight-lines, and a random point is taken on each of them, then the straight-line joining the points is in the same plane as the parallel (straight-lines) [Prop. 11.7]. So, for the same (reasons), each of the quadrilaterals $SPQT$ and $TQRU$ is also in one plane. And triangle URO is also in one plane [Prop. 11.2]. So, if we conceive straight-lines joining points P, S, Q, T, R , and U to A then some solid polyhedral figure will have been constructed between the circumferences BO and KO , being composed of pyramids whose bases (are) the quadrilaterals $KBPS, SPQT, TQRU$, and the triangle URO , and apex the point A . And if we also make the same construction on each of the sides KL, LM , and ME , just as on BK , and, further, (repeat the construction) in the remaining three quadrants, then some polyhedral figure which has been inscribed in the sphere will have been constructed, being contained by pyramids whose bases (are) the aforementioned quadrilaterals, and triangle URO , and the (quadrilaterals and triangles) similarly arranged to them, and apex the point A .

So, I say that the aforementioned polyhedron will not touch the lesser sphere on the surface on which the circle FGH is (situated).

Let the perpendicular (straight-line) AX have been drawn from point A to the plane $KBPS$, and let it meet the plane at point X [Prop. 11.11]. And let XB and XK have been joined. And since AX is at right-angles to the plane of quadrilateral $KBPS$, it is thus also at right-angles to all of the straight-lines joined to it which are also in the plane of the quadrilateral [Def. 11.3]. Thus, AX is at right-angles to each of BX and XK . And since AB is equal to AK , the (square) on AB is also equal to the (square) on AK . And the (sum of the squares) on AX and XB is equal to the (square) on AB . For the angle at X (is) a right-angle [Prop. 1.47]. And the (sum of the squares) on AX and XK is equal to the (square) on AK [Prop. 1.47]. Thus, the (sum of the squares) on AX and XB is equal to the (sum of the squares) on AX and XK . Let the (square) on AX have been subtracted from both. Thus, the remaining (square) on BX is equal to the remaining (square) on XK . Thus, BX (is) equal to XK . So, similarly, we can show that the straight-lines joined from X to P and S are equal to each of BX and XK .

ἐστι τῷ μὲν ἀπὸ τῆς BA ἴσα τὰ ἀπὸ τῶν $B\Psi$, ΨA , τῷ δὲ ἀπὸ τῆς KA ἴσα τὰ ἀπὸ τῶν $K\Omega$, ΩA . τὰ ἄρα ἀπὸ τῶν $B\Psi$, ΨA ἴσα ἐστὶ τοῖς ἀπὸ τῶν $K\Omega$, ΩA , ὣν τὸ ἀπὸ τῆς $K\Omega$ μείζον τοῦ ἀπὸ τῆς $B\Psi$. λοιπὸν ἄρα τὸ ἀπὸ τῆς ΩA ἑλάσσον ἐστὶ τοῦ ἀπὸ τῆς ΨA . μείζων ἄρα ἢ $A\Psi$ τῆς $A\Omega$. πολλῶν ἄρα ἢ $A\Psi$ μείζων ἐστὶ τῆς AH . καὶ ἐστὶν ἢ μὲν $A\Psi$ ἐπὶ μίαν τοῦ πολυέδρου βάσιν, ἢ δὲ AH ἐπὶ τὴν τῆς ἐλάσσονος σφαίρας ἐπιφάνειαν. ὥστε τὸ πολύεδρον οὐ ψάυσει τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν.

Δύο ἄρα σφαιρῶν περὶ τὸ αὐτὸ κέντρον οὐσῶν εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγέγραπται μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν. ὅπερ ἔδει ποιῆσαι.

Thus, a circle drawn (in the plane of the quadrilateral) with center X , and radius one of XB or XK , will also pass through P and S , and the quadrilateral $KBPS$ will be inside the circle.

And since KB is greater than WV , and WV (is) equal to SP , KB (is) thus greater than SP . And KB (is) equal to each of KS and BP . Thus, KS and BP are each greater than SP . And since quadrilateral $KBPS$ is in a circle, and KB , BP , and KS are equal (to one another), and PS (is) less (than them), and BX is the radius of the circle, the (square) on KB is thus greater than double the (square) on BX .[†] Let the perpendicular KY have been drawn from K to BV .[‡] And since BD is less than double DY , and as BD is to DY , so the (rectangle contained) by DB and BY (is) to the (rectangle contained) by DY and YB —a square being described on BY , and a (rectangular) parallelogram (with short side equal to BY) completed on YD —the (rectangle contained) by DB and BY is thus also less than double the (rectangle contained) by DY and YB . And, KD being joined, the (rectangle contained) by DB and BY is equal to the (square) on BK , and the (rectangle contained) by DY and YB equal to the (square) on KY [Props. 3.31, 6.8 corr.]. Thus, the (square) on KB is less than double the (square) on KY . But, the (square) on KB is greater than double the (square) on BX . Thus, the (square) on KY (is) greater than the (square) on BX . And since BA is equal to KA , the (square) on BA is equal to the (square) on KA . And the (sum of the squares) on BX and XA is equal to the (square) on BA , and the (sum of the squares) on KY and YA (is) equal to the (square) on KA [Prop. 1.47]. Thus, the (sum of the squares) on BX and XA is equal to the (sum of the squares) on KY and YA , of which the (square) on KY (is) greater than the (square) on BX . Thus, the remaining (square) on YA is less than the (square) on XA . Thus, AX (is) greater than AY . Thus, AX is much greater than AG .[§] And AX is (a perpendicular) on one of the bases of the polyhedron, and AG (is a perpendicular) on the surface of the lesser sphere. Hence, the polyhedron will not touch the lesser sphere on its surface.

Thus, there being two spheres about the same center, a polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere on its surface. (Which is) the very thing it was required to do.

[†] Since KB , BP , and KS are greater than the sides of an inscribed square, which are each of length $\sqrt{2}BX$.

[‡] Note that points Y and V are actually identical.

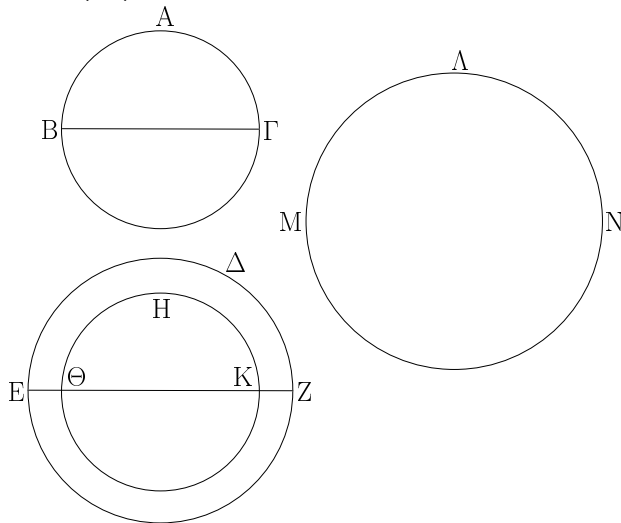
[§] This conclusion depends on the fact that the chord of the polygon in proposition 12.16 does not touch the inner circle.

Πόρισμα.

Ἐάν δὲ καὶ εἰς ἐτέραν σφαῖραν τῷ ἐν τῇ ΒΓΔΕ σφαίρᾳ στερεῶ πολυέδρῳ ὅμοιον στερεὸν πολυέδρον ἐγγραφῇ, τὸ ἐν τῇ ΒΓΔΕ σφαίρᾳ στερεὸν πολυέδρον πρὸς τὸ ἐν τῇ ἐτέρᾳ σφαίρᾳ στερεὸν πολυέδρον τριπλασίονα λόγον ἔχει, ἥπερ ἡ τῆς ΒΓΔΕ σφαίρας διάμετρος πρὸς τὴν τῆς ἐτέρας σφαίρας διάμετρον. διαιρεθέντων γὰρ τῶν στερεῶν εἰς τὰς ὁμοιοπληθεῖς καὶ ὁμοιοταγεῖς πυραμίδας ἔσονται αἱ πυραμίδες ὅμοιαι. αἱ δὲ ὅμοιαι πυραμίδες πρὸς ἀλλήλας ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ὁμολόγων πλευρῶν· ἡ ἄρα πυραμὶς, ἥς βάσις μὲν ἐστὶ τὸ ΚΒΟΣ τετράπλευρον, κορυφὴ δὲ τὸ Α σημεῖον, πρὸς τὴν ἐν τῇ ἐτέρᾳ σφαίρᾳ ὁμοιοταγῇ πυραμίδα τριπλασίονα λόγον ἔχει, ἥπερ ἡ ὁμόλογος πλευρὰ πρὸς τὴν ὁμόλογον πλευράν, τουτέστιν ἥπερ ἡ ΑΒ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περὶ κέντρον τὸ Α πρὸς τὴν ἐκ τοῦ κέντρου τῆς ἐτέρας σφαίρας. ὁμοίως καὶ ἐκάστη πυραμὶς τῶν ἐν τῇ περὶ κέντρον τὸ Α σφαίρᾳ πρὸς ἐκάστην ὁμοιοταγῇ πυραμίδα τῶν ἐν τῇ ἐτέρᾳ σφαίρᾳ τριπλασίονα λόγον ἔξει, ἥπερ ἡ ΑΒ πρὸς τὴν ἐκ τοῦ κέντρου τῆς ἐτέρας σφαίρας. καὶ ὥς ἐν τῶν ἡγουμένων πρὸς ἐν τῶν ἐπομένων, οὕτως ἅπαντα τὰ ἡγούμενα πρὸς ἅπαντα τὰ ἐπόμενα· ὥστε ὅλον τὸ ἐν τῇ περὶ κέντρον τὸ Α σφαίρᾳ στερεὸν πολυέδρον πρὸς ὅλον τὸ ἐν τῇ ἐτέρᾳ [σφαίρᾳ] στερεὸν πολυέδρον τριπλασίονα λόγον ἔξει, ἥπερ ἡ ΑΒ πρὸς τὴν ἐκ τοῦ κέντρου τῆς ἐτέρας σφαίρας, τουτέστιν ἥπερ ἡ ΒΔ διάμετρος πρὸς τὴν τῆς ἐτέρας σφαίρας διάμετρον· ὅπερ ἔδει δεῖξαι.

ιη'.

Αἱ σφαῖραι πρὸς ἀλλήλας ἐν τριπλασίονι λόγῳ εἰσὶ τῶν ἰδίων διαμέτρων.

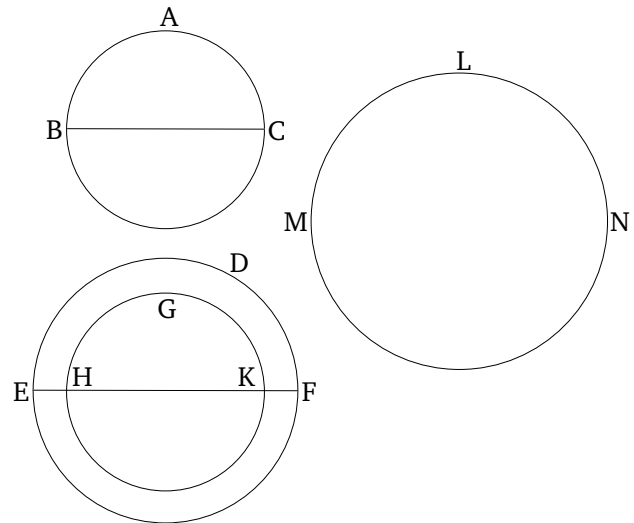


Corollary

And, also, if a similar polyhedral solid to that in sphere $BCDE$ is inscribed in another sphere then the polyhedral solid in sphere $BCDE$ has to the polyhedral solid in the other sphere the cubed ratio that the diameter of sphere $BCDE$ has to the diameter of the other sphere. For if the solids are divided into similarly numbered, and similarly situated, pyramids, then the pyramids will be similar. And similar pyramids are in the cubed ratio of corresponding sides [Prop. 12.8 corr.]. Thus, the pyramid whose base is quadrilateral $KBPS$, and apex the point A , will have to the similarly situated pyramid in the other sphere the cubed ratio that a corresponding side (has) to a corresponding side. That is to say, that of radius AB of the sphere about center A to the radius of the other sphere. And, similarly, each pyramid in the sphere about center A will have to each similarly situated pyramid in the other sphere the cubed ratio that AB (has) to the radius of the other sphere. And as one of the leading (magnitudes is) to one of the following (in two sets of proportional magnitudes), so (the sum of) all the leading (magnitudes is) to (the sum of) all of the following (magnitudes) [Prop. 5.12]. Hence, the whole polyhedral solid in the sphere about center A will have to the whole polyhedral solid in the other [sphere] the cubed ratio that (radius) AB (has) to the radius of the other sphere. That is to say, that diameter BD (has) to the diameter of the other sphere. (Which is) the very thing it was required to show.

Proposition 18

Spheres are to one another in the cubed ratio of their respective diameters.



Νενοήσθωσαν σφαῖραι αἱ $AB\Gamma$, ΔEZ , διάμετροι δὲ αὐτῶν αἱ $B\Gamma$, EZ · λέγω, ὅτι ἡ $AB\Gamma$ σφαῖρα πρὸς τὴν ΔEZ σφαῖραν τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Gamma$ πρὸς τὴν EZ .

Εἰ γὰρ μὴ ἡ $AB\Gamma$ σφαῖρα πρὸς τὴν ΔEZ σφαῖραν τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Gamma$ πρὸς τὴν EZ , ἔξει ἄρα ἡ $AB\Gamma$ σφαῖρα πρὸς ἐλάσσονά τινα τῆς ΔEZ σφαίρας τριπλασίονα λόγον ἢ πρὸς μείζονα ἥπερ ἡ $B\Gamma$ πρὸς τὴν EZ . ἐχέτω πρότερον πρὸς ἐλάσσονα τὴν $H\Theta K$, καὶ νενοήσθω ἡ ΔEZ τῇ $H\Theta K$ περὶ τὸ αὐτὸ κέντρον, καὶ ἐγγεγράφθω εἰς τὴν μείζονα σφαῖραν τὴν ΔEZ στερεὸν πολυέδρον μὴ ψαῦον τῆς ἐλάσσονος σφαίρας τῆς $H\Theta K$ κατὰ τὴν ἐπιφάνειαν, ἐγγεγράφθω δὲ καὶ εἰς τὴν $AB\Gamma$ σφαῖραν τῷ ἐν τῇ ΔEZ σφαίρᾳ στερεῷ πολυέδρῳ ὅμοιον στερεὸν πολυέδρον· τὸ ἄρα ἐν τῇ $AB\Gamma$ στερεὸν πολυέδρον πρὸς τὸ ἐν τῇ ΔEZ στερεὸν πολυέδρον τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Gamma$ πρὸς τὴν EZ . ἔχει δὲ καὶ ἡ $AB\Gamma$ σφαῖρα πρὸς τὴν $H\Theta K$ σφαῖραν τριπλασίονα λόγον ἥπερ ἡ $B\Gamma$ πρὸς τὴν EZ · ἔστιν ἄρα ὡς ἡ $AB\Gamma$ σφαῖρα πρὸς τὴν $H\Theta K$ σφαῖραν, οὕτως τὸ ἐν τῇ $AB\Gamma$ σφαίρᾳ στερεὸν πολυέδρον πρὸς τὸ ἐν τῇ ΔEZ σφαίρᾳ στερεὸν πολυέδρον· ἐναλλάξ [ἄρα] ὡς ἡ $AB\Gamma$ σφαῖρα πρὸς τὸ ἐν αὐτῇ πολυέδρον, οὕτως ἡ $H\Theta K$ σφαῖρα πρὸς τὸ ἐν τῇ ΔEZ σφαίρᾳ στερεὸν πολυέδρον. μείζων δὲ ἡ $AB\Gamma$ σφαῖρα τοῦ ἐν αὐτῇ πολυέδρου· μείζων ἄρα καὶ ἡ $H\Theta K$ σφαῖρα τοῦ ἐν τῇ ΔEZ σφαίρᾳ πολυέδρου. ἀλλὰ καὶ ἐλάττων· ἐμπεριέχεται γὰρ ὑπ' αὐτοῦ. οὐκ ἄρα ἡ $AB\Gamma$ σφαῖρα πρὸς ἐλάσσονα τῆς ΔEZ σφαίρας τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Gamma$ διάμετρος πρὸς τὴν EZ . ὁμοίως δὲ δεῖξομεν, ὅτι οὐδὲ ἡ ΔEZ σφαῖρα πρὸς ἐλάσσονα τῆς $AB\Gamma$ σφαίρας τριπλασίονα λόγον ἔχει ἥπερ ἡ EZ πρὸς τὴν $B\Gamma$.

Λέγω δὴ, ὅτι οὐδὲ ἡ $AB\Gamma$ σφαῖρα πρὸς μείζονά τινα τῆς ΔEZ σφαίρας τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Gamma$ πρὸς τὴν EZ .

Εἰ γὰρ δυνατόν, ἐχέτω πρὸς μείζονα τὴν AMN · ἀνάπαλιν ἄρα ἡ AMN σφαῖρα πρὸς τὴν $AB\Gamma$ σφαῖραν τριπλασίονα λόγον ἔχει ἥπερ ἡ EZ διάμετρος πρὸς τὴν $B\Gamma$ διάμετρον. ὡς δὲ ἡ AMN σφαῖρα πρὸς τὴν $AB\Gamma$ σφαῖραν, οὕτως ἡ ΔEZ σφαῖρα πρὸς ἐλάσσονά τινα τῆς $AB\Gamma$ σφαίρας, ἐπειδὴ περ μείζων ἐστὶν ἡ AMN τῆς ΔEZ , ὡς ἐμπροσθεν ἐδείχθη. καὶ ἡ ΔEZ ἄρα σφαῖρα πρὸς ἐλάσσονά τινα τῆς $AB\Gamma$ σφαίρας τριπλασίονα λόγον ἔχει ἥπερ ἡ EZ πρὸς τὴν $B\Gamma$ · ὅπερ ἀδύνατον ἐδείχθη. οὐκ ἄρα ἡ $AB\Gamma$ σφαῖρα πρὸς μείζονά τινα τῆς ΔEZ σφαίρας τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Gamma$ πρὸς τὴν EZ . ἐδείχθη δέ, ὅτι οὐδὲ πρὸς ἐλάσσονα. ἡ ἄρα $AB\Gamma$ σφαῖρα πρὸς τὴν ΔEZ σφαῖραν τριπλασίονα λόγον ἔχει ἥπερ ἡ $B\Gamma$ πρὸς τὴν EZ · ὅπερ ἔδει δεῖξαι.

Let the spheres ABC and DEF have been conceived, and (let) their diameters (be) BC and EF (respectively). I say that sphere ABC has to sphere DEF the cubed ratio that BC (has) to EF .

For if sphere ABC does not have to sphere DEF the cubed ratio that BC (has) to EF then sphere ABC will have to some (sphere) either less than, or greater than, sphere DEF the cubed ratio that BC (has) to EF . Let it, first of all, have (such a ratio) to a lesser (sphere), GHK . And let DEF have been conceived about the same center as GHK . And let a polyhedral solid have been inscribed in the greater sphere DEF , not touching the lesser sphere GHK on its surface [Prop. 12.17]. And let a polyhedral solid, similar to the polyhedral solid in sphere DEF , have also been inscribed in sphere ABC . Thus, the polyhedral solid in sphere ABC has to the polyhedral solid in sphere DEF the cubed ratio that BC (has) to EF [Prop. 12.17 corr.]. And sphere ABC also has to sphere GHK the cubed ratio that BC (has) to EF . Thus, as sphere ABC is to sphere GHK , so the polyhedral solid in sphere ABC (is) to the polyhedral solid in sphere DEF . [Thus], alternately, as sphere ABC (is) to the polygon within it, so sphere GHK (is) to the polyhedral solid within sphere DEF [Prop. 5.16]. And sphere ABC (is) greater than the polyhedron within it. Thus, sphere GHK (is) also greater than the polyhedron within sphere DEF [Prop. 5.14]. But, (it is) also less. For it is encompassed by it. Thus, sphere ABC does not have to (a sphere) less than sphere DEF the cubed ratio that diameter BC (has) to EF . So, similarly, we can show that sphere DEF does not have to (a sphere) less than sphere ABC the cubed ratio that EF (has) to BC either.

So, I say that sphere ABC does not have to some (sphere) greater than sphere DEF the cubed ratio that BC (has) to EF either.

For, if possible, let it have (the cubed ratio) to a greater (sphere), LMN . Thus, inversely, sphere LMN (has) to sphere ABC the cubed ratio that diameter EF (has) to diameter BC [Prop. 5.7 corr.]. And as sphere LMN (is) to sphere ABC , so sphere DEF (is) to some (sphere) less than sphere ABC , inasmuch as LMN is greater than DEF , as was shown before [Prop. 12.2 lem.]. And, thus, sphere DEF has to some (sphere) less than sphere ABC the cubed ratio that EF (has) to BC . The very thing was shown (to be) impossible. Thus, sphere ABC does not have to some (sphere) greater than sphere DEF the cubed ratio that BC (has) to EF . And it was shown that neither (does it have such a ratio) to a lesser (sphere). Thus, sphere ABC has to sphere DEF the cubed ratio that BC (has) to EF . (Which is) the very thing it was required to show.

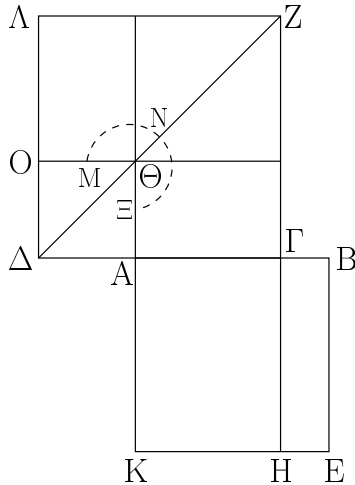
ELEMENTS BOOK 13

The Platonic Solids[†]

[†]The five regular solids—the cube, tetrahedron (*i.e.*, pyramid), octahedron, icosahedron, and dodecahedron—were probably discovered by the school of Pythagoras. They are generally termed “Platonic” solids because they feature prominently in Plato’s famous dialogue *Timaeus*. Many of the theorems contained in this book—particularly those which pertain to the last two solids—are ascribed to Theaetetus of Athens.

α'.

Ἐάν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ μείζον τμήμα προσλαβὼν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου.



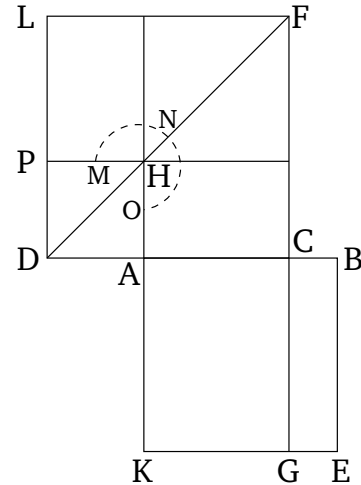
Εὐθεῖα γὰρ γραμμὴ ἡ AB ἄκρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μείζον τμήμα τὸ AG, καὶ ἐκβεβλήσθω ἐπ' εὐθείας τῇ ΓΑ εὐθεῖα ἡ ΑΔ, καὶ κείσθω τῆς AB ἡμίσεια ἡ ΑΔ· λέγω, ὅτι πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς ΓΔ τοῦ ἀπὸ τῆς ΔΑ.

Ἀναγεγράφθωσαν γὰρ ἀπὸ τῶν AB, ΔΓ τετράγωνα τὰ AE, ΔZ, καὶ καταγεγράφθω ἐν τῷ ΔZ τὸ σχῆμα, καὶ διήχθω ἡ ZΓ ἐπὶ τὸ H. καὶ ἐπεὶ ἡ AB ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ, τὸ ἄρα ὑπὸ τῶν ABΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς AG. καὶ ἐστὶ τὸ μὲν ὑπὸ τῶν ABΓ τὸ ΓΕ, τὸ δὲ ἀπὸ τῆς AG τὸ ZΘ· ἴσον ἄρα τὸ ΓΕ τῷ ZΘ. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ BA τῆς ΑΔ, ἴση δὲ ἡ μὲν BA τῇ ΚΑ, ἡ δὲ ΑΔ τῇ ΑΘ, διπλῇ ἄρα καὶ ἡ ΚΑ τῆς ΑΘ. ὥς δὲ ἡ ΚΑ πρὸς τὴν ΑΘ, οὕτως τὸ ΓΚ πρὸς τὸ ΓΘ· διπλάσιον ἄρα τὸ ΓΚ τοῦ ΓΘ. εἰσὶ δὲ καὶ τὰ ΛΘ, ΘΓ διπλάσια τοῦ ΓΘ. ἴσον ἄρα τὸ ΚΓ τοῖς ΛΘ, ΘΓ. ἐδείχθη δὲ καὶ τὸ ΓΕ τῷ ΘZ ἴσον· ὅλον ἄρα τὸ AE τετράγωνον ἴσον ἐστὶ τῷ MNΞ γνῶμονι. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ BA τῆς ΑΔ, τετραπλάσιόν ἐστι τὸ ἀπὸ τῆς BA τοῦ ἀπὸ τῆς ΑΔ, τουτέστι τὸ AE τοῦ ΔΘ. ἴσον δὲ τὸ AE τῷ MNΞ γνῶμονι· καὶ ὁ MNΞ ἄρα γνῶμων τετραπλάσιός ἐστι τοῦ AO· ὅλον ἄρα τὸ ΔZ πενταπλάσιόν ἐστι τοῦ AO. καὶ ἐστὶ τὸ μὲν ΔZ τὸ ἀπὸ τῆς ΔΓ, τὸ δὲ AO τὸ ἀπὸ τῆς ΔΑ· τὸ ἄρα ἀπὸ τῆς ΓΔ πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔΑ.

Ἐάν ἄρα εὐθεῖα ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ μείζον τμήμα προσλαβὼν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τετραγώνου· ὅπερ ἔδει δεῖξαι.

Proposition 1

If a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of the whole, is five times the square on the half.



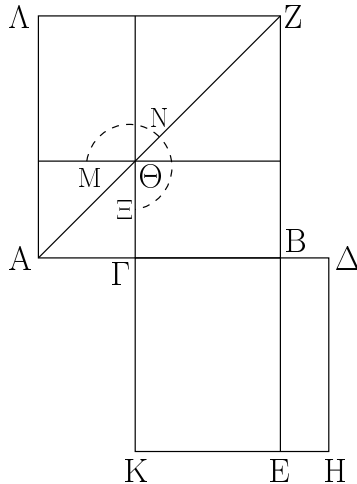
For let the straight-line AB have been cut in extreme and mean ratio at point C, and let AC be the greater piece. And let the straight-line AD have been produced in a straight-line with CA. And let AD be made (equal to) half of AB. I say that the (square) on CD is five times the (square) on DA.

For let the squares AE and DF have been described on AB and DC (respectively). And let the figure in DF have been drawn. And let FC have been drawn across to G. And since AB has been cut in extreme and mean ratio at C, the (rectangle contained) by ABC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. And CE is the (rectangle contained) by ABC, and FH the (square) on AC. Thus, CE (is) equal to FH. And since BA is double AD, and BA (is) equal to KA, and AD to AH, KA (is) thus also double AH. And as KA (is) to AH, so CK (is) to CH [Prop. 6.1]. Thus, CK (is) double CH. And LH plus HC is also double CH [Prop. 1.43]. Thus, KC (is) equal to LH plus HC. And CE was also shown (to be) equal to HF. Thus, the whole square AE is equal to the gnomon MNO. And since BA is double AD, the (square) on BA is four times the (square) on AD—that is to say, AE (is four times) DH. And AE (is) equal to gnomon MNO. And, thus, gnomon MNO is also four times AP. Thus, the whole of DF is five times AP. And DF is the (square) on DC, and AP the (square) on DA. Thus, the (square) on CD is five times the (square) on DA.

Thus, if a straight-line is cut in extreme and mean ratio then the square on the greater piece, added to half of

β'.

Ἐάν εὐθεῖα γραμμὴ τμήματος ἑαυτῆς πενταπλάσιον δύνηται, τῆς διπλασίας τοῦ εἰρημένου τμήματος ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζον τμήμα τὸ λοιπὸν μέρος ἐστὶ τῆς ἐξ ἄρχῃς εὐθείας.



Εὐθεῖα γὰρ γραμμὴ ἡ AB τμήματος ἑαυτῆς τοῦ AG πενταπλάσιον δυνάσθω, τῆς δὲ AG διπλῇ ἔστω ἡ $ΓΔ$. λέγω, ὅτι τῆς $ΓΔ$ ἄκρον καὶ μέσον λόγον τεμνομένου τὸ μείζον τμήμα ἐστὶν ἡ GB .

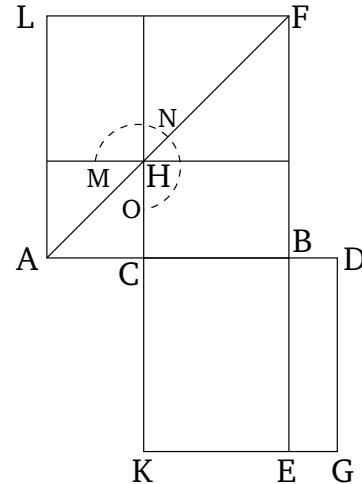
Ἀναγεγράφθω γὰρ ἀφ' ἐκατέρας τῶν AB , $ΓΔ$ τετράγωνα τὰ AZ , $ΓH$, καὶ καταγεγράφθω ἐν τῷ AZ τὸ σχῆμα, καὶ διήχθω ἡ BE . καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς BA τοῦ ἀπὸ τῆς AG , πενταπλάσιόν ἐστι τὸ AZ τοῦ $AΘ$. τετραπλάσιος ἄρα ὁ MNE γνῶμων τοῦ $AΘ$. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ $ΔΓ$ τῆς $ΓΑ$, τετραπλάσιος ἄρα ἐστὶ τὸ ἀπὸ $ΔΓ$ τοῦ ἀπὸ $ΓΑ$, τουτέστι τὸ $ΓH$ τοῦ $AΘ$. ἐδείχθη δὲ καὶ ὁ MNE γνῶμων τετραπλάσιος τοῦ $AΘ$. ἴσος ἄρα ὁ MNE γνῶμων τῷ $ΓH$. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ $ΔΓ$ τῆς $ΓΑ$, ἴση δὲ ἡ μὲν $ΔΓ$ τῇ $ΓΚ$, ἡ δὲ AG τῇ $ΓΘ$, [διπλῇ ἄρα καὶ ἡ $ΚΓ$ τῆς $ΓΘ$], διπλάσιος ἄρα καὶ τὸ KB τοῦ $BΘ$. εἰσὶ δὲ καὶ τὰ $ΛΘ$, $ΘB$ τοῦ $ΘB$ διπλάσια· ἴσον ἄρα τὸ KB τοῖς $ΛΘ$, $ΘB$. ἐδείχθη δὲ καὶ ὅλος ὁ MNE γνῶμων ὅλῳ τῷ $ΓH$ ἴσος· καὶ λοιπὸν ἄρα τὸ $ΘZ$ τῷ BH ἐστὶν ἴσον. καὶ ἐστὶ τὸ μὲν BH τὸ ὑπὸ τῶν $ΓΔB$. ἴση γὰρ ἡ $ΓΔ$ τῇ $ΔH$ · τὸ δὲ $ΘZ$ τὸ ἀπὸ τῆς $ΓB$ · τὸ ἄρα ὑπὸ τῶν $ΓΔB$ ἴσον ἐστὶ τῷ ἀπὸ τῆς $ΓB$. ἔστιν ἄρα ὡς ἡ $ΔΓ$ πρὸς τὴν $ΓB$, οὕτως ἡ $ΓB$ πρὸς τὴν BD . μείζων δὲ ἡ $ΔΓ$ τῆς $ΓB$ · μείζων ἄρα καὶ ἡ $ΓB$ τῆς BD . τῆς $ΓΔ$ ἄρα εὐθείας ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζον τμήμα ἐστὶν ἡ GB .

Ἐάν ἄρα εὐθεῖα γραμμὴ τμήματος ἑαυτῆς πενταπλάσιον δύνηται, τῆς διπλασίας τοῦ εἰρημένου τμήματος ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζον τμήμα τὸ λοιπὸν μέρος

the whole, is five times the square on the half. (Which is) the very thing it was required to show.

Proposition 2

If the square on a straight-line is five times the (square) on a piece of it, and double the aforementioned piece is cut in extreme and mean ratio, then the greater piece is the remaining part of the original straight-line.



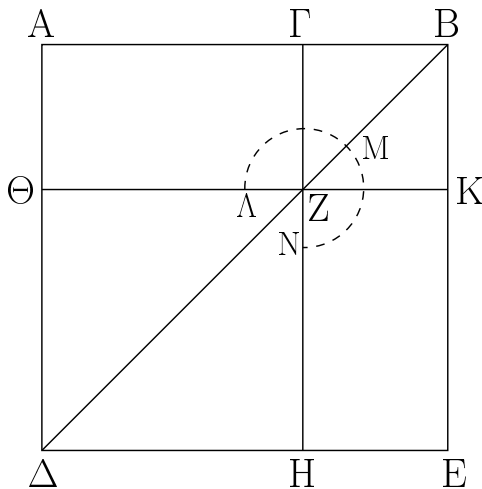
For let the square on the straight-line AB be five times the (square) on the piece of it, AC . And let CD be double AC . I say that if CD is cut in extreme and mean ratio then the greater piece is CB .

For let the squares AF and CG have been described on each of AB and CD (respectively). And let the figure in AF have been drawn. And let BE have been drawn across. And since the (square) on BA is five times the (square) on AC , AF is five times AH . Thus, gnomon MNO (is) four times AH . And since DC is double CA , the (square) on DC is thus four times the (square) on CA —that is to say, CG (is four times) AH . And the gnomon MNO was also shown (to be) four times AH . Thus, gnomon MNO (is) equal to CG . And since DC is double CA , and DC (is) equal to CK , and AC to CH , [KC (is) thus also double CH], (and) KB (is) also double BH [Prop. 6.1]. And LH plus HB is also double HB [Prop. 1.43]. Thus, KB (is) equal to LH plus HB . And the whole gnomon MNO was also shown (to be) equal to the whole of CG . Thus, the remainder HF is also equal to (the remainder) BG . And BG is the (rectangle contained) by CDB . For CD (is) equal to DG . And HF (is) the square on CB . Thus, the (rectangle contained) by CDB is equal to the (square) on CB . Thus, as DC is to CB , so CB (is) to BD [Prop. 6.17]. And DC (is) greater than CB (see lemma). Thus, CB (is) also greater than BD [Prop. 5.14]. Thus, if the straight-line CD is cut

καταγεγράφθω διπλοῦν τὸ σχῆμα. ἐπεὶ διπλῇ ἐστὶν ἡ ΑΓ τῆς ΔΓ, τετραπλάσιον ἄρα τὸ ἀπὸ τῆς ΑΓ τοῦ ἀπὸ τῆς ΔΓ, τουτέστι τὸ ΡΣ τοῦ ΖΗ. καὶ ἐπεὶ τὸ ὑπὸ τῶν ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ, καὶ ἐστὶ τὸ ὑπὸ τῶν ΑΒΓ τὸ ΓΕ, τὸ ἄρα ΓΕ ἴσον ἐστὶ τῷ ΡΣ. τετραπλάσιον δὲ τὸ ΡΣ τοῦ ΖΗ· τετραπλάσιον ἄρα καὶ τὸ ΓΕ τοῦ ΖΗ. πάλιν ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΔΓ, ἴση ἐστὶ καὶ ἡ ΘΚ τῇ ΚΖ. ὥστε καὶ τὸ ΗΖ τετράγωνον ἴσον ἐστὶ τῷ ΘΛ τετραγώνῳ. ἴση ἄρα ἡ ΗΚ τῇ ΚΛ, τουτέστιν ἡ ΜΝ τῇ ΝΕ· ὥστε καὶ τὸ ΜΖ τῷ ΖΕ ἐστὶν ἴσον. ἀλλὰ τὸ ΜΖ τῷ ΓΗ ἐστὶν ἴσον· καὶ τὸ ΓΗ ἄρα τῷ ΖΕ ἐστὶν ἴσον. κοινὸν προσκείσθω τὸ ΓΝ· ὁ ἄρα ΞΟΠ γνῶμων ἴσος ἐστὶ τῷ ΓΕ. ἀλλὰ τὸ ΓΕ τετραπλάσιον ἐδείχθη τοῦ ΗΖ· καὶ ὁ ΞΟΠ ἄρα γνῶμων τετραπλάσιός ἐστι τοῦ ΖΗ τετραγώνου. ὁ ΞΟΠ ἄρα γνῶμων καὶ τὸ ΖΗ τετράγωνον πενταπλάσιός ἐστι τοῦ ΖΗ. ἀλλὰ ὁ ΞΟΠ γνῶμων καὶ τὸ ΖΗ τετράγωνον ἐστὶ τὸ ΔΝ. καὶ ἐστὶ τὸ μὲν ΔΝ τὸ ἀπὸ τῆς ΔΒ, τὸ δὲ ΗΖ τὸ ἀπὸ τῆς ΔΓ. τὸ ἄρα ἀπὸ τῆς ΔΒ πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔΓ· ὅπερ ἔδει δεῖξαι.

δ'.

Ἐὰν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, τὸ ἀπὸ τῆς ὅλης καὶ τοῦ ἐλάσσονος τμήματος, τὰ συναμφότερα τετράγωνα, τριπλάσιά ἐστι τοῦ ἀπὸ τοῦ μείζονος τμήματος τετραγώνου.



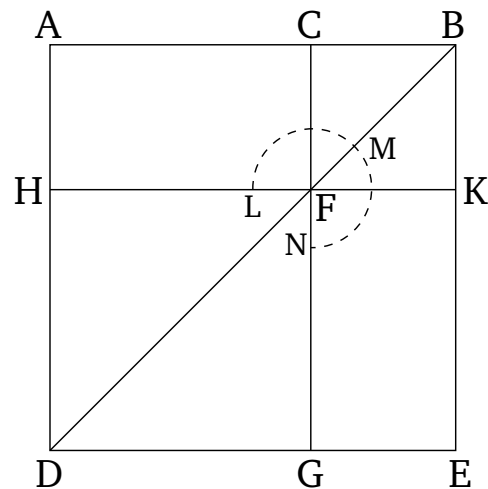
Ἐστω εὐθεῖα ἡ ΑΒ, καὶ τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ Γ, καὶ ἔστω μείζον τμήμα τὸ ΑΓ· λέγω, ὅτι τὰ ἀπὸ τῶν ΑΒ, ΒΓ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΓΑ.

Ἀναγεγράφθω γὰρ ἀπὸ τῆς ΑΒ τετράγωνον τὸ ΑΔΕΒ, καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ οὖν ἡ ΑΒ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ, καὶ τὸ μείζον τμήμα ἐστὶν ἡ ΑΓ, τὸ ἄρα ὑπὸ τῶν ΑΒΓ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΑΓ. καὶ ἐστὶ τὸ μὲν ὑπὸ τῶν ΑΒΓ τὸ ΑΚ, τὸ δὲ ἀπὸ τῆς ΑΓ τὸ ΘΗ·

For let the square AE have been described on AB . And let the figure have been drawn double. Since AC is double DC , the (square) on AC (is) thus four times the (square) on DC —that is to say, RS (is four times) FG . And since the (rectangle contained) by ABC is equal to the (square) on AC [Def. 6.3, Prop. 6.17], and CE is the (rectangle contained) by ABC , CE is thus equal to RS . And RS (is) four times FG . Thus, CE (is) also four times FG . Again, since AD is equal to DC , HK is also equal to KF . Hence, square GF is also equal to square HL . Thus, GK (is) equal to KL —that is to say, MN to NE . Hence, MF is also equal to FE . But, MF is equal to CG . Thus, CG is also equal to FE . Let CN have been added to both. Thus, gnomon OPQ is equal to CE . But, CE was shown (to be) equal to four times GF . Thus, gnomon OPQ is also four times square FG . Thus, gnomon OPQ plus square FG is five times FG . But, gnomon OPQ plus square FG is (square) DN . And DN is the (square) on DB , and GF the (square) on DC . Thus, the (square) on DB is five times the (square) on DC . (Which is) the very thing it was required to show.

Proposition 4

If a straight-line is cut in extreme and mean ratio then the sum of the squares on the whole and the lesser piece is three times the square on the greater piece.



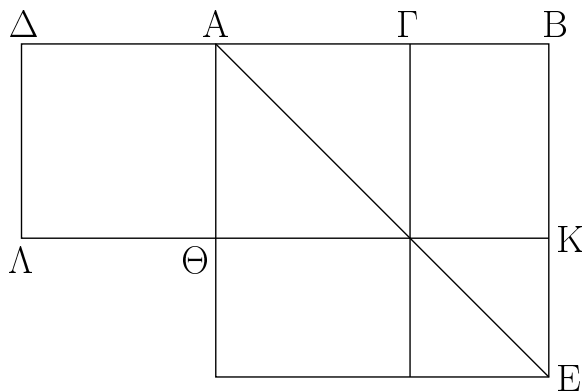
Let AB be a straight-line, and let it have been cut in extreme and mean ratio at C , and let AC be the greater piece. I say that the (sum of the squares) on AB and BC is three times the (square) on CA .

For let the square $ADEB$ have been described on AB , and let the (remainder of the) figure have been drawn. Therefore, since AB has been cut in extreme and mean ratio at C , and AC is the greater piece, the (rectangle

ἴσον ἄρα ἐστὶ τὸ AK τῷ ΘH . καὶ ἐπεὶ ἴσον ἐστὶ τὸ AZ τῷ ZE , κοινὸν προσκείσθω τὸ $ΓΚ$. ὅλον ἄρα τὸ AK ὅλῳ τῷ $ΓΕ$ ἐστὶν ἴσον· τὰ ἄρα AK , $ΓΕ$ τοῦ AK ἐστὶ διπλάσια. ἀλλὰ τὰ AK , $ΓΕ$ ὁ AMN γνῶμων ἐστὶ καὶ τὸ $ΓΚ$ τετράγωνον· ὁ ἄρα AMN γνῶμων καὶ τὸ $ΓΚ$ τετράγωνον διπλάσιόν ἐστι τοῦ AK . ἀλλὰ μὴν καὶ τὸ AK τῷ ΘH ἐδείχθη ἴσον· ὁ ἄρα AMN γνῶμων καὶ [τὸ $ΓΚ$ τετράγωνον διπλάσιόν ἐστι τοῦ ΘH · ὥστε ὁ AMN γνῶμων καὶ] τὰ $ΓΚ$, ΘH τετράγωνα τριπλάσιόν ἐστι τοῦ ΘH τετραγώνου. καὶ ἐστὶν ὁ [μὲν] AMN γνῶμων καὶ τὰ $ΓΚ$, ΘH τετράγωνα ὅλον τὸ AE καὶ τὸ $ΓΚ$, ἅπερ ἐστὶ τὰ ἀπὸ τῶν AB , $ΒΓ$ τετράγωνα, τὸ δὲ $H\Theta$ τὸ ἀπὸ τῆς $ΑΓ$ τετράγωνον. τὰ ἄρα ἀπὸ τῶν AB , $ΒΓ$ τετράγωνα τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς $ΑΓ$ τετραγώνου· ὅπερ ἔδει δεῖξαι.

ε'.

Ἐὰν εὐθεῖα γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, καὶ προστεθῇ αὐτῇ ἴση τῷ μείζονι τμήματι, ἡ ὅλη εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μείζον τμήμα ἐστὶν ἡ ἐξ ἀρχῆς εὐθεῖα.



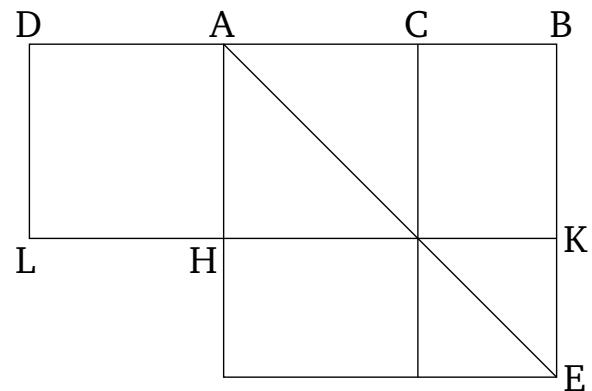
Εὐθεῖα γὰρ γραμμὴ ἡ AB ἄκρον καὶ μέσον λόγον τετμήσθω κατὰ τὸ Γ σημεῖον, καὶ ἔστω μείζον τμήμα ἡ $ΑΓ$, καὶ τῇ $ΑΓ$ ἴση [κείσθω] ἡ $ΑΔ$. λέγω, ὅτι ἡ $ΔΒ$ εὐθεῖα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ A , καὶ τὸ μείζον τμήμα ἐστὶν ἡ ἐξ ἀρχῆς εὐθεῖα ἡ AB .

Ἀναγεγράφθω γὰρ ἀπὸ τῆς AB τετράγωνον τὸ AE , καὶ καταγεγράφθω τὸ σχῆμα. ἐπεὶ ἡ AB ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Γ , τὸ ἄρα ὑπὸ $AB\Gamma$ ἴσον ἐστὶ τῷ ἀπὸ $ΑΓ$. καὶ ἐστὶ τὸ μὲν ὑπὸ $AB\Gamma$ τὸ $ΓΕ$, τὸ δὲ ἀπὸ τῆς $ΑΓ$ τὸ $\Gamma\Theta$. ἴσον ἄρα τὸ $ΓΕ$ τῷ $\Gamma\Theta$. ἀλλὰ τῷ μὲν $ΓΕ$ ἴσον ἐστὶ τὸ ΘE , τῷ δὲ $\Gamma\Theta$ ἴσον τὸ $\Delta\Theta$. καὶ τὸ $\Delta\Theta$ ἄρα ἴσον ἐστὶ τῷ ΘE [κοινὸν προσκείσθω τὸ ΘB]. ὅλον ἄρα τὸ ΔK ὅλῳ τῷ AE ἐστὶν ἴσον. καὶ ἐστὶ τὸ μὲν ΔK τὸ ὑπὸ τῶν $B\Delta$, ΔA ἴση

contained) by ABC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. And AK is the (rectangle contained) by ABC , and HG the (square) on AC . Thus, AK is equal to HG . And since AF is equal to FE [Prop. 1.43], let CK have been added to both. Thus, the whole of AK is equal to the whole of CE . Thus, AK plus CE is double AK . But, AK plus CE is the gnomon LMN plus the square CK . Thus, gnomon LMN plus square CK is double AK . But, indeed, AK was also shown (to be) equal to HG . Thus, gnomon LMN plus [square CK is double HG . Hence, gnomon LMN plus] the squares CK and HG is three times the square HG . And gnomon LMN plus the squares CK and HG is the whole of AE plus CK —which are the squares on AB and BC (respectively)—and GH (is) the square on AC . Thus, the (sum of the) squares on AB and BC is three times the square on AC . (Which is) the very thing it was required to show.

Proposition 5

If a straight-line is cut in extreme and mean ratio, and a (straight-line) equal to the greater piece is added to it, then the whole straight-line has been cut in extreme and mean ratio, and the original straight-line is the greater piece.



For let the straight-line AB have been cut in extreme and mean ratio at point C . And let AC be the greater piece. And let AD be [made] equal to AC . I say that the straight-line DB has been cut in extreme and mean ratio at A , and that the original straight-line AB is the greater piece.

For let the square AE have been described on AB , and let the (remainder of the) figure have been drawn. And since AB has been cut in extreme and mean ratio at C , the (rectangle contained) by ABC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. And CE is the (rectangle contained) by ABC , and CH the (square) on AC . But, HE is equal to CE [Prop. 1.43], and DH equal

γὰρ ἡ $\Delta\Delta$ τῇ $\Delta\Lambda$ · τὸ δὲ AE τὸ ἀπὸ τῆς AB · τὸ ἄρα ὑπὸ τῶν $B\Delta A$ ἴσον ἐστὶ τῷ ἀπὸ τῆς AB . ἔστιν ἄρα ὡς ἡ ΔB πρὸς τὴν BA , οὕτως ἡ BA πρὸς τὴν $\Delta\Delta$. μείζων δὲ ἡ ΔB τῆς BA · μείζων ἄρα καὶ ἡ BA τῆς $\Delta\Delta$.

Ἡ ἄρα ΔB ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ A , καὶ τὸ μείζον τμήμα ἐστὶν ἡ AB · ὅπερ ἔδει δεῖξαι.

to HC . Thus, DH is also equal to HE . [Let HB have been added to both.] Thus, the whole of DK is equal to the whole of AE . And DK is the (rectangle contained) by BD and DA . For AD (is) equal to DL . And AE (is) the (square) on AB . Thus, the (rectangle contained) by BDA is equal to the (square) on AB . Thus, as DB (is) to BA , so BA (is) to AD [Prop. 6.17]. And DB (is) greater than BA . Thus, BA (is) also greater than AD [Prop. 5.14].

Thus, DB has been cut in extreme and mean ratio at A , and the greater piece is AB . (Which is) the very thing it was required to show.

ε'.

Ἐὰν εὐθεῖα ῥητὴ ἄκρον καὶ μέσον λόγον τμηθῇ, ἐκάτερον τῶν τμημάτων ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.



Ἐστω εὐθεῖα ῥητὴ ἡ AB καὶ τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ Γ , καὶ ἔστω μείζον τμήμα ἡ AG · λέγω, ὅτι ἐκάτερά τῶν AG , GB ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐκβεβλήσθω γὰρ ἡ BA , καὶ κείσθω τῆς BA ἡμίσεια ἡ $\Delta\Delta$. ἐπεὶ οὖν εὐθεῖα ἡ AB τέτμηται ἄκρον καὶ μέσον λόγον κατὰ τὸ Γ , καὶ τῷ μείζονι τμήματι τῷ AG πρόσκειται ἡ $\Delta\Delta$ ἡμίσεια οὕσα τῆς AB , τὸ ἄρα ἀπὸ $\Gamma\Delta$ τοῦ ἀπὸ ΔA πενταπλάσιόν ἐστιν. τὸ ἄρα ἀπὸ $\Gamma\Delta$ πρὸς τὸ ἀπὸ ΔA λόγον ἔχει, ὃν ἀριθμὸς πρὸς ἀριθμὸν· σύμμετρον ἄρα τὸ ἀπὸ $\Gamma\Delta$ τῷ ἀπὸ ΔA . ῥητὸν δὲ τὸ ἀπὸ ΔA · ῥητὴ γάρ [ἐστὶν] ἡ ΔA ἡμίσεια οὕσα τῆς AB ῥητῆς οὕσης· ῥητὸν ἄρα καὶ τὸ ἀπὸ $\Gamma\Delta$ · ῥητὴ ἄρα ἐστὶ καὶ ἡ $\Gamma\Delta$. καὶ ἐπεὶ τὸ ἀπὸ $\Gamma\Delta$ πρὸς τὸ ἀπὸ ΔA λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμὸν, ἀσύμμετρος ἄρα μήκει ἡ $\Gamma\Delta$ τῇ ΔA · αἱ $\Gamma\Delta$, ΔA ἄρα ῥηταὶ εἰσι δυνάμει μόνον σύμμετροι· ἀποτομὴ ἄρα ἐστὶν ἡ AG . πάλιν, ἐπεὶ ἡ AB ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μείζον τμήμα ἐστὶν ἡ AG , τὸ ἄρα ὑπὸ AB , $B\Gamma$ τῷ ἀπὸ AG ἴσον ἐστίν. τὸ ἄρα ἀπὸ τῆς AG ἀποτομῆς παρὰ τὴν AB ῥητὴν παραβληθὲν πλάτος ποιεῖ τὴν $B\Gamma$. τὸ δὲ ἀπὸ ἀποτομῆς παρὰ ῥητὴν παραβαλλόμενον πλάτος ποιεῖ ἀποτομὴν πρώτην· ἀποτομὴ ἄρα πρώτη ἐστὶν ἡ GB . ἐδείχθη δὲ καὶ ἡ GA ἀποτομή.

Ἐὰν ἄρα εὐθεῖα ῥητὴ ἄκρον καὶ μέσον λόγον τμηθῇ, ἐκάτερον τῶν τμημάτων ἄλογός ἐστιν ἡ καλουμένη ἀποτομή· ὅπερ ἔδει δεῖξαι.

Proposition 6

If a rational straight-line is cut in extreme and mean ratio then each of the pieces is that irrational (straight-line) called an apotome.

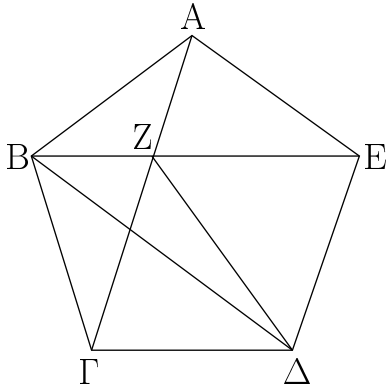


Let AB be a rational straight-line cut in extreme and mean ratio at C , and let AC be the greater piece. I say that AC and CB is each that irrational (straight-line) called an apotome.

For let BA have been produced, and let AD be made (equal) to half of BA . Therefore, since the straight-line AB has been cut in extreme and mean ratio at C , and AD , which is half of AB , has been added to the greater piece AC , the (square) on CD is thus five times the (square) on DA [Prop. 13.1]. Thus, the (square) on CD has to the (square) on DA the ratio which a number (has) to a number. The (square) on CD (is) thus commensurable with the (square) on DA [Prop. 10.6]. And the (square) on DA (is) rational. For DA [is] rational, being half of AB , which is rational. Thus, the (square) on CD (is) also rational [Def. 10.4]. Thus, CD is also rational. And since the (square) on CD does not have to the (square) on DA the ratio which a square number (has) to a square number, CD (is) thus incommensurable in length with DA [Prop. 10.9]. Thus, CD and DA are rational (straight-lines which are) commensurable in square only. Thus, AC is an apotome [Prop. 10.73]. Again, since AB has been cut in extreme and mean ratio, and AC is the greater piece, the (rectangle contained) by AB and BC is thus equal to the (square) on AC [Def. 6.3, Prop. 6.17]. Thus, the (square) on the apotome AC , applied to the rational (straight-line) AB , makes BC as width. And the (square) on an apotome, applied to a rational (straight-line), makes a first apotome as width [Prop. 10.97]. Thus, CB is a first apotome. And CA was also shown (to be) an apotome.

ζ'.

Ἐὰν πενταγώνου ἰσοπλεύρου αἱ τρεῖς γωνίαι ᾗτοι αἱ κατὰ τὸ ἐξῆς ἢ αἱ μὴ κατὰ τὸ ἐξῆς ἴσαι ᾧσιν, ἰσογώνιον ἔσται τὸ πεντάγωνον.



Πενταγώνου γὰρ ἰσοπλεύρου τοῦ ΑΒΓΔΕ αἱ τρεῖς γωνίαι πρότερον αἱ κατὰ τὸ ἐξῆς αἱ πρὸς τοῖς Α, Β, Γ ἴσαι ἀλλήλαις ἔστωσαν· λέγω, ὅτι ἰσογώνιον ἔστι τὸ ΑΒΓΔΕ πεντάγωνον.

Ἐπεζεύχθωσαν γὰρ αἱ ΑΓ, ΒΕ, ΖΔ. καὶ ἐπεὶ δύο αἱ ΓΒ, ΒΑ δυοὶ ταῖς ΒΑ, ΑΕ ἴσαι εἰσὶν ἑκατέρω ἑκατέρω, καὶ γωνία ἡ ὑπὸ ΓΒΑ γωνία τῇ ὑπὸ ΒΑΕ ἔστιν ἴση, βάσις ἄρα ἡ ΑΓ βάσει τῇ ΒΕ ἔστιν ἴση, καὶ τὸ ΑΒΓ τρίγωνον τῷ ΑΒΕ τριγώνῳ ἴσον, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν, ἡ μὲν ὑπὸ ΒΓΑ τῇ ὑπὸ ΒΕΑ, ἡ δὲ ὑπὸ ΑΒΕ τῇ ὑπὸ ΓΑΒ· ὥστε καὶ πλευρὰ ἡ ΑΖ πλευρᾷ τῇ ΒΖ ἔστιν ἴση. ἐδείχθη δὲ καὶ ὅλη ἡ ΑΓ ὅλη τῇ ΒΕ ἴση· καὶ λοιπὴ ἄρα ἡ ΖΓ λοιπῇ τῇ ΖΕ ἔστιν ἴση. ἔστι δὲ καὶ ἡ ΓΔ τῇ ΔΕ ἴση. δύο δὴ αἱ ΖΓ, ΓΔ δυοὶ ταῖς ΖΕ, ΕΔ ἴσαι εἰσὶν· καὶ βάσις αὐτῶν κοινὴ ἡ ΖΔ· γωνία ἄρα ἡ ὑπὸ ΖΓΔ γωνία τῇ ὑπὸ ΖΕΔ ἔστιν ἴση. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΒΓΑ τῇ ὑπὸ ΑΕΒ ἴση· καὶ ὅλη ἄρα ἡ ὑπὸ ΒΓΔ ὅλη τῇ ὑπὸ ΑΕΔ ἴση. ἀλλ' ἡ ὑπὸ ΒΓΔ ἴση ὑπόκειται ταῖς πρὸς τοῖς Α, Β γωνίαις· καὶ ἡ ὑπὸ ΑΕΔ ἄρα ταῖς πρὸς τοῖς Α, Β γωνίαις ἴση ἔστιν. ὁμοίως δὲ δείξομεν, ὅτι καὶ ἡ ὑπὸ ΓΔΕ γωνία ἴση ἔστι ταῖς πρὸς τοῖς Α, Β, Γ γωνίαις· ἰσογώνιον ἄρα ἔστι τὸ ΑΒΓΔΕ πεντάγωνον.

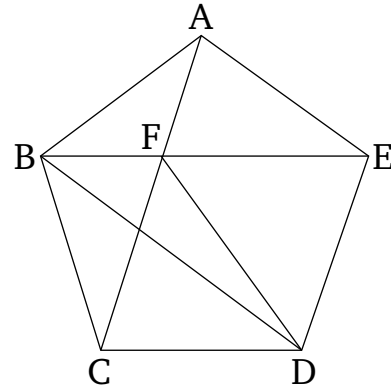
Ἀλλὰ δὴ μὴ ἔστωσαν ἴσαι αἱ κατὰ τὸ ἐξῆς γωνίαι, ἀλλ' ἔστωσαν ἴσαι αἱ πρὸς τοῖς Α, Γ, Δ σημείοις· λέγω, ὅτι καὶ οὕτως ἰσογώνιον ἔστι τὸ ΑΒΓΔΕ πεντάγωνον.

Ἐπεζεύχθω γὰρ ἡ ΒΔ. καὶ ἐπεὶ δύο αἱ ΒΑ, ΑΕ δυοὶ ταῖς ΒΓ, ΓΔ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ ΒΕ βάσει τῇ ΒΔ ἴση ἔστιν, καὶ τὸ ΑΒΕ τρίγωνον τῷ ΒΓΔ τριγώνῳ ἴσον ἔστιν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν·

Thus, if a rational straight-line is cut in extreme and mean ratio then each of the pieces is that irrational (straight-line) called an apotome.

Proposition 7

If three angles, either consecutive or not consecutive, of an equilateral pentagon are equal then the pentagon will be equiangular.



For let three angles of the equilateral pentagon $ABCDE$ —first of all, the consecutive (angles) at A , B , and C —be equal to one another. I say that pentagon $ABCDE$ is equiangular.

For let AC , BE , and FD have been joined. And since the two (straight-lines) CB and BA are equal to the two (straight-lines) BA and AE , respectively, and angle CBA is equal to angle BAE , base AC is thus equal to base BE , and triangle ABC equal to triangle ABE , and the remaining angles will be equal to the remaining angles which the equal sides subtend [Prop. 1.4], (that is), BCA (equal) to BEA , and ABE to CAB . And hence side AF is also equal to side BF [Prop. 1.6]. And the whole of AC was also shown (to be) equal to the whole of BE . Thus, the remainder FC is also equal to the remainder FE . And CD is also equal to DE . So, the two (straight-lines) FC and CD are equal to the two (straight-lines) FE and ED (respectively). And FD is their common base. Thus, angle FCD is equal to angle FED [Prop. 1.8]. And BCA was also shown (to be) equal to AEB . And thus the whole of BCD (is) equal to the whole of AED . But, (angle) BCD was assumed (to be) equal to the angles at A and B . Thus, (angle) AED is also equal to the angles at A and B . So, similarly, we can show that angle CDE is also equal to the angles at A , B , C . Thus, pentagon $ABCDE$ is equiangular.

And so let consecutive angles not be equal, but let the (angles) at points A , C , and D be equal. I say that pentagon $ABCDE$ is also equiangular in this case.

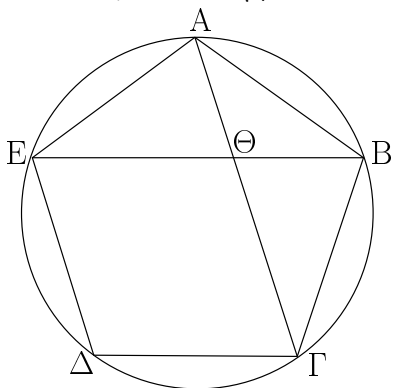
For let BD have been joined. And since the two

ἴση ἄρα ἐστὶν ἡ ὑπὸ AEB γωνία τῇ ὑπὸ $ΓΔΒ$. ἔστι δὲ καὶ ἡ ὑπὸ $ΒΕΔ$ γωνία τῇ ὑπὸ $ΒΔΕ$ ἴση, ἐπεὶ καὶ πλευρὰ ἡ $ΒΕ$ πλευρᾷ τῇ $ΒΔ$ ἐστὶν ἴση. καὶ ὅλη ἄρα ἡ ὑπὸ $ΑΕΔ$ γωνία ὅλη τῇ ὑπὸ $ΓΔΕ$ ἐστὶν ἴση. ἀλλὰ ἡ ὑπὸ $ΓΔΕ$ ταῖς πρὸς τοῖς $A, Γ$ γωνίαις ὑπόκειται ἴση· καὶ ἡ ὑπὸ $ΑΕΔ$ ἄρα γωνία ταῖς πρὸς τοῖς $A, Γ$ ἴση ἐστίν. διὰ τὰ αὐτὰ δὲ καὶ ἡ ὑπὸ $ΑΒΓ$ ἴση ἐστὶ ταῖς πρὸς τοῖς $A, Γ, Δ$ γωνίαις. ἰσογώνιον ἄρα ἐστὶ τὸ $ΑΒΓΔΕ$ πεντάγωνον· ὅπερ ἔδει δεῖξαι.

(straight-lines) BA and AE are equal to the (straight-lines) BC and CD , and they contain equal angles, base BE is thus equal to base BD , and triangle ABE is equal to triangle BCD , and the remaining angles will be equal to the remaining angles which the equal sides subtend [Prop. 1.4]. Thus, angle AEB is equal to (angle) CDB . And angle BED is also equal to (angle) BDE , since side BE is also equal to side BD [Prop. 1.5]. Thus, the whole angle AED is also equal to the whole (angle) CDE . But, (angle) CDE was assumed (to be) equal to the angles at A and C . Thus, angle AED is also equal to the (angles) at A and C . So, for the same (reasons), (angle) ABC is also equal to the angles at A, C , and D . Thus, pentagon $ABCDE$ is equiangular. (Which is) the very thing it was required to show.

η'.

Ἐὰν πενταγώνου ἰσοπλευροῦ καὶ ἰσογωνίου τὰς κατὰ τὸ ἐξῆς δύο γωνίας ὑποτείνωσιν εὐθεῖαι, ἄκρον καὶ μέσον λόγον τέμνουσιν ἀλλήλας, καὶ τὰ μείζονα αὐτῶν τμήματα ἴσα ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ.

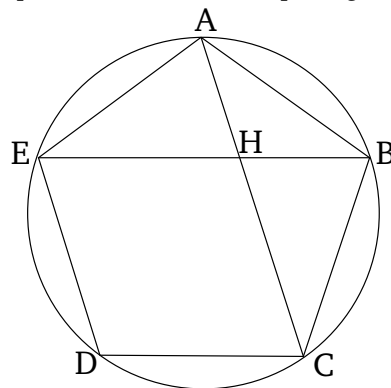


Πενταγώνου γὰρ ἰσοπλευρον καὶ ἰσογωνίου τοῦ $ΑΒΓΔΕ$ δύο γωνίας τὰς κατὰ τὸ ἐξῆς τὰς πρὸς τοῖς A, B ὑποτείνεωσαν εὐθεῖαι αἱ $ΑΓ, ΒΕ$ τέμνουσαι ἀλλήλας κατὰ τὸ $Θ$ σημεῖον· λέγω, ὅτι ἑκάτερα αὐτῶν ἄκρον καὶ μέσον λόγον τέμνεται κατὰ τὸ $Θ$ σημεῖον, καὶ τὰ μείζονα αὐτῶν τμήματα ἴσα ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ.

Περιγεγράφθω γὰρ περὶ τὸ $ΑΒΓΔΕ$ πεντάγωνον κύκλος ὁ $ΑΒΓΔΕ$. καὶ ἐπεὶ δύο εὐθεῖαι αἱ $ΕΑ, ΑΒ$ δυοὶ ταῖς $ΑΒ, ΒΓ$ ἴσαι εἰσὶ καὶ γωνίας ἴσας περιέχουσιν, βάσις ἄρα ἡ $ΒΕ$ βάσει τῇ $ΑΓ$ ἴση ἐστίν, καὶ τὸ $ΑΒΕ$ τρίγωνον τῷ $ΑΒΓ$ τριγώνῳ ἴσον ἐστίν, καὶ αἱ λοιπαὶ γωνίαι ταῖς λοιπαῖς γωνίαις ἴσαι ἔσονται ἑκάτερα ἑκατέρῃ, ὅφ' ἂς αἱ ἴσαι πλευραὶ ὑποτείνουσιν. ἴση ἄρα ἐστὶν ἡ ὑπὸ $ΒΑΓ$ γωνία τῇ ὑπὸ $ΑΒΕ$ · διπλῇ ἄρα ἡ ὑπὸ $ΑΘΕ$ τῇς ὑπὸ $ΒΑΘ$. ἔστι δὲ καὶ ἡ ὑπὸ $ΕΑΓ$ τῇς ὑπὸ $ΒΑΓ$ διπλῇ, ἐπειδὴ περ καὶ περιφέρεια ἡ $ΕΔΓ$ περιφερείας τῇς $ΓΒ$ ἐστὶ διπλῇ· ἴση ἄρα ἡ ὑπὸ $ΘΑΕ$ γωνία τῇ ὑπὸ $ΑΘΕ$ · ὥστε καὶ ἡ $ΘΕ$ εὐθεῖα τῇ $ΕΑ$, τουτέστι τῇ $ΑΒ$

Proposition 8

If straight-lines subtend two consecutive angles of an equilateral and equiangular pentagon then they cut one another in extreme and mean ratio, and their greater pieces are equal to the sides of the pentagon.



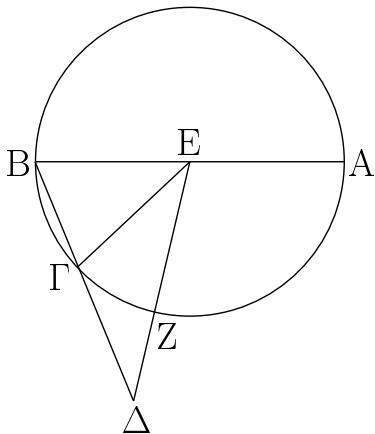
For let the two straight-lines, AC and BE , cutting one another at point H , have subtended two consecutive angles, at A and B (respectively), of the equilateral and equiangular pentagon $ABCDE$. I say that each of them has been cut in extreme and mean ratio at point H , and that their greater pieces are equal to the sides of the pentagon.

For let the circle $ABCDE$ have been circumscribed about pentagon $ABCDE$ [Prop. 4.14]. And since the two straight-lines EA and AB are equal to the two (straight-lines) AB and BC (respectively), and they contain equal angles, the base BE is thus equal to the base AC , and triangle ABE is equal to triangle ABC , and the remaining angles will be equal to the remaining angles, respectively, which the equal sides subtend [Prop. 1.4]. Thus, angle BAC is equal to (angle) ABE . Thus, (angle) AHE (is) double (angle) BAH [Prop. 1.32]. And EAC is also dou-

ἐστὶν ἴση. καὶ ἐπεὶ ἴση ἐστὶν ἡ BA εὐθεΐα τῇ AE , ἴση ἐστὶ καὶ γωνία ἡ ὑπὸ ABE τῇ ὑπὸ AEB . ἀλλὰ ἡ ὑπὸ ABE τῇ ὑπὸ $BA\Theta$ ἐδείχθη ἴση· καὶ ἡ ὑπὸ BEA ἄρα τῇ ὑπὸ $BA\Theta$ ἐστὶν ἴση. καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε ABE καὶ τοῦ $AB\Theta$ ἐστὶν ἡ ὑπὸ ABE · λοιπὴ ἄρα ἡ ὑπὸ BAE γωνία λοιπῇ τῇ ὑπὸ $A\Theta B$ ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ABE τρίγωνον τῷ $AB\Theta$ τριγώνῳ· ἀνάλογον ἄρα ἐστὶν ὡς ἡ EB πρὸς τὴν BA , οὕτως ἡ AB πρὸς τὴν $B\Theta$. ἴση δὲ ἡ BA τῇ $E\Theta$ · ὡς ἄρα ἡ BE πρὸς τὴν $E\Theta$, οὕτως ἡ $E\Theta$ πρὸς τὴν ΘB . μείζων δὲ ἡ BE τῆς $E\Theta$ · μείζων ἄρα καὶ ἡ $E\Theta$ τῆς ΘB . ἡ BE ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Θ , καὶ τὸ μείζον τμήμα τὸ ΘE ἴσον ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ. ὁμοίως δὲ δείξομεν, ὅτι καὶ ἡ AG ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Θ , καὶ τὸ μείζον αὐτῆς τμήμα ἡ $\Gamma\Theta$ ἴσον ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ· ὅπερ ἔδει δεῖξαι.

θ'.

Ἐὰν ἡ τοῦ ἑξαγώνου πλευρὰ καὶ ἡ τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων συντεθῶσιν, ἡ ὅλη εὐθεΐα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μείζον αὐτῆς τμήμα ἐστὶν ἡ τοῦ ἑξαγώνου πλευρὰ.



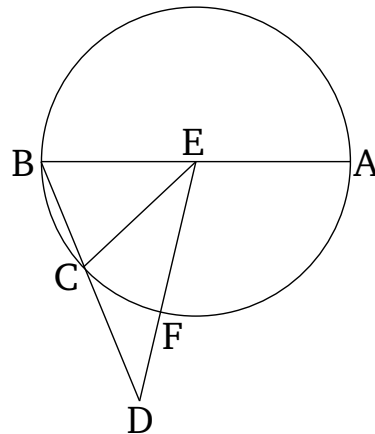
Ἐστω κύκλος ὁ $AB\Gamma$, καὶ τῶν εἰς τὸν $AB\Gamma$ κύκλον ἐγγραφομένων σχημάτων, δεκαγώνου μὲν ἔστω πλευρὰ ἡ $B\Gamma$, ἑξαγώνου δὲ ἡ $\Gamma\Delta$, καὶ ἔστωσαν ἐπ' εὐθείας· λέγω, ὅτι ἡ ὅλη εὐθεΐα ἡ BD ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μείζον αὐτῆς τμήμα ἐστὶν ἡ $\Gamma\Delta$.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ E σημεῖον, καὶ ἐπεζεύχθωσαν αἱ EB , $E\Gamma$, $E\Delta$, καὶ διήχθω ἡ BE ἐπὶ τὸ

ble BAC , inasmuch as circumference EDC is also double circumference CB [Props. 3.28, 6.33]. Thus, angle HAE (is) equal to (angle) AHE . Hence, straight-line HE is also equal to (straight-line) EA —that is to say, to (straight-line) AB [Prop. 1.6]. And since straight-line BA is equal to AE , angle ABE is also equal to AEB [Prop. 1.5]. But, ABE was shown (to be) equal to BAH . Thus, BEA is also equal to BAH . And (angle) ABE is common to the two triangles ABE and ABH . Thus, the remaining angle BAE is equal to the remaining (angle) AHB [Prop. 1.32]. Thus, triangle ABE is equiangular to triangle ABH . Thus, proportionally, as EB is to BA , so AB (is) to BH [Prop. 6.4]. And BA (is) equal to EH . Thus, as BE (is) to EH , so EH (is) to HB . And BE (is) greater than EH . EH (is) thus also greater than HB [Prop. 5.14]. Thus, BE has been cut in extreme and mean ratio at H , and the greater piece HE is equal to the side of the pentagon. So, similarly, we can show that AC has also been cut in extreme and mean ratio at H , and that its greater piece CH is equal to the side of the pentagon. (Which is) the very thing it was required to show.

Proposition 9

If the side of a hexagon and of a decagon inscribed in the same circle are added together then the whole straight-line has been cut in extreme and mean ratio (at the junction point), and its greater piece is the side of the hexagon.[†]



Let ABC be a circle. And of the figures inscribed in circle ABC , let BC be the side of a decagon, and CD (the side) of a hexagon. And let them be (laid down) straight-on (to one another). I say that the whole straight-line BD has been cut in extreme and mean ratio (at C), and that CD is its greater piece.

For let the center of the circle, point E , have been

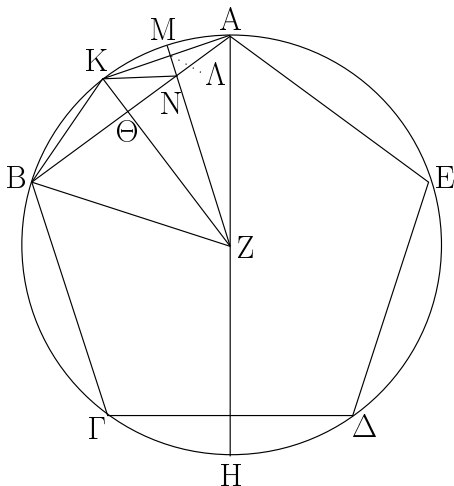
A. ἐπεὶ δεκαγώνου ἰσοπλευρον πλευρά ἐστὶν ἡ ΒΓ, πενταπλασίων ἄρα ἡ ΑΓΒ περιφέρεια τῆς ΒΓ περιφερείας· τετραπλασίων ἄρα ἡ ΑΓ περιφέρεια τῆς ΓΒ. ὥς δὲ ἡ ΑΓ περιφέρεια πρὸς τὴν ΓΒ, οὕτως ἡ ὑπὸ ΑΕΓ γωνία πρὸς τὴν ὑπὸ ΓΕΒ· τετραπλασίων ἄρα ἡ ὑπὸ ΑΕΓ τῆς ὑπὸ ΓΕΒ. καὶ ἐπεὶ ἴση ἡ ὑπὸ ΕΒΓ γωνία τῇ ὑπὸ ΕΓΒ, ἡ ἄρα ὑπὸ ΑΕΓ γωνία διπλασία ἐστὶ τῆς ὑπὸ ΕΓΒ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΕΓ εὐθεΐα τῇ ΓΔ· ἑκατέρα γὰρ αὐτῶν ἴση ἐστὶ τῇ τοῦ ἑξαγώνου πλευρᾷ τοῦ εἰς τὸν ΑΒΓ κύκλον [ἐγγραφομένου]· ἴση ἐστὶ καὶ ἡ ὑπὸ ΓΕΔ γωνία τῇ ὑπὸ ΓΔΕ γωνίᾳ· διπλασία ἄρα ἡ ὑπὸ ΕΓΒ γωνία τῆς ὑπὸ ΕΔΓ. ἀλλὰ τῆς ὑπὸ ΕΓΒ διπλασία ἐδείχθη ἡ ὑπὸ ΑΕΓ· τετραπλασία ἄρα ἡ ὑπὸ ΑΕΓ τῆς ὑπὸ ΕΔΓ. ἐδείχθη δὲ καὶ τῆς ὑπὸ ΒΕΓ τετραπλασία ἡ ὑπὸ ΑΕΓ· ἴση ἄρα ἡ ὑπὸ ΕΔΓ τῇ ὑπὸ ΒΕΓ. κοινὴ δὲ τῶν δύο τριγώνων, τοῦ τε ΒΕΓ καὶ τοῦ ΒΕΔ, ἡ ὑπὸ ΕΒΔ γωνία· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΒΕΔ τῇ ὑπὸ ΕΓΒ ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ΕΒΔ τρίγωνον τῷ ΕΒΓ τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΔΒ πρὸς τὴν ΒΕ, οὕτως ἡ ΕΒ πρὸς τὴν ΒΓ. ἴση δὲ ἡ ΕΒ τῇ ΓΔ. ἐστὶν ἄρα ὡς ἡ ΒΔ πρὸς τὴν ΔΓ, οὕτως ἡ ΔΓ πρὸς τὴν ΓΒ. μείζων δὲ ἡ ΒΔ τῆς ΔΓ· μείζων ἄρα καὶ ἡ ΔΓ τῆς ΓΒ. ἡ ΒΔ ἄρα εὐθεΐα ἄκρον καὶ μέσον λόγον τέτμηται [κατὰ τὸ Γ], καὶ τὸ μείζον τμήμα αὐτῆς ἐστὶν ἡ ΔΓ· ὅπερ ἔδει δεῖξαι.

found [Prop. 3.1], and let EB , EC , and ED have been joined, and let BE have been drawn across to A . Since BC is a side on an equilateral decagon, circumference ACB (is) thus five times circumference BC . Thus, circumference AC (is) four times CB . And as circumference AC (is) to CB , so angle AEC (is) to CEB [Prop. 6.33]. Thus, (angle) AEC (is) four times CEB . And since angle EBC (is) equal to ECB [Prop. 1.5], angle AEC is thus double ECB [Prop. 1.32]. And since straight-line EC is equal to CD —for each of them is equal to the side of the hexagon [inscribed] in circle ABC [Prop. 4.15 corr.]—angle CED is also equal to angle CDE [Prop. 1.5]. Thus, angle ECB (is) double EDC [Prop. 1.32]. But, AEC was shown (to be) double ECB . Thus, AEC (is) four times EDC . And AEC was also shown (to be) four times BEC . Thus, EDC (is) equal to BEC . And angle EBD (is) common to the two triangles BEC and BED . Thus, the remaining (angle) BED is equal to the (remaining angle) ECB [Prop. 1.32]. Thus, triangle EBD is equiangular to triangle EBC . Thus, proportionally, as DB is to BE , so EB (is) to BC [Prop. 6.4]. And EB (is) equal to CD . Thus, as BD is to DC , so DC (is) to CB . And BD (is) greater than DC . Thus, DC (is) also greater than CB [Prop. 5.14]. Thus, the straight-line BD has been cut in extreme and mean ratio [at C], and DC is its greater piece. (Which is), the very thing it was required to show.

† If the circle is of unit radius then the side of the hexagon is 1, whereas the side of the decagon is $(1/2)(\sqrt{5} - 1)$.

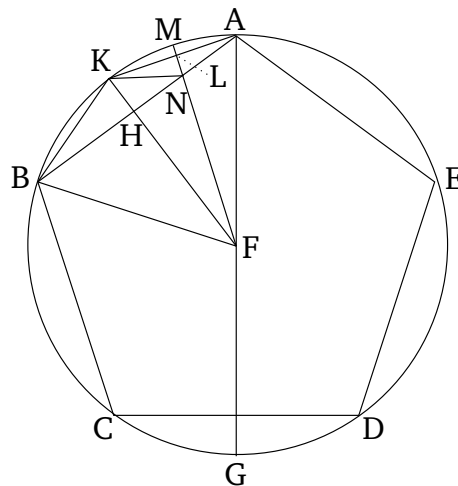
ι'.

Ἐὰν εἰς κύκλον πεντάγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ πενταγώνου πλευρὰ δύναται τὴν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων.



Proposition 10

If an equilateral pentagon is inscribed in a circle then the square on the side of the pentagon is (equal to) the (sum of the squares) on the (sides) of the hexagon and of the decagon inscribed in the same circle.†



Ἐστω κύκλος ὁ $ABΓΔΕ$, καὶ εἰς τὸ $ABΓΔΕ$ κύκλον πεντάγωνον ἰσοπλευρον ἐγγεγράφθω τὸ $ABΓΔΕ$. λέγω, ὅτι ἡ τοῦ $ABΓΔΕ$ πενταγώνου πλευρὰ δύναται τὴν τε τοῦ ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου πλευρὰν τῶν εἰς τὸν $ABΓΔΕ$ κύκλον ἐγγεγραμμένων.

Εἰλήφθω γάρ τὸ κέντρον τοῦ κύκλου τὸ Z σημεῖον, καὶ ἐπιζευχθεῖσα ἡ AZ διήχθω ἐπὶ τὸ H σημεῖον, καὶ ἐπεζεύχθω ἡ ZB , καὶ ἀπὸ τοῦ Z ἐπὶ τὴν AB κάθετος ἤχθω ἡ $ZΘ$, καὶ διήχθω ἐπὶ τὸ K , καὶ ἐπεζεύχθωσαν αἱ AK , KB , καὶ πάλιν ἀπὸ τοῦ Z ἐπὶ τὴν AK κάθετος ἤχθω ἡ $ZΛ$, καὶ διήχθω ἐπὶ τὸ M , καὶ ἐπεζεύχθω ἡ KN .

Ἐπεὶ ἴση ἐστὶν ἡ $ABΓH$ περιφέρεια τῇ $AEDH$ περιφέρειᾳ, ὧν ἡ $ABΓ$ τῇ AED ἐστὶν ἴση, λοιπὴ ἄρα ἡ $ΓH$ περιφέρεια λοιπῇ τῇ HD ἐστὶν ἴση. πενταγώνου δὲ ἡ $ΓΔ$ δεκαγώνου ἄρα ἡ $ΓH$. καὶ ἐπεὶ ἴση ἐστὶν ἡ ZA τῇ ZB , καὶ κάθετος ἡ $ZΘ$, ἴση ἄρα καὶ ἡ ὑπὸ AZK γωνία τῇ ὑπὸ KZB . ὥστε καὶ περιφέρεια ἡ AK τῇ KB ἐστὶν ἴση· διπλῇ ἄρα ἡ AB περιφέρεια τῆς BK περιφέρειας· δεκαγώνου ἄρα πλευρὰ ἐστὶν ἡ AK εὐθεῖα. διὰ τὰ αὐτὰ δὲ καὶ ἡ AK τῆς KM ἐστὶ διπλῇ. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ AB περιφέρεια τῆς BK περιφέρειας, ἴση δὲ ἡ $ΓΔ$ περιφέρεια τῇ AB περιφέρειᾳ, διπλῇ ἄρα καὶ ἡ $ΓΔ$ περιφέρεια τῆς BK περιφέρειας. ἐστὶ δὲ ἡ $ΓΔ$ περιφέρεια καὶ τῆς $ΓH$ διπλῇ· ἴση ἄρα ἡ $ΓH$ περιφέρεια τῇ BK περιφέρειᾳ. ἀλλὰ ἡ BK τῆς KM ἐστὶ διπλῇ, ἐπεὶ καὶ ἡ KA · καὶ ἡ $ΓH$ ἄρα τῆς KM ἐστὶ διπλῇ. ἀλλὰ μὴν καὶ ἡ $ΓB$ περιφέρεια τῆς BK περιφέρειας ἐστὶ διπλῇ· ἴση γὰρ ἡ $ΓB$ περιφέρεια τῇ BA . καὶ ὅλη ἄρα ἡ HB περιφέρεια τῆς BM ἐστὶ διπλῇ· ὥστε καὶ γωνία ἡ ὑπὸ HZB γωνίας τῆς ὑπὸ BZM [ἐστὶ] διπλῇ. ἐστὶ δὲ ἡ ὑπὸ HZB καὶ τῆς ὑπὸ ZAB διπλῇ· ἴση γὰρ ἡ ὑπὸ ZAB τῇ ὑπὸ ABZ . καὶ ἡ ὑπὸ BZN ἄρα τῇ ὑπὸ ZAB ἐστὶν ἴση. κοινὴ δὲ τῶν δύο τριγώνων, τοῦ τε ABZ καὶ τοῦ BZN , ἡ ὑπὸ ABZ γωνία· λοιπὴ ἄρα ἡ ὑπὸ AZB λοιπῇ τῇ ὑπὸ BNZ ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ ABZ τρίγωνον τῷ BZN τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ AB εὐθεῖα πρὸς τὴν BZ , οὕτως ἡ ZB πρὸς τὴν BN · τὸ ἄρα ὑπὸ τῶν ABN ἴσον ἐστὶ τῷ ἀπὸ BZ . πάλιν ἐπεὶ ἴση ἐστὶν ἡ AA τῇ AK , κοινὴ δὲ καὶ πρὸς ὀρθὰς ἡ AN , βάσις ἄρα ἡ KN βάσει τῇ AN ἐστὶν ἴση· καὶ γωνία ἄρα ἡ ὑπὸ AKN γωνία τῇ ὑπὸ LAN ἐστὶν ἴση. ἀλλὰ ἡ ὑπὸ LAN τῇ ὑπὸ KBN ἐστὶν ἴση· καὶ ἡ ὑπὸ AKN ἄρα τῇ ὑπὸ KBN ἐστὶν ἴση. καὶ κοινὴ τῶν δύο τριγώνων τοῦ τε AKB καὶ τοῦ AKN ἡ πρὸς τῷ A . λοιπὴ ἄρα ἡ ὑπὸ AKB λοιπῇ τῇ ὑπὸ KNA ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ KBA τρίγωνον τῷ KNA τριγώνῳ. ἀνάλογον ἄρα ἐστὶν ὡς ἡ BA εὐθεῖα πρὸς τὴν AK , οὕτως ἡ KA πρὸς τὴν AN · τὸ ἄρα ὑπὸ τῶν BAN ἴσον ἐστὶ τῷ ἀπὸ τῆς AK . ἐδείχθη δὲ καὶ τὸ ὑπὸ τῶν ABN ἴσον τῷ ἀπὸ τῆς BZ · τὸ ἄρα ὑπὸ τῶν ABN μετὰ τοῦ ὑπὸ BAN , ὅπερ ἐστὶ τὸ ἀπὸ τῆς BA , ἴσον ἐστὶ τῷ ἀπὸ τῆς BZ μετὰ τοῦ ἀπὸ τῆς AK . καὶ ἐστὶν ἡ μὲν BA πενταγώνου πλευρὰ, ἡ δὲ BZ ἑξαγώνου, ἡ δὲ AK δεκαγώνου.

Ἡ ἄρα τοῦ πενταγώνου πλευρὰ δύναται τὴν τε τοῦ

Let $ABCDE$ be a circle. And let the equilateral pentagon $ABCDE$ have been inscribed in circle $ABCDE$. I say that the square on the side of pentagon $ABCDE$ is the (sum of the squares) on the sides of the hexagon and of the decagon inscribed in circle $ABCDE$.

For let the center of the circle, point F , have been found [Prop. 3.1]. And, AF being joined, let it have been drawn across to point G . And let FB have been joined. And let FH have been drawn from F perpendicular to AB . And let it have been drawn across to K . And let AK and KB have been joined. And, again, let FL have been drawn from F perpendicular to AK . And let it have been drawn across to M . And let KN have been joined.

Since circumference $ABCG$ is equal to circumference $AEDG$, of which ABC is equal to AED , the remaining circumference CG is thus equal to the remaining (circumference) GD . And CD (is the side) of the pentagon. CG (is) thus (the side) of the decagon. And since FA is equal to FB , and FH is perpendicular (to AB), angle AFK (is) thus also equal to KFB [Props. 1.5, 1.26]. Hence, circumference AK is also equal to KB [Prop. 3.26]. Thus, circumference AB (is) double circumference BK . Thus, straight-line AK is the side of the decagon. So, for the same (reasons, circumference) AK is also double KM . And since circumference AB is double circumference BK , and circumference CD (is) equal to circumference AB , circumference CD (is) thus also double circumference BK . And circumference CD is also double CG . Thus, circumference CG (is) equal to circumference BK . But, BK is double KM , since KA (is) also (double KM). Thus, (circumference) CG is also double KM . But, indeed, circumference CB is also double circumference BK . For circumference CB (is) equal to BA . Thus, the whole circumference GB is also double BM . Hence, angle GFB [is] also double angle BFM [Prop. 6.33]. And GFB (is) also double FAB . For FAB (is) equal to ABF . Thus, BFN is also equal to FAB . And angle ABF (is) common to the two triangles ABF and BFN . Thus, the remaining (angle) AFB is equal to the remaining (angle) BNF [Prop. 1.32]. Thus, triangle ABF is equiangular to triangle BFN . Thus, proportionally, as straight-line AB (is) to BF , so FB (is) to BN [Prop. 6.4]. Thus, the (rectangle contained) by ABN is equal to the (square) on BF [Prop. 6.17]. Again, since AL is equal to LK , and LN is common and at right-angles (to KA), base KN is thus equal to base AN [Prop. 1.4]. And, thus, angle LKN is equal to angle LAN . But, LAN is equal to KBN [Props. 3.29, 1.5]. Thus, LKN is also equal to KBN . And the (angle) at A (is) common to the two triangles AKB and AKN . Thus, the remaining (angle) AKB is

ἑξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγεγραμμένων· ὅπερ ἔδει δεῖξαι.

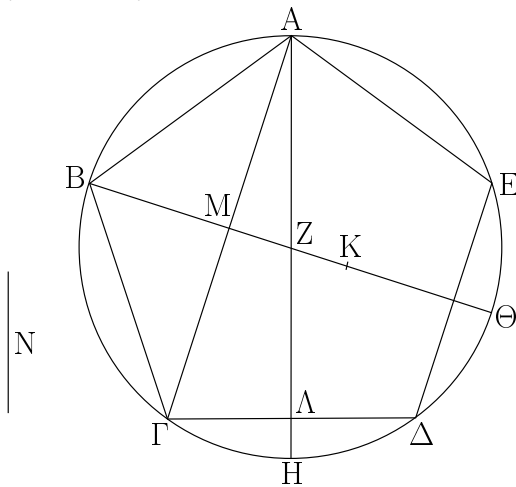
equal to the remaining (angle) KNA [Prop. 1.32]. Thus, triangle KBA is equiangular to triangle KNA . Thus, proportionally, as straight-line BA is to AK , so KA (is) to AN [Prop. 6.4]. Thus, the (rectangle contained) by BAN is equal to the (square) on AK [Prop. 6.17]. And the (rectangle contained) by ABN was also shown (to be) equal to the (square) on BF . Thus, the (rectangle contained) by ABN plus the (rectangle contained) by BAN , which is the (square) on BA [Prop. 2.2], is equal to the (square) on BF plus the (square) on AK . And BA is the side of the pentagon, and BF (the side) of the hexagon [Prop. 4.15 corr.], and AK (the side) of the decagon.

Thus, the square on the side of the pentagon (inscribed in a circle) is (equal to) the (sum of the squares) on the (sides) of the hexagon and of the decagon inscribed in the same circle.

† If the circle is of unit radius then the side of the pentagon is $(1/2) \sqrt{10 - 2\sqrt{5}}$.

ια'.

Ἐάν εἰς κύκλον ῥητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγεγράφῃ, ἡ τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

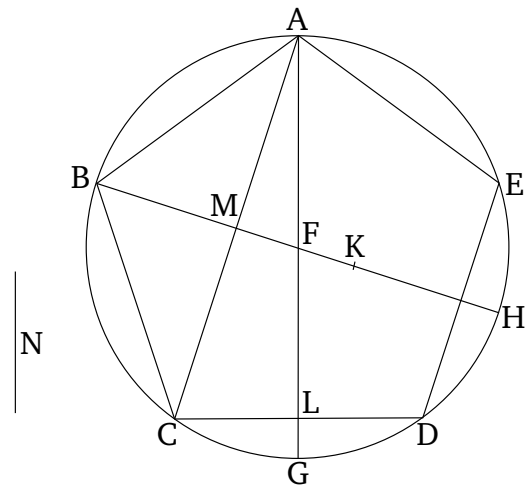


Εἰς γὰρ κύκλον τὸν $ABΓΔE$ ῥητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγεγράφθω τὸ $ABΓΔE$. λέγω, ὅτι ἡ τοῦ $[ABΓΔE]$ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Εἰλήφθω γὰρ τὸ κέντρον τοῦ κύκλου τὸ Z σημεῖον, καὶ ἐπεζεύχθωσαν αἱ AZ , ZB καὶ διήχθωσαν ἐπὶ τὰ H , Θ σημεῖα, καὶ ἐπεζεύχθω ἡ AG , καὶ κείσθω τῆς AZ τέταρτον μέρος ἡ ZK . ῥητὴ δὲ ἡ AZ . ῥητὴ ἄρα καὶ ἡ ZK . ἔστι δὲ καὶ ἡ BZ ῥητὴ· ὅλη ἄρα ἡ BK ῥητὴ ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ AGH περιφέρεια τῇ $AΔH$ περιφερείᾳ, ὥν ἡ $ABΓ$ τῇ $AEΔ$ ἐστὶν ἴση, λοιπὴ ἄρα ἡ $ΓH$ λοιπὴ τῇ $HΔ$ ἐστὶν ἴση. καὶ ἐὰν ἐπιζεύξωμεν τὴν $AΔ$, συνάγονται ὀρθαὶ αἱ

Proposition 11

If an equilateral pentagon is inscribed in a circle which has a rational diameter then the side of the pentagon is that irrational (straight-line) called minor.



For let the equilateral pentagon $ABCDE$ have been inscribed in the circle $ABCDE$ which has a rational diameter. I say that the side of pentagon $[ABCDE]$ is that irrational (straight-line) called minor.

For let the center of the circle, point F , have been found [Prop. 3.1]. And let AF and FB have been joined. And let them have been drawn across to points G and H (respectively). And let AC have been joined. And let FK made (equal) to the fourth part of AF . And AF (is) rational. FK (is) thus also rational. And BF is also rational. Thus, the whole of BK is rational. And since circumference ACG is equal to circumference ADG , of which

πρὸς τῷ Λ γωνίαι, καὶ διπλῇ ἢ $\Gamma\Delta$ τῆς $\Gamma\Lambda$. διὰ τὰ αὐτὰ δὴ καὶ αἱ πρὸς τῷ M ὀρθαί εἰσιν, καὶ διπλῇ ἢ $\Lambda\Gamma$ τῆς ΓM . ἐπεὶ οὖν ἴση ἐστὶν ἡ ὑπὸ $\Lambda\Lambda\Gamma$ γωνία τῇ ὑπὸ ΛMZ , κοινὴ δὲ τῶν δύο τριγώνων τοῦ τε $\Lambda\Gamma\Lambda$ καὶ τοῦ ΛMZ ἡ ὑπὸ $\Lambda\Lambda\Gamma$, λοιπὴ ἄρα ἡ ὑπὸ $\Lambda\Gamma\Lambda$ λοιπῇ τῇ ὑπὸ MZA ἐστὶν ἴση· ἰσογώνιον ἄρα ἐστὶ τὸ $\Lambda\Gamma\Lambda$ τρίγωνον τῷ ΛMZ τριγώνῳ· ἀνάλογον ἄρα ἐστὶν ὡς ἡ $\Lambda\Gamma$ πρὸς $\Gamma\Lambda$, οὕτως ἡ MZ πρὸς ZA · καὶ τῶν ἡγουμένων τὰ διπλάσια· ὡς ἄρα ἡ τῆς $\Lambda\Gamma$ διπλῇ πρὸς τὴν $\Gamma\Lambda$, οὕτως ἡ τῆς MZ διπλῇ πρὸς τὴν ZA . ὡς δὲ ἡ τῆς MZ διπλῇ πρὸς τὴν ZA , οὕτως ἡ MZ πρὸς τὴν ἡμίσειαν τῆς ZA · καὶ ὡς ἄρα ἡ τῆς $\Lambda\Gamma$ διπλῇ πρὸς τὴν $\Gamma\Lambda$, οὕτως ἡ MZ πρὸς τὴν ἡμίσειαν τῆς ZA · καὶ τῶν ἐπομένων τὰ ἡμίσεια· ὡς ἄρα ἡ τῆς $\Lambda\Gamma$ διπλῇ πρὸς τὴν ἡμίσειαν τῆς $\Gamma\Lambda$, οὕτως ἡ MZ πρὸς τὸ τέτατρον τῆς ZA . καὶ ἐστὶ τῆς μὲν $\Lambda\Gamma$ διπλῇ ἢ $\Delta\Gamma$, τῆς δὲ $\Gamma\Lambda$ ἡμίσεια ἢ ΓM , τῆς δὲ ZA τέτατρον μέρος ἢ ZK · ἐστὶν ἄρα ὡς ἡ $\Delta\Gamma$ πρὸς τὴν ΓM , οὕτως ἡ MZ πρὸς τὴν ZK . συνθέντι καὶ ὡς συναμφοτέρος ἢ $\Delta\Gamma M$ πρὸς τὴν ΓM , οὕτως ἢ MK πρὸς KZ · καὶ ὡς ἄρα τὸ ἀπὸ συναμφοτέρου τῆς $\Delta\Gamma M$ πρὸς τὸ ἀπὸ ΓM , οὕτως τὸ ἀπὸ MK πρὸς τὸ ἀπὸ KZ . καὶ ἐπεὶ τῆς ὑπὸ δύο πλευρᾶς τοῦ πενταγώνου ὑποτείνουσας, οἷον τῆς $\Lambda\Gamma$, ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζον τμήμα ἴσον ἐστὶ τῇ τοῦ πενταγώνου πλευρᾷ, τουτέστι τῇ $\Delta\Gamma$, τὸ δὲ μείζον τμήμα προσλαβὼν τὴν ἡμίσειαν τῆς ὅλης πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τῆς ὅλης, καὶ ἐστὶν ὅλης τῆς $\Lambda\Gamma$ ἡμίσεια ἢ ΓM , τὸ ἄρα ἀπὸ τῆς $\Delta\Gamma M$ ὡς μιᾶς πενταπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΓM . ὡς δὲ τὸ ἀπὸ τῆς $\Delta\Gamma M$ ὡς μιᾶς πρὸς τὸ ἀπὸ τῆς ΓM , οὕτως ἐδείχθη τὸ ἀπὸ τῆς MK πρὸς τὸ ἀπὸ τῆς KZ · πενταπλάσιον ἄρα τὸ ἀπὸ τῆς MK τοῦ ἀπὸ τῆς KZ . ῥητὸν δὲ τὸ ἀπὸ τῆς KZ · ῥητὴ γὰρ ἡ διάμετρος· ῥητὸν ἄρα καὶ τὸ ἀπὸ τῆς MK · ῥητὴ ἄρα ἐστὶν ἡ MK [δυνάμει μόνον]. καὶ ἐπεὶ τετραπλάσια ἐστὶν ἡ BZ τῆς ZK , πενταπλάσια ἄρα ἐστὶν ἡ BK τῆς KZ · εἰκοσιπενταπλάσιον ἄρα τὸ ἀπὸ τῆς BK τοῦ ἀπὸ τῆς KZ . πενταπλάσιον δὲ τὸ ἀπὸ τῆς MK τοῦ ἀπὸ τῆς KZ · πενταπλάσιον ἄρα τὸ ἀπὸ τῆς BK τοῦ ἀπὸ τῆς KM · τὸ ἄρα ἀπὸ τῆς BK πρὸς τὸ ἀπὸ KM λόγον οὐκ ἔχει, ὃν τετράγωνος ἀριθμὸς πρὸς τετράγωνον ἀριθμόν· ἀσύμμετρος ἄρα ἐστὶν ἡ BK τῇ KM μήκει. καὶ ἐστὶ ῥητὴ ἑκατέρα αὐτῶν. αἱ BK , KM ἄρα ῥηταί εἰσι δυνάμει μόνον σύμμετροι. ἐὰν δὲ ἀπὸ ῥητῆς ῥητὴ ἀφαιρεθῇ δυνάμει μόνον σύμμετρος οὕσα τῇ ὅλῃ, ἡ λοιπὴ ἄλογός ἐστιν ἀποτομή· ἀποτομή ἄρα ἐστὶν ἡ MB , προσαρμόζουσα δὲ αὐτῇ ἡ MK . λέγω δὴ, ὅτι καὶ τετάρτη. ὥ δὴ μείζον ἐστὶ τὸ ἀπὸ τῆς BK τοῦ ἀπὸ τῆς KM , ἐκείνῳ ἴσον ἔστω τὸ ἀπὸ τῆς N · ἢ BK ἄρα τῆς KM μείζον δύναται τῇ N . καὶ ἐπεὶ σύμμετρός ἐστὶν ἡ KZ τῇ ZB , καὶ συνθέντι σύμμετρός ἐστὶ ἡ KB τῇ ZB . ἀλλὰ ἡ BZ τῇ $B\Theta$ σύμμετρός ἐστιν· καὶ ἡ BK ἄρα τῇ $B\Theta$ σύμμετρός ἐστιν. καὶ ἐπεὶ πενταπλάσιόν ἐστι τὸ ἀπὸ τῆς BK τοῦ ἀπὸ τῆς KM , τὸ ἄρα ἀπὸ τῆς BK πρὸς τὸ ἀπὸ τῆς KM λόγον ἔχει, ὃν ϵ πρὸς ϵ . ἀναστρέψαντι ἄρα τὸ ἀπὸ τῆς BK πρὸς τὸ ἀπὸ τῆς N λόγον ἔχει, ὃν ϵ πρὸς

ABC is equal to AED , the remainder CG is thus equal to the remainder GD . And if we join AD then the angles at L are inferred (to be) right-angles, and CD (is inferred to be) double CL [Prop. 1.4]. So, for the same (reasons), the (angles) at M are also right-angles, and AC (is) double CM . Therefore, since angle ALC (is) equal to AMF , and (angle) LAC (is) common to the two triangles ACL and AMF , the remaining (angle) ACL is thus equal to the remaining (angle) MFA [Prop. 1.32]. Thus, triangle ACL is equiangular to triangle AMF . Thus, proportionally, as LC (is) to CA , so MF (is) to FA [Prop. 6.4]. And (we can take) the doubles of the leading (magnitudes). Thus, as double LC (is) to CA , so double MF (is) to FA . And as double MF (is) to FA , so MF (is) to half of FA . And, thus, as double LC (is) to CA , so MF (is) to half of FA . And (we can take) the halves of the following (magnitudes). Thus, as double LC (is) to half of CA , so MF (is) to the fourth of FA . And DC is double LC , and CM half of CA , and FK the fourth part of FA . Thus, as DC is to CM , so MF (is) to FK . Via composition, as the sum of DCM (i.e., DC and CM) (is) to CM , so MK (is) to KF [Prop. 5.18]. And, thus, as the (square) on the sum of DCM (is) to the (square) on CM , so the (square) on MK (is) to the (square) on KF . And since the greater piece of a (straight-line) subtending two sides of a pentagon, such as AC , (which is) cut in extreme and mean ratio is equal to the side of the pentagon [Prop. 13.8]—that is to say, to DC —and the square on the greater piece added to half of the whole is five times the (square) on half of the whole [Prop. 13.1], and CM (is) half of the whole, AC , thus the (square) on DCM , (taken) as one, is five times the (square) on CM . And the (square) on DCM , (taken) as one, (is) to the (square) on CM , so the (square) on MK was shown (to be) to the (square) on KF . Thus, the (square) on MK (is) five times the (square) on KF . And the square on KF (is) rational. For the diameter (is) rational. Thus, the (square) on MK (is) also rational. Thus, MK is rational [in square only]. And since BF is four times FK , BK is thus five times KF . Thus, the (square) on BK (is) twenty-five times the (square) on KF . And the (square) on MK (is) five times the square on KF . Thus, the (square) on BK (is) five times the (square) on KM . Thus, the (square) on BK does not have to the (square) on KM the ratio which a square number (has) to a square number. Thus, BK is incommensurable in length with KM [Prop. 10.9]. And each of them is a rational (straight-line). Thus, BK and KM are rational (straight-lines which are) commensurable in square only. And if from a rational (straight-line) a rational (straight-line) is subtracted, which is commensurable in square only with the

$\bar{\delta}$, οὐχ ὃν τετράγωνος πρὸς τετράγωνον· ἀσύμμετρος ἄρα ἐστὶν ἡ BK τῇ N · ἡ BK ἄρα τῆς KM μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ. ἐπεὶ οὖν ὅλη ἡ BK τῆς προσαρμοζούσης τῆς KM μείζον δύναται τῷ ἀπὸ ἀσύμμετρου ἑαυτῇ, καὶ ὅλη ἡ BK σύμμετρός ἐστι τῇ ἐκκειμένη ῥητῇ τῇ $B\Theta$, ἀποτομῇ ἄρα τετάρτη ἐστὶν ἡ MB . τὸ δὲ ὑπὸ ῥητῆς καὶ ἀποτομῆς τετάρτης περιεχόμενον ὀρθογώνιον ἄλογόν ἐστιν, καὶ ἡ δυναμένη αὐτὸ ἄλογός ἐστιν, καλεῖται δὲ ἐλάττων. δύναται δὲ τὸ ὑπὸ τῶν ΘBM ἢ AB διὰ τὸ ἐπιζευγνυμένης τῆς $A\Theta$ ἰσογώνιον γίνεσθαι τὸ $AB\Theta$ τρίγωνον τῷ ABM τριγώνῳ καὶ εἶναι ὡς τὴν ΘB πρὸς τὴν BA , οὕτως τὴν AB πρὸς τὴν BM .

Ἡ ἄρα AB τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων· ὁπερ ἔδει δεῖξαι.

whole, then the remainder is that irrational (straight-line called) an apotome [Prop. 10.73]. Thus, MB is an apotome, and MK its attachment. So, I say that (it is) also a fourth (apotome). So, let the (square) on N be (made) equal to that (magnitude) by which the (square) on BK is greater than the (square) on KM . Thus, the square on BK is greater than the (square) on KM by the (square) on N . And since KF is commensurable (in length) with FB then, via composition, KB is also commensurable (in length) with FB [Prop. 10.15]. But, BF is commensurable (in length) with BH . Thus, BK is also commensurable (in length) with BH [Prop. 10.12]. And since the (square) on BK is five times the (square) on KM , the (square) on BK thus has to the (square) on KM the ratio which 5 (has) to one. Thus, via conversion, the (square) on BK has to the (square) on N the ratio which 5 (has) to 4 [Prop. 5.19 corr.], which is not (that) of a square (number) to a square (number). BK is thus incommensurable (in length) with N [Prop. 10.9]. Thus, the square on BK is greater than the (square) on KM by the (square) on (some straight-line which is) incommensurable (in length) with (BK) . Therefore, since the square on the whole, BK , is greater than the (square) on the attachment, KM , by the (square) on (some straight-line which is) incommensurable (in length) with (BK) , and the whole, BK , is commensurable (in length) with the (previously) laid down rational (straight-line) BH , MB is thus a fourth apotome [Def. 10.14]. And the rectangle contained by a rational (straight-line) and a fourth apotome is irrational, and its square-root is that irrational (straight-line) called minor [Prop. 10.94]. And the square on AB is the rectangle contained by HBM , on account of joining AH , (so that) triangle ABH becomes equiangular with triangle ABM [Prop. 6.8], and (proportionally) as HB is to BA , so AB (is) to BM .

Thus, the side AB of the pentagon is that irrational (straight-line) called minor.[†] (Which is) the very thing it was required to show.

[†] If the circle has unit radius then the side of the pentagon is $(1/2)\sqrt{10 - 2\sqrt{5}}$. However, this length can be written in the “minor” form (see Prop. 10.94) $(\rho/\sqrt{2})\sqrt{1 + k/\sqrt{1 + k^2}} - (\rho/\sqrt{2})\sqrt{1 - k/\sqrt{1 + k^2}}$, with $\rho = \sqrt{5}/2$ and $k = 2$.

ιβ'.

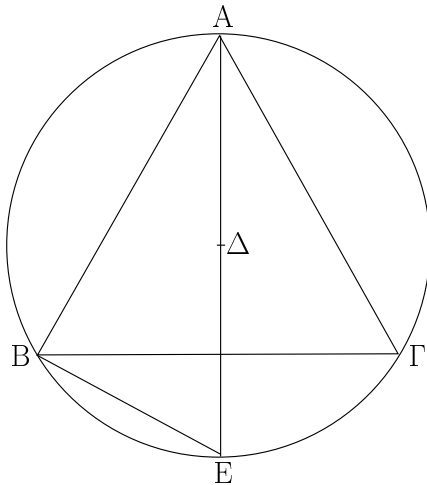
Ἐὰν εἰς κύκλον τρίγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ τριγώνου πλευρὰ δυνάμει τριπλασίῳ ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου.

Ἐστω κύκλος ὁ $AB\Gamma$, καὶ εἰς αὐτὸν τρίγωνον ἰσόπλευρον ἐγγεγράφῃ τὸ $AB\Gamma$. λέγω, ὅτι τοῦ $AB\Gamma$ τριγώνου μία πλευρὰ δυνάμει τριπλασίῳ ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ $AB\Gamma$ κύκλου.

Proposition 12

If an equilateral triangle is inscribed in a circle then the square on the side of the triangle is three times the (square) on the radius of the circle.

Let there be a circle ABC , and let the equilateral triangle ABC have been inscribed in it [Prop. 4.2]. I say that the square on one side of triangle ABC is three times the (square) on the radius of circle ABC .



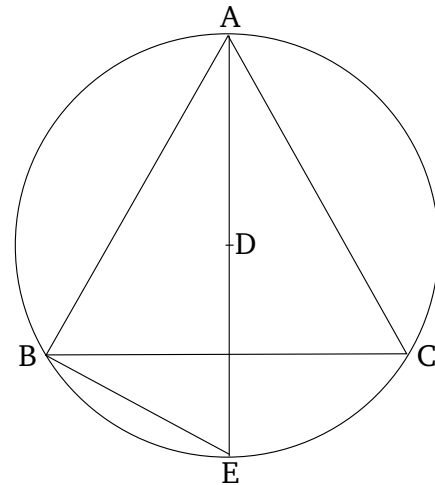
Εἰλήφθω γὰρ τὸ κέντρον τοῦ $AB\Gamma$ κύκλου τὸ Δ , καὶ ἐπιζευχθεῖσα ἡ $A\Delta$ διήχθω ἐπὶ τὸ E , καὶ ἐπεζεύχθω ἡ BE .

Καὶ ἐπεὶ ἰσόπλευρόν ἐστι τὸ $AB\Gamma$ τρίγωνον, ἡ BEG ἄρα περιφέρεια τρίτον μέρος ἐστὶ τῆς τοῦ $AB\Gamma$ κύκλου περιφέρειας. ἡ ἄρα BE περιφέρεια ἕκτον ἐστὶ μέρος τῆς τοῦ κύκλου περιφέρειας· ἐξαγώνου ἄρα ἐστὶν ἡ BE εὐθεῖα· ἴση ἄρα ἐστὶ τῇ ἐκ τοῦ κέντρου τῇ ΔE . καὶ ἐπεὶ διπλῇ ἐστὶν ἡ AE τῆς ΔE , τετραπλάσιον ἐστὶ τὸ ἀπὸ τῆς AE τοῦ ἀπὸ τῆς $E\Delta$, τούτέστι τοῦ ἀπὸ τῆς BE . ἴσον δὲ τὸ ἀπὸ τῆς AE τοῖς ἀπὸ τῶν AB , BE · τὰ ἄρα ἀπὸ τῶν AB , BE τετραπλάσιά ἐστι τοῦ ἀπὸ τῆς BE . διελόντι ἄρα τὸ ἀπὸ τῆς AB τριπλάσιόν ἐστι τοῦ ἀπὸ BE . ἴση δὲ ἡ BE τῇ ΔE · τὸ ἄρα ἀπὸ τῆς AB τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΔE .

Ἡ ἄρα τοῦ τριγώνου πλευρὰ δυνάμει τριπλασία ἐστὶ τῆς ἐκ τοῦ κέντρου [τοῦ κύκλου]· ὅπερ ἔδει δεῖξαι.

ιγ'.

Πυραμίδα συστήσασθαι καὶ σφαῖρα περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει ἡμιολία ἐστὶ τῆς πλευρᾶς τῆς πυραμίδος.



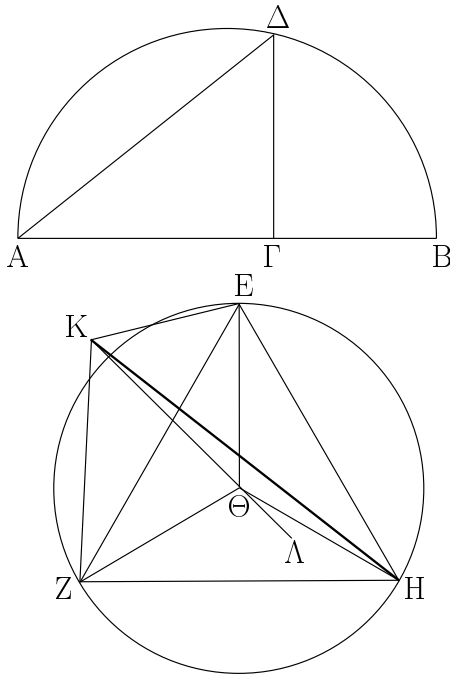
For let the center, D , of circle ABC have been found [Prop. 3.1]. And AD (being) joined, let it have been drawn across to E . And let BE have been joined.

And since triangle ABC is equilateral, circumference BEC is thus the third part of the circumference of circle ABC . Thus, circumference BE is the sixth part of the circumference of the circle. Thus, straight-line BE is (the side) of a hexagon. Thus, it is equal to the radius DE [Prop. 4.15 corr.]. And since AE is double DE , the (square) on AE is four times the (square) on ED —that is to say, of the (square) on BE . And the (square) on AE (is) equal to the (sum of the squares) on AB and BE [Props. 3.31, 1.47]. Thus, the (sum of the squares) on AB and BE is four times the (square) on BE . Thus, via separation, the (square) on AB is three times the (square) on BE . And BE (is) equal to DE . Thus, the (square) on AB is three times the (square) on DE .

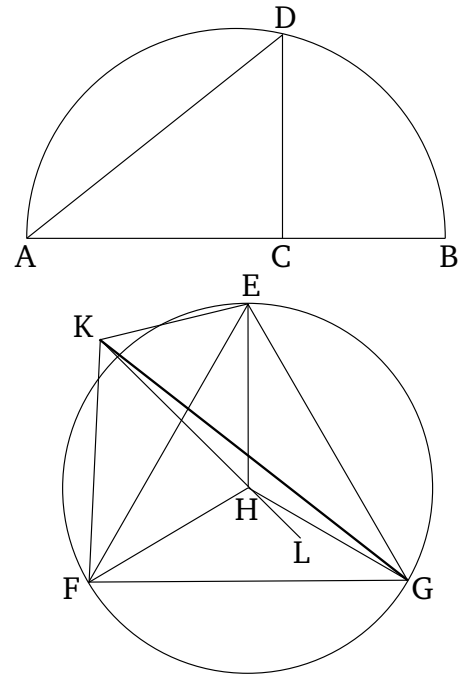
Thus, the square on the side of the triangle is three times the (square) on the radius [of the circle]. (Which is) the very thing it was required to show.

Proposition 13

To construct a (regular) pyramid (i.e., a tetrahedron), and to enclose (it) in a given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.



Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ AB , καὶ τετμηθῇ κατὰ τὸ Γ σημεῖον, ὥστε διπλασίαν εἶναι τὴν AG τῆς GB , καὶ γεγράφθω ἐπὶ τῆς AB ἡμικύκλιον τὸ ADB , καὶ ἤχθω ἀπὸ τοῦ Γ σημείου τῇ AB πρὸς ὀρθὰς ἡ $\Gamma\Delta$, καὶ ἐπεζεύχθω ἡ ΔA , καὶ ἐκκείσθω κύκλος ὁ EZH ἴσην ἔχων τὴν ἐκ τοῦ κέντρου τῇ $\Delta\Gamma$, καὶ ἐγγεγράφθω εἰς τὸν EZH κύκλον τρίγωνον ἰσοπλευρον τὸ EZH , καὶ εἰλήφθω τὸ κέντρον τοῦ κύκλου τὸ Θ σημεῖον, καὶ ἐπεζεύχθωσαν αἱ $E\Theta$, ΘZ , ΘH , καὶ ἀνεστάτω ἀπὸ τοῦ Θ σημείου τῷ τοῦ EZH κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἡ ΘK , καὶ ἀφηρήσθω ἀπὸ τῆς ΘK τῇ AG εὐθείᾳ ἴση ἡ ΘK , καὶ ἐπεζεύχθωσαν αἱ KE , KZ , KH , καὶ ἐπεὶ ἡ $K\Theta$ ὀρθὴ ἐστὶ πρὸς τὸ τοῦ EZH κύκλου ἐπίπεδον, καὶ πρὸς πάσας ἄρα τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ τοῦ EZH κύκλου ἐπιπέδῳ ὀρθὰς ποιήσει γωνίας. ἄπτεται δὲ αὐτῆς ἐκάστη τῶν ΘE , ΘZ , ΘH , ἡ ΘK ἄρα πρὸς ἐκάστη τῶν ΘE , ΘZ , ΘH ὀρθὴ ἐστίν. καὶ ἐπεὶ ἴση ἐστὶν ἡ μὲν AG τῇ ΘK , ἡ δὲ $\Gamma\Delta$ τῇ ΘE , καὶ ὀρθὰς γωνίας περιέχουσιν, βάσις ἄρα ἡ ΔA βάσει τῇ KE ἐστὶν ἴση. διὰ τὰ αὐτὰ δὴ καὶ ἐκατέρω τῶν KZ , KH τῇ ΔA ἐστὶν ἴση· αἱ τρεῖς ἄρα αἱ KE , KZ , KH ἴσαι ἀλλήλαις εἰσίν. καὶ ἐπεὶ διπλὴ ἐστὶν ἡ AG τῆς GB , τριπλὴ ἄρα ἡ AB τῆς BF . ὥς δὲ ἡ AB πρὸς τὴν BF , οὕτως τὸ ἀπὸ τῆς AD πρὸς τὸ ἀπὸ τῆς $\Delta\Gamma$, ὥς ἐξῆς δειχθήσεται. τριπλάσιον ἄρα τὸ ἀπὸ τῆς AD τοῦ ἀπὸ τῆς $\Delta\Gamma$. ἐστὶ δὲ καὶ τὸ ἀπὸ τῆς ZE τοῦ ἀπὸ τῆς $E\Theta$ τριπλάσιον, καὶ ἐστὶν ἴση ἡ $\Delta\Gamma$ τῇ $E\Theta$, ἴση ἄρα καὶ ἡ ΔA τῇ EZ . ἀλλὰ ἡ ΔA ἐκάστη τῶν KE , KZ , KH ἐδείχθη ἴση· καὶ ἐκάστη ἄρα τῶν EZ , ZH , HE ἐκάστη τῶν KE , KZ , KH ἐστὶν ἴση· ἰσοπλευρὰ ἄρα ἐστὶ τὰ τέσσαρα τρίγωνα τὰ EZH , KEZ , KZH , KEH . πυραμὶς ἄρα συνέσταται ἐκ τεσσάρων τριγώνων ἰσοπλευρῶν, ἧς βάσις μὲν ἐστὶ τὸ EZH τρίγωνον,



Let the diameter AB of the given sphere be laid out, and let it have been cut at point C such that AC is double CB [Prop. 6.10]. And let the semi-circle ADB have been drawn on AB . And let CD have been drawn from point C at right-angles to AB . And let DA have been joined. And let the circle EFG be laid down having a radius equal to DC , and let the equilateral triangle EFG have been inscribed in circle EFG [Prop. 4.2]. And let the center of the circle, point H , have been found [Prop. 3.1]. And let EH , HF , and HG have been joined. And let HK have been set up, at point H , at right-angles to the plane of circle EFG [Prop. 11.12]. And let HK , equal to the straight-line AC , have been cut off from HK . And let KE , KF , and KG have been joined. And since KH is at right-angles to the plane of circle EFG , it will thus also make right-angles with all of the straight-lines joining it (which are) also in the plane of circle EFG [Def. 11.3]. And HE , HF , and HG each join it. Thus, HK is at right-angles to each of HE , HF , and HG . And since AC is equal to HK , and CD to HE , and they contain right-angles, the base DA is thus equal to the base KE [Prop. 1.4]. So, for the same (reasons), KF and KG is each equal to DA . Thus, the three (straight-lines) KE , KF , and KG are equal to one another. And since AC is double CB , AB (is) thus triple BC . And as AB (is) to BC , so the (square) on AD (is) to the (square) on DC , as will be shown later [see lemma]. Thus, the (square) on AD (is) three times the (square) on DC . And the (square) on FE is also three times the (square) on EH [Prop. 13.12], and DC is equal to EH . Thus, DA (is)

κορυφή δὲ τὸ Κ σημεῖον.

Δεῖ δὴ αὐτὴν καὶ σφαῖρα περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ δυνάμει τῆς πλευρᾶς τῆς πυραμίδος.

Ἐκβεβλήσθω γὰρ ἐπ' εὐθείας τῇ ΚΘ εὐθεῖα ἡ ΘΑ, καὶ κείσθω τῇ ΓΒ ἴση ἡ ΘΑ. καὶ ἐπεὶ ἐστὶν ὡς ἡ ΑΓ πρὸς τὴν ΓΔ, οὕτως ἡ ΓΔ πρὸς τὴν ΓΒ, ἴση δὲ ἡ μὲν ΑΓ τῇ ΚΘ, ἡ δὲ ΓΔ τῇ ΘΕ, ἡ δὲ ΓΒ τῇ ΘΑ, ἔστιν ἄρα ὡς ἡ ΚΘ πρὸς τὴν ΘΕ, οὕτως ἡ ΕΘ πρὸς τὴν ΘΑ· τὸ ἄρα ὑπὸ τῶν ΚΘ, ΘΑ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ. καὶ ἐστὶν ὀρθὴ ἑκατέρα τῶν ὑπὸ ΚΘΕ, ΕΘΑ γωνιῶν· τὸ ἄρα ἐπὶ τῆς ΚΑ γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Ε [ἐπειδὴ περ ἐὰν ἐπιζεύζωμεν τὴν ΕΑ, ὀρθὴ γίνεται ἡ ὑπὸ ΑΕΚ γωνία διὰ τὸ ἰσογώνιον γίνεσθαι τὸ ΕΑΚ τρίγωνον ἑκατέρω τῶν ΕΑΘ, ΕΘΚ τριγώνων]. ἐὰν δὴ μενούσης τῆς ΚΑ περιεγεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἥξει καὶ διὰ τῶν Ζ, Η σημείων ἐπιζευγνυμένων τῶν ΖΑ, ΛΗ καὶ ὀρθῶν ὁμοίως γινομένων τῶν πρὸς τοῖς Ζ, Η γωνιῶν· καὶ ἔσται ἡ πυραμὶς σφαῖρα περιειλημμένη τῇ δοθείσῃ. ἡ γὰρ ΚΑ τῆς σφαίρας διάμετρος ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμετρῷ τῇ ΑΒ, ἐπειδὴ περ τῇ μὲν ΑΓ ἴση κεῖται ἡ ΚΘ, τῇ δὲ ΓΒ ἡ ΘΑ.

Λέγω δὴ, ὅτι ἡ τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ δυνάμει τῆς πλευρᾶς τῆς πυραμίδος.

Ἐπεὶ γὰρ διπλῇ ἐστὶν ἡ ΑΓ τῆς ΓΒ, τριπλῇ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΓ· ἀναστρέψαντι ἡμιολία ἄρα ἐστὶν ἡ ΒΑ τῆς ΑΓ. ὡς δὲ ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως τὸ ἀπὸ τῆς ΒΑ πρὸς τὸ ἀπὸ τῆς ΑΔ [ἐπειδὴ περ ἐπιζευγνυμένης τῆς ΔΒ ἐστὶν ὡς ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως ἡ ΔΑ πρὸς τὴν ΑΓ διὰ τὴν ὁμοιότητα τῶν ΔΑΒ, ΔΑΓ τριγώνων, καὶ εἶναι ὡς τὴν πρώτην πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας]. ἡμιόλιον ἄρα καὶ τὸ ἀπὸ τῆς ΒΑ τοῦ ἀπὸ τῆς ΑΔ. καὶ ἐστὶν ἡ μὲν ΒΑ ἡ τῆς δοθείσης σφαίρας διάμετρος, ἡ δὲ ΑΔ ἴση τῇ πλευρᾷ τῆς πυραμίδος.

Ἡ ἄρα τῆς σφαίρας διάμετρος ἡμιολία ἐστὶ τῆς πλευρᾶς τῆς πυραμίδος· ὅπερ ἔδει δεῖξαι.

also equal to EF . But, DA was shown (to be) equal to each of KE , KF , and KG . Thus, EF , FG , and GE are equal to KE , KF , and KG , respectively. Thus, the four triangles EFG , KEF , KFG , and KEG are equilateral. Thus, a pyramid, whose base is triangle EFG , and apex the point K , has been constructed from four equilateral triangles.

So, it is also necessary to enclose it in the given sphere, and to show that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

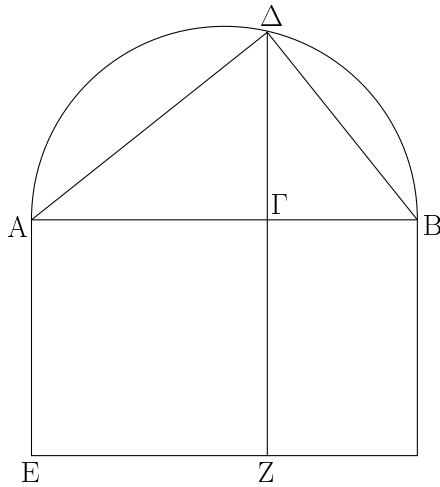
For let the straight-line HL have been produced in a straight-line with KH , and let HL be made equal to CB . And since as AC (is) to CD , so CD (is) to CB [Prop. 6.8 corr.], and AC (is) equal to KH , and CD to HE , and CB to HL , thus as KH is to HE , so EH (is) to HL . Thus, the (rectangle contained) by KH and HL is equal to the (square) on EH [Prop. 6.17]. And each of the angles KHE and EHL is a right-angle. Thus, the semi-circle drawn on KL will also pass through E [inasmuch as if we join EL then the angle LEK becomes a right-angle, on account of triangle ELK becoming equiangular to each of the triangles ELH and EHK [Props. 6.8, 3.31]]. So, if KL remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, it will also pass through points F and G , (because) if FL and LG are joined, the angles at F and G will similarly become right-angles. And the pyramid will have been enclosed by the given sphere. For the diameter, KL , of the sphere is equal to the diameter, AB , of the given sphere—inasmuch as KH was made equal to AC , and HL to CB .

So, I say that the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.

For since AC is double CB , AB is thus triple BC . Thus, via conversion, BA is one and a half times AC . And as BA (is) to AC , so the (square) on BA (is) to the (square) on AD [inasmuch as if DB is joined then as BA is to AD , so DA (is) to AC , on account of the similarity of triangles DAB and DAC . And as the first is to the third (of four proportional magnitudes), so the (square) on the first (is) to the (square) on the second.] Thus, the (square) on BA (is) also one and a half times the (square) on AD . And BA is the diameter of the given sphere, and AD (is) equal to the side of the pyramid.

Thus, the square on the diameter of the sphere is one and a half times the (square) on the side of the pyramid.[†] (Which is) the very thing it was required to show.

[†] If the radius of the sphere is unity then the side of the pyramid (i.e., tetrahedron) is $\sqrt{8/3}$.



Λήμμα.

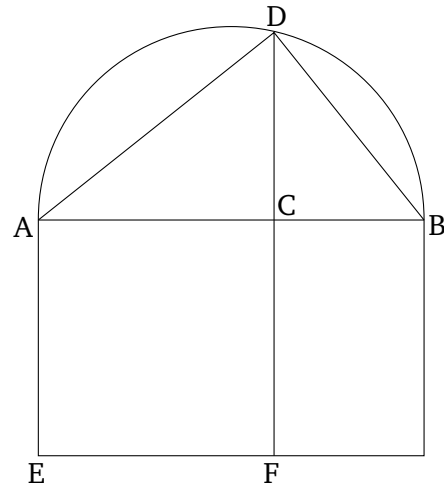
Δεικτέον, ὅτι ἐστὶν ὡς ἡ AB πρὸς τὴν $B\Gamma$, οὕτως τὸ ἀπὸ τῆς $A\Delta$ πρὸς τὸ ἀπὸ τῆς $\Delta\Gamma$.

Ἐκκείσθω γὰρ ἡ τοῦ ἡμικυκλίου καταγραφὴ, καὶ ἐπεζεύχθω ἡ ΔB , καὶ ἀναγεγράφθω ἀπὸ τῆς $A\Gamma$ τετράγωνον τὸ $E\Gamma$, καὶ συμπληρώσθω τὸ ZB παραλληλόγραμμον. ἔπει οὖν διὰ τὸ ἰσογώνιον εἶναι τὸ ΔAB τρίγωνον τῷ $\Delta A\Gamma$ τριγώνῳ ἐστὶν ὡς ἡ BA πρὸς τὴν $A\Delta$, οὕτως ἡ ΔA πρὸς τὴν $A\Gamma$, τὸ ἄρα ὑπὸ τῶν BA , $A\Gamma$ ἴσον ἐστὶ τῷ ἀπὸ τῆς $A\Delta$. καὶ ἐπεὶ ἐστὶν ὡς ἡ AB πρὸς τὴν $B\Gamma$, οὕτως τὸ EB πρὸς τὸ BZ , καὶ ἐστὶ τὸ μὲν EB τὸ ὑπὸ τῶν BA , $A\Gamma$ ἴση γὰρ ἡ EA τῇ $A\Gamma$. τὸ δὲ BZ τὸ ὑπὸ τῶν $A\Gamma$, ΓB , ὡς ἄρα ἡ AB πρὸς τὴν $B\Gamma$, οὕτως τὸ ὑπὸ τῶν BA , $A\Gamma$ πρὸς τὸ ὑπὸ τῶν $A\Gamma$, ΓB . καὶ ἐστὶ τὸ μὲν ὑπὸ τῶν BA , $A\Gamma$ ἴσον τῷ ἀπὸ τῆς $A\Delta$, τὸ δὲ ὑπὸ τῶν $A\Gamma B$ ἴσον τῷ ἀπὸ τῆς $\Delta\Gamma$. ἡ γὰρ $\Delta\Gamma$ κάθετος τῶν τῆς βάσεως τμημάτων τῶν $A\Gamma$, ΓB μέση ἀνάλογόν ἐστι διὰ τὸ ὀρθὴν εἶναι τὴν ὑπὸ $A\Delta B$. ὡς ἄρα ἡ AB πρὸς τὴν $B\Gamma$, οὕτως τὸ ἀπὸ τῆς $A\Delta$ πρὸς τὸ ἀπὸ τῆς $\Delta\Gamma$. ὅπερ ἔδει δεῖξαι.

ιδ'.

Ὅκταέδρον συστήσασθαι καὶ σφαῖρα περιλαβεῖν, ἥ καὶ τὰ πρότερα, καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασία ἐστὶ τῆς πλευρᾶς τοῦ ὀκταέδρου.

Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ AB , καὶ τεμήσθω δίχα κατὰ τὸ Γ , καὶ γεγράφθω ἐπὶ τῆς AB ἡμικύκλιον τὸ $A\Delta B$, καὶ ἦχθω ἀπὸ τοῦ Γ τῇ AB πρὸς ὀρθὰς ἡ $\Gamma\Delta$, καὶ ἐπεζεύχθω ἡ ΔB , καὶ ἐκκείσθω τετράγωνον τὸ $EZH\Theta$ ἴσην ἔχον ἐκάστην τῶν πλευρῶν τῇ ΔB , καὶ



Lemma

It must be shown that as AB is to BC , so the (square) on AD (is) to the (square) on DC .

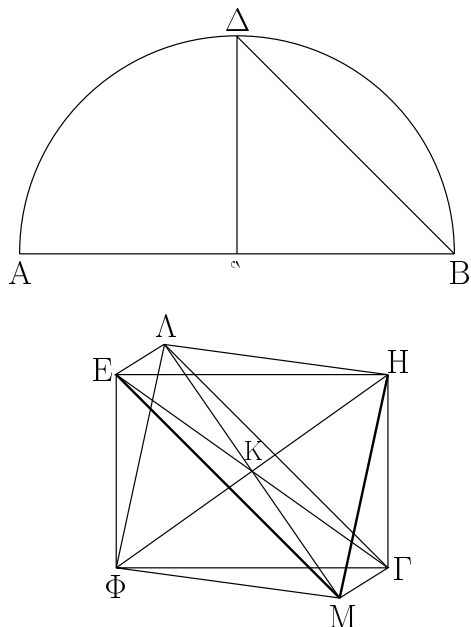
For, let the figure of the semi-circle have been set out, and let DB have been joined. And let the square EC have been described on AC . And let the parallelogram FB have been completed. Therefore, since, on account of triangle DAB being equiangular to triangle DAC [Props. 6.8, 6.4], (proportionally) as BA is to AD , so DA (is) to AC , the (rectangle contained) by BA and AC is thus equal to the (square) on AD [Prop. 6.17]. And since as AB is to BC , so EB (is) to BF [Prop. 6.1]. And EB is the (rectangle contained) by BA and AC —for EA (is) equal to AC . And BF the (rectangle contained) by AC and CB . Thus, as AB (is) to BC , so the (rectangle contained) by BA and AC (is) to the (rectangle contained) by AC and CB . And the (rectangle contained) by BA and AC is equal to the (square) on AD , and the (rectangle contained) by ACB (is) equal to the (square) on DC . For the perpendicular DC is the mean proportional to the pieces of the base, AC and CB , on account of ADB being a right-angle [Prop. 6.8 corr.]. Thus, as AB (is) to BC , so the (square) on AD (is) to the (square) on DC . (Which is) the very thing it was required to show.

Proposition 14

To construct an octahedron, and to enclose (it) in a (given) sphere, like in the preceding (proposition), and to show that the square on the diameter of the sphere is double the (square) on the side of the octahedron.

Let the diameter AB of the given sphere be laid out, and let it have been cut in half at C . And let the semi-circle ADB have been drawn on AB . And let CD be drawn from C at right-angles to AB . And let DB have

ἐπεξεύχθωσαν αἱ ΘΖ, ΕΗ, καὶ ἀνεστάτω ἀπὸ τοῦ Κ σημείου τῷ τοῦ ΕΖΗΘ τετραγώνου ἐπιπέδῳ πρὸς ὀρθὰς εὐθεΐα ἡ ΚΛ καὶ διήχθω ἐπὶ τὰ ἕτερα μέρη τοῦ ἐπιπέδου ὡς ἡ ΚΜ, καὶ ἀφηρήσθω ἀφ' ἑκατέρας τῶν ΚΛ, ΚΜ μιᾶ τῶν ΕΚ, ΖΚ, ΗΚ, ΘΚ ἴση ἑκατέρα τῶν ΚΛ, ΚΜ, καὶ ἐπεξεύχθωσαν αἱ ΑΕ, ΑΖ, ΑΗ, ΑΘ, ΜΕ, ΜΖ, ΜΗ, ΜΘ.

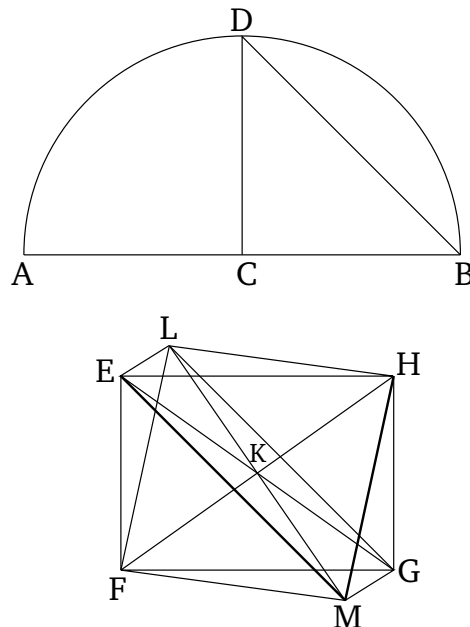


Καὶ ἐπεὶ ἴση ἐστὶν ἡ ΚΕ τῇ ΚΘ, καὶ ἐστὶν ὀρθὴ ἡ ὑπὸ ΕΚΘ γωνία, τὸ ἄρα ἀπὸ τῆς ΘΕ διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΕΚ. πάλιν, ἐπεὶ ἴση ἐστὶν ἡ ΑΚ τῇ ΚΕ, καὶ ἐστὶν ὀρθὴ ἡ ὑπὸ ΑΚΕ γωνία, τὸ ἄρα ἀπὸ τῆς ΕΑ διπλάσιόν ἐστι τοῦ ἀπὸ ΕΚ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΘΕ διπλάσιον τοῦ ἀπὸ τῆς ΕΚ· τὸ ἄρα ἀπὸ τῆς ΑΕ ἴσον ἐστὶ τῷ ἀπὸ τῆς ΕΘ· ἴση ἄρα ἐστὶν ἡ ΑΕ τῇ ΕΘ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΑΘ τῇ ΘΕ ἐστὶν ἴση· ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΕΘ τρίγωνον. ὁμοίως δὴ δείξομεν, ὅτι καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μὲν εἰσιν αἱ τοῦ ΕΖΗΘ τετραγώνου πλευраί, κορυφαὶ δὲ τὰ Α, Μ σημεία, ἰσόπλευρόν ἐστιν· ὀκτάεδρον ἄρα συνέσταται ὑπὸ ὀκτὼ τριγώνων ἰσοπλευρῶν περιεχόμενον.

Δεῖ δὴ αὐτὸ καὶ σφαῖρα περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίῳ ἐστὶ τῆς τοῦ ὀκταέδρου πλευρᾶς.

Ἐπεὶ γάρ αἱ τρεῖς αἱ ΑΚ, ΚΜ, ΚΕ ἴσαι ἀλλήλαις εἰσίν, τὸ ἄρα ἐπὶ τῆς ΑΜ γραφόμενον ἡμικύκλιον ἤξει καὶ διὰ τοῦ Ε. καὶ διὰ τὰ αὐτά, ἐὰν μενούσης τῆς ΑΜ περιεγεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἤξει καὶ διὰ τῶν Ζ, Η, Θ σημείων, καὶ ἔσται σφαῖρα περιελημμένη τὸ ὀκτάεδρον. λέγω δὴ, ὅτι καὶ τῇ δοθείσῃ. ἐπεὶ γάρ ἴση ἐστὶν ἡ ΑΚ τῇ ΚΜ, κοινὴ δὲ ἡ ΚΕ,

been joined. And let the square $EFGH$, having each of its sides equal to DB , be laid out. And let HF and EG have been joined. And let the straight-line KL have been set up, at point K , at right-angles to the plane of square $EFGH$ [Prop. 11.12]. And let it have been drawn across on the other side of the plane, like KM . And let KL and KM , equal to one of EK , FK , GK , and HK , have been cut off from KL and KM , respectively. And let LE , LF , LG , LH , ME , MF , MG , and MH have been joined.



And since KE is equal to KH , and angle EKH is a right-angle, the (square) on the HE is thus double the (square) on EK [Prop. 1.47]. Again, since LK is equal to KE , and angle LKE is a right-angle, the (square) on EL is thus double the (square) on EK [Prop. 1.47]. And the (square) on HE was also shown (to be) double the (square) on EK . Thus, the (square) on LE is equal to the (square) on EH . Thus, LE is equal to EH . So, for the same (reasons), LH is also equal to HE . Triangle LEH is thus equilateral. So, similarly, we can show that each of the remaining triangles, whose bases are the sides of the square $EFGH$, and apexes the points L and M , are equilateral. Thus, an octahedron contained by eight equilateral triangles has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is double the (square) on the side of the octahedron.

For since the three (straight-lines) LK , KM , and KE are equal to one another, the semi-circle drawn on LM will thus also pass through E . And, for the same (reasons), if LM remains (fixed), and the semi-circle is car-

καὶ γωνίας ὀρθὰς περιέχουσιν, βάσις ἄρα ἡ ΛE βάσει τῇ EM ἐστὶν ἴση. καὶ ἐπεὶ ὀρθὴ ἐστὶν ἡ ὑπὸ ΛEM γωνία· ἐν ἡμικυκλίῳ γάρ· τὸ ἄρα ἀπὸ τῆς ΛM διπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΛE . πάλιν, ἐπεὶ ἴση ἐστὶν ἡ AG τῇ GB , διπλασία ἐστὶν ἡ AB τῆς BG . ὥς δὲ ἡ AB πρὸς τὴν BG , οὕτως τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς BD · διπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς BD . ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΛM διπλάσιον τοῦ ἀπὸ τῆς ΛE . καὶ ἐστὶν ἴσον τὸ ἀπὸ τῆς ΔB τῷ ἀπὸ τῆς ΛE · ἴση γὰρ κεῖται ἡ $E\Theta$ τῇ ΔB . ἴσον ἄρα καὶ τὸ ἀπὸ τῆς AB τῷ ἀπὸ τῆς ΛM · ἴση ἄρα ἡ AB τῇ ΛM . καὶ ἐστὶν ἡ AB ἡ τῆς δοθείσης σφαίρας διάμετρος· ἡ ΛM ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ.

Περιεῖληπται ἄρα τὸ ὀκτάεδρον τῇ δοθείσῃ σφαίρᾳ. καὶ συναποδέδεικται, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίῳ ἐστὶ τῆς τοῦ ὀκταέδρου πλευρᾶς· ὅπερ ἔδει δεῖξαι.

ried around, and again established at the same (position) from which it began to be moved, then it will also pass through points F , G , and H , and the octahedron will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since LK is equal to KM , and KE (is) common, and they contain right-angles, the base LE is thus equal to the base EM [Prop. 1.4]. And since angle LEM is a right-angle—for (it is) in a semi-circle [Prop. 3.31]—the (square) on LM is thus double the (square) on LE [Prop. 1.47]. Again, since AC is equal to CB , AB is double BC . And as AB (is) to BC , so the (square) on AB (is) to the (square) on BC [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is double the (square) on BC . And the (square) on LM was also shown (to be) double the (square) on LE . And the (square) on DB is equal to the (square) on LE . For EH was made equal to DB . Thus, the (square) on AB (is) also equal to the (square) on LM . Thus, AB (is) equal to LM . And AB is the diameter of the given sphere. Thus, LM is equal to the diameter of the given sphere.

Thus, the octahedron has been enclosed by the given sphere, and it has been simultaneously proved that the square on the diameter of the sphere is double the (square) on the side of the octahedron.[†] (Which is) the very thing it was required to show.

[†] If the radius of the sphere is unity then the side of octahedron is $\sqrt{2}$.

ιε'.

Proposition 15

Κύβον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ἥ καὶ τὴν πυραμίδα, καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίῳ ἐστὶ τῆς τοῦ κύβου πλευρᾶς.

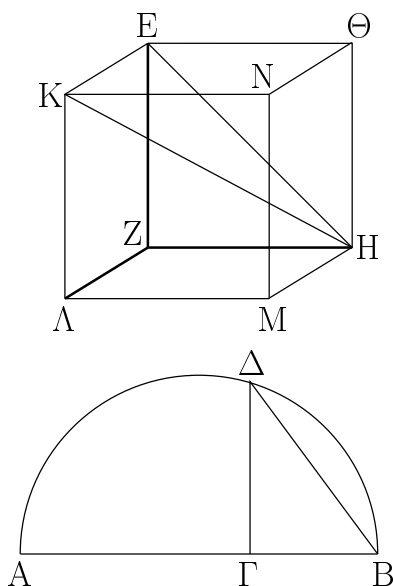
Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ AB καὶ τετμήσθω κατὰ τὸ Γ ὥστε διπλὴν εἶναι τὴν AG τῆς GB , καὶ γεγράψθω ἐπὶ τῆς AB ἡμικύκλιον τὸ $A\Delta B$, καὶ ἀπὸ τοῦ Γ τῇ AB πρὸς ὀρθὰς ἤχθω ἡ $\Gamma\Delta$, καὶ ἐπεζεύχθω ἡ ΔB , καὶ ἐκκείσθω τετράγωνον τὸ $EZH\Theta$ ἴσην ἔχον τὴν πλευρὰν τῇ ΔB , καὶ ἀπὸ τῶν E , Z , H , Θ τῷ τοῦ $EZH\Theta$ τετραγώνου ἐπιπέδῳ πρὸς ὀρθὰς ἤχθωσαν αἱ EK , $Z\Lambda$, HM , ΘN , καὶ ἀφῆρήσθω ἀπὸ ἐκάστης τῶν EK , $Z\Lambda$, HM , ΘN μίαν τῶν EZ , ZH , $H\Theta$, ΘE ἴσην ἐκάστη τῶν EK , $Z\Lambda$, HM , ΘN , καὶ ἐπεζεύχθωσαν αἱ $K\Lambda$, ΛM , MN , NK · κύβος ἄρα συνέσταται ὁ ZN ὑπὸ ἑξ τετραγώνων ἴσων περιεχόμενος.

Δεῖ δὴ αὐτὸν καὶ σφαίρᾳ περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασία ἐστὶ τῆς πλευρᾶς τοῦ κύβου.

To construct a cube, and to enclose (it) in a sphere, like in the (case of the) pyramid, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.

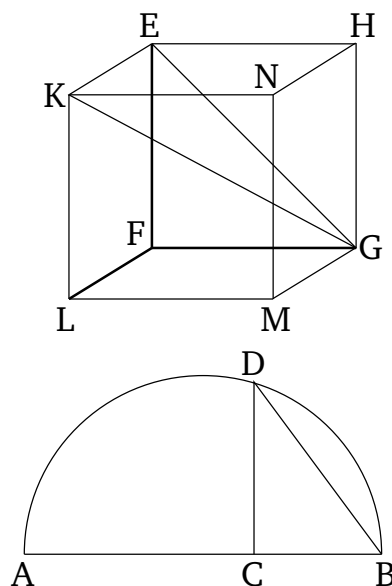
Let the diameter AB of the given sphere be laid out, and let it have been cut at C such that AC is double CB . And let the semi-circle ADB have been drawn on AB . And let CD have been drawn from C at right-angles to AB . And let DB have been joined. And let the square $EFGH$, having (its) side equal to DB , be laid out. And let EK , FL , GM , and HN have been drawn from (points) E , F , G , and H , (respectively), at right-angles to the plane of square $EFGH$. And let EK , FL , GM , and HN , equal to one of EF , FG , GH , and HE , have been cut off from EK , FL , GM , and HN , respectively. And let KL , LM , MN , and NK have been joined. Thus, a cube contained by six equal squares has been constructed.

So, it is also necessary to enclose it by the given sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube.



Ἐπεζεύχθωσαν γὰρ αἱ KH, EH. καὶ ἐπεὶ ὀρθὴ ἐστὶν ἡ ὑπὸ KEH γωνία διὰ τὸ καὶ τὴν KE ὀρθὴν εἶναι πρὸς τὸ EH ἐπίπεδον δηλαδὴ καὶ πρὸς τὴν EH εὐθεῖαν, τὸ ἄρα ἐπὶ τῆς KH γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ E σημείου. πάλιν, ἐπεὶ ἡ HZ ὀρθὴ ἐστὶ πρὸς ἑκατέραν τῶν ZΛ, ZE, καὶ πρὸς τὸ ZK ἄρα ἐπίπεδον ὀρθὴ ἐστὶν ἡ HZ· ὥστε καὶ ἐὰν ἐπιζεύσωμεν τὴν ZK, ἡ HZ ὀρθὴ ἔσται καὶ πρὸς τὴν ZK· καὶ διὰ τοῦτο πάλιν τὸ ἐπὶ τῆς HK γραφόμενον ἡμικύκλιον ἥξει καὶ διὰ τοῦ Z. ὁμοίως καὶ διὰ τῶν λοιπῶν τοῦ κύβου σημείων ἥξει. ἐὰν δὴ μενούσης τῆς KH περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἔσται σφαῖρα περιειλημμένης ὁ κύβος. λέγω δὴ, ὅτι καὶ τῇ δοθείσῃ. ἐπεὶ γὰρ ἴση ἐστὶν ἡ HZ τῇ ZE, καὶ ἐστὶν ὀρθὴ ἡ πρὸς τῷ Z γωνία, τὸ ἄρα ἀπὸ τῆς EH διπλάσιόν ἐστι τοῦ ἀπὸ τῆς EZ. ἴση δὲ ἡ EZ τῇ EK· τὸ ἄρα ἀπὸ τῆς EH διπλάσιόν ἐστι τοῦ ἀπὸ τῆς EK· ὥστε τὰ ἀπὸ τῶν HE, EK, τοῦτέστι τὸ ἀπὸ τῆς HK, τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς EK. καὶ ἐπεὶ τριπλασίον ἐστὶν ἡ AB τῆς BG, ὥς δὲ ἡ AB πρὸς τὴν BG, οὕτως τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς BD, τριπλάσιον ἄρα τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς BD. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς HK τοῦ ἀπὸ τῆς KE τριπλάσιον. καὶ κεῖται ἴση ἡ KE τῇ ΔB· ἴση ἄρα καὶ ἡ KH τῇ AB. καὶ ἐστὶν ἡ AB τῆς δοθείσης σφαίρας διάμετρος· καὶ ἡ KH ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ.

Τῇ δοθείσῃ ἄρα σφαῖρα περιεῖληπται ὁ κύβος· καὶ συναποδεδείκται, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίον ἐστὶ τῆς τοῦ κύβου πλευρᾶς· ὅπερ ἔδει δεῖξαι.



For let KG and EG have been joined. And since angle KEG is a right-angle—on account of KE also being at right-angles to the plane EG , and manifestly also to the straight-line EG [Def. 11.3]—the semi-circle drawn on KG will thus also pass through point E . Again, since GF is at right-angles to each of FL and FE , GF is thus also at right-angles to the plane FK [Prop. 11.4]. Hence, if we also join FK then GF will also be at right-angles to FK . And, again, on account of this, the semi-circle drawn on GK will also pass through point F . Similarly, it will also pass through the remaining (angular) points of the cube. So, if KG remains (fixed), and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then the cube will have been enclosed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For since GF is equal to FE , and the angle at F is a right-angle, the (square) on EG is thus double the (square) on EF [Prop. 1.47]. And EF (is) equal to EK . Thus, the (square) on EG is double the (square) on EK . Hence, the (sum of the squares) on GE and EK —that is to say, the (square) on GK [Prop. 1.47]—is three times the (square) on EK . And since AB is three times BC , and as AB (is) to BC , so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9], the (square) on AB (is) thus three times the (square) on BD . And the (square) on GK was also shown (to be) three times the (square) on KE . And KE was made equal to DB . Thus, KG (is) also equal to AB . And AB is the radius of the given sphere. Thus, KG is also equal to the diameter of the given sphere.

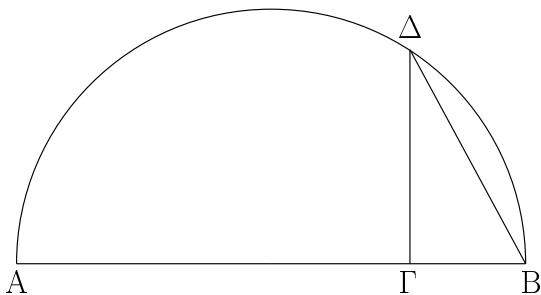
Thus, the cube has been enclosed by the given sphere. And it has simultaneously been shown that the square on the diameter of the sphere is three times the (square) on

the side of the cube.[†] (Which is) the very thing it was required to show.

[†] If the radius of the sphere is unity then the side of the cube is $\sqrt{4/3}$.

ιγ'.

Εἰκοσάεδρον συστήσασθαι καὶ σφαῖρα περιλαβεῖν, ἥ καὶ τὰ προειρημένα σχήματα, καὶ δεῖξαι, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων.

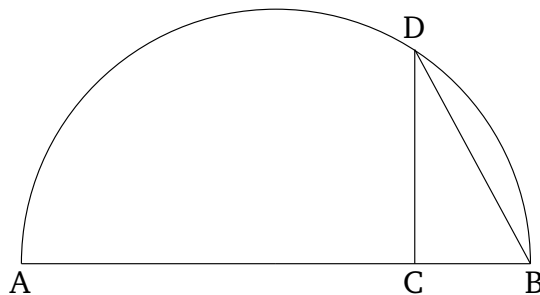


Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ καὶ τετμήσθω κατὰ τὸ Γ ὥστε τετραπλῆν εἶναι τὴν ΑΓ τῆς ΓΒ, καὶ γεγράψθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΔΒ, καὶ ἥχθω ἀπὸ τοῦ Γ τῇ ΑΒ πρὸς ὀρθὰς γωνίας εὐθεῖα γραμμὴ ἡ ΓΔ, καὶ ἐπεξεύχθω ἡ ΔΒ, καὶ ἐκκείσθω κύκλος ὁ ΕΖΗΘΚ, οὗ ἡ ἐν τοῦ κέντρου ἴση ἔστω τῇ ΔΒ, καὶ ἐγγεγράφθω εἰς τὸν ΕΖΗΘΚ κύκλον πεντάγωνον ἰσόπλευρόν τε καὶ ἰσογώνιον τὸ ΕΖΗΘΚ, καὶ τετμήσθωσαν αἱ ΕΖ, ΖΗ, ΗΘ, ΘΚ, ΚΕ περιφέρειαι δίχα κατὰ τὰ Λ, Μ, Ν, Ξ, Ο σημεία, καὶ ἐπεξεύχθωσαν αἱ ΑΜ, ΜΝ, ΝΞ, ΞΟ, ΟΛ, ΕΟ. ἰσόπλευρον ἄρα ἐστὶ καὶ τὸ ΑΜΝΞΟ πεντάγωνον, καὶ δεκαγώνου ἡ ΕΟ εὐθεῖα. καὶ ἀνεστάτωσαν ἀπὸ τῶν Ε, Ζ, Η, Θ, Κ σημείων τῶ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς γωνίας εὐθεῖαι αἱ ΕΠ, ΖΡ, ΗΣ, ΘΤ, ΚΥ ἴσαι οὖσαι τῇ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου, καὶ ἐπεξεύχθωσαν αἱ ΠΡ, ΡΣ, ΣΤ, ΤΥ, ΥΠ, ΠΛ, ΑΡ, ΡΜ, ΜΣ, ΣΝ, ΝΤ, ΤΞ, ΞΥ, ΥΟ, ΟΠ.

Καὶ ἐπεὶ ἑκάτερα τῶν ΕΠ, ΚΥ τῶ αὐτῷ ἐπιπέδῳ πρὸς ὀρθὰς ἐστίν, παράλληλος ἄρα ἐστὶν ἡ ΕΠ τῇ ΚΥ. ἐστὶ δὲ αὐτῇ καὶ ἴση· αἱ δὲ τὰς ἴσας τε καὶ παραλλήλους ἐπιζευγνύουσαι ἐπὶ τὰ αὐτὰ μέρη εὐθεῖαι ἴσαι τε καὶ παράλληλοί εἰσιν. ἡ ΠΥ ἄρα τῇ ΕΚ ἴση τε καὶ παράλληλός ἐστιν. πενταγώνου δὲ ἰσοπλεύρου ἡ ΕΚ· πενταγώνου ἄρα ἰσοπλεύρου καὶ ἡ ΠΥ τοῦ εἰς τὸν ΕΖΗΘΚ κύκλον ἐγγεγραμμένου. διὰ τὰ αὐτὰ δὴ καὶ ἑκάστη τῶν ΠΡ, ΡΣ, ΣΤ, ΤΥ πενταγώνου ἐστὶν ἰσοπλεύρου τοῦ εἰς τὸν ΕΖΗΘΚ κύκλον ἐγγεγραμμένου· ἰσόπλευρον ἄρα τὸ ΠΡΣΤΥ πεντάγωνον. καὶ ἐπεὶ ἐξαγώνου μὲν ἐστὶν ἡ ΠΕ, δεκαγώνου δὲ ἡ ΕΟ, καὶ ἐστὶν ὀρθὴ ἡ ὑπὸ ΠΕΟ, πενταγώνου ἄρα ἐστὶν ἡ ΠΟ· ἡ γὰρ τοῦ πενταγώνου πλευρὰ δύναται τὴν τε τοῦ ἐξαγώνου καὶ τὴν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγεγραμμένων. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΟΥ πενταγώνου ἐστὶ

Proposition 16

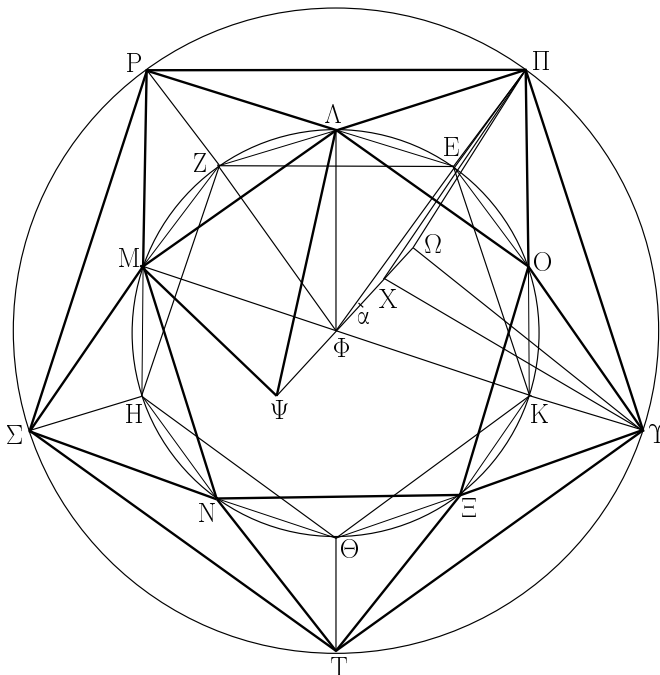
To construct an icosahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the icosahedron is that irrational (straight-line) called minor.



Let the diameter AB of the given sphere be laid out, and let it have been cut at C such that AC is four times CB [Prop. 6.10]. And let the semi-circle ADB have been drawn on AB . And let the straight-line CD have been drawn from C at right-angles to AB . And let DB have been joined. And let the circle $EFGHK$ be set down, and let its radius be equal to DB . And let the equilateral and equiangular pentagon $EFGHK$ have been inscribed in circle $EFGHK$ [Prop. 4.11]. And let the circumferences EF , FG , GH , HK , and KE have been cut in half at points L , M , N , O , and P (respectively). And let LM , MN , NO , OP , PL , and EP have been joined. Thus, pentagon $LMNOP$ is also equilateral, and EP (is) the side of the decagon (inscribed in the circle). And let the straight-lines EQ , FR , GS , HT , and KU , which are equal to the radius of circle $EFGHK$, have been set up at right-angles to the plane of the circle, at points E , F , G , H , and K (respectively). And let QR , RS , ST , TU , UQ , QL , LR , RM , MS , SN , NT , TO , OU , UP , and PQ have been joined.

And since EQ and KU are each at right-angles to the same plane, EQ is thus parallel to KU [Prop. 11.6]. And it is also equal to it. And straight-lines joining equal and parallel (straight-lines) on the same side are (themselves) equal and parallel [Prop. 1.33]. Thus, QU is equal and parallel to EK . And EK (is the side) of an equilateral pentagon (inscribed in circle $EFGHK$). Thus, QU (is) also the side of an equilateral pentagon inscribed in circle $EFGHK$. So, for the same (reasons), QR , RS , ST , and TU are also the sides of an equilateral pentagon inscribed in circle $EFGHK$. Pentagon $QRSTU$ (is) thus equilat-

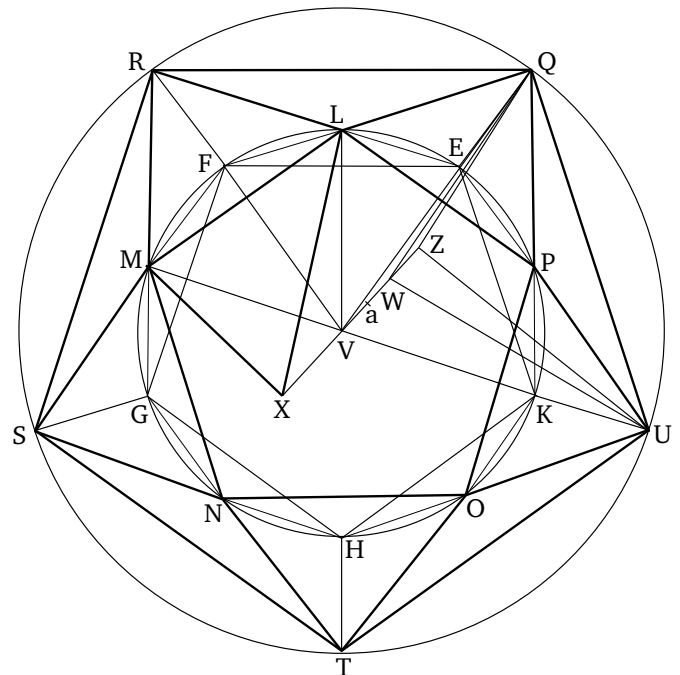
πλευρά. ἔστι δὲ καὶ ἡ ΠΥ πενταγώνου· ἰσόπλευρον ἄρα ἔστι τὸ ΠΟΥ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΠΑΡ, ΡΜΣ, ΣΝΤ, ΤΞΥ ἰσόπλευρόν ἐστιν. καὶ ἐπεὶ πενταγώνου ἐδείχθη ἑκατέρα τῶν ΠΛ, ΠΟ, ἔστι δὲ καὶ ἡ ΛΟ πενταγώνου, ἰσόπλευρον ἄρα ἔστι τὸ ΠΛΟ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΛΡΜ, ΜΣΝ, ΝΤΞ, ΞΥΟ τριγώνων ἰσόπλευρόν ἐστιν.



Εἰλήφθω τὸ κέντρον τοῦ ΕΖΗΘΚ κύκλου τὸ Φ σημεῖον· καὶ ἀπὸ τοῦ Φ τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἀνεστάντω ἡ ΦΩ, καὶ ἐκβεβλήσθω ἐπὶ τὰ ἔτερα μέρη ὡς ἡ ΦΨ, καὶ ἀφῆρησθω ἑξαγώνου μὲν ἡ ΦΧ, δεκαγώνου δὲ ἑκατέρα τῶν ΦΨ, ΧΩ, καὶ ἐπεξεύχθωσαν αἱ ΠΩ, ΠΧ, ΥΩ, ΕΦ, ΛΦ, ΛΨ, ΨΜ.

Καὶ ἐπεὶ ἑκατέρα τῶν ΦΧ, ΠΕ τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἐστίν, παράλληλος ἄρα ἐστὶν ἡ ΦΧ τῇ ΠΕ. εἰσὶ δὲ καὶ ἴσαι· καὶ αἱ ΕΦ, ΠΧ ἄρα ἴσαι τε καὶ παράλληλοί εἰσιν. ἑξαγώνου δὲ ἡ ΕΦ· ἑξαγώνου ἄρα καὶ ἡ ΠΧ. καὶ ἐπεὶ ἑξαγώνου μὲν ἐστὶν ἡ ΠΧ, δεκαγώνου δὲ ἡ ΧΩ, καὶ ὀρθὴ ἐστὶν ἡ ὑπὸ ΠΧΩ γωνία, πενταγώνου ἄρα ἐστὶν ἡ ΠΩ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΥΩ πενταγώνου ἐστίν, ἐπειδὴ περ,

eral. And side QE is (the side) of a hexagon (inscribed in circle $EFGHK$), and EP (the side) of a decagon, and (angle) QEP is a right-angle, thus QP is (the side) of a pentagon (inscribed in the same circle). For the square on the side of a pentagon is (equal to the sum of) the (squares) on (the sides of) a hexagon and a decagon inscribed in the same circle [Prop. 13.10]. So, for the same (reasons), PU is also the side of a pentagon. And QU is also (the side) of a pentagon. Thus, triangle QPU is equilateral. So, for the same (reasons), (triangles) QLR , RMS , SNT , and TOU are each also equilateral. And since QL and QP were each shown (to be the sides) of a pentagon, and LP is also (the side) of a pentagon, triangle QLP is thus equilateral. So, for the same (reasons), triangles LRM , MSN , NTO , and OUP are each also equilateral.



Let the center, point V , of circle $EFGHK$ have been found [Prop. 3.1]. And let VZ have been set up, at (point) V , at right-angles to the plane of the circle. And let it have been produced on the other side (of the circle), like VX . And let VW have been cut off (from XZ so as to be equal to the side) of a hexagon, and each of VX and WZ (so as to be equal to the side) of a decagon. And let QZ , QW , UZ , EV , LV , LX , and XM have been joined.

And since VW and QE are each at right-angles to the plane of the circle, VW is thus parallel to QE [Prop. 11.6]. And they are also equal. EV and QW are thus equal and parallel (to one another) [Prop. 1.33].

ἐὰν ἐπιζεύξωμεν τὰς ΦΚ, ΧΥ, ἴσαι καὶ ἀπεναντίον ἔσονται, καὶ ἔστιν ἡ ΦΚ ἐκ τοῦ κέντρου οὕσα ἐξαγώνου. ἐξαγώνου ἄρα καὶ ἡ ΧΥ. δεκαγώνου δὲ ἡ ΧΩ, καὶ ὀρθή ἡ ὑπὸ ΥΧΩ· πενταγώνου ἄρα ἡ ΥΩ. ἔστι δὲ καὶ ἡ ΠΥ πενταγώνου· ἰσόπλευρον ἄρα ἔστί τὸ ΠΥΩ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μὲν εἰσιν αἱ ΠΡ, ΡΣ, ΣΤ, ΤΥ εὐθεῖαι, κορυφή δὲ τὸ Ω σημεῖον, ἰσόπλευρόν ἐστιν. πάλιν, ἐπεὶ ἐξαγώνου μὲν ἡ ΦΛ, δεκαγώνου δὲ ἡ ΦΨ, καὶ ὀρθή ἔστιν ἡ ὑπὸ ΛΦΨ γωνία, πενταγώνου ἄρα ἔστιν ἡ ΛΨ. διὰ τὰ αὐτὰ δὴ ἐὰν ἐπιζεύξωμεν τὴν ΜΦ οὕσαν ἐξαγώνου, συνάγεται καὶ ἡ ΜΨ πενταγώνου. ἔστι δὲ καὶ ἡ ΑΜ πενταγώνου· ἰσόπλευρον ἄρα ἔστί τὸ ΑΜΨ τρίγωνον. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἕκαστον τῶν λοιπῶν τριγώνων, ὧν βάσεις μὲν εἰσιν αἱ ΜΝ, ΝΞ, ΞΟ, ΟΛ, κορυφή δὲ τὸ Ψ σημεῖον, ἰσόπλευρόν ἐστιν. συνέσταται ἄρα εἰκοσάεδρον ὑπὸ εἴκοσι τριγώνων ἰσοπλευρῶν περιεχόμενον.

Δεῖ δὴ αὐτὸ καὶ σφαῖρα περιλαβεῖν τῇ δοθείσῃ καὶ δεῖξαι, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάσσων.

Ἐπεὶ γὰρ ἐξαγώνου ἔστιν ἡ ΦΧ, δεκαγώνου δὲ ἡ ΧΩ, ἡ ΦΩ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Χ, καὶ τὸ μείζον αὐτῆς τμήμα ἐστιν ἡ ΦΧ· ἔστιν ἄρα ὡς ἡ ΩΦ πρὸς τὴν ΦΧ, οὕτως ἡ ΦΧ πρὸς τὴν ΧΩ. ἴση δὲ ἡ μὲν ΦΧ τῇ ΦΕ, ἡ δὲ ΧΩ τῇ ΦΨ· ἔστιν ἄρα ὡς ἡ ΩΦ πρὸς τὴν ΦΕ, οὕτως ἡ ΕΦ πρὸς τὴν ΦΨ. καὶ εἰσιν ὀρθαὶ αἱ ὑπὸ ΩΦΕ, ΕΦΨ γωνίαι· ἐὰν ἄρα ἐπιζεύξωμεν τὴν ΕΩ εὐθεῖαν, ὀρθή ἔσται ἡ ὑπὸ ΨΕΩ γωνία διὰ τὴν ὁμοιότητα τῶν ΨΕΩ, ΦΕΩ τριγώνων. διὰ τὰ αὐτὰ δὴ ἐπεὶ ἔστιν ὡς ἡ ΩΦ πρὸς τὴν ΦΧ, οὕτως ἡ ΦΧ πρὸς τὴν ΧΩ, ἴση δὲ ἡ μὲν ΩΦ τῇ ΨΧ, ἡ δὲ ΦΧ τῇ ΧΠ, ἔστιν ἄρα ὡς ἡ ΨΧ πρὸς τὴν ΧΠ, οὕτως ἡ ΠΧ πρὸς τὴν ΧΩ. καὶ διὰ τοῦτο πάλιν ἐὰν ἐπιζεύξωμεν τὴν ΠΨ, ὀρθή ἔσται ἡ πρὸς τῷ Π γωνία· τὸ ἄρα ἐπὶ τῆς ΨΩ γραφόμενον ἡμικύκλιον ἦξει καὶ διὰ τοῦ Π. καὶ ἐὰν μενούσης τῆς ΨΩ περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, ἦξει καὶ διὰ τοῦ Π καὶ τῶν λοιπῶν σημείων τοῦ εἰκοσαέδρου, καὶ ἔσται σφαῖρα περιειλημμένον τὸ εἰκοσάεδρον. λέγω δὴ, ὅτι καὶ τῇ δοθείσῃ. τετμησθῶ γὰρ ἡ ΦΧ δίχα κατὰ τὸ α. καὶ ἐπεὶ εὐθεῖα γραμμὴ ἡ ΦΩ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Χ, καὶ τὸ ἐλάσσον αὐτῆς τμήμα ἐστιν ἡ ΩΧ, ἡ ἄρα ΩΧ προσλαβοῦσα τὴν ἡμίσειαν τοῦ μείζονος τμήματος τὴν Χα πενταπλάσιον δύναται τοῦ ἀπὸ τῆς ἡμισείας τοῦ μείζονος τμήματος· πενταπλάσιον ἄρα ἔστί τὸ ἀπὸ τῆς Ωα τοῦ ἀπὸ τῆς αΧ. καὶ ἔστι τῆς μὲν Ωα διπλῆ ἡ ΩΨ, τῆς δὲ αΧ διπλῆ ἡ ΦΧ· πενταπλάσιον ἄρα ἔστί τὸ ἀπὸ τῆς ΩΨ τοῦ ἀπὸ τῆς ΧΦ. καὶ ἐπεὶ τετραπλῆ ἔστιν ἡ ΑΓ τῆς ΓΒ, πενταπλῆ ἄρα ἔστιν ἡ ΑΒ τῆς ΒΓ. ὡς δὲ ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως τὸ ἀπὸ τῆς ΑΒ πρὸς τὸ ἀπὸ τῆς ΒΔ· πενταπλάσιον ἄρα ἔστί τὸ ἀπὸ τῆς ΑΒ τοῦ ἀπὸ τῆς ΒΔ. ἐδείχθη δὲ καὶ τὸ ἀπὸ τῆς ΩΨ πενταπλάσιον τοῦ ἀπὸ τῆς ΦΧ. καὶ ἔστιν ἴση ἡ ΔΒ τῇ

And EV (is the side) of a hexagon. Thus, QW (is) also (the side) of a hexagon. And since QW is (the side) of a hexagon, and WZ (the side) of a decagon, and angle QWZ is a right-angle [Def. 11.3, Prop. 1.29], QZ is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), UZ is also (the side) of a pentagon—inasmuch as, if we join VK and WU then they will be equal and opposite. And VK , being (equal) to the radius (of the circle), is (the side) of a hexagon [Prop. 4.15 corr.]. Thus, WU (is) also the side of a hexagon. And WZ (is the side) of a decagon, and (angle) UWZ (is) a right-angle. Thus, UZ (is the side) of a pentagon [Prop. 13.10]. And QU is also (the side) of a pentagon. Triangle QUZ is thus equilateral. So, for the same (reasons), each of the remaining triangles, whose bases are the straight-lines QR , RS , ST , and TU , and apexes the point Z , are also equilateral. Again, since VL (is the side) of a hexagon, and VX (the side) of a decagon, and angle LVX is a right-angle, LX is thus (the side) of a pentagon [Prop. 13.10]. So, for the same (reasons), if we join MV , which is (the side) of a hexagon, MX is also inferred (to be the side) of a pentagon. And LM is also (the side) of a pentagon. Thus, triangle LMX is equilateral. So, similarly, it can be shown that each of the remaining triangles, whose bases are the (straight-lines) MN , NO , OP , and PL , and apexes the point X , are also equilateral. Thus, an icosahedron contained by twenty equilateral triangles has been constructed.

So, it is also necessary to enclose it in the given sphere, and to show that the side of the icosahedron is that irrational (straight-line) called minor.

For, since VW is (the side) of a hexagon, and WZ (the side) of a decagon, VZ has thus been cut in extreme and mean ratio at W , and VW is its greater piece [Prop. 13.9]. Thus, as ZV is to VW , so VW (is) to WZ . And VW (is) equal to VE , and WZ to VX . Thus, as ZV is to VE , so EV (is) to VX . And angles ZVE and EVX are right-angles. Thus, if we join straight-line EZ then angle XEZ will be a right-angle, on account of the similarity of triangles XEZ and VEZ . [Prop. 6.8]. So, for the same (reasons), since as ZV is to VW , so VW (is) to WZ , and ZV (is) equal to XW , and VW to WQ , thus as XW is to WQ , so QW (is) to WZ . And, again, on account of this, if we join QX then the angle at Q will be a right-angle [Prop. 6.8]. Thus, the semi-circle drawn on XZ will also pass through Q [Prop. 3.31]. And if XZ remains fixed, and the semi-circle is carried around, and again established at the same (position) from which it began to be moved, then it will also pass through (point) Q , and (through) the remaining (angular) points of the icosahedron. And the icosahedron will have been en-

ΦΧ· ἑκατέρα γὰρ αὐτῶν ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου· ἴση ἄρα καὶ ἡ ΑΒ τῇ ΨΩ. καὶ ἐστὶν ἡ ΑΒ ἡ τῆς δοθείσης σφαίρας διάμετρος· καὶ ἡ ΨΩ ἄρα ἴση ἐστὶ τῇ τῆς δοθείσης σφαίρας διαμέτρῳ· τῇ ἄρα δοθείσῃ σφαίρᾳ περιεῖληπται τὸ εἰκοσάεδρον.

Λέγω δὴ, ὅτι ἡ τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων. ἐπεὶ γὰρ ῥητὴ ἐστὶν ἡ τῆς σφαίρας διάμετρος, καὶ ἐστὶ δυνάμει πενταπλασίῳ τῆς ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου, ῥητὴ ἄρα ἐστὶ καὶ ἡ ἐκ τοῦ κέντρου τοῦ ΕΖΗΘΚ κύκλου· ὥστε καὶ ἡ διάμετρος αὐτοῦ ῥητὴ ἐστὶν. ἐὰν δὲ εἰς κύκλον ῥητὴν ἔχοντα τὴν διάμετρον πεντάγωνον ἰσόπλευρον ἐγγραφῇ, ἡ τοῦ πενταγώνου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων. ἡ δὲ τοῦ ΕΖΗΘΚ πενταγώνου πλευρὰ ἡ τοῦ εἰκοσαέδρου ἐστίν. ἡ ἄρα τοῦ εἰκοσαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἐλάττων.

closed by a sphere. So, I say that (it is) also (enclosed) by the given (sphere). For let VW have been cut in half at a . And since the straight-line VZ has been cut in extreme and mean ratio at W , and ZW is its lesser piece, then the square on ZW added to half of the greater piece, Wa , is five times the (square) on half of the greater piece [Prop. 13.3]. Thus, the (square) on Za is five times the (square) on aW . And ZX is double Za , and VW double aW . Thus, the (square) on ZX is five times the (square) on WV . And since AC is four times CB , AB is thus five times BC . And as AB (is) to BC , so the (square) on AB (is) to the (square) on BD [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is five times the (square) on BD . And the (square) on ZX was also shown (to be) five times the (square) on VW . And DB is equal to VW . For each of them is equal to the radius of circle $EFGHK$. Thus, AB (is) also equal to XZ . And AB is the diameter of the given sphere. Thus, XZ is equal to the diameter of the given sphere. Thus, the icosahedron has been enclosed by the given sphere.

So, I say that the side of the icosahedron is that irrational (straight-line) called minor. For since the diameter of the sphere is rational, and the square on it is five times the (square) on the radius of circle $EFGHK$, the radius of circle $EFGHK$ is thus also rational. Hence, its diameter is also rational. And if an equilateral pentagon is inscribed in a circle having a rational diameter then the side of the pentagon is that irrational (straight-line) called minor [Prop. 13.11]. And the side of pentagon $EFGHK$ is (the side) of the icosahedron. Thus, the side of the icosahedron is that irrational (straight-line) called minor.

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει πενταπλασίῳ ἐστὶ τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ἀφ' οὗ τὸ εἰκοσάεδρον ἀναγέγραπται, καὶ ὅτι ἡ τῆς σφαίρας διάμετρος σύγκειται ἐκ τε τῆς τοῦ ἑξαγώνου καὶ δύο τῶν τοῦ δεκαγώνου τῶν εἰς τὸν αὐτὸν κύκλον ἐγγραφομένων. ὅπερ ἔδει δεῖξαι.

Corollary

So, (it is) clear, from this, that the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described, and that the diameter of the sphere is the sum of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the same circle.[†]

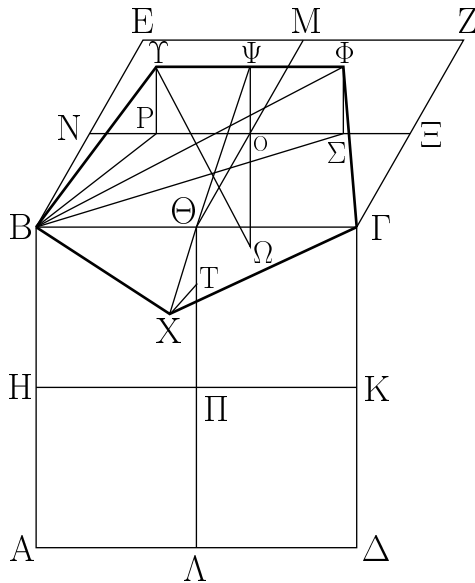
[†] If the radius of the sphere is unity then the radius of the circle is $2/\sqrt{5}$, and the sides of the hexagon, decagon, and pentagon/icosahedron are $2/\sqrt{5}$, $1 - 1/\sqrt{5}$, and $(1/\sqrt{5})\sqrt{10 - 2\sqrt{5}}$, respectively.

ιζ'.

Δωδεκάεδρον συστήσασθαι καὶ σφαίρᾳ περιλαβεῖν, ἥ καὶ τὰ προειρημένα σχήματα, καὶ δεῖξαι, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

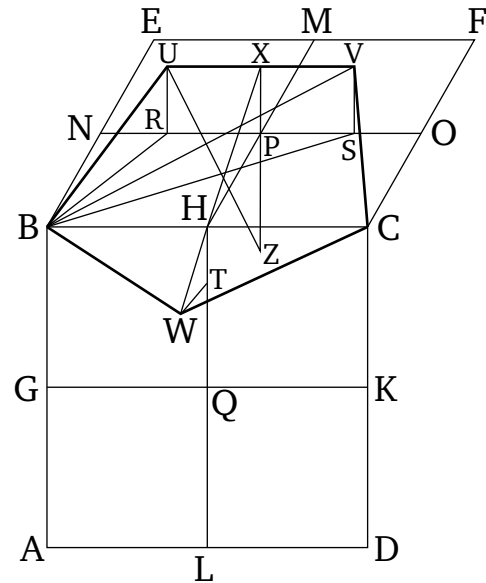
Proposition 17

To construct a dodecahedron, and to enclose (it) in a sphere, like the aforementioned figures, and to show that the side of the dodecahedron is that irrational (straight-line) called an apotome.



Ἐκκείσθωσαν τοῦ προειρημένου κύβου δύο ἐπίπεδα πρὸς ὀρθὰς ἀλλήλοις τὰ $AB\Gamma\Delta$, ΓBEZ , καὶ τετμήσθω ἑκάστη τῶν AB , $B\Gamma$, $\Gamma\Delta$, ΔA , EZ , EB , $Z\Gamma$ πλευρῶν δίχα κατὰ τὰ H , Θ , K , Λ , M , N , Ξ , καὶ ἐπεζεύχθωσαν αἱ HK , $\Theta\Lambda$, $M\Theta$, $N\Xi$, καὶ τετμήσθω ἑκάστη τῶν NO , $O\Xi$, $\Theta\Pi$ ἄκρον καὶ μέσον λόγον κατὰ τὰ P , Σ , T σημεῖα, καὶ ἔστω αὐτῶν μείζονα τμήματα τὰ PO , $O\Sigma$, $T\Pi$, καὶ ἀνεστάτωσαν ἀπὸ τῶν P , Σ , T σημείων τοῖς τοῦ κύβου ἐπιπέδοις πρὸς ὀρθὰς ἐπὶ τὰ ἐκτὸς μέρη τοῦ κύβου αἱ PY , $\Sigma\Phi$, TX , καὶ κείσθωσαν ἴσαι ταῖς PO , $O\Sigma$, $T\Pi$, καὶ ἐπεζεύχθωσαν αἱ YB , BX , $X\Gamma$, $\Gamma\Phi$, ΦY .

Λέγω, ὅτι τὸ $YBX\Gamma\Phi$ πεντάγωνον ἰσόπλευρόν τε καὶ ἐν ἐνὶ ἐπιπέδῳ καὶ ἔτι ἰσογώνιον ἐστίν. ἐπεζεύχθωσαν γὰρ αἱ PB , ΣB , ΦB . καὶ ἐπεὶ εὐθεῖα ἡ NO ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ P , καὶ τὸ μείζον τμήμα ἐστὶν ἡ PO , τὰ ἄρα ἀπὸ τῶν ON , NP τριπλάσια ἐστὶ τοῦ ἀπὸ τῆς PO . ἴση δὲ ἡ μὲν ON τῇ NB , ἡ δὲ OP τῇ PY . τὰ ἄρα ἀπὸ τῶν BN , NP τριπλάσια ἐστὶ τοῦ ἀπὸ τῆς PY . τοῖς δὲ ἀπὸ τῶν BN , NP τὸ ἀπὸ τῆς BP ἐστὶν ἴσον· τὸ ἄρα ἀπὸ τῆς BP τριπλάσιόν ἐστὶ τοῦ ἀπὸ τῆς PY . ὥστε τὰ ἀπὸ τῶν BP , PY τετραπλάσια ἐστὶ τοῦ ἀπὸ τῆς PY . τοῖς δὲ ἀπὸ τῶν BP , PY ἴσον ἐστὶ τὸ ἀπὸ τῆς BY . τὸ ἄρα ἀπὸ τῆς BY τετραπλάσιόν ἐστὶ τοῦ ἀπὸ τῆς YP . διπλῇ ἄρα ἐστὶν ἡ BY τῆς PY . ἐστὶ δὲ καὶ ἡ ΦY τῆς YP διπλῇ, ἐπειδὴ περ καὶ ἡ ΣP τῆς OP , τουτέστι τῆς PY , ἐστὶ διπλῇ· ἴση ἄρα ἡ BY τῇ $Y\Phi$. ὁμοίως δὲ δεῖχθήσεται, ὅτι καὶ ἑκάστη τῶν BX , $X\Gamma$, $\Gamma\Phi$ ἑκατέρω τῶν BY , $Y\Phi$ ἐστὶν ἴση. ἰσόπλευρον ἄρα ἐστὶ τὸ $YBX\Gamma\Phi$ πεντάγωνον. λέγω δὴ, ὅτι καὶ ἐν ἐνὶ ἐστὶν ἐπιπέδῳ. ἤχθω γὰρ ἀπὸ τοῦ O ἑκατέρω τῶν PY , $\Sigma\Phi$ παράλληλος ἐπὶ τὰ ἐκτὸς τοῦ κύβου μέρη ἡ $O\Psi$, καὶ ἐπεζεύχθωσαν αἱ $\Psi\Theta$, ΘX . λέγω, ὅτι ἡ $\Psi\Theta X$ εὐθεῖα ἐστίν. ἐπεὶ γὰρ ἡ $\Theta\Pi$ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ T , καὶ τὸ μείζον αὐτῆς τμήμα ἐστὶν ἡ $T\Pi$, ἔστιν ἄρα ὡς ἡ $\Theta\Pi$ πρὸς τὴν $T\Pi$, οὕτως ἡ $T\Pi$ πρὸς τὴν



Let two planes of the aforementioned cube [Prop. 13.15], $ABCD$ and $CBEF$, (which are) at right-angles to one another, be laid out. And let the sides AB , BC , CD , DA , EF , EB , and FC have each been cut in half at points G , H , K , L , M , N , and O (respectively). And let GK , HL , MH , and NO have been joined. And let NP , PO , and HQ have each been cut in extreme and mean ratio at points R , S , and T (respectively). And let their greater pieces be RP , PS , and TQ (respectively). And let RU , SV , and TW have been set up on the exterior side of the cube, at points R , S , and T (respectively), at right-angles to the planes of the cube. And let them be made equal to RP , PS , and TQ . And let UB , BW , WC , CV , and VU have been joined.

I say that the pentagon $UBWCV$ is equilateral, and in one plane, and, further, equiangular. For let RB , SB , and VB have been joined. And since the straight-line NP has been cut in extreme and mean ratio at R , and RP is the greater piece, the (sum of the squares) on PN and NR is thus three times the (square) on RP [Prop. 13.4]. And PN (is) equal to NB , and PR to RU . Thus, the (sum of the squares) on BN and NR is three times the (square) on RU . And the (square) on BR is equal to the (sum of the squares) on BN and NR [Prop. 1.47]. Thus, the (square) on BR is three times the (square) on RU . Hence, the (sum of the squares) on BR and RU is four times the (square) on RU . And the (square) on BU is equal to the (sum of the squares) on BR and RU [Prop. 1.47]. Thus, the (square) on BU is four times the (square) on RU . Thus, BU is double RU . And VU is also double UR , inasmuch as SR is also double PR —that is to say, RU . Thus, BU (is) equal to UV . So, similarly, it can be shown that each of BW , WC , CV is equal to each

ΤΘ. ἴση δὲ ἡ μὲν ΘΠ τῇ ΘΟ, ἡ δὲ ΠΤ ἑκατέρᾳ τῶν ΤΧ, ΟΨ· ἔστιν ἄρα ὡς ἡ ΘΟ πρὸς τὴν ΟΨ, οὕτως ἡ ΧΤ πρὸς τὴν ΤΘ. καὶ ἔστι παράλληλος ἡ μὲν ΘΟ τῇ ΤΧ· ἑκατέρα γὰρ αὐτῶν τῷ ΒΔ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν· ἡ δὲ ΤΘ τῇ ΟΨ· ἑκατέρα γὰρ αὐτῶν τῷ ΒΖ ἐπιπέδῳ πρὸς ὀρθάς ἐστιν. ἐὰν δὲ δύο τρίγωνα συντεθῇ κατὰ μίαν γωνίαν, ὡς τὰ ΨΟΘ, ΘΤΧ, τὰς δύο πλευρὰς ταῖς δυὸν ἀνάλογον ἔχοντα, ὥστε τὰς ὁμολόγους αὐτῶν πλευρὰς καὶ παραλλήλους εἶναι, αἱ λοιπαὶ εὐθεῖαι ἐπ' εὐθείας ἔσσονται· ἐπ' εὐθείας ἄρα ἐστὶν ἡ ΨΘ τῇ ΘΧ. πᾶσα δὲ εὐθεῖα ἐν ἐνὶ ἐστὶν ἐπιπέδῳ· ἐν ἐνὶ ἄρα ἐπιπέδῳ ἐστὶ τὸ ΥΒΧΓΦ πεντάγωνον.

Λέγω δὴ, ὅτι καὶ ἰσογώνιον ἐστίν.

Ἐπεὶ γὰρ εὐθεῖα γραμμὴ ἡ ΝΟ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ρ, καὶ τὸ μείζον τμήμα ἐστὶν ἡ ΟΡ [ἔστιν ἄρα ὡς συναμφοτέρος ἡ ΝΟ, ΟΡ πρὸς τὴν ΟΝ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΡ], ἴση δὲ ἡ ΟΡ τῇ ΟΣ [ἔστιν ἄρα ὡς ἡ ΣΝ πρὸς τὴν ΝΟ, οὕτως ἡ ΝΟ πρὸς τὴν ΟΣ], ἡ ΝΣ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ο, καὶ τὸ μείζον τμήμα ἐστὶν ἡ ΝΟ· τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΟ τριπλάσια ἐστὶ τοῦ ἀπὸ τῆς ΝΟ. ἴση δὲ ἡ μὲν ΝΟ τῇ ΝΒ, ἡ δὲ ΟΣ τῇ ΣΦ· τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΦ τετράγωνα τριπλάσια ἐστὶ τοῦ ἀπὸ τῆς ΝΒ· ὥστε τὰ ἀπὸ τῶν ΦΣ, ΣΝ, ΝΒ τετραπλάσια ἐστὶ τοῦ ἀπὸ τῆς ΝΒ. τοῖς δὲ ἀπὸ τῶν ΣΝ, ΝΒ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΣΒ· τὰ ἄρα ἀπὸ τῶν ΒΣ, ΣΦ, τουτέστι τὸ ἀπὸ τῆς ΒΦ [ὀρθὴ γὰρ ἡ ὑπὸ ΦΣΒ γωνία], τετραπλάσιόν ἐστὶ τοῦ ἀπὸ τῆς ΝΒ· διπλῇ ἄρα ἐστὶν ἡ ΦΒ τῆς ΒΝ. ἔστι δὲ καὶ ἡ ΒΓ τῆς ΒΝ διπλῇ· ἴση ἄρα ἐστὶν ἡ ΒΦ τῇ ΒΓ. καὶ ἐπεὶ δύο αἱ ΒΥ, ΥΦ δυοὶ ταῖς ΒΧ, ΧΓ ἴσαι εἰσίν, καὶ βάσις ἡ ΒΦ βάσει τῇ ΒΓ ἴση, γωνία ἄρα ἡ ὑπὸ ΒΥΦ γωνία τῇ ὑπὸ ΒΧΓ ἐστὶν ἴση. ὁμοίως δὲ δεῖξομεν, ὅτι καὶ ἡ ὑπὸ ΥΦΓ γωνία ἴση ἐστὶ τῇ ὑπὸ ΒΧΓ· αἱ ἄρα ὑπὸ ΒΧΓ, ΒΥΦ, ΥΦΓ τρεῖς γωνίαι ἴσαι ἀλλήλαις εἰσίν. ἐὰν δὲ πενταγώνου ἰσοπλευροῦ αἱ τρεῖς γωνίαι ἴσαι ἀλλήλαις ὦσιν, ἰσογώνιον ἔσται τὸ πεντάγωνον· ἰσογώνιον ἄρα ἐστὶ τὸ ΒΥΦΓΧ πεντάγωνον. ἐδείχθη δὲ καὶ ἰσόπλευρον· τὸ ἄρα ΒΥΦΓΧ πεντάγωνον ἰσόπλευρόν ἐστι καὶ ἰσογώνιον, καὶ ἐστὶν ἐπὶ μιᾷ τοῦ κύβου πλευρᾶς τῆς ΒΓ. ἐὰν ἄρα ἐφ' ἑκάστης τῶν τοῦ κύβου δώδεκα πλευρῶν τὰ αὐτὰ κατασκευάσωμεν, συσταθήσεται τι σχῆμα στερεὸν ὑπὸ δώδεκα πενταγόνων ἰσοπλευρῶν τε καὶ ἰσογώνιων περιεχόμενον, ὃ καλεῖται δωδεκάεδρον.

Δεῖ δὴ αὐτὸ καὶ σφαῖρα περιλαβεῖν τῇ δοθείῃ καὶ δεῖξαι, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐκβεβλήσθω γὰρ ἡ ΨΟ, καὶ ἔστω ἡ ΨΩ· συμβάλλει ἄρα ἡ ΟΩ τῇ τοῦ κύβου διαμέτρῳ, καὶ δίχα τέμνουσιν ἀλλήλας· τοῦτο γὰρ δέδεικται ἐν τῷ παρατελείτῳ θεωρηματι τοῦ ἐνδεκάτου βιβλίου. τεμνέτωσαν κατὰ τὸ Ω· τὸ Ω ἄρα κέντρον ἐστὶ τῆς σφαίρας τῆς περιλαμβανούσης τὸν κύβον, καὶ ἡ ΩΟ ἡμίσεια τῆς πλευρᾶς τοῦ κύβου. ἐπεζεύχθω δὲ ἡ ΥΩ. καὶ ἐπεὶ εὐθεῖα γραμμὴ ἡ ΝΣ ἄκρον καὶ μέσον λόγον τέτμηται κατὰ τὸ Ο, καὶ τὸ μείζον αὐτῆς τμήμα ἐστὶν ἡ ΝΟ,

of BU and UV . Thus, pentagon $BUVCW$ is equilateral. So, I say that it is also in one plane. For let PX have been drawn from P , parallel to each of RU and SV , on the exterior side of the cube. And let XH and HW have been joined. I say that XHW is a straight-line. For since HQ has been cut in extreme and mean ratio at T , and QT is its greater piece, thus as HQ is to QT , so QT (is) to TH . And HQ (is) equal to HP , and QT to each of TW and PX . Thus, as HP is to PX , so WT (is) to TH . And HP is parallel to TW . For of each of them is at right-angles to the plane BD [Prop. 11.6]. And TH (is parallel) to PX . For each of them is at right-angles to the plane BF [Prop. 11.6]. And if two triangles, like XPH and HTW , having two sides proportional to two sides, are placed together at a single angle such that their corresponding sides are also parallel then the remaining sides will be straight-on (to one another) [Prop. 6.32]. Thus, XH is straight-on to HW . And every straight-line is in one plane [Prop. 11.1]. Thus, pentagon $UBWCV$ is in one plane.

So, I say that it is also equiangular.

For since the straight-line NP has been cut in extreme and mean ratio at R , and PR is the greater piece [thus as the sum of NP and PR is to PN , so NP (is) to PR], and PR (is) equal to PS [thus as SN is to NP , so NP (is) to PS], NS has thus also been cut in extreme and mean ratio at P , and NP is the greater piece [Prop. 13.5]. Thus, the (sum of the squares) on NS and SP is three times the (square) on NP [Prop. 13.4]. And NP (is) equal to NB , and PS to SV . Thus, the (sum of the) squares on NS and SV is three times the (square) on NB . Hence, the (sum of the squares) on VS , SN , and NB is four times the (square) on NB . And the (square) on SB is equal to the (sum of the squares) on SN and NB [Prop. 1.47]. Thus, the (sum of the squares) on BS and SV —that is to say, the (square) on BV [for angle $VSΒ$ (is) a right-angle]—is four times the (square) on NB [Def. 11.3, Prop. 1.47]. Thus, VB is double BN . And BC (is) also double BN . Thus, BV is equal to BC . And since the two (straight-lines) BU and UV are equal to the two (straight-lines) BW and WC (respectively), and the base BV (is) equal to the base BC , angle BUV is thus equal to angle BWC [Prop. 1.8]. So, similarly, we can show that angle UVC is equal to angle BWC . Thus, the three angles BWC , BUV , and UVC are equal to one another. And if three angles of an equilateral pentagon are equal to one another then the pentagon is equiangular [Prop. 13.7]. Thus, pentagon $BUVCW$ is equiangular. And it was also shown (to be) equilateral. Thus, pentagon $BUVCW$ is equilateral and equiangular, and it is on one of the sides, BC , of the cube. Thus, if we make the

τὰ ἄρα ἀπὸ τῶν ΝΣ, ΣΟ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἴση δὲ ἡ μὲν ΝΣ τῇ ΨΩ, ἐπειδὴ περ καὶ ἡ μὲν ΝΟ τῇ ΟΩ ἐστὶν ἴση, ἡ δὲ ΨΟ τῇ ΟΣ. ἀλλὰ μὴν καὶ ἡ ΟΣ τῇ ΨΥ, ἐπεὶ καὶ τῇ ΡΟ· τὰ ἄρα ἀπὸ τῶν ΩΨ, ΨΥ τριπλάσιά ἐστι τοῦ ἀπὸ τῆς ΝΟ. τοῖς δὲ ἀπὸ τῶν ΩΨ, ΨΥ ἴσον ἐστὶ τὸ ἀπὸ τῆς ΥΩ· τὸ ἄρα ἀπὸ τῆς ΥΩ τριπλάσιόν ἐστι τοῦ ἀπὸ τῆς ΝΟ. ἐστὶ δὲ καὶ ἡ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περιλαμβανοῦσης τὸν κύβον δυνάμει τριπλασίον τῆς ἡμισείας τῆς τοῦ κύβου πλευρᾶς· προδεδεικται γὰρ κύβον συστήσασθαι καὶ σφαίρα περιλαβεῖν καὶ δεῖξαι, ὅτι ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίον ἐστὶ τῆς πλευρᾶς τοῦ κύβου. εἰ δὲ ὅλη τῆς ὅλης, καὶ [ἡ] ἡμίσεια τῆς ἡμισείας· καὶ ἐστὶν ἡ ΝΟ ἡμίσεια τῆς τοῦ κύβου πλευρᾶς· ἡ ἄρα ΥΩ ἴση ἐστὶ τῇ ἐκ τοῦ κέντρου τῆς σφαίρας τῆς περιλαμβανοῦσης τὸν κύβον. καὶ ἐστὶ τὸ Ω κέντρον τῆς σφαίρας τῆς περιλαμβανοῦσης τὸν κύβον· τὸ Υ ἄρα σημεῖον πρὸς τῇ ἐπιφανείᾳ ἐστὶ τῆς σφαίρας. ὁμοίως δὲ δεῖξομεν, ὅτι καὶ ἐκάστη τῶν λοιπῶν γωνιῶν τοῦ δωδεκαέδρου πρὸς τῇ ἐπιφανείᾳ ἐστὶ τῆς σφαίρας· περιεῖληπται ἄρα τὸ δωδεκαέδρον τῇ δοθείσῃ σφαίρᾳ.

Λέγω δὴ, ὅτι ἡ τοῦ δωδεκαέδρου πλευρὰ ἄλογός ἐστιν ἡ καλουμένη ἀποτομή.

Ἐπεὶ γὰρ τῆς ΝΟ ἄκρον καὶ μέσον λόγον τετμημένης τὸ μείζον τμημὰ ἐστὶν ὁ ΡΟ, τῆς δὲ ΟΞ ἄκρον καὶ μέσον λόγον τετμημένης τὸ μείζον τμημὰ ἐστὶν ἡ ΟΣ, ὅλης ἄρα τῆς ΝΞ ἄκρον καὶ μέσον λόγον τετμημένης τὸ μείζον τμημὰ ἐστὶν ἡ ΡΣ. [οἷον ἐπεὶ ἐστὶν ὡς ἡ ΝΟ πρὸς τὴν ΟΡ, ἡ ΟΡ πρὸς τὴν ΡΝ, καὶ τὰ διπλάσια· τὰ γὰρ μέρη τοῖς ἰσάκεις πολλαπλασίοις τὸν αὐτὸν ἔχει λόγον· ὡς ἄρα ἡ ΝΞ πρὸς τὴν ΡΣ, οὕτως ἡ ΡΣ πρὸς συναμφοτέρον τὴν ΝΡ, ΣΞ. μείζων δὲ ἡ ΝΞ τῆς ΡΣ· μείζων ἄρα καὶ ἡ ΡΣ συναμφοτέρου τῆς ΝΡ, ΣΞ· ἡ ΝΞ ἄρα ἄκρον καὶ μέσον λόγον τέτμηται, καὶ τὸ μείζον αὐτῆς τμημὰ ἐστὶν ἡ ΡΣ.] ἴση δὲ ἡ ΡΣ τῇ ΥΦ· τῆς ἄρα ΝΞ ἄκρον καὶ μέσον λόγον τετμημένης τὸ μείζον τμημὰ ἐστὶν ἡ ΥΦ. καὶ ἐπεὶ ῥητὴ ἐστὶν τῆς σφαίρας διάμετρος καὶ ἐστὶ δυνάμει τριπλασίον τῆς τοῦ κύβου πλευρᾶς, ῥητὴ ἄρα ἐστὶν ἡ ΝΞ πλευρὰ οὔσα τοῦ κύβου. ἐὰν δὲ ῥητὴ γραμμὴ ἄκρον καὶ μέσον λόγον τμηθῇ, ἐκάτερον τῶν τμημάτων ἄλογός ἐστιν ἀποτομή.

Ἡ ΥΦ ἄρα πλευρὰ οὔσα τοῦ δωδεκαέδρου ἄλογός ἐστιν ἀποτομή.

same construction on each of the twelve sides of the cube then some solid figure contained by twelve equilateral and equiangular pentagons will have been constructed, which is called a dodecahedron.

So, it is necessary to enclose it in the given sphere, and to show that the side of the dodecahedron is that irrational (straight-line) called an apotome.

For let XP have been produced, and let (the produced straight-line) be XZ . Thus, PZ meets the diameter of the cube, and they cut one another in half. For, this has been proved in the penultimate theorem of the eleventh book [Prop. 11.38]. Let them cut (one another) at Z . Thus, Z is the center of the sphere enclosing the cube, and ZP (is) half the side of the cube. So, let UZ have been joined. And since the straight-line NS has been cut in extreme and mean ratio at P , and its greater piece is NP , the (sum of the squares) on NS and SP is thus three times the (square) on NP [Prop. 13.4]. And NS (is) equal to XZ , inasmuch as NP is also equal to PZ , and XP to PS . But, indeed, PS (is) also (equal) to XU , since (it is) also (equal) to RP . Thus, the (sum of the squares) on ZX and XU is three times the (square) on NP . And the (square) on UZ is equal to the (sum of the squares) on ZX and XU [Prop. 1.47]. Thus, the (square) on UZ is three times the (square) on NP . And the square on the radius of the sphere enclosing the cube is also three times the (square) on half the side of the cube. For it has previously been demonstrated (how to) construct the cube, and to enclose (it) in a sphere, and to show that the square on the diameter of the sphere is three times the (square) on the side of the cube [Prop. 13.15]. And if the (square on the) whole (is three times) the (square on the) whole, then the (square on the) half (is) also (three times) the (square on the) half. And NP is half of the side of the cube. Thus, UZ is equal to the radius of the sphere enclosing the cube. And Z is the center of the sphere enclosing the cube. Thus, point U is on the surface of the sphere. So, similarly, we can show that each of the remaining angles of the dodecahedron is also on the surface of the sphere. Thus, the dodecahedron has been enclosed by the given sphere.

So, I say that the side of the dodecahedron is that irrational straight-line called an apotome.

For since RP is the greater piece of NP , which has been cut in extreme and mean ratio, and PS is the greater piece of PO , which has been cut in extreme and mean ratio, RS is thus the greater piece of the whole of NO , which has been cut in extreme and mean ratio. [Thus, since as NP is to PR , (so) PR (is) to RN , and (the same is also true) of the doubles. For parts have the same ratio as similar multiples (taken in corresponding

order) [Prop. 5.15]. Thus, as NO (is) to RS , so RS (is) to the sum of NR and SO . And NO (is) greater than RS . Thus, RS (is) also greater than the sum of NR and SO [Prop. 5.14]. Thus, NO has been cut in extreme and mean ratio, and RS is its greater piece.] And RS (is) equal to UV . Thus, UV is the greater piece of NO , which has been cut in extreme and mean ratio. And since the diameter of the sphere is rational, and the square on it is three times the (square) on the side of the cube, NO , which is the side of the cube, is thus rational. And if a rational (straight)-line is cut in extreme and mean ratio then each of the pieces is the irrational (straight-line called) an apotome.

Thus, UV , which is the side of the dodecahedron, is the irrational (straight-line called) an apotome [Prop. 13.6].

Πόρισμα.

Ἐκ δὴ τούτου φανερόν, ὅτι τῆς τοῦ κύβου πλευρᾶς ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζον τμημᾶ ἐστὶν ἡ τοῦ δωδεκαέδρου πλευρά. ὅπερ ἔδει δεῖξαι.

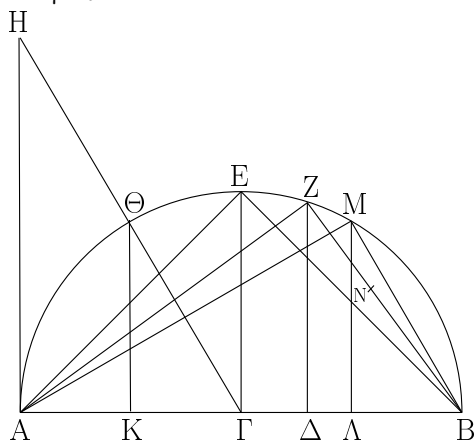
Corollary

So, (it is) clear, from this, that the side of the dodecahedron is the greater piece of the side of the cube, when it is cut in extreme and mean ratio.[†] (Which is) the very thing it was required to show.

[†] If the radius of the circumscribed sphere is unity then the side of the cube is $\sqrt{4/3}$, and the side of the dodecahedron is $(1/3)(\sqrt{15} - \sqrt{3})$.

ကေ'.

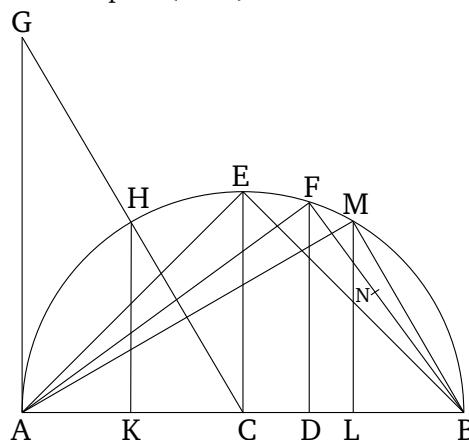
Τὰς πλευρὰς τῶν πέντε σχημάτων ἐκθέσθαι καὶ συγκρί-
ναι πρὸς ἀλλήλας.



Ἐκκείσθω ἡ τῆς δοθείσης σφαίρας διάμετρος ἡ ΑΒ, καὶ τεμνῆσθω κατὰ τὸ Γ ὥστε ἴσῃν εἶναι τὴν ΑΓ τῇ ΓΒ, κατὰ δὲ τὸ Δ ὥστε διπλασίονα εἶναι τὴν ΑΔ τῆς ΔΒ, καὶ γεγράφθω ἐπὶ τῆς ΑΒ ἡμικύκλιον τὸ ΑΕΒ, καὶ ἀπὸ τῶν Γ, Δ τῇ ΑΒ πρὸς ὀρθὰς ἤχθωσαν αἱ ΓΕ, ΔΖ, καὶ ἐπεζεύχθωσαν αἱ ΑΖ, ΖΒ, ΕΒ. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ ΑΔ τῆς ΔΒ, τριπλῇ ἄρα ἐστὶν ἡ ΑΒ τῆς ΒΔ. ἀναστρέφαντι ἡμιολία ἄρα ἐστὶν ἡ ΒΑ τῆς ΑΔ. ὥς δὲ ἡ ΒΑ πρὸς τὴν ΑΔ, οὕτως τὸ ἀπὸ τῆς ΒΑ

Proposition 18

To set out the sides of the five (aforementioned) figures, and to compare (them) with one another.[†]



Let the diameter, AB , of the given sphere be laid out. And let it have been cut at C , such that AC is equal to CB , and at D , such that AD is double DB . And let the semi-circle AEB have been drawn on AB . And let CE and DF have been drawn from C and D (respectively), at right-angles to AB . And let AF , FB , and EB have been joined. And since AD is double DB , AB is thus triple BD . Thus, via conversion, BA is one and a half

πρὸς τὸ ἀπὸ τῆς AZ · ἰσογώνιον γάρ ἐστι τὸ AZB τρίγωνον τῷ $AZ\Delta$ τριγώνῳ· ἡμιόλιον ἄρα ἐστὶ τὸ ἀπὸ τῆς BA τοῦ ἀπὸ τῆς AZ . ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει ἡμιολία τῆς πλευρᾶς τῆς πυραμίδος. καὶ ἐστὶν ἡ AB ἡ τῆς σφαίρας διάμετρος· ἡ AZ ἄρα ἴση ἐστὶ τῇ πλευρᾷ τῆς πυραμίδος.

Πάλιν, ἐπεὶ διπλασίον ἐστὶν ἡ AD τῆς ΔB , τριπλῇ ἄρα ἐστὶν ἡ AB τῆς $B\Delta$. ὥς δὲ ἡ AB πρὸς τὴν $B\Delta$, οὕτως τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς BZ · τριπλασίον ἄρα ἐστὶ τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς BZ . ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει τριπλασίον τῆς τοῦ κύβου πλευρᾶς. καὶ ἐστὶν ἡ AB ἡ τῆς σφαίρας διάμετρος· ἡ BZ ἄρα τοῦ κύβου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἴση ἐστὶν ἡ AG τῇ GB , διπλῇ ἄρα ἐστὶν ἡ AB τῆς BG . ὥς δὲ ἡ AB πρὸς τὴν BG , οὕτως τὸ ἀπὸ τῆς AB πρὸς τὸ ἀπὸ τῆς BE · διπλασίον ἄρα ἐστὶ τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς BE . ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει διπλασίον τῆς τοῦ ὀκταέδρου πλευρᾶς. καὶ ἐστὶν ἡ AB ἡ τῆς δοθείσης σφαίρας διάμετρος· ἡ BE ἄρα τοῦ ὀκταέδρου ἐστὶ πλευρά.

Ἦχθω δὲ ἀπὸ τοῦ A σημείου τῇ AB εὐθείᾳ πρὸς ὀρθὰς ἡ AH , καὶ κείσθω ἡ AH ἴση τῇ AB , καὶ ἐπεζεύχθω ἡ $H\Gamma$, καὶ ἀπὸ τοῦ Θ ἐπὶ τὴν AB κάθετος ἡ $\chi\theta\omega$ ἡ ΘK . καὶ ἐπεὶ διπλῇ ἐστὶν ἡ HA τῆς AG · ἴση γὰρ ἡ HA τῇ AB · ὥς δὲ ἡ HA πρὸς τὴν AG , οὕτως ἡ ΘK πρὸς τὴν $K\Gamma$, διπλῇ ἄρα καὶ ἡ ΘK τῆς $K\Gamma$. τετραπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς ΘK τοῦ ἀπὸ τῆς $K\Gamma$ · τὰ ἄρα ἀπὸ τῶν ΘK , $K\Gamma$, ὅπερ ἐστὶ τὸ ἀπὸ τῆς $\Theta\Gamma$, πενταπλάσιον ἐστὶ τοῦ ἀπὸ τῆς $K\Gamma$. ἴση δὲ ἡ $\Theta\Gamma$ τῇ GB · πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς $B\Gamma$ τοῦ ἀπὸ τῆς $ΓK$. καὶ ἐπεὶ διπλῇ ἐστὶν ἡ AB τῆς GB , ὦν ἡ AD τῆς ΔB ἐστὶ διπλῇ, λοιπὴ ἄρα ἡ $B\Delta$ λοιπῆς τῆς $\Delta\Gamma$ ἐστὶ διπλῇ. τριπλῇ ἄρα ἡ $B\Gamma$ τῆς $\Gamma\Delta$ · ἐνναπλάσιον ἄρα τὸ ἀπὸ τῆς $B\Gamma$ τοῦ ἀπὸ τῆς $\Gamma\Delta$. πενταπλάσιον δὲ τὸ ἀπὸ τῆς $B\Gamma$ τοῦ ἀπὸ τῆς ΓK · μείζον ἄρα τὸ ἀπὸ τῆς ΓK τοῦ ἀπὸ τῆς $\Gamma\Delta$. μείζων ἄρα ἐστὶν ἡ ΓK τῆς $\Gamma\Delta$. κείσθω τῇ ΓK ἴση ἡ $\Gamma\Lambda$, καὶ ἀπὸ τοῦ Λ τῇ AB πρὸς ὀρθὰς ἡ $\chi\theta\omega$ ἡ ΛM , καὶ ἐπεζεύχθω ἡ MB . καὶ ἐπεὶ πενταπλάσιον ἐστὶ τὸ ἀπὸ τῆς $B\Gamma$ τοῦ ἀπὸ τῆς ΓK , καὶ ἐστὶ τῆς μὲν $B\Gamma$ διπλῇ ἡ AB , τῆς δὲ ΓK διπλῇ ἡ $K\Lambda$, πενταπλάσιον ἄρα ἐστὶ τὸ ἀπὸ τῆς AB τοῦ ἀπὸ τῆς $K\Lambda$. ἔστι δὲ καὶ ἡ τῆς σφαίρας διάμετρος δυνάμει πενταπλάσιον τῆς ἐκ τοῦ κέντρου τοῦ κύκλου, ἀφ' οὗ τὸ εἰκοσάεδρον ἀναγέγραπται. καὶ ἐστὶν ἡ AB ἡ τῆς σφαίρας διάμετρος· ἡ $K\Lambda$ ἄρα ἐκ τοῦ κέντρου ἐστὶ τοῦ κύκλου, ἀφ' οὗ τὸ εἰκοσάεδρον ἀναγέγραπται· ἡ $K\Lambda$ ἄρα ἐξαγώνου ἐστὶ πλευρά τοῦ εἰρημένου κύκλου. καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος σύγκειται ἐκ τε τῆς τοῦ ἐξαγώνου καὶ δύο τῶν τοῦ δεκαγώνου τῶν εἰς τὸν εἰρημένον κύκλον ἐγγραφομένων, καὶ ἐστὶν ἡ μὲν AB ἡ τῆς σφαίρας διάμετρος, ἡ δὲ $K\Lambda$ ἐξαγώνου πλευρά, καὶ ἴση ἡ AK τῇ AB , ἑκατέρα ἄρα τῶν AK , AB δεκαγώνου ἐστὶ πλευρά τοῦ ἐγγραφομένου εἰς τὸν κύκλον, ἀφ' οὗ τὸ εἰκοσάεδρον ἀναγέγραπται. καὶ ἐπεὶ δεκαγώνου μὲν ἡ AB , ἐξαγώνου

times AD . And as BA (is) to AD , so the (square) on BA (is) to the (square) on AF [Def. 5.9]. For triangle AFB is equiangular to triangle AFD [Prop. 6.8]. Thus, the (square) on BA is one and a half times the (square) on AF . And the square on the diameter of the sphere is also one and a half times the (square) on the side of the pyramid [Prop. 13.13]. And AB is the diameter of the sphere. Thus, AF is equal to the side of the pyramid.

Again, since AD is double DB , AB is thus triple BD . And as AB (is) to BD , so the (square) on AB (is) to the (square) on BF [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is three times the (square) on BF . And the square on the diameter of the sphere is also three times the (square) on the side of the cube [Prop. 13.15]. And AB is the diameter of the sphere. Thus, BF is the side of the cube.

And since AC is equal to CB , AB is thus double BC . And as AB (is) to BC , so the (square) on AB (is) to the (square) on BE [Prop. 6.8, Def. 5.9]. Thus, the (square) on AB is double the (square) on BE . And the square on the diameter of the sphere is also double the (square) on the side of the octagon [Prop. 13.14]. And AB is the diameter of the given sphere. Thus, BE is the side of the octagon.

So let AG have been drawn from point A at right-angles to the straight-line AB . And let AG be made equal to AB . And let GC have been joined. And let HK have been drawn from H , perpendicular to AB . And since GA is double AC . For GA (is) equal to AB . And as GA (is) to AC , so HK (is) to KC [Prop. 6.4]. HK (is) thus also double KC . Thus, the (square) on HK is four times the (square) on KC . Thus, the (sum of the squares) on HK and KC , which is the (square) on HC [Prop. 1.47], is five times the (square) on KC . And HC (is) equal to CB . Thus, the (square) on BC (is) five times the (square) on CK . And since AB is double CB , of which AD is double DB , the remainder BD is thus double the remainder DC . BC (is) thus triple CD . The (square) on BC (is) thus nine times the (square) on CD . And the (square) on BC (is) five times the (square) on CK . Thus, the (square) on CK (is) greater than the (square) on CD . CK is thus greater than CD . Let CL be made equal to CK . And let LM have been drawn from L at right-angles to AB . And let MB have been joined. And since the (square) on BC is five times the (square) on CK , and AB is double BC , and KL double CK , the (square) on AB is thus five times the (square) on KL . And the square on the diameter of the sphere is also five times the (square) on the radius of the circle from which the icosahedron has been described [Prop. 13.16 corr.]. And AB is the diameter of the sphere. Thus, KL is the radius of the circle from

δὲ ἡ $ΜΑ$ ἴση γὰρ ἐστὶ τῇ $ΚΛ$, ἐπεὶ καὶ τῇ $ΘΚ$ ἴσον γὰρ ἀπέχουσιν ἀπὸ τοῦ κέντρου· καὶ ἐστὶν ἑκατέρα τῶν $ΘΚ$, $ΚΛ$ διπλασίων τῆς $ΚΓ$ · πενταγώνου ἄρα ἐστὶν ἡ $ΜΒ$. ἡ δὲ τοῦ πενταγώνου ἐστὶν ἡ τοῦ εἰκοσαέδρου· εἰκοσαέδρου ἄρα ἐστὶν ἡ $ΜΒ$.

Καὶ ἐπεὶ ἡ ZB κύβου ἐστὶ πλευρά, τετμήσθω ἄκρον καὶ μέσον λόγον κατὰ τὸ N , καὶ ἔστω μείζων τμήμα τὸ NB · ἡ NB ἄρα δωδεκαέδρου ἐστὶ πλευρά.

Καὶ ἐπεὶ ἡ τῆς σφαίρας διάμετρος ἐδείχθη τῆς μὲν AZ πλευρᾶς τῆς πυραμίδος δυνάμει ἡμιολία, τῆς δὲ τοῦ ὀκταέδρου τῆς BE δυνάμει διπλασίων, τῆς δὲ τοῦ κύβου τῆς ZB δυνάμει τριπλασίων, οἷων ἄρα ἡ τῆς σφαίρας διάμετρος δυνάμει ἕξ, τοιούτων ἡ μὲν τῆς πυραμίδος τεσσάρων, ἡ δὲ τοῦ ὀκταέδρου τριῶν, ἡ δὲ τοῦ κύβου δύο. ἡ μὲν ἄρα τῆς πυραμίδος πλευρὰ τῆς μὲν τοῦ ὀκταέδρου πλευρᾶς δυνάμει ἐστὶν ἐπίτριστος, τῆς δὲ τοῦ κύβου δυνάμει διπλῇ, ἡ δὲ τοῦ ὀκταέδρου τῆς τοῦ κύβου δυνάμει ἡμιολία. αἱ μὲν οὖν εἰρημέναι τῶν τριῶν σχημάτων πλευραί, λέγω δὴ πυραμίδος καὶ ὀκταέδρου καὶ κύβου, πρὸς ἀλλήλας εἰσὶν ἐν λόγοις ῥητοῖς. αἱ δὲ λοιπαὶ δύο, λέγω δὴ ἡ τε τοῦ εἰκοσαέδρου καὶ ἡ τοῦ δωδεκαέδρου, οὔτε πρὸς ἀλλήλας οὔτε πρὸς τὰς προειρημένους εἰσὶν ἐν λόγοις ῥητοῖς· ἄλογοι γὰρ εἰσιν, ἡ μὲν ἐλάττω, ἡ δὲ ἀποτομή.

Ὅτι μείζων ἐστὶν ἡ τοῦ εἰκοσαέδρου πλευρὰ ἡ $ΜΒ$ τῆς τοῦ δωδεκαέδρου τῆς NB , δείξομεν οὕτως.

Ἐπεὶ γὰρ ἰσογώνιον ἐστὶ τὸ $ZΔB$ τρίγωνον τῷ ZAB τριγώνῳ, ἀνάλογόν ἐστιν ὡς ἡ $ΔB$ πρὸς τὴν BZ , οὕτως ἡ BZ πρὸς τὴν BA . καὶ ἐπεὶ τρεῖς εὐθεῖαι ἀνάλογόν εἰσιν, ἐστὶν ὡς ἡ πρώτη πρὸς τὴν τρίτην, οὕτως τὸ ἀπὸ τῆς πρώτης πρὸς τὸ ἀπὸ τῆς δευτέρας· ἐστὶν ἄρα ὡς ἡ $ΔB$ πρὸς τὴν BA , οὕτως τὸ ἀπὸ τῆς $ΔB$ πρὸς τὸ ἀπὸ τῆς BZ · ἀνάπαλιν ἄρα ὡς ἡ AB πρὸς τὴν $BΔ$, οὕτως τὸ ἀπὸ τῆς ZB πρὸς τὸ ἀπὸ τῆς $BΔ$. τριπλῇ δὲ ἡ AB τῆς $BΔ$ · τριπλάσιον ἄρα τὸ ἀπὸ τῆς ZB τοῦ ἀπὸ τῆς $BΔ$. ἐστὶ δὲ καὶ τὸ ἀπὸ τῆς $AΔ$ τοῦ ἀπὸ τῆς $ΔB$ τετραπλάσιον· διπλῇ γὰρ ἡ $AΔ$ τῆς $ΔB$ · μείζων ἄρα τὸ ἀπὸ τῆς $AΔ$ τοῦ ἀπὸ τῆς ZB · μείζων ἄρα ἡ $AΔ$ τῆς ZB · πολλῶ ἄρα ἡ $ΑΔ$ τῆς ZB μείζων ἐστὶν. καὶ τῆς μὲν $ΑΔ$ ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζων τμήμα ἐστὶν ἡ $ΚΛ$, ἐπειδὴ περ ἡ μὲν $ΑΚ$ ἐξαγώνου ἐστὶν, ἡ δὲ $ΚΑ$ δεκαγώνου· τῆς δὲ ZB ἄκρον καὶ μέσον λόγον τεμνομένης τὸ μείζων τμήμα ἐστὶν ἡ NB · μείζων ἄρα ἡ $ΚΛ$ τῆς NB . ἴση δὲ ἡ $ΚΛ$ τῇ $ΑΜ$ · μείζων ἄρα ἡ $ΑΜ$ τῆς NB [τῆς δὲ $ΑΜ$ μείζων ἐστὶν ἡ $ΜΒ$]. πολλῶ ἄρα ἡ $ΜΒ$ πλευρὰ οὔσα τοῦ εἰκοσαέδρου μείζων ἐστὶ τῆς NB πλευρᾶς οὔσης τοῦ δωδεκαέδρου· ὅπερ ἔδει δεῖξαι.

which the icosahedron has been described. Thus, KL is (the side) of the hexagon (inscribed) in the aforementioned circle [Prop. 4.15 corr.]. And since the diameter of the sphere is composed of (the side) of the hexagon, and two of (the sides) of the decagon, inscribed in the aforementioned circle, and AB is the diameter of the sphere, and KL the side of the hexagon, and AK (is) equal to LB , thus AK and LB are each sides of the decagon inscribed in the circle from which the icosahedron has been described. And since LB is (the side) of the decagon. And ML (is the side) of the hexagon—for (it is) equal to KL , since (it is) also (equal) to HK , for they are equally far from the center. And HK and KL are each double KC . MB is thus (the side) of the pentagon (inscribed in the circle) [Props. 13.10, 1.47]. And (the side) of the pentagon is (the side) of the icosahedron [Prop. 13.16]. Thus, MB is (the side) of the icosahedron.

And since FB is the side of the cube, let it have been cut in extreme and mean ratio at N , and let NB be the greater piece. Thus, NB is the side of the dodecahedron [Prop. 13.17 corr.].

And since the (square) on the diameter of the sphere was shown (to be) one and a half times the square on the side, AF , of the pyramid, and twice the square on (the side), BE , of the octagon, and three times the square on (the side), FB , of the cube, thus, of whatever (parts) the (square) on the diameter of the sphere (makes) six, of such (parts) the (square) on (the side) of the pyramid (makes) four, and (the square) on (the side) of the octagon three, and (the square) on (the side) of the cube two. Thus, the (square) on the side of the pyramid is one and a third times the square on the side of the octagon, and double the square on (the side) of the cube. And the (square) on (the side) of the octahedron is one and a half times the square on (the side) of the cube. Therefore, the aforementioned sides of the three figures—I mean, of the pyramid, and of the octahedron, and of the cube—are in rational ratios to one another. And (the sides of) the remaining two (figures)—I mean, of the icosahedron, and of the dodecahedron—are neither in rational ratios to one another, nor to the (sides) of the aforementioned (three figures). For they are irrational (straight-lines): (namely), a minor [Prop. 13.16], and an apotome [Prop. 13.17].

(And), we can show that the side, MB , of the icosahedron is greater than the (side), NB , or the dodecahedron, as follows.

For, since triangle FDB is equiangular to triangle FAB [Prop. 6.8], proportionally, as DB is to BF , so BF (is) to BA [Prop. 6.4]. And since three straight-lines are (continually) proportional, as the first (is) to the third,

so the (square) on the first (is) to the (square) on the second [Def. 5.9, Prop. 6.20 corr.]. Thus, as DB is to BA , so the (square) on DB (is) to the (square) on BF . Thus, inversely, as AB (is) to BD , so the (square) on FB (is) to the (square) on BD . And AB (is) triple BD . Thus, the (square) on FB (is) three times the (square) on BD . And the (square) on AD is also four times the (square) on DB . For AD (is) double DB . Thus, the (square) on AD (is) greater than the (square) on FB . Thus, AD (is) greater than FB . Thus, AL is much greater than FB . And KL is the greater piece of AL , which is cut in extreme and mean ratio—inasmuch as LK is (the side) of the hexagon, and KA (the side) of the decagon [Prop. 13.9]. And NB is the greater piece of FB , which is cut in extreme and mean ratio. Thus, KL (is) greater than NB . And KL (is) equal to LM . Thus, LM (is) greater than NB [and MB is greater than LM]. Thus, MB , which is (the side) of the icosahedron, is much greater than NB , which is (the side) of the dodecahedron. (Which is) the very thing it was required to show.

† If the radius of the given sphere is unity then the sides of the pyramid (i.e., tetrahedron), octahedron, cube, icosahedron, and dodecahedron, respectively, satisfy the following inequality: $\sqrt{8/3} > \sqrt{2} > \sqrt{4/3} > (1/\sqrt{5}) \sqrt{10 - 2\sqrt{5}} > (1/3) (\sqrt{15} - \sqrt{3})$.

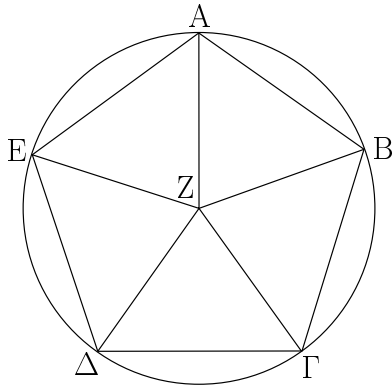
Λέγω δὴ, ὅτι παρὰ τὰ εἰρημένα πέντε σχήματα οὐ συσταθήσεται ἕτερον σχῆμα περιεχόμενον ὑπὸ ἰσοπλευρῶν τε καὶ ἰσογωνίων ἴσων ἀλλήλοις.

Ὑπὸ μὲν γὰρ δύο τριγώνων ἢ ὅλως ἐπιπέδων στερεὰ γωνία οὐ συνίσταται. ὑπὸ δὲ τριῶν τριγώνων ἢ τῆς πυραμίδος, ὑπὸ δὲ τεσσάρων ἢ τοῦ ὀκταέδρου, ὑπὸ δὲ πέντε ἢ τοῦ εἰκοσαέδρου· ὑπὸ δὲ ἑξ τριγώνων ἰσοπλευρῶν τε καὶ ἰσογωνίων πρὸς ἐνὶ σημείῳ συνισταμένων οὐκ ἔσται στερεὰ γωνία· οὕσης γὰρ τῆς τοῦ ἰσοπλευροῦ τριγώνου γωνίας διμοίρου ὀρθῆς ἔσονται αἱ ἑξ τέσσαρσιν ὀρθαῖς ἴσαι· ὅπερ ἀδύνατον· ἅπαντα γὰρ στερεὰ γωνία ὑπὸ ἐλασσόνων ἢ τεσσάρων ὀρθῶν περέχεται. διὰ τὰ αὐτὰ δὲ οὐδὲ ὑπὸ πλειόνων ἢ ἑξ γωνιῶν ἐπιπέδων στερεὰ γωνία συνίσταται. ὑπὸ δὲ τετραγώνων τριῶν ἢ τοῦ κύβου γωνία περιέχεται· ὑπὸ δὲ τεσσάρων ἀδύνατον· ἔσονται γὰρ πάλιν τέσσαρες ὀρθαί. ὑπὸ δὲ πενταγώνων ἰσοπλευρῶν καὶ ἰσογωνίων, ὑπὸ μὲν τριῶν ἢ τοῦ δωδεκαέδρου· ὑπὸ δὲ τεσσάρων ἀδύνατον· οὕσης γὰρ τῆς τοῦ πενταγώνου ἰσοπλευροῦ γωνίας ὀρθῆς καὶ πέμπτου, ἔσονται αἱ τέσσαρες γωνίαι τεσσάρων ὀρθῶν μείζους· ὅπερ ἀδύνατον. οὐδὲ μὴν ὑπὸ πολυγώνων ἐτέρων σχημάτων περισχεθήσεται στερεὰ γωνία διὰ τὸ αὐτὸ ἄτοπον.

Οὐκ ἄρα παρὰ τὰ εἰρημένα πέντε σχήματα ἕτερον σχῆμα στερεὸν συσταθήσεται ὑπὸ ἰσοπλευρῶν τε καὶ ἰσογωνίων περιεχόμενον· ὅπερ εἶδει δεῖξαι.

So, I say that, beside the five aforementioned figures, no other (solid) figure can be constructed (which is) contained by equilateral and equiangular (planes), equal to one another.

For a solid angle cannot be constructed from two triangles, or indeed (two) planes (of any sort) [Def. 11.11]. And (the solid angle) of the pyramid (is constructed) from three (equiangular) triangles, and (that) of the octahedron from four (triangles), and (that) of the icosahedron from (five) triangles. And a solid angle cannot be (made) from six equilateral and equiangular triangles set up together at one point. For, since the angles of a equilateral triangle are (each) two-thirds of a right-angle, the (sum of the) six (plane) angles (containing the solid angle) will be four right-angles. The very thing (is) impossible. For every solid angle is contained by (plane angles whose sum is) less than four right-angles [Prop. 11.21]. So, for the same (reasons), a solid angle cannot be constructed from more than six plane angles (equal to two-thirds of a right-angle) either. And the (solid) angle of a cube is contained by three squares. And (a solid angle contained) by four (squares is) impossible. For, again, the (sum of the plane angles containing the solid angle) will be four right-angles. And (the solid angle) of a dodecahedron (is contained) by three equilateral and equiangular pentagons. And (a solid angle contained) by four



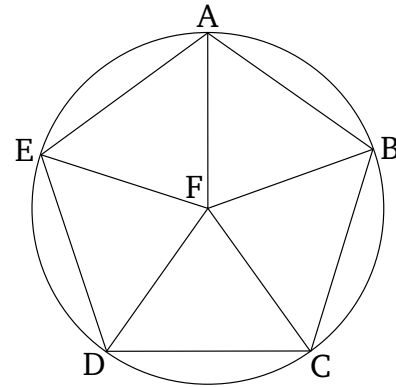
Λήμμα.

Ὅτι δὲ ἡ τοῦ ἰσοπλεύρου καὶ ἰσογωνίου πενταγώνου γωνία ὀρθή ἐστι καὶ πέμπτου, οὕτω δεικτέον.

Ἐστω γὰρ πεντάγωνον ἰσόπλευρον καὶ ἰσογώνιον τὸ ΑΒΓΔΕ, καὶ περιγεγράφθω περὶ αὐτὸ κύκλος ὁ ΑΒΓΔΕ, καὶ εἰληφθῇ αὐτοῦ τὸ κέντρον τὸ Ζ, καὶ ἐπεζεύχθωσαν αἱ ΖΑ, ΖΒ, ΖΓ, ΖΔ, ΖΕ. δίχα ἄρα τέμνουσι τὰς πρὸς τοῖς Α, Β, Γ, Δ, Ε τοῦ πενταγώνου γωνίας. καὶ ἐπεὶ αἱ πρὸς τῷ Ζ πέντε γωνίαι τέσσαρσιν ὀρθαῖς ἴσαι εἰσὶ καὶ εἰσιν ἴσαι, μία ἄρα αὐτῶν, ὡς ἡ ὑπὸ ΑΖΒ, μιᾶς ὀρθῆς ἐστὶ παρὰ πέμπτου· λοιπαὶ ἄρα αἱ ὑπὸ ΖΑΒ, ΑΒΖ μιᾶς εἰσιν ὀρθῆς καὶ πέμπτου. ἴση δὲ ἡ ὑπὸ ΖΑΒ τῇ ὑπὸ ΖΒΓ· καὶ ὅλη ἄρα ἡ ὑπὸ ΑΒΓ τοῦ πενταγώνου γωνία μιᾶς ἐστὶν ὀρθῆς καὶ πέμπτου· ὅπερ ἔδει δεῖξαι.

(equiangular pentagons is) impossible. For, the angle of an equilateral pentagon being one and one-fifth of right-angle, four (such) angles will be greater (in sum) than four right-angles. The very thing (is) impossible. And, on account of the same absurdity, a solid angle cannot be constructed from any other (equiangular) polygonal figures either.

Thus, beside the five aforementioned figures, no other solid figure can be constructed (which is) contained by equilateral and equiangular (planes). (Which is) the very thing it was required to show.



Lemma

It can be shown that the angle of an equilateral and equiangular pentagon is one and one-fifth of a right-angle, as follows.

For let $ABCDE$ be an equilateral and equiangular pentagon, and let the circle $ABCDE$ have been circumscribed about it [Prop. 4.14]. And let its center, F , have been found [Prop. 3.1]. And let FA , FB , FC , FD , and FE have been joined. Thus, they cut the angles of the pentagon in half at (points) A , B , C , D , and E [Prop. 1.4]. And since the five angles at F are equal (in sum) to four right-angles, and are also equal (to one another), (any) one of them, like AFB , is thus one less a fifth of a right-angle. Thus, the (sum of the) remaining (angles in triangle ABF), FAB and ABF , is one plus a fifth of a right-angle [Prop. 1.32]. And FAB (is) equal to FBC . Thus, the whole angle, ABC , of the pentagon is also one and one-fifth of a right-angle. (Which is) the very thing it was required to show.